Remark. Our results cover those of [4, 7], which can be obtained by setting $p_1(x) = bx^\beta$, $q_0(x) = ax^\alpha$. and $p_0(x) = cx^\gamma$, where α, β, γ, a, b, and c are real constants.

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Weak Solutions to Burgers-Type Equations with Nonsmooth Data

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We give a simple proof of the existence and uniqueness of the solution $u \in C\big((0, T) \times \mathbb{R}\big)$ to the Cauchy problem

$$(\partial_t - \partial_x^2)u + \partial_x f(u) = 0 \quad \text{in} \quad \mathcal{D}'\big((0, T) \times \mathbb{R}\big), \qquad u(0, \cdot) = u_0(\cdot) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),$$

for the case in which the function $f \colon \mathbb{R} \to \mathbb{R}$ is only assumed to be continuous.

§1. Introduction. We are chiefly interested in the uniqueness of weak solutions to the following Cauchy problem for a quasilinear parabolic Burgers-type equation:

$$(\partial_t - \partial_x^2)u(t, x) + \partial_x f\big(u(t, x)\big) = 0, \qquad (t, x) \in (0, T) \times \mathbb{R}, \quad T > 0, \tag{1}$$

$$u(0, x) = u_0(x), \qquad x \in \mathbb{R}, \tag{2}$$

when the problem data $f \colon \mathbb{R} \to \mathbb{R}$ and $u_0 \colon \mathbb{R} \to \mathbb{R}$ are nonsmooth functions (here ∂_t, ∂_x, and ∂_x^2 stand for the first- and second-order partial derivatives with respect to t and x). Our main observation is that the regularization of the difference between two solutions to the original equation satisfies a linear parabolic equation with two spatial variables and that the maximum principle estimate of a solution to the latter equation does not use any information about the coefficients of the first derivatives of the solution (these coefficients are given by the derivative of the regularization of f in the intermediate point between the regularizations of the two solutions).

The classical Burgers equation has the form

$$(\partial_t - \partial_x^2)u + \partial_x(u^2/2) = 0\,;$$

it was thoroughly studied in [1]. Equations of the form (1) naturally arise as "parabolic approximations" in studying generalized (discontinuous) global solutions for quasilinear hyperbolic conservation laws [2–4]. Let us also point out that the Burgers equation and its generalizations were studied in various directions in [5–7].

Throughout the paper we use the standard designations of function spaces.

§2. Theorem. *Let $T > 0$, let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, and let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then problem (1), (2) has a unique (weak) solution*

$$u = u(t, x) \in C\big((0, T) \times \mathbb{R}\big)$$

with the following properties:

(i) *u, $\partial_x u \in L^2_{\mathrm{loc}}\big((0,T)\times\mathbb{R}\big)$; u is bounded and uniformly continuous on $(\delta, T)\times\mathbb{R}$ for any $\delta \in (0, T)$;*

(ii) *$(\partial_t - \partial_x^2)u + \partial_x f(u) = 0$ in $\mathcal{D}'\big((0, T) \times \mathbb{R}\big)$, that is,*

$$\int_0^T \int_{\mathbb{R}} \big(u(\partial_t + \partial_x^2)\psi + f(u)\partial_x\psi\big)\, dx\, dt = 0 \quad \forall \psi \in C_0^\infty\big((0, T) \times \mathbb{R}\big)\,; \tag{3}$$

(iii) *$\displaystyle \lim_{t\downarrow 0}\, \sup_{a,b\in\mathbb{R}}\left|\int_a^b \big(u(t, x) - u_0(x)\big)\, dx\right| = 0.$*

Moreover, if $u, v \in C\big((0, \infty) \times \mathbb{R}\big)$ are weak solutions to problem (1), (2) on $(0, \infty) \times \mathbb{R}$ that satisfy (i) and (ii) with $T = \infty$ and assume the initial values $u_0, v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ in the sense of (iii), then

$$\sup_{a,b\in\mathbb{R}}\left|\int_a^b \big(u(t, x) - v(t, x)\big)\, dx\right| \leq \sup_{a,b\in\mathbb{R}}\left|\int_a^b \big(u_0(x) - v_0(x)\big)\, dx\right| \tag{4}$$

for any $t > 0$.

The proof is in two parts.

