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Suboptimal Control of Nonlinear Object: Problem of Keeping Tabs on Reference Trajectory

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Abstract: An optimal control problem is formulated for a class of nonlinear systems which can be presented by system with linear structure and state-dependent coefficients (SDC). The system being under the influence of uncontrollable disturbance is supposed. The linearity of the transformed system structure and the quadratic functional make it possible to pass over from the Hamilton-Jacoby-Bellman equation (HJB) to the state dependent Riccati equation (SDRE) upon the control synthesis. In thus paper the optimal control problem by nonlinear system in a task of Keeping Tabs on Reference Trajectory we decide in a key of differential game. The presented example illustrates the application of the proposed control method.

Keywords: Nonlinear Systems; Optimal Control; Differential Game; Riccati Equation

1. INTRODUCTION

Methods for the synthesis of optimal linear systems with quadratic cost functional are well known and have been successfully used in the design of relevant facilities. In various formulations of control problems objects can be both stationary and non-stationary. For problems of controlling objects in a given time interval can be specified condition at the right end. For the problem of keeping tabs of an objects on reference trajectory synthesis of optimal linear regulators available only for problems with the specified ending time of transitional process ($t \in [0, T]$). For linear time-invariant systems in the case when the desired output is constant vector, and time tends to infinity, all the results are approximate and valid for very large values of the transition process. To date, "there is no theory that considers the limiting case $T = \infty$ " [2].

Infinite-time horizon nonlinear optimal control (ITHNOC) presents a "viable option" [1] for synthesizing controllers for nonlinear systems by making a state-input tradeoff, where the objective is to minimize the cost given by a performance index. In the least few years, algorithms using state-dependent Riccati equation (SDRE) have been proposed, in the main part, for solving stabilization nonlinear problem. This method, first proposed by Pearson [7] and later expended by Wernli & Cook [8], was independently studied by Mracek & Clouter [6]. The method entails factorization of the nonlinear dynamics into the state vector and the product of a matrix-valued function that depends on the state itself (SDC). The theoretical contribution in Mrasek, Clouter has initiated an increasing use of SDRE techniques in a wide variety of nonlinear control applications [3]. The method seems to work well in applications but between any non-decided problems we have the nonlinear object's optimal control in task of Keeping Tabs on Reference Trajectory (KTRT).

In this paper an optimal control problem of KTRT is formulated for a class of nonlinear systems being under the influence of uncontrollable disturbance for which there exists a representation transforming the original system into a system with a linear structure and state-dependent coefficients. Perturbations in dynamic systems may arise from unknown disturbance signals, model uncertainty, component ageing, etc. Bounds on the perturbations variables are typically known and may be used to obtain, under certain conditions, ultimate bounds on the perturbed system trajectories.

Considering disturbance as actions of some player counteracting successful performance of a control problem, we will consider a task in a key of differential game with two players. Examining the problem of synthesis of the control law as the differential game of two players we introduce the quadratic functional. The linearity of the transformed system structure and the quadratic functional make it possible to pass over from the Hamilton-Jacoby-Bellman equation (HJB) to the state-dependent Riccati equation (SDRE) and special vector with the state-dependent coefficients and given trajectories upon the control synthesis. Existing of this special vector and the arbitrary unit vector witch provide of system's stability difference main result this research from a known results using of SDRE-method.

Note, just like in the case of linear systems synthesized infinite-time horizon control for nonlinear objects in the problem of keeping tabs on the reference trajectory is only suboptimal and valid for very large values of the transition process.

This paper is organized as follows. In Section 2 we derive the optimal full-state-feedback solution of a deterministic affine nonlinear system. In Section 3 we discuss the stability characteristic of the derived optimal control. In Section 4 illustrates the result with an example.

