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Macdonald polynomials and BGG reciprocity for current algebras

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Abstract We study the category $\mathcal{I}_{\mathrm{gr}}$ of graded representations with finite-dimensional graded pieces for the current algebra $\mathfrak{g} \otimes \mathbf{C}[t]$ where $\mathfrak{g}$ is a simple Lie algebra. This category has many similarities with the category $\mathcal{O}$ of modules for $\mathfrak{g}$, and in this paper, we prove an analog of the famous BGG duality in the case of $\mathfrak{sl}_{n+1}$.

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1 Introduction

The current algebra associated with a simple Lie algebra is just the Lie algebra of polynomial maps from $\mathbf{C} \to \mathfrak{g}$ and can be identified with the space $\mathfrak{g} \otimes \mathbf{C}[t]$ with the obvious commutator. Another way of thinking of this is as a maximal parabolic subalgebra in the corresponding untwisted affine Kac–Moody algebra. The Lie algebra and its universal enveloping algebra inherit a grading coming from the natural grading on $\mathbf{C}[t]$. We are interested in the category $\mathcal{I}$ of $\mathbf{Z}$-graded modules for $\mathfrak{g}[t]$ with the restriction that the graded pieces are finite-dimensional. Originally, the study of this category was largely motivated by its relationship with the representation theory of affine and quantum affine algebras associated with a simple Lie algebra $\mathfrak{g}$. However, it is also now of independent interest since it yields connections with problems arising in mathematical physics, for instance the $X = M$ conjectures, see [1,8,13].

The category $\mathcal{I}$ is a non-semisimple category and has many similarities with other well-known categories of representations in Lie theory. However, there are many essential differences in the theory as we shall see below, which makes it quite remarkable that one can formulate (see [3]) the famous Bernstein–Gelfand–Gelfand (BGG)-reciprocity result for the category $\mathcal{O}$. In [3], the result was proved for $\mathfrak{sl}_2$ by different methods. In the current paper, we use the combinatorics of Macdonald polynomials to extend the result to $\mathfrak{sl}_{n+1}$.

The main ingredients, in the original theorem of Bernstein–Gelfand–Gelfand, were the irreducible modules $V(\lambda)$ for a simple Lie algebra, the Verma module $M(\lambda)$ and the projective cover $P(\lambda)$ of $V(\lambda)$ where λ is a linear functional on a Cartan subalgebra of $\mathfrak{g}$. The Verma modules have a nice freeness property, and it is relatively easy to prove that the projective module has a filtration by Verma modules. Further, the Verma modules have a Jordan–Hölder series, and the BGG-theorem states that the filtration multiplicity of the Verma module $M(\mu)$ in the projective $P(\lambda)$ is equal to the Jordan–Hölder multiplicity of $V(\lambda)$ in $M(\mu)$.

In our context, the irreducible objects of $\mathcal{I}$ are indexed by two parameters, (λ, r) where λ varies over the index set of irreducible finite-dimensional representations of $\mathfrak{g}$ and r varies over the integers. The category $\mathcal{I}$ also contains the projective covers $P(\lambda, r)$ of the simple object $V(\lambda, r)$. The appropriate analog of the Verma module is the global Weyl module $W(\lambda, r)$ defined originally in [7] via generators and relations. It is in fact the maximal quotient of $P(\lambda, r)$ with respect to the property that the eigenvalues of $\mathfrak{h}$ lie in a certain finite set. At this stage, however, two points of similarity with the object in the category $\mathcal{O}$, fail: the global Weyl modules do have a nice freeness property, but it is for a much smaller algebra than in the case of $\mathfrak{g}$. Thus, we have to work much harder to prove that the projective modules have a filtration by global Weyl modules. We use an idea from algebraic groups (see [9]) and define a filtration on any object of $\mathcal{I}$. This filtration is canonically defined once we fix a total order on the set of dominant integral weights of $\mathfrak{g}$, and so, we call this a canonical o-filtration of the object. We show that the successive quotients of the filtration are isomorphic to a quotient of a direct sum of global Weyl modules.

