Higher-order (2 + 4) Korteweg-de Vries-like equation for interfacial waves in a symmetric three-layer fluid

O. E. Kurkina,[1,2,3] A. A. Kurkin,[3] T. Soomere,[1] E. N. Pelinovsky,[2,3,4] and E. A. Rouvinskaya[2,4]

[1]*Wave Engineering Laboratory, Institute of Cybernetics, Tallinn University of Technology, Tallinn, Estonia*
[2]*Department of Mathematics, National Research University Higher School of Economics, Nizhny Novgorod Branch, Nizhny Novgorod, Russia*
[3]*Department of Applied Mathematics, Nizhny Novgorod State Technical University, Nizhny Novgorod, Russia*
[4]*Department of Nonlinear Geophysical Processes, Institute of Applied Physics of Russian Academy of Sciences, Nizhny Novgorod, Russia*

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We address a specific but possible situation in natural water bodies when the three-layer stratification has a symmetric nature, with equal depths of the uppermost and the lowermost layers. In such case, the coefficients at the leading nonlinear terms of the modified Korteweg-de Vries (mKdV) equation vanish simultaneously. It is shown that in such cases there exists a specific balance between the leading nonlinear and dispersive terms. An extension to the mKdV equation is derived by means of combination of a sequence of asymptotic methods. The resulting equation contains a cubic and a quintic nonlinearity of the same magnitude and possesses solitary wave solutions of different polarity. The properties of smaller solutions resemble those for the solutions of the mKdV equation, whereas the height of the taller solutions is limited and they become table-like. It is demonstrated numerically that the collisions of solitary wave solutions to the resulting equation are weakly inelastic: the basic properties of the counterparts experience very limited changes but the interactions are certainly accompanied by a certain level of radiation of small-amplitude waves. © *2011 American Institute of Physics*. [doi:10.1063/1.3657816]

I. INTRODUCTION

Internal gravity waves serve as one of the most important constituent of nonlinear wave motions in stratified environment.[1] Internal waves often play the decisive role in various processes in geophysical and other similar media where they may serve as major agents carrying energy from remote areas to specific domains in the ocean. For example, energy release into local turbulent motions and subsequent mixing caused by internal waves apparently is an important source of the potential energy that is needed to bring deep, dense bottom water to the surface.[2–4] Their impact on the functioning of the ocean has been relatively well understood in shelf regions where high-amplitude, nonlinear, or breaking internal waves frequently contribute to localized highly energetic motions that not only are able to substantially affect offshore engineering activities[5] but also cause resuspension and transport of bottom sediments[6] or release of nutrients and pollution into the water column. Extensive patches of turbulence associated with propagation and breaking of such waves are hypothesized to be an important source of vertical mixing of water masses and, consequently, a key mechanism supporting penetration of various substances (including adverse impacts) from bottom layers up to the ocean surface. Changes to the properties of water at some depths may substantially modify acoustic properties of the environment.

The most interesting phenomena in this respect are internal solitons—localized, long-living disturbances that can carry wave energy and momentum far from the place of their generation, survive collisions with similar entities along their journey, and release the energy in certain specific conditions. The existence such phenomena has been recognized for a long time and their generic representatives have received a lot of attention.[7,8] Still there are major gaps in our understanding of the conditions of existence, properties, appearance, and dynamics of long-living solitonic internal waves and wave packets. Most of the relevant research has addressed their properties in a greatly simplified environment; usually in the framework of different versions of the two-layer fluid and/or solitonic solutions of integrable evolution equations. The simplest equation of this class is the well-known Korteweg-de Vries (KdV) equation that describes the motion of weakly nonlinear internal waves in the long-wave limit.[9] Later on, a variety of its extensions have been introduced, for example, the modified Korteweg-de Vries (mKdV) equation[10,11] and Gardner equation (sometimes called the KdV equation with combined nonlinearity[12–14] as it accounts for both quadratic and cubic nonlinearity).

The need for systematically accounting for higher nonlinearities stems from the nature of these equations. While the coefficients of similar equations in some other environments can be expressed in terms of simple combinations of governing scales for the particular problem,[15,16] the coefficients of nonlinear evolution equations for internal waves are defined by the particular vertical distribution of water density, properties of shear flow, and boundary conditions at the water surface.[17–20] A specific property of such equations is that some coefficients at the nonlinear terms may vanish for certain symmetric situations.[10,21] In such cases, it is necessary to account for higher-order nonlinearities to adequately describe the motion. This process not only leads to the necessity of inclusion of some additional terms in the governing equations but also naturally highlights a variety of qualitatively new phenomena in the dynamics of localized, solitonic (non-radiating) internal waves – solutions to such equations.[12–14,22,23]

In this paper, we shall analyze properties of internal solitons in the framework of the widely used concept of layered fluid. Models of wave motion for such environments are attractive for both theoretical research and applications because of their ability to mirror the basic properties of the actual internal wave systems using model equations of stratified media containing small number parameters and in many cases allowing for extensive analytical studies of the properties of solutions. As mentioned above, the simplest system basically representing the key properties of internal waves is the two-layer model. Various properties of internal waves in this approximation have been addressed in numerous analytical and numerical studies as well as by means of *in situ* observations and laboratory experiments.[17,21,24–33] These studies cover a wide range of different regimes of wave motion, from ideally linear systems and weakly nonlinear models up to fully nonlinear phenomena.

The two-layer model and the family of the KdV equation and its generalizations have been used as a simple but instructive model to demonstrate the richness of internal wave phenomena compared to long weakly nonlinear surface waves. Namely, for internal waves, the coefficient at the quadratic nonlinearity in the KdV equation vanishes for the naturally occurring symmetric situation when the layers have equal thicknesses in Boussinesq approximation. The leading nonlinear term in almost symmetric situations is the cubic one and the Gardner equation (or its generalizations) has to be used to describe the wave motion.[21,24,25,34–37] The inclusion of only one (cubic) nonlinear term gives rise to several principally new phenomena that do not occur in the KdV environment. For example, if the coefficient at the cubic term is negative (this is the case for the two-layer fluid), the amplitude of solitons has an upper limit. While the increase in the energy for the KdV soliton is associated with a higher and narrower wave profile, the similar process for the relevant class of solutions of the Gardner equation results in a widening of the wave profile and formation of a plateau-like entity with steep front and back and very gently sloping upper part. Such appearance of large-amplitude solitary waves has been repeatedly observed in both laboratory and field conditions.[11,38,39]

The two-layer fluid is, in fact, quite a simplified representation of the natural stratified flows. For example, in many areas of the World Ocean, the vertical stratification has a clearly pronounced three-layer structure, with well-defined seasonal thermocline at a depth of $\sim$100 m and the main thermocline at much larger depths.[40,41] Several basins such as the Baltic Sea host more or less continuously three-layer vertical structure.[42] In order to reveal the basic features of the internal wave field in such environments it is necessary to introduce a three-layer model. Such models are considerably more complex than the two-layer systems; however, they allow for much more analytical progress compared to the fully stratified situation.

In this paper, we address a generalization of the mKdV equation for the three-layer stratification. This procedure is basically straightforward, albeit cumbersome and technically complicated. The resulting equation admits solitary wave solutions for a certain range of parameters. The focus of the study is almost symmetric situations in which the lower-order nonlinear terms vanish, and the higher-order contributions govern the behavior of wave phenomena in the system. A simple symmetric situation corresponds to the equal thicknesses of the uppermost and the lowermost layers provided the density differences between the layers are also equal. This situation can naturally occur in shallow strongly stratified seas as the Baltic Sea where the interplay of fresh water discharge to the surface and irregular salt water intrusion in the bottom layers frequently give rise to two density jumps of comparable size and sharpness and an almost symmetric three-layer structure and may lead to vanishing of several interactions between baroclinic Rossby waves.[42,43] The key development through the research into such an environment (that is impossible in the two-layer medium) is the possibility of having the cubic nonlinearity with a positive coefficient. Moreover, this coefficient may change its sign in different domains and may even vanish under certain conditions.[10] As a result, the dynamics of internal waves in such environments is much richer in content and reveals several features that cannot become evident in two-layer flows. In particular, the possibility of simultaneous vanishing of the coefficients at both the quadratic and cubic nonlinear terms makes it possible to naturally generalize the mKdV equation towards accounting for the quadric nonlinearity and towards even more detailed analytical description of the properties of internal wave dynamics in layered fluids.