2. NONLINEAR OPTIMAL REGULATION

2.1 Statement of the problem

Consider the continuous-time, deterministic, full-state feedback, infinite-time horizon nonlinear optimal regulation problem, where the system is nonlinear in the state, and affine in the input, represented in the form

$$\dot{x}(t) = f(x) + g_1(x)w(t) + g_2(x)u(t), \; x(t_0) = x_0, \quad y(t) = Cx(t) . \tag{1}$$

Here $x(t) \in R^n$ – state of systems; $x \in \Omega_x$, $X_0 \in \Omega_x$ – domain of possible initial conditions of system; $y \in R^m$, $m \le n$ – system exit; $u \in R^r$ – control; $w \in R^k$ – disturbance variable; $f(x),\ g_1(x),\ g_2(x)$ – matrixes are real and continuous. It is supposed that at all (x) system (1) is controllable and observable [1], $t \in R^+$. Besides, functions $f(x),\ g_1(x),\ g_2(x)$ we will assume to be rather smooth (C_∞) that through any $(0, x_0) \in t \times \Omega_x$ passed one and only one decision (1) $x(t,0,x_0)$ and there would be the unique corresponding exit of system $y(t) = Cx(t, x_0)$.

The disturbance variable $w(t)$ are assumed to be bounded as follows: $|w_i(t)| \le \sigma_i(x(t)),\ i = 1,...,k,\ t \ge 0$, where $\sigma_i(x(t)) \ge 0$ for all $x(t) \in \Omega_x$. This condition we will write down in a look

$$|w(t)| \prec \sigma(x(t)),\ \forall\ t \ge 0. \quad (2)$$

Let us suppose that $z(t) \in R^m$ is desire (given) trajectory of the system exit $y(t)$ and

$$\dot{z}(t) = G(t)z(t),\ z(0) = z_0, \quad (3)$$

where matrix $G(t)$ is real and continuous.

Then in general case we'll have:

$$\varepsilon(t) = y(t) - z(t). \quad (4)$$

Considering disturbance $w(t)$ as actions of some player counteracting successful performance of a control problem, we will consider a task in a key of differential game with two players *U* and *W*.

The organization of controls $u(t)$ and $w(t)$ with use of a principle of feedback is supposed. Examining the problem of synthesis of the control law as the differential game of two players *U* and *W* we introduce the functional

$$J(x,u,w) = = \min_u \max_w \frac{1}{2} \int_0^\infty \left\{ \varepsilon^T(t) Q \varepsilon(t) + u^T(t) R u(t) - w^T(t) P(\sigma(x(t))) w(t) \right\} dt. \quad (5)$$

Here, the matrix Q can be positive semi-definite; the matrices R, P are positive definite. Positively definite matrix $P(\sigma(x(t)))$ is created so that to consider the greatest possible disturbance of a look (2) operating on system. Additional requirements to values of parameters of these matrixes will be defined further.

2.2 The Hamilton-Jacobi-Bellman equation

From the outset of Section 2.1, the ITHNOC problem on the set Ω_x is to minimize (5) with respect $u \in U$ and maximize (5) with respect $w \in W$ [1]. In particular, a solution to this problem is said to exist on the set Ω_x if there exist a finite continuous positive-definite value function $V : \Omega_x \to R^+$ defined by

$$V(x) = \inf_{u \in U} \sup_{w \in W} J(x,u,w) \quad (6)$$

for all $x \in \Omega_x$, the infinum-suprenum being the given set of admissible controls U and W. Ideally, the desire value function *V* is a stationary solution to the Cauchy problem for the associated dynamic programming equation, represented by the first-order nonlinear Hamilton-Jacobi partial differential equation (PDE)

$$\frac{\partial V(x)}{\partial t} + H\left\{ x, u, w, \frac{\partial V(x)}{\partial x(t)} \right\} = 0,$$

where *H* is Hamiltonian, and $\partial V(x)/\partial x$ denotes the row-vector of partial derivatives of $V(x)$ with respect to x. Applying Pontryagin principle to the nonlinear optimum controls problem (1)-(5) gives the Hamiltonian

$$H = \inf_u \sup_w \left\{ \frac{\partial V(x)}{\partial x(t)} \left[f(x) + g_1(x)w(t) + g_2(x)u(t) \right] + \frac{1}{2}\left[\left[y(t) - z(t) \right]^T Q \left[y(t) - z(t) \right] + u^T(t) R u(t) - w^T(t) P(\sigma(x(t))) w^T(t) \right] \right\}. \quad (7)$$

For infinite-time formalization, $V(x)$ is assumed stationary ($\partial V(x)/\partial t = 0$) such that the HJB equation becomes

$$\inf_u \sup_w \left\{ \frac{\partial V(x)}{\partial x(t)} \left[f(x) + g_1(x)w(t) + g_2(x)u(t) \right] + \frac{1}{2}\left[\left[y(t) - z(t) \right]^T Q \left[y(t) - z(t) \right] + u^T(t) R u(t) - w^T(t) P(\sigma(x(t))) w(t) \right] \right\} = 0 \quad (8)$$

with boundary conditions $V(0) = 0$.