§3. Existence. For smooth initial data, the solution to problem (1), (2) is the x-derivative of the solution $v = v(t, x)$ to the problem

$$(\partial_t - \partial_x^2)v + f(\partial_x v) = 0 \quad \text{in } (0, T) \times \mathbb{R}, \tag{5}$$

$$v(0, x) = v_0(x) := \int_0^x u_0(\xi)\, d\xi \quad \text{on } \mathbb{R}. \tag{6}$$

In our case v_0 is bounded and Lipschitz continuous on $\mathbb{R}$ with Lipschitz constant $\|u_0\|_{L^\infty(\mathbb{R})}$; thus [8], there exists a unique bounded continuous function $v = v(t, x)$ on $[0, T] \times \mathbb{R}$ such that $\partial_t v$, $\partial_x v$, $\partial_x^2 v \in L^2_{\mathrm{loc}}\big((0, T) \times \mathbb{R}\big)$, v and $\partial_x v$ are bounded and uniformly continuous on $(\delta, T) \times \mathbb{R}$ for any $\delta \in (0, T)$, and v satisfies Eq. (5) almost everywhere on $(0, T) \times \mathbb{R}$ and assumes the initial values (6) in the sense of $C\big([0, T]; L^\infty(\mathbb{R})\big)$:

$$\lim_{t\downarrow 0}\, \sup_{x\in\mathbb{R}}\big|v(t, x) - v_0(x)\big| = 0$$

(note that since f in (5) depends only on $p = \partial_x v$, the restriction imposed in [8] on the growth of $f(p)$ with respect to p can be removed).

Let us verify that the function $u := \partial_x v$ satisfies (i)–(iii). Obviously, $u \in C\big((0, T) \times \mathbb{R}\big)$ and (i) is satisfied. If $\psi \in C_0^\infty\big((0, T) \times \mathbb{R}\big)$, then we multiply Eq. (5) by $\partial_x\psi$, integrate it over $(0, T) \times \mathbb{R}$, use integration by parts to transpose one derivative from $\partial_x^2 v$ to $\partial_x\psi$, and take account of the identity

$$\int_0^T \int_{\mathbb{R}} \partial_t v\, \partial_x\psi\, dx\, dt = \int_0^T \int_{\mathbb{R}} \partial_x v\, \partial_t\psi\, dx\, dt,$$

thus obtaining (ii) with $u = \partial_x v$. Finally, the inequality

$$\left| \int_a^b \partial_x v(t,x)\,dx - \int_a^b u_0(x)\,dx \right| \leq \big|v(t,b) - v_0(b)\big| + \big|v(t,a) - v_0(a)\big|,$$

valid for $a, b \in \mathbb{R}$, shows that $u = \partial_x v$ also satisfies (iii). The existence of a weak solution to problem (1), (2) is thereby proved.

§4. Uniqueness. Let u, $v \in C\big((0,T) \times \mathbb{R}\big)$ be two solutions to problem (1), (2) that satisfy (i) and (ii) with $T = \infty$ and assume the initial values u_0, $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, respectively, in the sense of (iii). Let us choose an arbitrary $T > 0$ and consider a function $\varphi \in C_0^\infty(\mathbb{R})$ (a smoothing kernel) such that

$$\varphi \geq 0, \quad \operatorname{supp}\varphi \subset (-1,1), \quad \text{and} \quad \int_{\mathbb{R}} \varphi(x)\,dx = 1.$$

Set

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0, \quad x \in \mathbb{R}, \qquad (\varphi \otimes \varphi)(t,x) := \varphi(t)\varphi(x), \quad t, x \in \mathbb{R}.$$

Next, we fix a $\delta \in (0, T/2)$ and, assuming that $\varepsilon \in (0, \delta)$, define the following functions on $[\delta, T - \delta] \times \mathbb{R}$:

$$u^\varepsilon := u * (\varphi_\varepsilon \otimes \varphi_\varepsilon), \qquad f(u)^\varepsilon := (f \circ u) * (\varphi_\varepsilon \otimes \varphi_\varepsilon)$$