Almost zero values of the coefficients of quadratic and cubic nonlinear terms of the mKdV equation and its generalizations (see Eq. (6) below for an example), albeit not very usual in the World Ocean, can still be found for real sea stratifications in certain regions. The maps of the geographical points where the coefficient α of quadratic nonlinearity is small ($|\alpha| < 1 \times 10^{-4}$, while the usual values of this parameter are of order of 1×10^{-2}), and at the same time the coefficient α_1 of cubic nonlinearity is also small and non-negative ($0 \leq \alpha_1 \leq 1 \times 10^{-5}$, usual values of order of 1×10^{-3}) (Fig. 1) reveals that such situations occur during some months in selected shelf seas and in the North Atlantic. Hydrological data to calculate the numerical values of the nonlinear parameters on a base of integral expressions for continuous stratification[44] were taken from the climatological atlas GDEM V3.0.[45] Horizontal shear flow was not accounted for. The maps in Figs. 1 and 2 are drawn using OCEAN DATA VIEW software.[46]

Three-layer density stratification (and its symmetric examples as well) frequently appears in the Baltic Sea. The density structure of the central part often consists of a mixed upper layer, well-defined seasonal thermocline, an intermediate layer, and main halocline (Fig. 2). Importantly, the density changes between the layers are more or less equal.[42] There is evidence that this stratification will be more frequent in the Baltic Sea for some climate change scenarios.

The paper is organized as follows. Section II presents the basic equations of motion for the three-layer model.

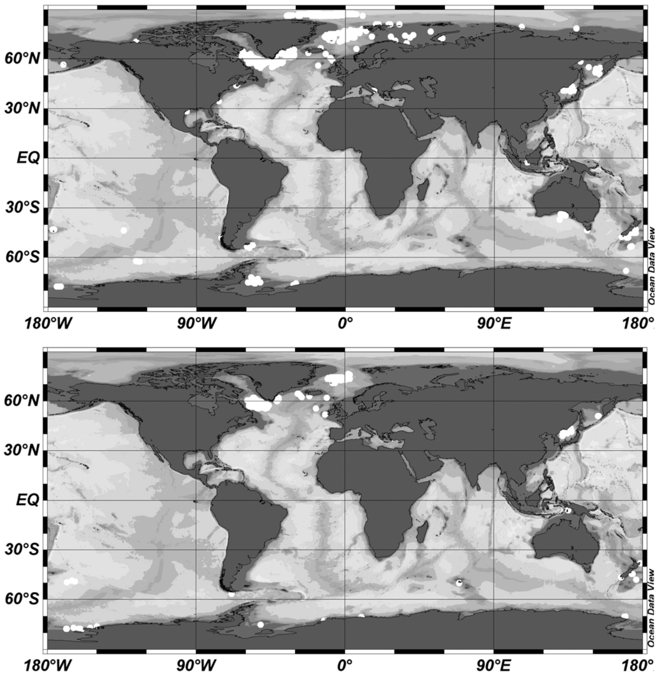

FIG. 1. Maps of the geographical locations (white points) where the coefficient of quadratic nonlinearity is small and the coefficient of cubic nonlinearity is small and non-negative ($|\alpha| < 1\times10^{-4}$, $0 \le \alpha_1 < 1\times10^{-5}$): upper panel – January, lower panel – July.

Asymptotic analysis of the equations for the interfaces of the three-layer model is provided in Sec. III up to the 4th order with respect to the small parameters in the system. Section IV focuses on the modifications of the derivation of the mKdV equation towards expressing the balance between nonlinear and dispersive terms in the case when the coefficients of the leading nonlinear terms vanish. The basic properties of solitary solutions to the resulting equation are discussed in Sec. V and their nonlinear interactions in Sec. VI. Section VII presents discussion and conclusions and Appendix—the analytical expression of the coefficients of the derived equations and parameters of numerical simulations.

II. NONLINEAR EQUATIONS OF MOTION

Let us consider a model situation of irrotational motions in a three-layer inviscid fluid of total thickness H overlying a flat horizontal bottom in the approximation of a rigid lid on the surface of the fluid (Fig. 3). We consider the symmetric case in which the thicknesses h of the surface and bottom layer are equal and assume that the density differences between the layers are also equal; then densities in the layers are $\rho_1 = \rho$, $\rho_2 = \rho + \Delta\rho$, $\rho_3 = \rho_2 + \Delta\rho = \rho + 2\Delta\rho$, where ρ is the density in the bottom layer. As usual, we employ the Boussinesq approximation and assume that densities in the layer differ insignificantly ($\Delta\rho/\rho \ll 1$). In this case, the equations of the motion are Laplace equations for the velocity potential in each layer

$$\nabla^2\Phi_i = 0, \quad i = 1,2,3. \tag{1}$$

The vertical velocities at the bottom and at the fixed upper boundary obviously vanish

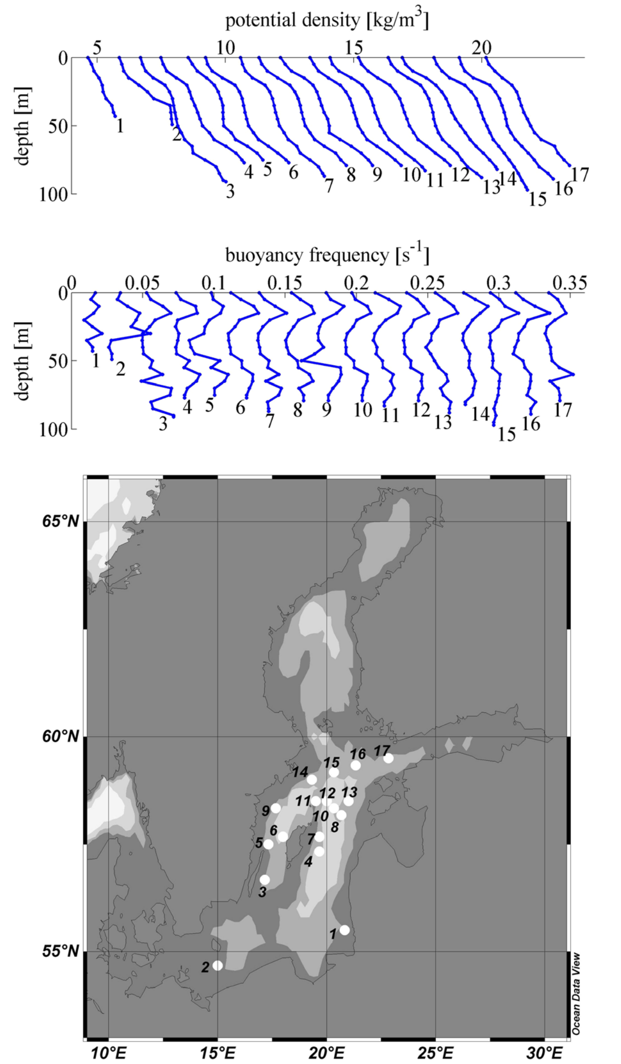

FIG. 2. (Color online) Upper panel: vertical density profiles in the Baltic Sea (based on GDEM V3.0 climatology for July). The profile number M is shifted to the right by M–1 density units. Middle panel: corresponding vertical buoyancy frequency profiles. The profile number M is shifted to the right by $0.02(M–1)$ s^{-1}. Lower panel: map of the geographical locations (white points) of the profiles above.

$$\Phi_{1z}(z=0) = 0, \quad \Phi_{3z}(z=H) = 0, \tag{2}$$

where the subscript x, y, or z denotes a partial derivative. The classical kinematic and dynamic boundary conditions at the interfaces between the layers complete the setup of the problem

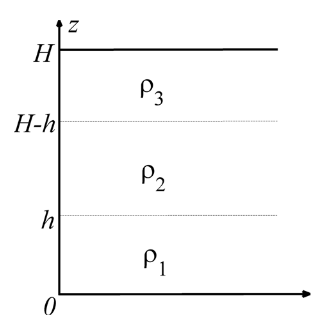

FIG. 3. Definition sketch of the three-layer fluid.

$$\left.\begin{aligned} \eta_t + \Phi_{1_x}\eta_x - \Phi_{1_z} = 0 \\ \eta_t + \Phi_{2_x}\eta_x - \Phi_{2_z} = 0 \\ \rho_1\left(\Phi_{1t} + \frac{1}{2}(\nabla\Phi_1)^2 + g\eta\right) = \rho_2\left(\Phi_{2t} + \frac{1}{2}(\nabla\Phi_2)^2 + g\eta\right) \end{aligned}\right\} z = h + \eta(x,t),$$

$$\left.\begin{aligned} \zeta_t + \Phi_{2_x}\zeta_x - \Phi_{2_z} = 0 \\ \zeta_t + \Phi_{3_x}\zeta_x - \Phi_{3_z} = 0 \\ \rho_2\left(\Phi_{2t} + \frac{1}{2}(\nabla\Phi_2)^2 + g\zeta\right) = \rho_3\left(\Phi_{3t} + \frac{1}{2}(\nabla\Phi_3)^2 + g\zeta\right) \end{aligned}\right\} z = H - h + \zeta(x,t). \quad (3)$$

Here, the unknown functions $\eta(x, t)$ and $\zeta(x, t)$ denote the instantaneous position of the interface between the bottom and the middle layer and between the upper and middle layer, respectively.