The optimal controls w and u satisfies

$$\frac{\partial H}{\partial w} = -w^T(t) P(\sigma(x(t))) + \frac{\partial V(x)}{\partial x(t)} g_1(x) = 0,$$

$$\partial^2 H / \partial w^2 = -P(\sigma(x(t))) < 0,$$

$$\frac{\partial H}{\partial u} = u^T(t) R + \frac{\partial V(x)}{\partial x(t)} g_2(x) = 0,\ \ \partial^2 H / \partial u^2 = R > 0,$$

$$w(t) = P^{-1}(\sigma(x(t))) g_1^T(x) \left(\frac{\partial V(x)}{\partial x(t)} \right)^T, \quad (9)$$

$$u(t) = -R^{-1} g_2^T(x) \left(\frac{\partial V(x)}{\partial x(t)} \right)^T, \quad (10)$$

where $\partial V(x)/\partial x$ is solution of the follows equation

$$\frac{\partial V(x)}{\partial x(t)} f(x) + \frac{1}{2}\left[y(t) - z(t) \right]^T Q \left[y(t) - z(t) \right] + \frac{1}{2} \frac{\partial V(x)}{\partial x(t)} \left[g_1(x) P^{-1}(\sigma(x(t))) g_1^T(x) - g_2(x) R^{-1} g_2^T(x) \right] \left(\frac{\partial V(x)}{\partial x(t)} \right)^T = 0.$$

2.3 SDRE Solution

The SDRE methodology uses extended linearization, also known as state dependent coefficient (SDC) parameterization, as the key design concept in formulation the nonlinear optimal control problem [3].

Condition 1. Vector-valued function $f(x)$ is continuously of x on Ω_x, such that $f(0) = 0$.

Condition 2. Continuously vector-valued matrices $g_1(x),\ g_2(x)$ of x on Ω_x, such that $g_1(x) \ne 0$, $g_2(x) \ne 0$, $\forall x$.

Under conditions 1 and 2, using extended linearization, the nonlinear system (1) can be represented in SDC form

$$\dot{x}(t) = A(x)x(t) + g_1(x)w(t) + g_2(x)u(t),\ x(t_0) = x_0, \quad y(t) = Cx(t), \quad (12)$$

which has a linear structure and with SDC matrix $A(x)$ such that $f(x) = A(x)x(t)$.

HJB equation (11) can be rewrite in form

$$\frac{1}{2} \frac{\partial V(x)}{\partial x(t)} A(x)x(t) + \frac{1}{2} x^T(t) A^T(x) \left(\frac{\partial V(x)}{\partial x(t)} \right)^T + \frac{1}{2} \frac{\partial V(x)}{\partial x(t)} \left[g_1(x) P^{-1}(\sigma(x(t))) g_1^T(x) - g_2(x) R^{-1} g_2^T(x) \right] \left(\frac{\partial V(x)}{\partial x(t)} \right)^T + \frac{1}{2} x^T(t) C^T Q C x(t) + \frac{1}{2} z^T(t) Q z(t) - x^T(t) C^T Q z(t) = 0. \quad (13)$$

Let us define $\left[\partial V(x)/\partial x \right]^T$ as

$$\left[\frac{\partial V(x)}{\partial x} \right]^T = S(x)x(t) + q(x). \quad (14)$$

Expression (13) becomes

$$\frac{1}{2}[S(x)x(t)+q(x)]^T A(x)x(t)+\frac{1}{2}x^T(t)A^T(x)[S(x)x(t)+q(x)]+$$
$$+\frac{1}{2}[S(x)x(t)+q(x)]^T\left[g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)-g_2(x)R^{-1}g_2^T(x)\right]\times$$
$$\times[S(x)x(t)+q(x)]+$$
$$+\frac{1}{2}x^T(t)C^TQCx(t)+\frac{1}{2}z^T(t)Qz(t)-x^T(t)C^TQz(t)=0.$$

