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GENERALIZED BANKRUPTCY PROBLEM

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GENERALIZED BANKRUPTCY PROBLEM[3]

A new approach to the bankruptcy problem is considered. We take into account the system of intercreditor debts. The traditional approach is a special case of our generalized model. Utilizing a cooperative game approach we prove the core existence of the corresponding game. The sequential nucleolus solution is applied.

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1. Introduction

Originally the bankruptcy problem assumes only one debtor and many creditors. This statement arose from O'Neill (1982) and Aumann, Mashler (1985). This approach is useful and it is applied to many related problems, e.g. radio resource management (Lucas-Estañ, et.al., (2012)), museum pass problem (Casas-Méndez et.al. (2011)), airport problem (Hu et.al. (2012)) and other cost and resource sharing problems. The traditional statement of the problem was extended to multi-issue bankruptcy problems (see e.g. Bergantiños et.al. (2010) and Calleja et.al.(2005)).

Traditional example assumes that, a firm goes bankrupt and not all creditors recover the whole debt. The bankruptcy causes creditor wealth reduction. We generalized this basic situation by adding claim connections into the set of agents. Each creditor has debts to other creditors. An interesting situation lies behind such a generalization: bankruptcy means that there is at least one agent, which cannot repay his debt in full, therefore some creditors may have problems with their own debt service and a new bankruptcy may appear. This is the reason why it is important to help big corporations or banks during the world economic crisis.

Intercreditor debts make the bankruptcy problem more complicated. The objective of this paper is to find a solution to this generalized bankruptcy problem.

In the next section the generalized bankruptcy model is described. Section 3 provides a cooperative game approach and utilizes a sequential nucleolus solution. Section 4 concludes.

2. The model

Agent 0 goes bankrupt. He has an estate $E_0 \in \mathbb{R}_+$ (an amount of perfectly divisible homogeneous commodity, or money). There are n creditors, $\mathcal{N} \coloneqq \{1, \dots, n\}$, with debt vector $c \in \mathbb{R}_+^n$. A bankruptcy problem is a couple $(E_0, c) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ such that $\sum_{i \in \mathcal{N}} c_i > E_0$. The bankruptcy problem set with $|\mathcal{N}| = n$ agents is denoted by $\mathcal{C}^n$. A comprehensive survey of this type problem can be found in (Thomson (2003)).

Our goal is to generalize the bankruptcy problem in the case where we have a set of agents, a set of claims and where everyone is a creditor and a borrower at the same time. Moreover, every agent has his own estate. Suppose, we have one bankrupt agent, who owes some sum of money to every other agent, but all agents have debts to each other. Every agent is rational, so they do not borrow more than the sum of their own estate, the debt of the bankrupt agent, and the total debt from other agents to him.

The generalized bankruptcy problem is a triple (E, c, C), where $E \in \mathbb{R}_+^{n+1}$ is a vector of estates of agents, $(E_0, c) \in \mathcal{C}^n$, C is a non-negative matrix of claims ($c_{ij} \geq 0$ is a sum that agent i owes to agent j) and nobody owes more than the sum he will have if all other agents return all the debts. In addition, each agent endowed with its own E_i. The following condition holds

$$E_i + c_i + \sum_{p\in\mathcal{N}} c_{pi} \geq \sum_{p\in\mathcal{N}} c_{ip},\ i \in \mathcal{N}.$$

The whole set of those triples is denoted as $\mathcal{L}^n$. When C is a zero matrix we get a classical bankruptcy game for any problem from $\mathcal{C}^n$ with little addition of agents estates.

We also define pure debt matrix $D = C - C^T$. It is antisymmetric matrix. The sum of all matrix D elements is equal to zero.

An agent *i* balance is denoted by

$$B_i \coloneqq E_i + c_i + \sum_{p\in\mathcal{N}} c_{pi} - \sum_{p\in\mathcal{N}} c_{ip} = E_i + c_i + \sum_{p\in\mathcal{N}} d_{pi} \geq 0.$$

Total balance is equal to

$$\sum_{i\in\mathcal{N}} B_i = \sum_{i\in\mathcal{N}} (E_i + c_i).$$

All creditors before agent 0 bankruptcy expect to have this sum.