III. ASYMPTOTIC EXPANSION

In order to derive an evolution equation for wavelike motions of relatively small amplitude in the described environment, it is convenient to introduce characteristic scales of both the environment and the waves, and to perform the analysis in nondimensional form. The system of motions has one obvious length scale—the fluid depth H. Wave motions provide two more natural scales—the typical horizontal scale of motions L and amplitude a (understood here as the typical magnitude of deviation of the interfaces from their undisturbed location). Following the underlying assumptions usually made in derivation of the KdV equation and its generalizations, we focus on long waves, that is, on the situation $L \gg H$ where the typical wave length considerably exceeds the depth of the fluid. In this case, small parameter $\bar{\mu} = H/L$ naturally enters the system. Long-living wave motions usually exist in the system if the amplitude of such motions is small compared to the fluid depth, equivalently, when the parameter of nonlinearity $\varepsilon = a/H \ll 1$. Finally, the parameter characterizing the role of dispersion in wave propagation $\mu = \bar{\mu}^2$ becomes evident in the process of nondimensionalisation.

As long-living wave motions and solitary waves usually exist when the magnitude of the terms representing nonlinear and dispersive is balanced, we assume that $\varepsilon \sim \mu$. This implies that $\bar{\mu} \sim \sqrt{\varepsilon}$.

The asymptotic analysis is straightforward under assumption that all the above-discussed parameters are small. As it is quite cumbersome and largely follows that in Ref. 24, we omit the details and depict only the major steps. All unknown functions are expanded into Taylor series in the vicinity of one of the interfaces. The constituents of the resulting series are then expanded into power series with respect to powers of $\varepsilon = a/H \ll 1$. In order to use the technique of multiple time and spatial scales,[43,47,48] we introduce the "slow" time and "stretched" coordinate along the x-axis

$$\xi = \varepsilon^{1/2}(x - ct), \quad \tau = \varepsilon^{3/2}t, \quad (4)$$

where c is the phase speed. Substitution of these expansions and expressions (4) into nondimensionalised Eqs. (3) leads to an infinite system of equations. This system can be solved recursively until any desired order. Details of the relevant derivation are available in Ref. 49.

The system of equations for the leading order ($\sim\varepsilon^0$) terms has a relatively simple form and describes the field of linear internal waves in the three-layer environment in question. One of the two wave modes that exist in this system has a completely symmetric nature and results in synchronous in-phase movements of both the interfaces $\eta(x, t) = \zeta(x, t)$. The other mode is antisymmetric: the motions of the interfaces are synchronous but have opposite signs $\eta(x, t) = -\zeta(x, t)$.

Below, we shall consider only the nonlinear motions of the symmetric mode corresponding to the following expression for the phase speed (similar to that of the classical interfacial waves in two-layer medium):

$$c^2 = \frac{gh\Delta\rho}{\rho}. \quad (5)$$

The outcome of the asymptotic procedure at the first order ($\sim\varepsilon^1$) is, as expected, the well-known KdV equation that describes the motion of both the interfaces.

Combining of lower-order equations leads to the following generalization of the KdV equation for the interfaces (presented here in the original coordinates for ζ):

$$\begin{aligned} &\zeta_t + c\zeta_x + \alpha\zeta\zeta_x + \beta\zeta_{xxx} + \alpha_1\zeta^2\zeta_x + \beta_1\zeta_{5x} + \gamma_1\zeta\zeta_{xxx} + \gamma_2^*\zeta_x\zeta_{xx} \\ &+ \alpha_2^*\zeta^3\zeta_x + \beta_2\zeta_{7x} + \gamma_{21}\zeta_{xx}\zeta_{xxx} + \gamma_{22}\zeta_x\zeta_{xxxx} + \gamma_{23}\zeta\zeta_{5x} \\ &+ \gamma_{31}\zeta_x^3 + \gamma_{32}\zeta\zeta_x\zeta_{xx} + \gamma_{33}\zeta^2\zeta_{xxx} + \alpha_3\zeta^4\zeta_x + \beta_3\zeta_{9x} \\ &+ \gamma_{41}^*\zeta_x\zeta_{6x} + \gamma_{42}^*\zeta_{xx}\zeta_{5x} + \gamma_{42}^*\zeta_{xxx}\zeta_{xxxx} + \gamma_{51}\zeta\zeta_{xx}\zeta_{xxx} \\ &+ \gamma_{52}\int\zeta_x\zeta_{xx}\zeta_{xxx}dx + \gamma_{53}\zeta\zeta_x\zeta_{xxxx} + \gamma_{54}\zeta^2\zeta_{5x} + \gamma_{55}\zeta_x^2\zeta_{xxx} \\ &+ \gamma_{56}\zeta_x\zeta_{xx}^2 + \gamma_{61}^*\zeta\zeta_x^3 + \gamma_{62}^*\zeta^2\zeta_x\zeta_{xx} + \gamma_{63}^*\zeta^3\zeta_{xxx} = 0. \end{aligned} \quad (6)$$

The coefficients of Eq. (6) are given in Appendix. The equation for η has a similar structure and differs from Eq. (6) only by signs of a few additives marked by *.

An interesting, important, and rich in content particular case occurs when the coefficients α at quadratic terms and the coefficient reflecting the role of nonlinear dispersion γ_1 vanish simultaneously. This always happens for the symmetric distribution of layers in the three-layer medium for an arbitrary ratio of the layers' thicknesses. This case, as mentioned above, is not very common, but eventually occurs in natural conditions in some domains of micro-tidal estuaries and shelf seas and thus needs more detailed analysis.

IV. MODIFIED KdV EQUATION (mKdV)

In essence, the vanishing of the coefficient α at quadratic terms in the resulting evolution equations means that some other terms largely govern the motion and that the straightforward asymptotic expansion introduced above becomes invalid. It is easy to see that, formally, the reason for the failure is that the assumption of equivalence of the contributions of nonlinearity and dispersion ($\varepsilon \sim \mu$) to the formation of motion patterns is no more valid. In such situations, it is first necessary to establish which from the higher-order terms provides the largest contribution to the motion. As the presence of solitonic solutions in the system in question normally is associated with a specific balance of nonlinear and dispersive terms, it is natural to keep this balance also in the higher-order modifications to the KdV equation. The appearance of Eqs. (6) suggests that such a balance in symmetric stratified media is possible if $\varepsilon^2 \sim \mu$. This property actually means that the higher-order nonlinear terms provide a relatively large contribution to the governing equation compared with the dispersive terms.