From this equation we will have

$$x^T(t)\Big\{ S(x)A(x)+A^T(x)S(x)-$$
$$-S(x)\left[g_2(x)R^{-1}g_2^T(x)-g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)\right]S(x)+C^TQC\Big\}x(t)-$$
$$-q^T(x)\left[g_2(x)R^{-1}g_2^T(x)-g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)\right]q(x)+$$
$$+2q^T(t)\left\{A(x)-\left[g_2(x)R^{-1}g_2^T(x)-g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)\right]S(x)\right\}x(t)+$$
$$+z^T(t)Qz(t)-2z^T(t)QCx(t)=0.$$

Define of $S(x)$ and $q(x)$ such that

$$S(x)A(x)+A^T(x)S(x)+C^TQC-$$
$$-S(x)\left[g_2(x)R^{-1}g_2^T(x)-g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)\right]S(x)=0, \quad (15)$$

$$-q^T(x)\left[g_2(x)R^{-1}g_2^T(x)-g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)\right]q(x)+$$
$$+2q^T(t)\left\{A(x)-\left[g_2(x)R^{-1}g_2^T(x)-g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)\right]S(x)\right\}x(t)+$$
$$+z^T(t)Qz(t)-2z^T(t)QCx(t)=0. \quad (16)$$

Equation (15) is SDRE. To find the solution of equation (16) we use a version of Lemma by [9].

Lemma. Let $x\in R^n$ be a real vector, $\gamma(x)\in R^n$ and $\mu(x)\in R^n$ be real vector functions, and $\alpha(z,x)$ be a scalar real function. Then satisfies the condition

$$\gamma^T(x)\Pi(x)\gamma(x)+2\gamma^T(x)\mu(x)+\alpha(z,x)=0 \quad (17)$$

if and only if

$$\mu^T(x)\Pi^+(x)\mu(x)\ge\alpha(z,x) \quad (18)$$

where $\Pi^+(x)$ is the pseudo-inverse on $\Pi(x)$. In this case, the set of all solutions to (18) is represented by

$$\gamma(x)=-\Pi^+(x)\mu(x)+K^+(x)k(x)\beta(x) \quad (19)$$

where

$$\beta(x)=\left[\mu^T(x)\Pi^+(x)\mu(x)-\alpha(z,x)\right]^{1/2}, \quad (20)$$

and $K(x)$ is a square matrix, such that $\Pi(x)=\mathrm{K}(x)\mathrm{K}^T(x)$, and $k(x)$ is arbitrary unit vector.

Proof. The sufficiency is shown by direct substitution. Suppressing the arguments and substituting Eq. (19) into Eq. (17) gives

$$\left[\Pi^+(x)\mu(x)+\mathrm{K}^+(x)k(x)\beta(x)\right]^T\Pi(x)\left[\Pi^+(x)\mu(x)+\mathrm{K}^+(x)k(x)\beta(x)\right]-$$
$$-2\left[\Pi^+(x)\mu(x)+\mathrm{K}^+(x)k(x)\beta(x)\right]^T\mu(x)+\alpha(z,x)=0.$$

After some manipulations using the pseudo-inverse property,

$$\left[\Pi^+\right]^T\Pi\Pi^+=\Pi^+,\quad \left[\mathrm{K}^+\right]^T\Pi\mathrm{K}^+=\left\{\left[\mathrm{K}^+\right]^T K^T\right\}\left\{\left[\mathrm{K}^+\right]^T\mathrm{K}^T\right\}=\mathrm{I},$$

$$\left[\mathrm{K}^+k\right]^T\Pi\Pi^+=k^T\left\{\mathrm{K}\mathrm{K}^+\right\}^T\mathrm{K}\mathrm{K}^+\left[\mathrm{K}^+\right]^T=k^T\left[\mathrm{K}^+\right]^T$$

we obtain $-\mu^T(x)\Pi^+(x)\mu(x)+a(z,x)+\beta^T(x)\beta(x)=0$.

By substituting (20) into the above equation we establish the sufficiently.