3. Cooperative games

According to some solution of the bankruptcy problem agent i will receive f_i instead of c_i. Because $c_i - f_i \geq 0$ resulting balance

$$\Delta_i = E_i + f_i + \sum_{p\in\mathcal{N}} c_{pi} - \sum_{p\in\mathcal{N}} c_{ip}$$

takes a value of any sign. If this balance is negative, the agent becomes insolvent.

Consider a simple game “Solvent/Insolvent”.

$$v_{\pm}(S) = \begin{cases} 1, & if \ \sum_{i\in S} \Delta_i \geq 0 \\ 0, & otherwise. \end{cases}$$

If a coalition has a negative total resulting balance then it will go bankrupt. By coalescing, players cancel internal debts and unite their endowments, payments from agent 0, other players’ debts to the coalition and coalition’s debt.

Since $\sum_{i\in\mathcal{N}} \Delta_i = \sum_{i\in\mathcal{N}} (E_i + f_i) = \sum_{i\in\mathcal{N}} E_i + E_0 \geq 0$ we have $v_{\pm}(\mathcal{N}) = 1$, (we assume that sharing is effective $\sum_{i\in\mathcal{N}} f_i = E_0$). There is enough money in the economy to save the solvency of all agents, but some agents do not have incentives to offset of debts.

Theorem 1. The core of game "Solvent/Insolvent" is empty.

Proof: Non-emptiness of the core condition for simple games is the existence of a veto player (Nakamura (1979)). Suppose there is a veto player i^* then

$$v_{\pm}(\mathcal{N}\backslash\{i^*\}) = 0.$$

This implies

$$\sum_{i\in\mathcal{N}\backslash\{i^*\}} \Delta_i = \sum_{i\in\mathcal{N}} \Delta_i - \Delta_{i^*} = \sum_{i\in\mathcal{N}}(E_i + f_i) - \Delta_{i^*} = \sum_{i\in\mathcal{N}} E_i + E_0 - \Delta_{i^*} < 0,$$

$$\sum_{i\in\mathcal{N}} E_i + E_0 < \Delta_{i^*}.$$

The resulting balance of player i^* should be higher than the sum of all estates in the economy which is impossible, then $(\mathcal{N}, v_{\pm}) = \emptyset$. ■

This simple game shows a lack of acceptable solution. It happens because the payments from bankrupt f_i were defined out of the game. The approach in which sharing of the estate of a bankrupt agent was prior to allowing the other agents to negotiatedoesn't lead to good solution.

Let internalize bankrupt agent (player 0) into the game and divide E_0 taking into account the global claims structure (define $B_0 \coloneqq 0$). The natural way to do that lies behind the next reasoning. Agents unite, their mutual claims disappear, they take all estate E_0, but return to the counter-coalition the sum of money, which they wanted from the beginning. Now we can define a game, associated with any problem in $\mathcal{L}^n$. If player 0 has enough money to pay ($E_0 \geq \sum_{i\in\mathcal{N}} c_i$), then all players will get back their debts. Player *i* will have B_i, coalition $S \subset \mathcal{N}$ will have $\sum_{i\in S} B_i$. The case is more interesting when player 0 does not have enough money.

Let $\mathcal{N}^+ \coloneqq \mathcal{N} \cup \{0\}$.

Coalition S$\in \mathcal{N}$ pays off a debt. If coalition S has enough money a counter-coalition will have $\sum_{i\in\mathcal{N}^+\backslash S} B_i$. (The net payment $\sum_{i\in\mathcal{N}^+\backslash S}(B_i - E_i)$ can be negative).