Introducing variables $\eta = \varepsilon\tilde{\eta}$, $\zeta = \varepsilon\tilde{\zeta}$, $\tilde{x} = \bar{\mu}x$, $\tilde{t} = \bar{\mu}t$, Eq. (6) can be presented as follows (for simplicity we omit tildes):

$$\begin{aligned}
&\zeta_t + c\zeta_x + \varepsilon\alpha\zeta\zeta_x + \mu\beta\zeta_{xxx} + \varepsilon^2\alpha_1\zeta^2\zeta_x + \mu^2\beta_1\zeta_{5x} \\
&\quad + \varepsilon\mu\left(\gamma_1\zeta\zeta_{xxx} + \gamma_2^*\zeta_x\zeta_{xx}\right) + \varepsilon^3\alpha_2^*\zeta^3\zeta_x + \mu^3\beta_2\zeta_{7x} \\
&\quad + \varepsilon\mu^2(\gamma_{21}\zeta_{xx}\zeta_{xxx} + \gamma_{22}\zeta_x\zeta_{xxxx} + \gamma_{23}\zeta\zeta_{5x}) \\
&\quad + \varepsilon^2\mu\left(\gamma_{31}\zeta_x^3 + \gamma_{32}\zeta\zeta_x\zeta_{xx} + \gamma_{33}\zeta^2\zeta_{xxx}\right) + \varepsilon^4\alpha_3\zeta^4\zeta_x + \mu^4\beta_3\zeta_{9x} \\
&\quad + \varepsilon\mu^3\left(\gamma_{41}^*\zeta_x\zeta_{6x} + \gamma_{42}^*\zeta_{xx}\zeta_{5x} + \gamma_{43}^*\zeta_{xxx}\zeta_{xxxx}\right) \\
&\quad + \varepsilon^2\mu^2\Bigg(\gamma_{51}\zeta\zeta_{xx}\zeta_{xxx} + \gamma_{52}\int\zeta_x\zeta_{xx}\zeta_{xxx}dx + \gamma_{53}\zeta\zeta_x\zeta_{xxxx} \\
&\quad + \gamma_{54}\zeta^2\zeta_{5x} + \gamma_{55}\zeta_x^2\zeta_{xxx} + \gamma_{56}\zeta_x\zeta_{xx}^2\Bigg) \\
&\quad + \varepsilon^3\mu\left(\gamma_{61}^*\zeta\zeta_x^3 + \gamma_{62}^*\zeta^2\zeta_x\zeta_{xx} + \gamma_{63}^*\zeta^3\zeta_{xxx}\right) = 0. \qquad (7)
\end{aligned}$$

Making Making use of assumption $\varepsilon^2 = \mu$ leads to the following equation:

$$\begin{aligned}
&\zeta_t + c\zeta_x + \alpha_1\zeta^2\zeta_x + \beta\zeta_{xxx} + \varepsilon\left(\alpha_2^*\zeta^3\zeta_x + \gamma_2^*\zeta_x\zeta_{xx}\right) \\
&\quad + \varepsilon^2\left(\alpha_3\zeta^4\zeta_x + \gamma_{31}\zeta_x^3 + \gamma_{32}\zeta\zeta_x\zeta_{xx} + \gamma_{33}\zeta^2\zeta_{xxx} + \beta_1\zeta_{5x}\right) \\
&\quad + O\left(\varepsilon^3\right) = 0, \qquad (8)
\end{aligned}$$

where the analytical expressions for the coefficients are presented in Appendix and, as above, in the equation for $\eta(x, t)$ the signs of coefficients α_2^* and γ_2^* are reversed. The lowest order terms in Eq. (8) form well-known modified KdV equation (mKdV) that are complemented with terms describing the contribution of higher-order ($O(\varepsilon)$ and $O(\varepsilon^2)$) nonlinearities, nonlinear dispersion and also linear dispersion $\beta_1\zeta_{5x}$.

It is interesting and instructive to analyze the dependence of the coefficients of Eq. (8) on the ratio of the layers' thicknesses in our case of symmetric stratification. Figure 4 demonstrates that the coefficient α_1 at the cubic nonlinear term and the coefficient α_2 at the highest-order quadratic nonlinearity vanish simultaneously at $h/H = 9/26 \equiv h_{cr}/H$. For both coefficients, this is the zero-crossing point at which their sign is reversed. Only the coefficient α_3 at the quintic nonlinearity is always negative. The coefficients β, β_1 reflecting the contribution from linear dispersive terms as well as the coefficient γ_2 characterizing nonlinear dispersion are always positive. Finally, the coefficients at nonlinear dispersive terms γ_{31}, γ_{32}, and γ_{33} have one zero-crossing point, whereas these points coincide neither with each other nor with the point h_{cr}/H.

The performed analysis shows that although the higher-order terms of the mKdV are usually small in comparison with the leading-order nonlinear and dispersive terms, in the symmetric stratification the coefficients at the two leading nonlinear terms vanish simultaneously. Although this situation evidently does not happen very often, its presence is likely in some domains in natural conditions[43] and it is, therefore, important to analyze the behavior of internal waves in conditions when the contribution from these two terms is negligible and the motion is governed by higher order additives.

It is convenient to start the detailed analysis of this case from expansion of the coefficients of Eq. (8) into Taylor series in the vicinity of the zero-crossing point for the coefficients of the leading-order nonlinearities h_{cr} ($\Delta = (h - h_{cr})/H$)

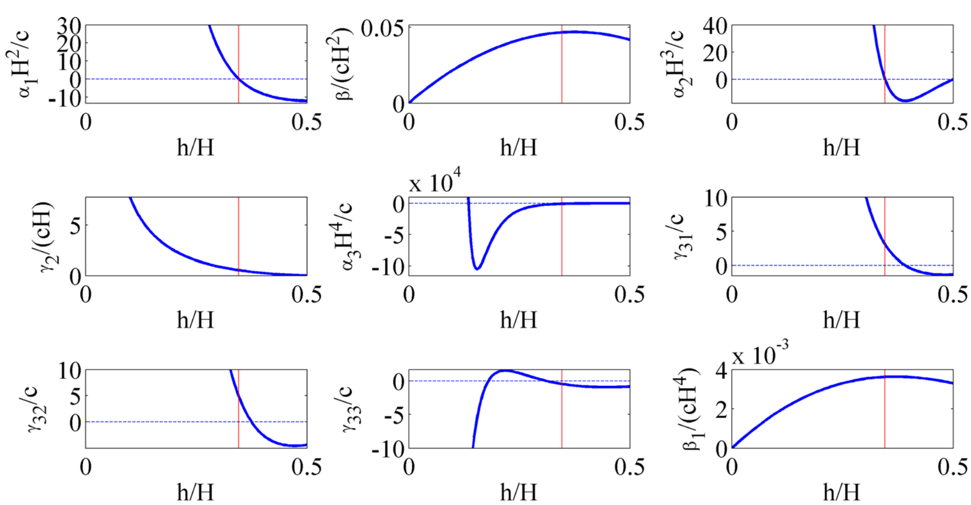

FIG. 4. (Color online) The dependence of coefficients of Eq. (8) on the ratio of the thicknesses of the layers in the case of symmetric stratification. Horizontal dotted lines indicate the zero-crossing points of the relevant coefficients and the vertical dotted lines—the point h_{cr}/H.

$$\alpha_1 = -\frac{c}{H^2}\frac{57122}{243}\Delta + O(\Delta^2) = \tilde{\alpha}_1\Delta + O(\Delta^2),$$
$$\beta = cH^2\left(\frac{63}{1352} + \frac{1}{52}\Delta\right) + O(\Delta^2) = \tilde{\beta} + O(\Delta),$$
$$\alpha_2^* = \frac{c}{H^3}\frac{5940688}{6561}\Delta + O(\Delta^2) = \tilde{\alpha}_2\Delta + O(\Delta^2),$$
$$\frac{\gamma_2^*}{cH} = \frac{7}{13} - \frac{115}{18}\Delta + O(\Delta^2),$$
$$\frac{\alpha_3 H^4}{c} = -\frac{47411260}{59049} + \frac{12084101978}{531441}\Delta + O(\Delta^2),$$
$$\frac{\gamma_{31}}{c} = \frac{1483}{486} - \frac{298285}{2916}\Delta + O(\Delta^2),$$
$$\frac{\gamma_{32}}{c} = \frac{2341}{486} - \frac{641017}{2916}\Delta + O(\Delta^2),$$
$$\frac{\gamma_{33}}{c} = -\frac{109}{243} - \frac{30589}{2916}\Delta + O(\Delta^2),$$
$$\frac{\beta_1}{cH^4} = \frac{66291}{18279040} + \frac{1099}{1054560}\Delta + O(\Delta^2).$$

The extension of the mKdV equation towards inclusion of the higher-order terms for small values of Δ (equivalently, in the vicinity of the point h_{cr}) is as follows:

$$\zeta_t + c\zeta_x + \tilde{\alpha}_1\Delta\zeta^2\zeta_x + \tilde{\beta}\zeta_{xxx} + \varepsilon\left(\tilde{\alpha}_2^*\Delta\zeta^3\zeta_x + \tilde{\gamma}_2^*\zeta_x\zeta_{xx}\right) + \varepsilon^2\left(\tilde{\alpha}_3\zeta^4\zeta_x + \tilde{\gamma}_{31}\zeta_x^3 + \tilde{\gamma}_{32}\zeta\zeta_x\zeta_{xx} + \tilde{\gamma}_{33}\zeta^2\zeta_{xxx} + \tilde{\beta}_1\zeta_{5x}\right) + O(\varepsilon^3) = 0. \quad (9)$$