To show the conditions are necessary replacement by $\gamma(x)=\mathrm{K}^+(x)\tilde{\gamma}(x)$, $\mu(x)=\mathrm{K}^T(x)\tilde{\mu}(x)$. Then equation (17) can be rewritten in the following form

$\tilde{\gamma}^T(x)\tilde{\gamma}(x)+2\tilde{\gamma}^T(x)\tilde{\mu}(x)+\alpha(z,x)=0$, or take into account (19)

$\tilde{\gamma}^T(x)\tilde{\gamma}(x)+2\tilde{\gamma}^T(x)\tilde{\mu}(x)+\tilde{\mu}^T(x)\tilde{\mu}(x)\le 0$.

Then $\left[\tilde{\gamma}(x)+\tilde{\mu}(x)\right]^T\left[\tilde{\gamma}(x)+\tilde{\mu}(x)\right]=\beta^2(x)=\tilde{\gamma}^T(x)\tilde{\gamma}(x)+a(z,x)$, and $\left[\tilde{\gamma}(x)+\tilde{\mu}(x)\right]=k(x)\beta(x)$,

$\gamma(x)=-\Pi^+(x)\mu(x)+\mathrm{K}^+(x)k(x)\beta(x)$. Consequently, (19) follows.

This completes the proof of Lemma.

Let us use this Lemma to find the solution of the equation (16) and lead follow designations:

$$q(x)=-\gamma(x), \quad (21)$$

$$\Pi(x)=\left[g_2(x)R^{-1}g_2^T(x)-g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)\right], \quad (22)$$

$$\mu(x)=$$
$$=\left\{A(x)-\left[g_2(x)R^{-1}g_2^T(x)-g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)\right]S(x)\right\}x(t), \quad (23)$$

$$\alpha(z,x)=z^T(t)Qz(t)-2z^T(t)QCx(t), \quad (24)$$

$$\beta(x)=\left[\mu^T(x)\Pi^+(x)\mu(x)-z^T(t)Qz(t)+2z^T(t)QCx(t)\right]^{1/2}. \quad (25)$$

Condition 3. Solution of equations (15) and (16) existence if for all $z\in\Omega_z$, $x\in\Omega_x$

i) matrix

$$\Pi(x)=\left[g_2(x)R^{-1}g_2^T(x)-g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)\right]\ge 0$$

is positive semi-definite and symmetric real matrix for all $x\in\Omega_x$;

ii) $\mu^T(x)\Pi^+(x)\mu(x)\ge z^T(t)Qz(t)-2z^T(t)QCx(t)$;

iii) $k(x)=sign\left[\mathrm{K}^+(x)\right]^T C^T\varepsilon(t)$.

So solution of equation (17) will have

$$q(x)=\Pi^+(x)\mu(x)-$$
$$-\mathrm{K}^+(x)k(x)\left[\mu^T(x)\Pi^+(x)\mu(x)-z^T(t)Qz(t)+2z^T(t)QCx(t)\right]^{1/2} \quad (26)$$

and controls (9) and (10)

$$w(t)=P^{-1}(\sigma(x(t)))g_1^T(x)\left[S(x)x(t)+q(x)\right], \quad (27)$$

$$u(t)=-R^{-1}g_2^T(x)\left[S(x)x(t)+q(x)\right], \quad (28)$$

where $S(x)$ is solution SDRE (15) and $q(x)$ is solution of the equation (26).

Optimal trajectory is defined as solution of equation

$$\dot{x}(t)=f(x)-\Pi(x)\left[S(x)x(t)+q(x)\right],$$
$$x(t_0)=x_0, \quad (29)$$
$$y(t)=Cx(t).$$

Theorem 1. Let's consider system (1), being under the influence of the uncontrollable disturbance satisfying to bound (2). Let's assume existence of an extended linearization transforming system into the system with linear structure with SDC matrices. Then controls at performance of the condition 3 and delivering a minimum to a functional (5) are defined by expressions (27) and (28) where the matrix $S(x)$ is the solution of the state dependent Riccati equation (15) and vector-function $q(x)$ is the solution of the equation (26).

Note. Let reference trajectory be $z(t) = 0$. In this case $q(x) = 0$. It is not difficult to obtain if we substitute (23) in (26).