Coalition S has $\sum_{i\in S} E_i - \sum_{i\in\mathcal{N}^+\backslash S}(B_i - E_i) = \sum_{i\in\mathcal{N}^+} E_i - \sum_{i\in\mathcal{N}^+\backslash S} B_i$, if this number is positive and nothing otherwise

$$v_{\mathcal{L}}(S) \coloneqq \left(\sum_{i\in\mathcal{N}^+} E_i - \sum_{i\in\mathcal{N}^+\backslash S} B_i\right)_+.$$

Coalition S takes all the money but gives back balances to all other players. To prove the existence of the core of this game we will prove its convexity.

Game is convex if its characteristic function is supermodular:

$$\forall A, B \subseteq \mathcal{N},\ v(A) + v(B) \leq v(A \cap B) + v(A \cup B).$$

Theorem 2. Game $(\mathcal{N}, v_{\mathcal{L}})$ is convex.

Proof: We rewrite the supermodularity condition for $v_{\mathcal{L}}$

$$\left(\sum_{i\in\mathcal{N}^+} E_i - \sum_{i\in\mathcal{N}^+\setminus A} B_i\right)_+ + \left(\sum_{i\in\mathcal{N}^+} E_i - \sum_{i\in\mathcal{N}^+\setminus B} B_i\right)_+ \le$$
$$\left(\sum_{i\in\mathcal{N}^+} E_i - \sum_{i\in\mathcal{N}^+\setminus (A\cap B)} B_i\right)_+ + \left(\sum_{i\in\mathcal{N}^+} E_i - \sum_{i\in\mathcal{N}^+\setminus (A\cup B)} B_i\right)_+.$$

Taking into account

$$\sum_{i\in\mathcal{N}^+} B_i = \sum_{i\in\mathcal{N}^+} E_i + \sum_{i\in\mathcal{N}} c_i,$$

we have

$$\sum_{i\in\mathcal{N}^+} E_i - \sum_{i\in\mathcal{N}^+\setminus A} B_i = \sum_{i\in\mathcal{N}^+} E_i - \sum_{i\in\mathcal{N}^+\setminus A} B_i + \sum_{i\in\mathcal{N}^+} B_i - \sum_{i\in\mathcal{N}^+} E_i - \sum_{i\in\mathcal{N}} c_i,$$

and

$$\left(\sum_{i\in A} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+ + \left(\sum_{i\in B} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+ \le$$
$$\left(\sum_{i\in A\cap B} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+ + \left(\sum_{i\in A\cup B} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+.$$

There are three cases:

1. $\sum_{i\in A\cap B} B_i - \sum_{i\in\mathcal{N}} c_i \ge 0$ and $\sum_{i\in A\cup B} B_i - \sum_{i\in\mathcal{N}} c_i \ge 0;$
2. $\sum_{i\in A\cap B} B_i - \sum_{i\in\mathcal{N}} c_i \le 0$ and $\sum_{i\in A\cup B} B_i - \sum_{i\in\mathcal{N}} c_i \ge 0;$
3. $\sum_{i\in A\cap B} B_i - \sum_{i\in\mathcal{N}} c_i \le 0$ and $\sum_{i\in A\cup B} B_i - \sum_{i\in\mathcal{N}} c_i \le 0.$

In the first case we have an equality

$$\sum_{i\in A} B_i - \sum_{i\in\mathcal{N}} c_i + \sum_{i\in B} B_i - \sum_{\mathcal{N}} c_i = \sum_{i\in A\cap B} B_i - \sum_{i\in\mathcal{N}} c_i + \sum_{i\in A\cup B} B_i - \sum_{i\in\mathcal{N}} c_i.$$

In the second case $\left(\sum_{i\in A} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+$ and $\left(\sum_{i\in B} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+$ are always less than $\sum_{i\in A\cup B} B_i - \sum_{i\in\mathcal{N}} c_i$. The only case which can violate inequality is the following

$$\sum_{i\in A} B_i - \sum_{i\in\mathcal{N}} c_i \ge 0 \text{ and } \sum_{i\in B} B_i - \sum_{i\in\mathcal{N}} c_i \ge 0.$$

Then

$$\left(\sum_{i\in A} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+ + \left(\sum_{i\in B} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+ = \sum_{i\in A} B_i - \sum_{i\in\mathcal{N}} c_i + \sum_{i\in B} B_i -$$
$$\sum_{i\in\mathcal{N}} c_i = \sum_{i\in A\cap B} B_i - \sum_{i\in\mathcal{N}} c_i + \sum_{i\in A\cup B} B_i - \sum_{i\in\mathcal{N}} c_i \le \sum_{i\in A\cup B} B_i - \sum_{i\in\mathcal{N}} c_i.$$

The second case does not violate inequality.