The second difficulty we encounter is that the global Weyl modules are not of finite length. To circumvent this, we recall that they have a unique maximal finite-dimensional quotient called the local Weyl modules (see [4,7]), and this allows us to formulate the desired result. Namely, we prove that the projective module $P(\lambda, r)$ has a filtration by global Weyl modules, and the multiplicity of $W(\mu, s)$ in $P(\lambda, r)$ is the multiplicity of $V(\lambda, s)$ in the local Weyl module $W_{\mathrm{loc}}(\mu, r)$. This result was proved in [3] in the case of $\mathfrak{sl}_2$ and conjectured to be true in general.

In this paper, we establish the conjecture $\mathfrak{sl}_{n+1}$, the conjecture is true by showing that the canonical o-filtration of $P(\lambda, r)$ is actually a filtration by global Weyl modules. Our proof differs from the proof of BGG reciprocity in the category $\mathcal{O}$, which relies on homological properties of Verma modules and their duals. In our context, this approach is problematic and needs a lot of modification. Partial results may be found in [2].

To explain the restriction to $\mathfrak{sl}_{n+1}$ and the connection with Macdonald polynomials, we need some further comments on local Weyl modules. It was proved in [6] that for $\mathfrak{sl}_{n+1}$, the local Weyl module is isomorphic to a Demazure module in a level one representation of the affine Kac–Moody algebra. In [14], it was shown that the character of such a Demazure module is given by specialization of a Macdonald polynomial at $t = 0$. In Sect. 5, we use several properties of Macdonald polynomials to establish certain combinatorial identities. In Sect. 6, we prove that these identities have a representation theoretic interpretation, namely they give a relation between the Hilbert series of $P(\lambda, r)$ and a sum of Hilbert series of global Weyl modules (with multiplicity). This is enough to establish the reciprocity result. In the general simply laced case, it is still true that the local Weyl modules are Demazure modules, and their characters are given in [11] via non-symmetric Macdonald polynomials. In the non-simply laced case, it was proved in [13] that local Weyl modules have a filtration by Demazure modules and the characters are known. The missing piece in the case when $\mathfrak{g}$ is not of type $\mathfrak{sl}_{n+1}$ is thus the combinatorial problem studied in Sect. 5. It is necessary to establish the correct version of Lemma 5, and we will return to this elsewhere.

2 Preliminaries

2.1. Throughout this paper, we denote by $\mathbf{C}$ the field of complex numbers and $\mathbf{Z}$ (resp. $\mathbf{Z}_+$) the set of integers (resp. non-negative integers). For a Lie algebra, $\mathfrak{a}$ denote by $\mathbf{U}(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$. If t is an indeterminate, let $\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbf{C}[t]$ be the Lie algebra with commutator given by,

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg, \quad a, b \in \mathfrak{a}, \quad f, g, \in \mathbf{C}[t].$$

We identify $\mathfrak{a}$ with the Lie subalgebra $\mathfrak{a} \otimes 1$ of $\mathfrak{a}[t]$. The Lie algebra $\mathfrak{a}[t]$ has a natural $\mathbf{Z}_+$-grading given by the powers of t and this also induces a $\mathbf{Z}_+$-grading on $\mathbf{U}(\mathfrak{a}[t])$, and $\mathbf{U}(\mathfrak{a}[t])[0] = \mathbf{U}(\mathfrak{a})$. The graded pieces of $\mathbf{U}(\mathfrak{a}[t])$ are $\mathfrak{a}$-modules under left and right multiplication by elements of $\mathfrak{a}$ and hence also under the adjoint action of $\mathfrak{a}$.

2.2. Throughout the paper, $\mathfrak{g}$ denotes a finite-dimensional complex simple Lie algebra of rank n and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$. Let $I = \{1, \ldots, n\}$ and fix a set $\{\alpha_i : i \in I\}$ of simple roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and a set $\{\omega_i : i \in I\}$ of fundamental weights. Let Q (resp. Q^+) be the integer span (resp. the non-negative integer span) of $\{\alpha_i : i \in I\}$ and similarly define P (resp. P^+) to be the $\mathbf{Z}$ (resp. $\mathbf{Z}_+$) span of

$\{\omega_i : i \in I\}$. Let $\{x_i^{\pm}, h_i : i \in I\}$ be a set of Chevalley generators of $\mathfrak{g}$ and let $\mathfrak{n}^{\pm}$ be the Lie subalgebra of $\mathfrak{g}$ generated by the elements $x_i^{\pm}$, $i \in I$. We have,

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathbf{U}(\mathfrak{g}) = \mathbf{U}(\mathfrak{n}^-) \otimes \mathbf{U}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{n}^+).$$

Let W be the Weyl group of $\mathfrak{g}$ and let $w_0 \in W$ be the longest element of W. Given $\lambda, \mu \in \mathfrak{h}^*$, we say that $\lambda \le \mu$ iff $\lambda - \mu \in Q^+$.