3. ANALYSIS OF STABILITY

Let us use the second Lyapunov method for investigation of the system stability [1]. The function $\tilde{V}(x) = 2V(x)$ where $V(x)$ is the Bellman function for system (1) and cost function (5), is the Lyapunov function. Let $\omega_i\{|x|\}, i = 1,2,3$, be the scalar nondecreasing functions, such that $\omega_i(0) = 0,\ \omega_i\{|x|\} > 0$ for $x \neq 0$. The function $\tilde{V}(x)$ satisfies the condition $\omega_1\{|x|\} \le \tilde{V}(x) \le \omega_2\{|x|\}, \forall x$. It follows from the second Lyapunov theorem that if the following condition is proved:

$$\frac{\partial \tilde{V}(x)}{\partial x}\frac{dx(t)}{dt} \le -\omega_3\{|x|\}, \tag{30}$$

then the system is stable.
Considering (29), from the last expression in view of (30) we will have

$$\frac{\partial \tilde{V}(x)}{\partial x}\times 2\{f(x) - \Pi(x)[S(x)x(t) + q(x)]\} \le -\omega_3\{|x|\}. \tag{31}$$

Let us appoint $\omega_3\{|x|\}$ and $\omega_3\{|x|\} = \varepsilon^T(t)Q\varepsilon(t) + q^T(t)\Pi(x)q(t)$, where matrices Q and $\Pi(x) = g_2(x)R^{-1}g_2^T(x) - g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)$ are the positively semi-definite. After all transformations (31) takes the following form:

$$\begin{aligned} &x^T(t)\{S(x)A(x) + A^T(x)S(x) - S(x)\Pi(x)S(x)\}x(t) - \\ &- x^T(t)S(x)\Pi(x)S(x)x(t) + \\ &+ 2q^T(t)\{A(x) - \Pi(x)S(x)\}x(t) \le \\ &\le -\left[z^T(t)Qz(t) + q^T(t)\Pi(x)q(t)\right]. \end{aligned} \tag{32}$$

If the right hand side is moved to the left, in view of equations (15) and (16), inequality (32) is simplified,

$$-x^T(t)S(x)\Pi(x)S(x)x(t) < 0 \tag{33}$$

at all $\forall x \neq 0$. Therefore the matrix

$$\Pi(x) = \left[g_2(x)R^{-1}g_2^T(x) - g_1(x)P^{-1}(\sigma(x(t)))g_1^T(x)\right],\ \forall x \neq 0$$

is positively semi-definite at all $x \in \Omega_x$.
Now let us define arbitrary unit vector $k(x)$ which provides stability to system (29).
Using (22) and (23) rewriting (26)

$$q(x) = \Pi^+(x)[f(x) - \Pi S(x)x(t)] - \mathrm{K}^+(x)k(x)\beta(x). \tag{34}$$

Substituting (34) to (29) we obtain

$$\dot{x}(t) = -\mathrm{K}^+(x)k(x)\beta(x) . \tag{35}$$

Define the arbitrary unit vector $k(x)$. Let $V_k(\varepsilon)$ be the Lyapunov function such that

$$V_k(\varepsilon) = \varepsilon^T(t)\varepsilon(t) = [Cx(t) - z(t)]^T[Cx(t) - z(t)].$$

Than system (29) with respect to $\beta(x) \ge 0$ is stable if

$$\dot{V}(\varepsilon) = -\varepsilon^T(t)C\mathrm{K}^+(x)k(x)\beta(x) - \varepsilon^T(x)G(t)z(t) < 0, \quad x \neq 0.$$

Under Condition 3 we can see that next inequality

$$\begin{aligned} &\left|\varepsilon^T(t)C\mathrm{K}^+(x)k(x)\times\right. \\ &\times\left[x^T(t)\{A(x) - \Pi(x)S(x)\}^T\Pi^+\{A(x) - \Pi(x)S(x)\}x(t) + \right. \\ &\left.\left. + \varepsilon^T(t)Qz(t) + z^T(t)QCx(t)\right]^{1/2}\right| \ge \left|\varepsilon^T(x)G(t)z(t)\right| \end{aligned}$$

is well defined. From the last condition we get

$$k(x) = sign\left\{\left[\mathrm{K}^+(x)\right]^T C^T\varepsilon(t)\right\} \tag{36}$$

under conditions that

$$\mu^T(x)\Pi^+(x)\mu(x) \ge z^T(t)Qz(t) - 2z^T(t)QCx(t)$$

and $\left[\mu^T(x)\Pi^+(x)\mu(x) - z^T(t)Qz(t) + 2z^T(t)QCx(t)\right]^{1/2} \ge 0$.