In the third case we have equality

$$\left(\sum_{i\in A} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+ + \left(\sum_{i\in B} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+ = \quad 0 \quad = \left(\sum_{i\in A\cap B} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+ +$$
$$\left(\sum_{i\in A\cup B} B_i - \sum_{i\in\mathcal{N}} c_i\right)_+.$$

Supermodularity implies convexity of the game. Because of the convexity of the game the core (natural compromise division) exists (Shapley, (1971)).■

Bankruptcy laws usually define some ordered partition of creditors (and their claims) $\preccurlyeq$ or/and weights w. For example, Article 134 of Federal Law No. 127-FZ of the Russian Federation of October 26, 2002 “Concerning Insolvency (Bankruptcy)” describes the priority for the satisfaction of creditors’ claims. First of all current obligations shall be settled out of the bankrupt estate outside the order of priority: court expenses, expenses associated with the payment of remunerations to the arbitration manager and register keeper; current utility and operational payments which are essential for the conduct of the debtor's activities, etc. Creditors' claims shall be satisfied in the following order: first in order of priority shall be settlements in respect to claims of citizens before whom the debtor bears liability for damage caused to life or health; second in order of priority shall be settlements in respect to the payment of redundancy allowances and payment for the labour of persons who work or worked under an employment agreement, and in respect of remunerations under authors' agreements; third in order of priority shall be settlements with other creditors.

Hokari (2005) developed a solution to such a problem, which is called sequential nucleolus relative to $\preccurlyeq$ and w. Let $v^{\mathcal{N}}$ be a convex game with a finite number of players. Let $(P_1, P_2, \dots, P_K)$ be the ordered partition of $\mathcal{N}$ induced by $\preccurlyeq$. For each $k \in \{1,2,\dots,K\}$, let v_k be the associated contribution game for P_k. Then $Nu^{(\preccurlyeq,w)}(v)$ is defined to be the payoff vector x^* obtained as follows:

For $k \in \{1,2,\dots,K\}$,

(i) if $P_k = \{i\}$, then $x_i^* \equiv v_k(P_k)$,

(ii) if $P_k = \{i,j\}$, then

$$\begin{cases} x_i^* \equiv v_k(\{i\}) + \frac{w_i}{w_i+w_j}[v_k(\{i,j\}) - v_k(\{i\}) - v_k(\{j\})], \\ x_j^* \equiv v_k(\{j\}) + \frac{w_j}{w_i+w_j}[v_k(\{i,j\}) - v_k(\{i\}) - v_k(\{j\})], \end{cases}$$

(iii) if $|P_k| \geq 3$, then if $x_{pk}^* \equiv Nu(v_k)$.

$Nu(v_k)$ is the nucleolus solution (Schmeidler, (1969)). Nucleolus is the solution which lexicographically minimizes vector of excesses. The excess of x for a coalition $S \subseteq \mathcal{N}$ is the quantity $v(S) - \sum_{i \in S} x_i$.

The contribution game for P_k associated with v and $(P_1, P_2, \dots, P_K)$ is defined by setting for all $S \subseteq P_k$,

$$v_k(S) \equiv \begin{cases} v(S) \text{ if } k = 1, \\ v(P_1 \cup \dots \cup P_{k-1} \cup S) - v(P_1 \cup \dots \cup P_{k-1}) \text{ if } k \geq 2. \end{cases}$$

By using a sequential nucleolus solution each creditor obtains a nonnegative payment (there would be no subsequent bankruptcies). Furthermore a sequential nucleolus satisfies several good properties.