It is easy to see that in this particular case of the symmetric three-layer stratification a balance between nonlinear and dispersive effects exists when the small parameters Δ and ε are related as follows $\Delta \sim \varepsilon^2$. After performing rescaling $\hat{x} = \bar{\mu}x$, $\hat{t} = \bar{\mu}t$ and removing the "hats," we have

$$\zeta_t + \varepsilon^2\left(\tilde{\alpha}_1\zeta^2\zeta_x + \tilde{\alpha}_3\zeta^4\zeta_x\right) + \mu\tilde{\beta}\zeta_{xxx} + \varepsilon\mu\tilde{\gamma}_2^*\zeta_x\zeta_{xx} + \varepsilon^2\mu\left(\tilde{\gamma}_{31}\zeta_x^3 + \tilde{\gamma}_{32}\zeta\zeta_x\zeta_{xx} + \tilde{\gamma}_{33}\zeta^2\zeta_{xxx}\right) + \mu^2\tilde{\beta}_1\zeta_{5x} + O\left(\varepsilon^3, \mu^3, \varepsilon\mu^2\right) = 0,$$

where, as above, the equation for $\eta(x, t)$ only differs from the above one by the reversed sign of the coefficient $\tilde{\gamma}_2^*$.

Summarizing the above considerations, in this particular case the balance between the leading (nonzero) dispersive and nonlinear terms occurs when $\varepsilon^2 = \mu$. In the coordinate system consisting of slow time and stretched x-axis $\hat{x} = x - ct$, $\hat{t} = \varepsilon^2 t$, the relevant equation for any of the interfaces is (the "hats" omitted)

$$\zeta_t + \tilde{\alpha}_1\zeta^2\zeta_x + \tilde{\alpha}_3\zeta^4\zeta_x + \tilde{\beta}\zeta_{xxx} = 0. \quad (10)$$

Equation (10) provides a description of motions and phenomena occurring in situations when the leading-order nonlinear and dispersive terms of the mKdV equation are negligible. In this case, the resulting equation contains two leading nonlinear terms of the same magnitude: the cubic and quintic nonlinearities. Both the terms consist of the product of a power (2nd or 4th) of the unknown function and its first-order spatial derivative. The resulting equation differs from the classical Gardner equation (that is frequently used for motions in situations containing zero-crossing of the coefficient at the quadratic nonlinearity, for example, in two-layer media with almost equal layers) by the presence of the quintic nonlinearity. This equation, obviously, belongs to the family of generalized Gardner equations[50,51] used to describe different properties of internal waves. Following the nomenclature of different extensions of the KdV equation, Eq. (10) may be called $2+4$ Korteweg-de Vries-like equation, abbreviated $2+4$ KdV.

The above has shown that auxiliary equations derived in the process of the asymptotic analysis had different appearances for different interfaces. Remarkably, Eq. (10) is universal in the sense that it is identical for both the upper and the lower interfaces. This feature is not completely unexpected as the classical mKdV equation is also universal for both the interfaces. Therefore, we can analyze only one equation (10) without the loss of generality. We shall do this for the upper interface $\zeta(x, t)$.

As equations $\zeta_t + \zeta^n\zeta_x + \zeta_{xxx} = 0$ with $n > 2$ are non-integrable (in particular, with respect to the Zakharov-Shabat method[52,53]), it is likely that Eq. (10) is also non-integrable. The question about its integrability is, however, out of the scope of the current study.

Differently from Eq. (8), Eq. (10) has two conservation laws of mass and energy

$$M = \int_{-\infty}^{\infty}\zeta dx, \quad E = \int_{-\infty}^{\infty}\zeta^2 dx. \quad (11)$$

The decisive parameters for wave motion in media described by Eq. (10) are the signs of the coefficients at its nonlinear terms. As mentioned above, the coefficient α_1 at the cubic nonlinearity is sign-variable in the vicinity of h_{cr}/H. The coefficient α_3 at the quintic nonlinearity is negative in this region but may change its sign at $h/H < (423 - 3\sqrt{6641})/1324 \approx 0.1348$. The analysis of the stratification corresponding to these values of the layers' thicknesses is out of the scope of this paper, mostly because it corresponds to quite large values of Δ but Eq. (10) is valid in the neighborhood of $\Delta = 0$.

An important feature of Eq. (10) is that it has solitary wave solutions. The relevant analysis of their existence and appearance (including the impact of the quintic nonlinearity on their shape compared to that of the classical mKdV equation) is presented in the following sections. It is well-known that solitons of the extensions of KdV equations of the form $\zeta_t + \zeta^n\zeta_x + \zeta_{xxx} = 0$ with $n \geq 4$ are unstable,[53] and that for such equations both smooth and localized initial conditions develop singularities within finite time (or at finite distances). To our knowledge, neither integrability of Eq. (10) nor stability of solitary wave solutions for Eq. (10) (or its analogues with combined (cubic and quintic) nonlinearity) has been addressed in literature. But all our numerical simulations of the initial problem for Eq. (10) with smooth and localized initial conditions showed a stable wave dynamics, with no evidence of instabilities or collapses even in interactions. Therefore, it seems plausible that the cubic nonlinearity plays a stabilizing role. This conjecture, however, requires a more detailed study, which is out of scope of this paper.

V. SOLITARY WAVE SOLUTIONS

Let us consider steady-state (in a properly chosen moving coordinate system) localized solutions $\zeta_s(x - Vt)$ to Eq. (10). Such solutions can be expressed in implicit form in terms of the integral

$$Y - Y_0 = \sqrt{\beta}\int_{v_0}^{v_S}\left(V\zeta^2 - \frac{\alpha_1}{6}\zeta^4 - \varepsilon\frac{\alpha_3}{15}\zeta^6\right)^{-1/2} d\zeta, \qquad (12)$$

where V is the propagation speed of the disturbance and $Y = x - Vt$. The substitution $A = \zeta^2$ would reduce Eq. (12) to a similar expression for the Gardner equation, and the solitary wave solutions can be derived. For $\alpha_1 > 0$ these solutions can be expressed explicitly

$$\zeta(Y) = \pm\sqrt{\frac{60V}{5\alpha_1 + \sqrt{25\alpha_1^2 + 240\alpha_3 V}\cosh\left(2Y\sqrt{\frac{V}{\beta}}\right)}}. \qquad (13)$$

It is easy to demonstrate that Eq. (13) describes localized solutions to Eq. (10). The maximum amplitude a of the excursions of the interface is obviously

$$a = \sqrt{\frac{60V}{5\alpha_1 + \sqrt{25\alpha_1^2 + 240\alpha_3 V}}}. \qquad (14)$$

Similarly to the solutions of the Gardner equation, the propagation speed of such solutions depends on their amplitude a

$$V = \frac{\alpha_1}{6}a^2 + \frac{\alpha_3}{15}a^4.$$

The natural limit for the wave speed stems from the request that expressions under the square root in Eqs. (13) and (14) must be non-negative for physically meaningful solutions. This condition means that the wave speed V has an upper limit

$$V_{\text{lim}} = -\frac{5\alpha_1^2}{48\alpha_3}. \qquad (15)$$

Consequently, the amplitude of solutions is also limited by the following value:

$$a_{\text{lim}} = \frac{1}{2}\sqrt{\frac{5\alpha_1}{-\alpha_3}}. \qquad (16)$$

The limiting amplitude in Eq. (16) can be expressed using the relative thickness of the lower and upper layers (equivalently, using the ratio $l = h/H$):

$$\frac{a_{\text{lim}}}{H} = \frac{2}{3}l^2\sqrt{15}\sqrt{\frac{9 - 26l}{1324l^3 - 1508l^2 + 513l - 45}}. \qquad (17)$$

The limiting amplitude reaches its maximum values $a_{\text{lim}} \approx 0.0864\,H$ at $h/H \approx 0.286$. This corresponds to the highest possible location $h + a_{\text{lim}} \approx 0.397\,H$ of the lower interface,