(convolutions of functions of two variables). We set $f^\varepsilon := f * \varphi_\varepsilon$, $x \in \mathbb{R}$ (the convolution of functions of one variable). Since convolution commutes with the differentiation operator, it follows that

$$\partial_x(u^\varepsilon) = (\partial_x u)^\varepsilon = (\partial_x u) * (\varphi_\varepsilon \otimes \varphi_\varepsilon).$$

Set

$$\psi(\tau, \xi) := (\varphi_\varepsilon \otimes \varphi_\varepsilon)(t - \tau, x - \xi), \qquad t \in [\delta, T - \delta], \quad \tau, x, \xi \in \mathbb{R}.$$

Then $\psi \in C_0\big((0,T) \times \mathbb{R}\big)$ for fixed (t, x); by substituting this function into Eq. (3) with t replaced by τ and x by ξ, we find that u^ε satisfies the equation

$$(\partial_t - \partial_x^2)u^\varepsilon + \partial_x f(u)^\varepsilon = 0 \quad \text{everywhere on } (\delta, T - \delta] \times \mathbb{R}. \tag{7}$$

Consider the function

$$U^\varepsilon(t, a, b) := \int_a^b u^\varepsilon(t, x)\,dx, \qquad t \in [\delta, T - \delta], \quad (a,b) \in \mathbb{R}^2.$$

We have

$$\partial_a U^\varepsilon(t, a, b) = -u^\varepsilon(t, a), \qquad \partial_a^2 U^\varepsilon(t, a, b) = -\partial_x u^\varepsilon(t, a),$$
$$\partial_b U^\varepsilon(t, a, b) = u^\varepsilon(t, b), \qquad \partial_b^2 U^\varepsilon(t, a, b) = \partial_x u^\varepsilon(t, b);$$

thus, by (7),

$$\partial_t U^\varepsilon(t, a, b) = \int_a^b \partial_t u^\varepsilon(t, x)\,dx = \int_a^b \partial_x\big(\partial_x u^\varepsilon - f(u)^\varepsilon\big)(t, x)\,dx$$
$$= \partial_x u^\varepsilon(t, b) - \partial_x u^\varepsilon(t, a) - f(u)^\varepsilon(t, b) + f(u)^\varepsilon(t, a)$$
$$= \partial_b^2 U^\varepsilon(t, a, b) + \partial_a^2 U^\varepsilon(t, a, b) + \Big[f^\varepsilon\big(u^\varepsilon(t, b)\big) - f(u)^\varepsilon(t, b)\Big]$$
$$- f^\varepsilon\big(\partial_b U^\varepsilon(t, a, b)\big) + \Big[f(u)^\varepsilon(t, a) - f^\varepsilon\big(u^\varepsilon(t, a)\big)\Big] + f^\varepsilon\big(-\partial_a U^\varepsilon(t, a, b)\big).$$

Consequently, U^ε satisfies the following problem:

$$(\partial_t - \partial_a^2 - \partial_b^2)U^\varepsilon + f^\varepsilon(\partial_b U^\varepsilon) - f^\varepsilon(-\partial_a U^\varepsilon) = F_u^\varepsilon(t, a) - F_u^\varepsilon(t, b)$$

everywhere on $(\delta, T - \delta] \times \mathbb{R}_a \times \mathbb{R}_b$,

$$U^\varepsilon(\delta, a, b) = \int_a^b u^\varepsilon(\delta, x)\, dx, \qquad (a, b) \in \mathbb{R}^2 = \mathbb{R}_a \times \mathbb{R}_b,$$

where

$$F_u^\varepsilon(t, x) := f(u)^\varepsilon(t, x) - f^\varepsilon\big(u^\varepsilon(t, x)\big), \qquad (t, x) \in [\delta, T - \delta] \times \mathbb{R}.$$

Let us carry out a similar argument for v and introduce similar notation. Then for the difference

$$W^\varepsilon(t, a, b) = U^\varepsilon(t, a, b) - V^\varepsilon(t, a, b) = \int_a^b \big(u^\varepsilon(t, x) - v^\varepsilon(t, x)\big)\, dx$$

we obtain the parabolic equation

$$(\partial_t - \partial_a^2 - \partial_b^2)W^\varepsilon - B^\varepsilon(t, a)\partial_a W^\varepsilon + B^\varepsilon(t, b)\partial_b W^\varepsilon = F^\varepsilon(t, a, b) \quad \text{everywhere on } (\delta, T - \delta] \times \mathbb{R}^2, \quad (8)$$

where

$$B^\varepsilon(t, x) := \int_0^1 (f^\varepsilon)'\big(\theta u^\varepsilon(t, x) + (1 - \theta)v^\varepsilon(t, x)\big)\, d\theta, \qquad (t, x) \in [\delta, T - \delta] \times \mathbb{R},$$

$$F^\varepsilon(t, a, b) := F_u^\varepsilon(t, a) - F_v^\varepsilon(t, a) + F_v^\varepsilon(t, b) - F_u^\varepsilon(t, b).$$