Corollary 1. (Hokari (2005)). In the set of convex games, a single valued solution satisfies efficiency, homogeneity, individual rationality, zero-independence and max consistency if and only if it is a sequential nucleolus.

Let us discuss these properties with respect to our problem.

Efficiency means full distribution of resources. Homogeneity implies scale invariance. Individual rationality: $x_i^* \geq v(\{i\})$. No doubt these properties are necessary to our problem.

Zero-independence and max consistency are more sophisticated. Zero-independence is defined as follows. If for all $x \in \sigma(v)$ and all $\beta \in \mathbb{R}^n$, if for all $S \subseteq \mathcal{N}$, $v'^{(S)} = v(S) + \sum_{i \in S} \beta_i$, then $x + \beta \in \sigma(v')$.

If we change the payoff of each individual coalition ($v(\{i\})$ by some β_i and payoffs of other coalitions by the sum of the individual β_i then in the solution σ the difference in individual payoffs should be also equal to β_i.

In other words, if we change the zero point of the game, the result should be changed in the same way. In the generalized bankruptcy problem the change in the balance vector modifies its characteristic function in such way. In our problem $v(\{i\})$ can be positive and became $v(\{i\})+\beta_i$ if the agent has a positive balance and his own estate is increased. It seems reasonable that in such a case an agent's payoff should increased by the same amount as the change of his own estate.

In the generalized bankruptcy problem the first bankruptcy causes the consequent bankruptcies and the new problem needs to be solved. In this case the application of the max consistency property (Davis, Mashler (1965)) is natural. Imagine a set of agents S, who have already got theirs payoffs, according to the solution σ, and quit the game. Now we have a new game defined on $\mathcal{N} \backslash S$ and it is a desirable property of σ if the payoffs in that new game are the same as in the bigger game.

If $X' \coloneqq \{x \in \mathbb{R}^n : x(\mathcal{N}) \leq v(\mathcal{N})\}$ is the set of all feasible allocations, the max-reduced game $r^x_{\mathcal{N}'}(v)$ of $(\mathcal{N}, v)$ on the coalition $\mathcal{N}'$ for an allocation $x \in X'$ is defined as

$$r^x_{\mathcal{N}'}(v)(T) = \begin{cases} v(\mathcal{N}) - v(\mathcal{N} \backslash \mathcal{N}'), & T = \mathcal{N}' \\ \max\limits_{Q \subseteq \mathcal{N} \backslash \mathcal{N}'} [v(T \cup Q) - x(Q)], & \emptyset \neq T \subsetneq \mathcal{N}' \\ 0, & T = \emptyset \end{cases}$$

The solution $\sigma(\mathcal{N}, v)$ is consistent if for any coalition $\mathcal{N}' \subset \mathcal{N}$ we have $x \in \sigma(\mathcal{N}, v) \Rightarrow x_{\mathcal{N}'} \in \sigma(\mathcal{N}', r^x_{\mathcal{N}'}(v))$, where $x_{\mathcal{N}'}$ is a vector without coordinates, corresponding to agents from $\mathcal{N} \backslash \mathcal{N}'$. By Chang (2007) a max-reduced operator is transitive. An example that illustrates reduced game property to the bankruptcy problem can be found in (Dreissen, (1991)).

A sequential nucleolus is based on nucleolus solution. Maschler et al. (1979) characterize the nucleolus as lexicographic center of the game and show that it belongs to the core (if the core is not empty). Prenucleolus (a variant of the nucleolus) was justified by Sobolev (1975) and Orshan (1993) as the unique solution concept, defined over the class of cooperative games that satisfies single valuedness, anonymity, covariance under strategic equivalence and reduced game property.

4. Conclusion

The paper provides a model of the generalized bankruptcy problem. The new statement is a better model of the real world problem than the traditional one because it incorporates the system of intercreditor debts. Applying Bankruptcy law conditions into the model we move closer to a practical application. Utilizing a sequential nucleolus we achieve a single valued solution for the new problem.

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