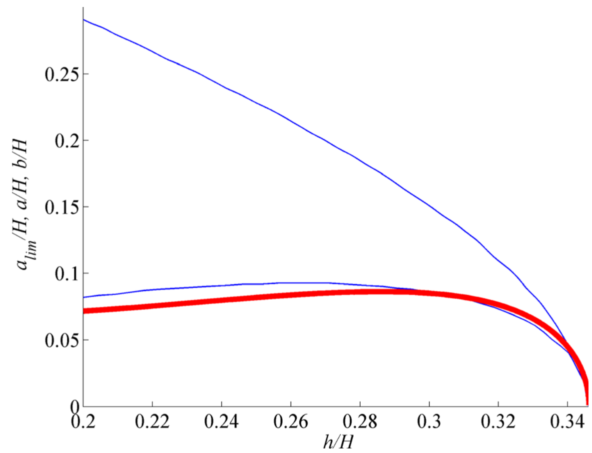

FIG. 5. (Color online) Nondimensional values of limiting amplitudes of solitary wave solutions to Eq. (10) (thick line) and nondimensional amplitudes of deviations of interfaces in three-layer symmetric flows with conjugate flows (thin lines, $a(b)$ in Ref. 60). The upper and lower thin lines show the deviation of the lower and the upper interfaces, respectively, for the positive disturbances. The situation is reversed for the negative solitary waves.

whereas the relevant $h \approx 0.3243\,H$ insignificantly differs from h_{cr}.

The dependence of the limiting amplitude a_{lim} on the relative thickness of layers h/H has quite a complicated appearance (Fig. 5). This amplitude increases slowly with the increase in h/H and reveals a very gently sloping maximum at $h/H \approx 0.286$. Further increase in h/H is first accompanied with a gentle decrease in a_{lim} that is replaced by almost explosive decrease in the limiting amplitude near the zero-crossing point $h/H = 9/26$.

For relatively low solitary wave speeds (equivalently, relatively low amplitudes), the profiles of solitary solutions to Eq. (10) resemble similar solutions to the mKdV equation (Fig. 6; see the expressions for the relevant parameters in

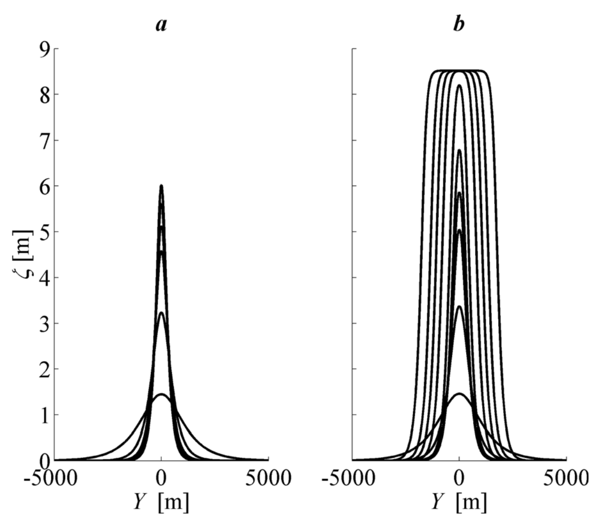

FIG. 6. The shape of solitary wave solutions to the classical mKdV equation (a) and for Eq. (10) (b), for solitary wave speeds of $V = 0.001$, 0.005, 0.01, 0.0125, 0.015, 0.0172, 0.017294, 0.01729412, 0.01729412063, 0.01729412063364 (m/s). The smaller speeds correspond to the lower-amplitude waves. The parameters for the calculations are indicated in Appendix. Note that several lines for large-amplitude waves in panel (a) are not separable from each other.

Appendix). For larger wave speeds (larger amplitudes), the two sets of solutions considerable differ from each other. The key difference is that large-amplitude solutions to Eq. (10) form a table-like wide signal that may, theoretically, infinitely widen in the process $V \to V_{\lim}$ whereas their amplitude asymptotically tends to $a_{\lim}$. Both positive and negative solitary wave solutions are possible for each combination of the signs of the coefficients of Eq. (10).

Similar table-like solutions are characteristic for some other equations containing higher-order nonlinear terms, for example, they exist for the Gardner equation that describes motions in the two-layer fluid and where the cubic nonlinearity is of the leading order. The existence of such wide table-like solitons and the possibility of their propagation in combinations with other solitons have been demonstrated for several other classes of nonlinear wave models.[27,38,54,55]

The solutions in question have several features similar to Gardner solitons; in particular, their amplitude is limited and the increase in the amplitude is accompanied by virtually unlimited widening of the wave profile. The set of Gardner solitons has much more rich in content variety of properties compared to the KdV and mKdV solitons. For example, this set allows for propagation of smaller solitons along the wide table-like crest of large solitons the amplitude of which is close to the limiting one, or the possibility of formation of two trains of solitons of different polarity during the process of dispersion of disturbances with amplitudes exceeding the limiting amplitude. Moreover, the development of the system of solitons is very much different for rectangular and smooth initial pulses of otherwise equivalent properties. It is natural to expect that the set of solutions to Eq. (10) also possesses a larger variety of different features compared to the ensembles of KdV or mKdV solitons.

The described table-like appearance of solutions for the relatively large-amplitude and rapidly moving disturbances with steep fronts may have substantial consequences in practical applications. The propagation of such disturbances is similar to the motion of smooth bores that possess considerable danger for objects on their way. Such horizontal motions (smooth bores, sometimes called conjugate flows) are frequent in vertically inhomogeneous fluids.[56–58] The performed analysis shows that such flows can naturally occur in three-layer fluids for some combinations of the layers' thicknesses and density variations.

Numerical simulations of flows in media with continuous vertical stratification with two pycnoclines and with a simplified three-layer model have been performed by several authors.[59–61] A comparison of the results of the theory of conjugate fluxes in symmetric three-layer environment with the values of the limiting amplitude based on Eqs. (16) and (17) is presented in Fig. 5. For the situations in which the thicknesses of the upper and lower layers are close to h_{cr}, the amplitudes of deviations of the interfaces from their undisturbed locations are relatively small and almost equal for all the presented cases, for conjugate fluxes in the nonlinear model and for the above derived estimate of $a_{\lim}$ in the weakly nonlinear model. If the thicknesses of the layers are considerably different from h_{cr}, the amplitudes of deviations of the upper and lower interfaces become substantially different. It is remarkable that the estimate of the deviation $a_{\lim}$ in the weakly nonlinear framework practically coincides with one of those for the fully nonlinear case for a wide range of h/H. The similarity occurs for the deviation of the interface that bends into the thinner outer (the uppermost or the lowermost) layer.

For the range of layers' depths $h > h_{cr}$ ($\Delta > 0$), a solution exists neither for our weakly nonlinear approach nor in the theory of conjugate flows.

VI. INTERACTIONS OF SOLITARY SOLUTIONS OF THE 2 + 4 KdV EQUATION

Generally speaking, interactions and collisions of solitary solutions to non-integrable evolution equations are inelastic. It is, therefore, not unexpected that solitary solutions to Eq. (10) interact inelastically with each other and with the background wave fields. As a demonstration of this feature, we present an example of numerically simulated collision of two solitonic solutions corresponding to the values of coefficients given in Appendix.