To conclude the define condition of the system stability, it remain to note that the arbitrary unit vector $k(x)$ must be satisfy of condition (36). Thus, performance of a condition of positive definiteness of matrix $\Pi(x)$ that was supposed at synthesis of optimum controls (27) and (28), and vector $k(x)$ (36) provides stability to nonlinear system.
Theorem 2. Let us consider the system (1) being under the influence of uncontrollable disturbance satisfying to bound (2). Let us assume existence of an extended linearization transforming system into the system with linear structure with SDC matrices. Then controls at performance of Conditions 1 and 2 and delivering a minimum to a functional (5) are defined by expressions (27) and (28) where the matrix $S(x)$ is the solution of the state dependent Riccati equation (15) and vector-function $q(x)$ is the solution of the equation (16), provide to system stability if matrix $\Pi(x)$ (22) would be, at least, positively semi-definite at all $x \in \Omega_x$ and arbitrary unit vector $k(x)$ defined by (36).

4. EXAMPLE

4.1 Mathematical model of a large nuclear Pressurized Heavy Water Reactor (PHWR) and statement of the control problem

The behavior of a large nuclear reactor can be described with sufficient accuracy using a nodal model, like the spatial model of a 540 MWe PHWR [8] :

$$\begin{aligned} &\dot{P}_i = \frac{1}{l_i}\sum_{j=1}^{14}(a_{ji}P_j - a_{ij}P_i) + \left(\frac{-K_i(H_i - H_{i0}) - \dfrac{\bar{\sigma}_{X_i}X_i}{\Sigma_{ai}} - \beta}{l_i}\right)P_i + \lambda C_i, \\ &\dot{C}_i = -\lambda C_i + \frac{\beta}{l_i}P_i, \\ &\dot{I}_i = \gamma_I\Sigma_{fi}P_i - \lambda_I I_i, \\ &\dot{X}_i = \gamma_X\Sigma_{fi}P_i + \lambda_I I_i - (\lambda_X + \bar{\sigma}_{X_i}P_i)X_i, \\ &\dot{H}_i = -m_i q_i,\ (i = 1,2...,14), \end{aligned} \tag{37}$$

where P_i indicates the power level, C_i denotes effective one group delayed neutron precursor concentration, I_i denotes iodine concentration, X_i denotes xenon concentration and H_i is the water level of the zone control compartments in the i^{th} zone of the reactor. The description and the values of the several reactor parameters and the steady state values of zonal powers are as given in [9].
The coupling coefficients a_{ij} are dependent upon the geometry, material composition and the characteristic distance between the zones and is mathematically can be expressed as

$$a_{ij} = \begin{cases} \dfrac{D\nu l\psi_{ij}}{d_{ij}V}, & if \;\; i \neq j, \\ 0, & if \;\; i = j. \end{cases}$$ Physical data for the 540 MWe PHWR for all zones [5]

$l = 7{,}9\times10^{-4}$, *s;* $\beta = 7{,}5\times10^{-3}$; $\lambda = 9{,}1\times10^{-2}$, s^{-1};

$\lambda_I = 2{,}878\times10^{-5}$, s^{-1}; $\lambda_X = 2{,}1\times10^{-5}$, s^{-1};

$\Sigma_\alpha = 3{,}2341\times10^{-3}$, cm^{-1}; $\Sigma_f = 1{,}262\times10^{-3}$, cm^{-1};

$\gamma_I = 6{,}18\times10^{-2}$; $\gamma_X = 6\times10^{-3}$; $\sigma_X = 1{,}2\times10^{-18}$, cm^2;

$\Sigma_{eff} = 3{,}2\times10^{-17}$, *MJ;* $D = 0{,}9328$, *cm;*

$\nu = 3{,}19\times10^5$, *cm/s;* $K = -3{,}5\times10^{-5}$; $m = 2$; $H_0 = 100, cm$.

Since the reactivity in the core is directly proportional to water level in the zone control compartments (ZCC), in the above nodal core model, instead water level in the ZCCs, in the above nodal core model, instead is directly considered as input to the respective zones.