By continuity, for $t = \delta$ the function W^ε satisfies the initial conditions

$$W^\varepsilon(\delta, a, b) = \int_a^b \big(u^\varepsilon(\delta, x) - v^\varepsilon(\delta, x)\big)\, dx, \qquad (a, b) \in \mathbb{R}^2. \quad (9)$$

By the maximum principle [9, §1, Theorem 10], the solution W^ε to problem (8), (9) satisfies the estimate

$$\big|W^\varepsilon(t, a, b)\big| \leq \sup_{(a,b)\in\mathbb{R}^2} \big|W^\varepsilon(\delta, a, b)\big| + (t - \delta) \sup_{(t,a,b)\in[\delta,T-\delta]\times\mathbb{R}^2} \big|F^\varepsilon(t, a, b)\big|$$

for $(t, a, b) \in [\delta, T - \delta] \times \mathbb{R}^2$, whence it follows that

$$\left|\int_a^b \big(u^\varepsilon(t, x) - v^\varepsilon(t, x)\big)\, dx\right| \leq \sup_{a,b\in\mathbb{R}} \left|\int_a^b \big(u^\varepsilon(\delta, x) - u_0(x)\big)\, dx\right|$$

$$+ \sup_{a,b\in\mathbb{R}} \left|\int_a^b \big(u_0(x) - v_0(x)\big)\, dx\right| + \sup_{a,b\in\mathbb{R}} \left|\int_a^b \big(v^\varepsilon(\delta, x) - v_0(x)\big)\, dx\right|$$

$$+ 2T \sup_{(t,x)\in[\delta,T-\delta]\times\mathbb{R}} \Big(\big|F_u^\varepsilon(t, x)\big| + \big|F_v^\varepsilon(t, x)\big|\Big). \quad (10)$$

Since u and v are uniformly continuous on $[\delta, T] \times \mathbb{R}$, the properties of convolution [10, Lemma 2.18] imply that the fourth term on the right-hand side in Eq. (10) tends to 0 as $\varepsilon \to +0$, and hence in the limit as $\varepsilon \to +0$ inequality (10) yields

$$\left|\int_a^b \big(u(t, x) - v(t, x)\big)\, dx\right| \leq \sup_{a,b\in\mathbb{R}} \left|\int_a^b \big(u(\delta, x) - u_0(x)\big)\, dx\right|$$

$$+ \sup_{a,b\in\mathbb{R}} \left|\int_a^b \big(u_0(x) - v_0(x)\big)\, dx\right| + \sup_{a,b\in\mathbb{R}} \left|\int_a^b \big(v(\delta, x) - v_0(x)\big)\, dx\right|$$

for $t \in [\delta, T - \delta]$, $a, b \in \mathbb{R}$. In view of (iii), by passing to the limit in the latter inequality as $\delta \to +0$, we obtain (4) for $t \in (0, T)$. Since $T > 0$ is arbitrary, we conclude that (4) is valid for all $t > 0$.

Now if $u_0 = v_0$, then, by Eq. (4) and by the continuity of u and v on $(0, \infty) \times \mathbb{R}$, we find that $u = v$ everywhere on $(0, \infty) \times \mathbb{R}$, thus completing the proof of the theorem. $\quad\square$

§5. Open question. Is it possible to solve problem (1), (2) and prove the uniqueness of the solution for the case in which the state function f is discontinuous, say, if f is the Heaviside function: $f(u) = 0$ for $u < 0$ and $f(u) = 1$ for $u > 0$? What role is played by the value $f(0)$ in this case? The solution to this problem is of interest also for hyperbolic conservation laws with discontinuous state function (see also [11]).

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