The numerical code used in integration of Eq. (10) employs an implicit pseudo-spectral method[62] that conserves the integrals presented in Eq. (11). The spatial domain was chosen based on the analytically estimated propagation speed

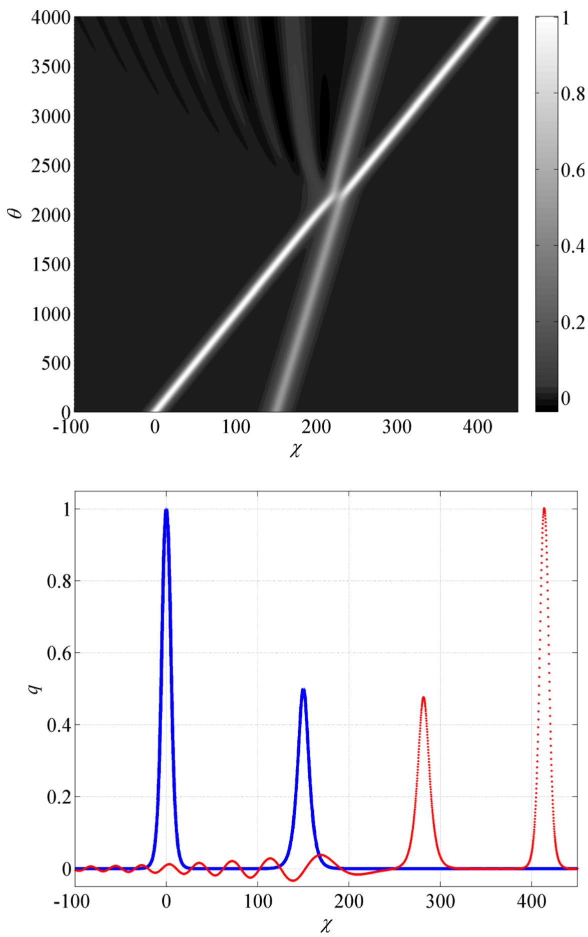

FIG. 7. (Color online) Interaction of solitary wave solutions of elevation to Eq. (10) in nondimensional coordinates: space-time plot (above); cross-section of the wave field at $\theta = 0$ (solid lines), and after the collision ($\theta = 4000$, dotted line, below).

of solitons and interaction time and was extended as occasion required. The numerical code was tested against exact analytical multisoliton solutions to the classical mKdV equation and by means of long-term tracking of propagation of exact solitary solutions to Eq. (10) in the absence of other disturbances. In particular, the results of the latter test did not change when the spatial step was decreased by a factor of two. For simplicity, we use the nondimensional form of Eq. (10)

$$q_\theta + q^2 q_\chi - q^4 q_\chi + q_{\chi\chi\chi} = 0, \tag{18}$$

where

$$\theta = \left|\frac{\alpha_1^3}{\alpha_3}\right|^{3/4} (\beta)^{-1/2} t, \quad \chi = \left|\frac{\alpha_1^3}{\alpha_3}\right|^{1/4} (\beta)^{-1/2} x, \quad q = \left|\frac{\alpha_1}{\alpha_3}\right|^{1/2} \varsigma. \tag{19}$$

The step in the x-direction was set to 0.2 and the step in time to 0.1. The performed tests indicated that the numerical inaccuracies for this choice of parameters is of the order of 10^{-6} whereas the value of the small parameter $\mu > 0.04$.

The initial state was composed from two solitary waves with nondimensional amplitudes of 1 and ± 0.5. The nondimensional limiting amplitude for the parameters in use is $\sqrt{5}/2 \approx 1.118$. The corresponding dimensional amplitudes would have been 7.62 m and ± 3.81 m, respectively. Before integrating this constellation, evolution of each of the counterparts was integrated until $\theta = 4000$. During this interval, the total error of the numerical solution (caused, e.g., by small distortions of the amplitude of the numerical solution owing to its discrete representation and by radiation of wave energy) did not exceed 1×10^{-6}.

The initial state for studies of interaction of these solitary waves was composed simply as a linear superposition of the counterparts. The smaller wave was placed ahead of the larger one, at a distance (counted as the distance between the relevant maxima) of 150 nondimensional units of length. The simulation was performed until $\theta = 4000$, that is, quite a long time after the interaction of the highest parts of the waves was terminated (Fig. 7).

The evolution of solitary waves of elevation resembled the typical scenarios for soliton interactions of similar type in the classical KdV and mKdV frameworks in which the taller wave overtakes the smaller one.[63,64] In such interactions, the counterparts usually lose their identity and merge into a composite structure at a certain instant. After a while,

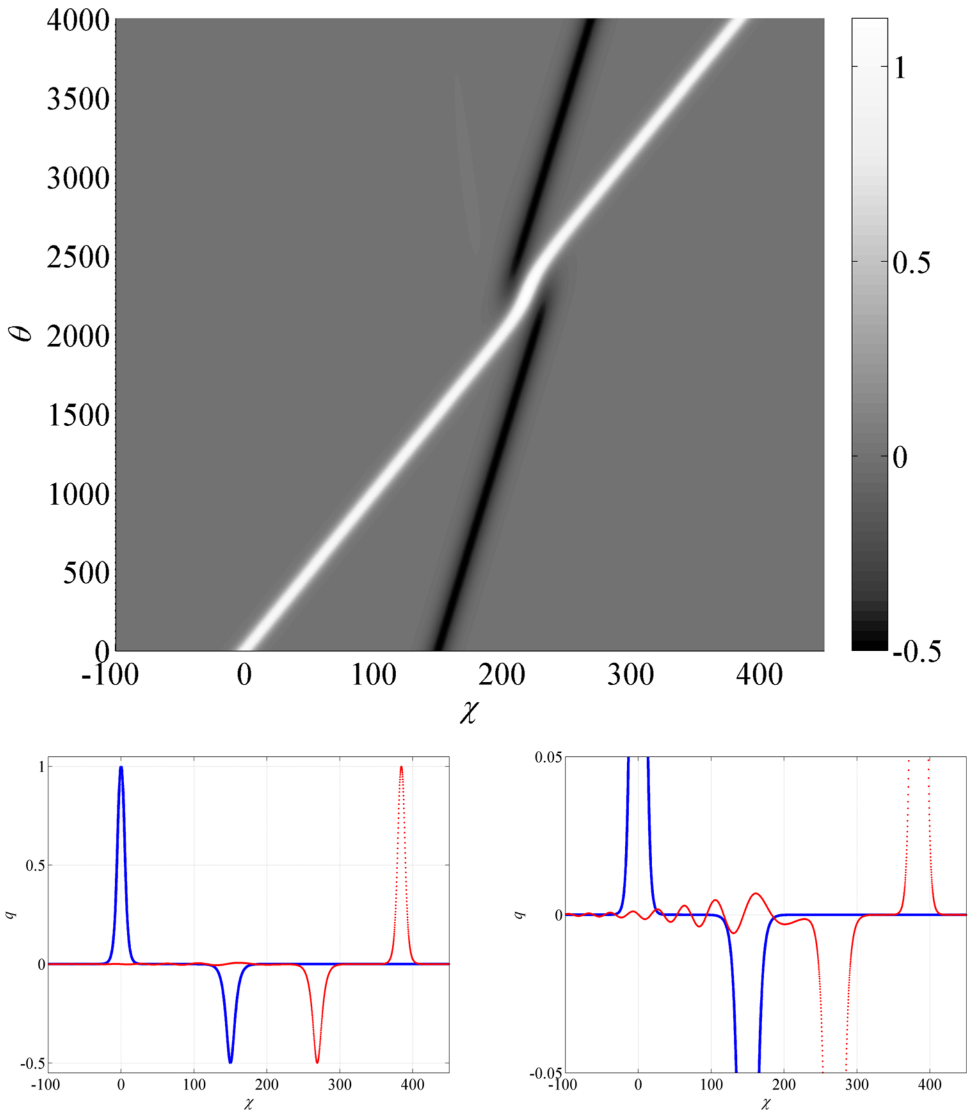

FIG. 8. (Color online) Interaction of solitary waves of different polarity in the framework of Eq. (10) in nondimensional coordinates: space-time plot (above); cross-section of the wave field at $\theta = 0$ (solid lines), and after the collision ($\theta = 4000$, dotted line, below). The right bottom panel is a zoomed version of the left bottom panel.

the counterparts emerge again whereas one cannot say whether the counterparts propagate through each other as waves do or collide as particles do. The interaction process is accompanied, as in the case of KdV solitons, by a clear decrease of the amplitude of the composite structure during the merging phase and by a substantial phase shift.

Differently from processes governed by integrable equations, the collision was accompanied by generation of very long and almost stationary localized depression area. The entire process was also accompanied by a modest radiation of wave energy from the interaction region. The amplitude of wavelike disturbances was about 4×10^{-2}, that is, by several orders of magnitude larger than the level of numerical errors ($<1 \times 10^{-6}$). This level of wave generation signifies the effects of dispersion on the evolution of the system. Note that the amplitude of the disturbances matches the magnitude of the relevant dispersive terms $O(\mu) = O(\Delta)$ that also is about 4×10^{-2}.

The collision of solitary waves of different polarities has a similar appearance. Both the counterparts largely survive the collision but the phase shift for the wave of depression is more pronounced (Fig. 8). The amplitude of radiated waves is, however, much smaller than in the above case and does not exceed 1×10^{-2}. Test simulations with a twice higher spatial resolution led to practically identical results. Therefore, the described side effects such as wave radiation in both cases and the formation of a long depression area in the collision of waves of elevation evidently are an inherent part of interactions of solitary waves in the framework of Eq. (10).