Let reference power of the reactor be a solution

$\dot{z}(t) = az(t) + P^*_{14},\; z(t_0) = z_0$, where $a = -0.2$ and $P* = const$.

Let us rewrite (40) in the form

$$\dot{x}(t) = f(x) + Gu(t),\;\; x(t_0) = x_0 \qquad (38)$$

where $x(t) = col[I\,X\,C\,H\,P]$.

Factorization of the nonlinear dynamics (38) into the state vector and the product of a matrix-valued function that depends on the state itself :

$$\dot{x}(t) = A(x)x(t) + Gu(t),\;\; x(t_0) = x_0,$$

where

$$A_i(x) = \begin{bmatrix} -\lambda_I & 0 & 0 & 0 & \gamma_I\Sigma_f \\ \lambda_I & -\lambda_X - \sigma_X P_i & 0 & 0 & \gamma_X\Sigma_f \\ 0 & 0 & -\lambda & 0 & \dfrac{\beta}{l} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{-l\sigma_X P_i}{\Sigma_a} & \lambda & \dfrac{-KP_i}{l} & \dfrac{KH_0 - \beta - a_{ij}}{l} \end{bmatrix},$$

$$G_i = col\begin{bmatrix}0 & 0 & 0 & -m & 0\end{bmatrix},\;\; (i = 1,2...,14).$$

The steady state values of zonal powers and zonal volumes are given in Table [5].

Zone	Power (MW)	Volume, m^3
1,6,8,13	132,75	14,7
2,7,9,14	135,99	14,7
3,10	123,30	17,6
4,11	98,55	8,8
5,12	123,30	17,6

Examining the problem of synthesis of the control law for object (37) we introduce the functional

$$J(x,u) = \frac{1}{2}\int_0^\infty \left\{\varepsilon^T(t)Q\,\varepsilon(t) + u^T(t)Ru(t)\right\}dt,$$

$Q = 250,\;\; R = 10000$.

The guaranteed control $u(t)$ is possible to present as

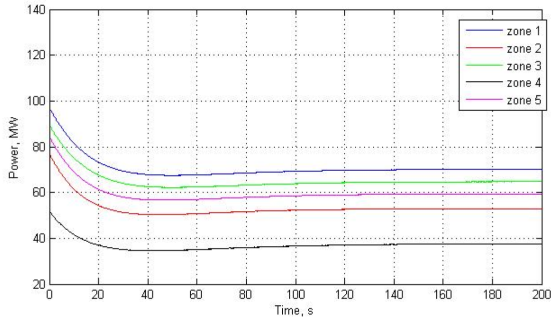
$$u(t) = -R^{-1}G^T\left[S^* x(t) + q(x)\right].$$

4.2 Result of modelling

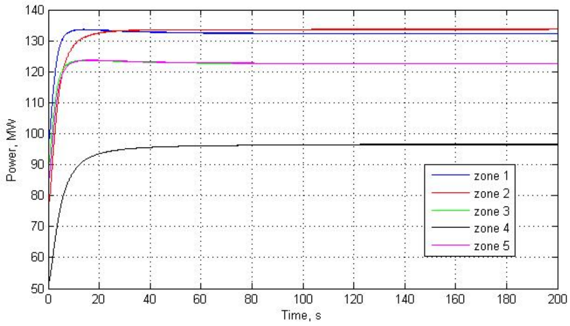
The simulation was performed in Simulink of MATLAB software with the following plant parameters (5 zones).

$$I_0 = \begin{bmatrix}359{,}742\\368{,}523\\334{,}133\\267{,}063\\334{,}133\end{bmatrix},\; X_0 = \begin{bmatrix}39{,}314\\39{,}382\\46{,}115\\23{,}922\\46{,}115\end{bmatrix},\; C_0 = 10^4\begin{bmatrix}1{,}385\\1{,}419\\1{,}286\\1{,}028\\1{,}286\end{bmatrix}.$$

Plot of power on 5 zone of reactor ($P^*_{14} = 280\,MW$):



Plot of power on 5 zone of reactor ($P^*_{14} = 610\,MW$):



Results of modeling of system with guaranteeing control well will be coordinated with schedules of transients from [5].

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