Consequently, collisions of solitary wave solutions of Eq. (10) basically have inelastic nature although both the intensity of wave radiation and changes to the amplitudes of the solitons are fairly minor. For example, the collision of waves of elevation led to the increase in the amplitude of the taller soliton from 1 to 1.002 and an accompanying decrease in the smaller solution from 0.5 to 0.477. The collision of waves of different polarity led to much smaller changes: the post-collision amplitudes of the waves were 1.001 and −0.499, respectively. Note that this effect also does not exceed the order of $O(\mu)$. The effects caused by interactions of different solitary waves with similar entities and with the wave background could, of course, be much larger during longer time intervals and/or caused by multiple collisions. As expected for non-integrable equations, such interactions should finally lead to damping of solitary waves to a level of magnitude at which they are either practically linear or are governed by a different balance of nonlinear and dispersive terms.

VII. CONCLUSIONS AND DISCUSSION

Although contemporary numerical methods and fully nonlinear approaches such as the method of conjugate flows allow for extensive studies into properties of highly nonlinear internal waves, many specific features can still be recognized, analyzed, and understood using classical methods for analytical studies into internal waves in ideal layered fluids and in the weakly nonlinear framework. Such fully analytical methods make it possible to exactly establish qualitative appearance of disturbances of different shapes and amplitudes, and, more importantly, to understand the specific features of the behavior of waves corresponding to situation where a substantial change in the overall regime of wave propagation is possible.

The performed analytical investigation of different regimes of wave propagation in a relatively simple but frequently occurring in nature symmetric three-layer environment reveals several interesting feature of wave shapes that are usually hidden in the analysis of non-symmetric situations. The key development is the derivation of a new nonlinear evolution equation that describes the wave motion in situations where all the leading-order nonlinear terms in the classical modified Korteweg-de Vries (mKdV) equation vanish simultaneously. Such situations may happen quite frequently in relatively shallow non-tidal strongly stratified basins such as the Baltic Sea. In this case, the evolution equation governing wave motion contains two nonlinear terms (cubic and quintic nonlinearities) of the same magnitude. This equation is obtained using the basically standard asymptotic procedure that is widely used in similar problems and is of the second order of accuracy as the mKdV equation for the non-symmetric situation.

The resulting equation differs from the mKdV equation in two important aspects. First, it reflects a different balance between the (higher-order) nonlinear terms and the dispersive terms compared to that exploited in the mKdV equation. More importantly, this equation contains two nonlinear terms of the same magnitude—the cubic and quintic nonlinearities, the latter one distinguishing the resulting equation from the mKdV equation. The resulting equation has solitary wave solutions. As this equation probably is not integrable, the possible solitonic nature of these solutions, the persistence of these solutions in interactions and the existence of multisoliton solutions is a subject of further research.

The presence of the quintic nonlinearity does not substantially modify the shape of the solitons of relatively small amplitude but leads to radical changes in the appearance of larger-amplitude solitary waves. Their amplitude and propagation speed are limited. Larger-amplitude solitons have a table-like shape with very steep fronts. The motion of such solitary waves may be accompanied by high water speeds and strong hydrodynamic loads in the areas where the structure of medium is favorable for their existence. It is likely that the classical solitary solutions to the mKdV equation are transformed into such structures when they approach sea areas where the coefficients at the lower-order nonlinear terms vanish.

ACKNOWLEDGMENTS

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APPENDIX: COEFFICIENTS OF THE EXTENDED KdV EQUATION

The coefficients of, Eq. (6), expressed in nondimensional form in terms of $l = h/H$ are as follows:

$$c^2 = \frac{gh\Delta\rho}{\rho}; \quad \alpha = 0;$$

$$\frac{\beta}{cH^2} = -l\frac{4l-3}{12}; \quad \frac{\alpha_1}{c}H^2 = -\frac{3(26l-9)}{8l^3};$$

$$\frac{\beta_1}{cH^4} = l\frac{16l^3-45l+30}{1440}; \quad \gamma_1 = 0;$$

$$\frac{\gamma_2^*}{cH} = \frac{3(8l^2-10l+3)}{8l}; \quad \frac{\alpha_2^*}{c}H^3 = \frac{9(52l^2-44l+9)}{16l^5};$$

$$\frac{\beta_2}{cH^6} = -\frac{c\left(320l^5-1008l^4+1260l^3-945l^2+630l-252\right)l}{120\,960};$$

$$\frac{\gamma_{21}}{cH^3} = -\frac{256l^4-200l^3-198l^2+231l-57}{96l};$$

$$\frac{\gamma_{22}}{cH^3} = -\frac{352l^4-704l^3+486l^2-123H^3l+6}{192l};$$

$$\gamma_{23} = 0; \quad \frac{\gamma_{31}}{c} = \frac{40l^3+726l^2-819l+207}{96l^3};$$

$$\frac{\gamma_{32}}{c} = \frac{1912l^3-678l^2-927l+342}{96l^3};$$

$$\frac{\gamma_{33}}{c} = \frac{952l^3-1110l^2+369l-36}{96l^3};$$

$$\frac{\alpha_3}{c}H^4 = -\frac{9(1324l^3-1508l^2+513l-45)}{128l^7};$$

$$\frac{\beta_3}{cH^8} = -\frac{\left(17152l^7-76800l^6+151200l^5-171360l^4+121275l^3+56700l^2+21420l-6120\right)l}{29030400};$$

$$\frac{\gamma_{41}^*}{cH^5} = \frac{11264l^6-33376l^5+37824l^4-19350l^3+4335l^2-684l+180}{34560l};$$

$$\frac{\gamma_{42}^*}{cH^5} = \frac{5504l^6-4192l^5+40152l^4-55030l^3+31005l^2-7218l+444}{11520l};$$

$$\frac{\gamma_{43}^*}{cH^5} = \frac{4864l^6-14240l^5+7968l^4-13578l^3+21219l^2-10809l+1926}{6912l};$$

$$\frac{\gamma_{51}}{cH^2} = -\frac{16544l^5-18784l^4+16842l^3-25377l^2+18504l-3996}{2304l^3};$$

$$\frac{\gamma_{52}}{cH^2} = -\frac{3(32l^3-64l^2+42l-9)}{256l};$$

$$\frac{\gamma_{53}}{cH^2} = -\frac{56864l^5+93936l^4-330570l^3+26485l^2-76320l+6804}{11520l^3};$$

$$\frac{\gamma_{54}}{cH^2} = -\frac{48672l^5-94256l^4+64590l^3-18255l^2+2430l-216}{11520l^3};$$

$$\frac{\gamma_{55}}{cH^2} = \frac{189280l^5-646880l^4+743190l^3-334395l^2+33030l+6804}{11520l^3};$$

$$\frac{\gamma_{56}}{cH^2} = \frac{121120l^5-275560l^4+152040l^3+70800l^2-91035l+20844}{5760l^3};$$

$$\frac{\gamma_{61}^*}{c}H = -\frac{1424l^4 + 332l^3 - 3444l^2 + 2271l - 405}{64l^5};$$

$$\frac{\gamma_{62}^*}{c}H = -\frac{10928l^4 - 11476l^3 - 192l^2 + 2787l - 594}{128l^5};$$

$$\frac{\gamma_{63}^*}{c}H = -\frac{1904l^4 - 3172l^3 + 1848l^2 - 441l + 36}{64l^5}.$$

The coefficients for the first-order terms and for the quadratic nonlinearity for the symmetric three-layer medium have been obtained in Ref. 22 and presented here only for completeness.

In calculations, we use the following parameters of the medium: total depth $H = 100$ м, depth of the uppermost and lowermost layers $h = 30$ m and the relative change in the density $\Delta\rho/\rho = 1 \times 10^{-2}$. The corresponding values of the linear wave speed and the coefficients of Eq. (10) are as follows:

Parameter	Value
c (m/s)	1.72
α (s^{-1})	0.0
β (m^3/s)	771.98
α_1 (m·s)$^{-1}$	0.002859
α_3 (m^3·s)$^{-1}$	−0.00004924
$a_{\rm lim}$ (m)	8.5194
$V_{\rm lim}$ (m/s)	0.01729

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