2.3. For any $\mathfrak{g}$-module M and $\mu \in \mathfrak{h}^*$, set

$$M_\mu = \{m \in M \ : \ hm = \mu(h)m, \quad h \in \mathfrak{h}\}, \quad \mathrm{wt}(M) = \left\{\mu \in \mathfrak{h}^* \ : \ M_\mu \neq 0\right\}.$$

We say M is a *weight module* for $\mathfrak{g}$ if

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu.$$

Any finite-dimensional $\mathfrak{g}$-module is a weight module. It is well known that the set of isomorphism classes of irreducible finite-dimensional $\mathfrak{g}$-modules is in bijective correspondence with P^+. For $\lambda \in P^+$, we denote by $V(\lambda)$ a representative of the corresponding isomorphism class. Then, $V(\lambda)$ is generated as a $\mathfrak{g}$-module by a vector v_λ with defining relations

$$\mathfrak{n}^+ v_\lambda = 0, \quad h v_\lambda = \lambda(h) v_\lambda, \quad (x_i^-)^{\lambda(h_i)+1} v_\lambda = 0, \quad h \in \mathfrak{h}, \ \ i \in I.$$

and recall that $\mathrm{wt}\, V(\lambda) \subset \lambda - Q^+$. The module $V(0)$ is the trivial module for $\mathfrak{g}$, and we shall write it as $\mathbf{C}$. Let $\mathbf{Z}[P]$ be the integral group ring $\mathbf{Z}[P]$ spanned by elements $e(\mu)$, $\mu \in P$ and given a finite-dimensional $\mathfrak{g}$-module, let

$$\mathrm{ch}_{\mathfrak{g}} M = \sum_{\mu \in P} \dim_{\mathbf{C}} M_\mu e(\mu).$$

The set $\{\mathrm{ch}_{\mathfrak{g}}\, V(\mu) : \mu \in P^+\}$ is a linearly independent subset of $\mathbf{Z}[P]$.

We say that M is a *locally finite-dimensional* $\mathfrak{g}$-module if it is a direct sum of finite-dimensional $\mathfrak{g}$-modules, in which case M is necessarily a weight module. Using Weyl's theorem, one knows that a locally finite-dimensional $\mathfrak{g}$-module M is isomorphic to a direct sum of modules of the form $V(\lambda)$, $\lambda \in P^+$ and hence $\mathrm{wt}\, M \subset P$.

2.4. Let $\mathcal{I}$ be the category whose objects are graded $\mathfrak{g}[t]$-modules V with finite-dimensional graded components and where the morphisms are maps of graded $\mathfrak{g}[t]$-modules. Thus, an object V of $\mathcal{I}$ is a $\mathbf{Z}$-graded vector space $V = \oplus_{s \in \mathbf{Z}} V[s]$, $\dim V[s] < \infty$, which admits a left action of $\mathfrak{g}[t]$ satisfying

$$(\mathfrak{g} \otimes t^r) V[s] \subset V[s+r], \quad s, r \in \mathbf{Z}.$$

For all $r \in \mathbf{Z}$, the subspace $V[r]$ is a finite-dimensional $\mathfrak{g}$-module. A morphism between two objects V and W of $\mathcal{I}$ is a degree zero map of graded $\mathfrak{g}[t]$-modules.

Clearly, $\mathcal{I}$ is closed under taking submodules, quotients and finite direct sums. For any $r \in \mathbf{Z}$ let $\tau_r : \mathcal{I} \to \mathcal{I}$ be the grade shifting operator given by $V \mapsto \tau_r V$, where $(\tau_r V)[s] = V[s-r]$ for all $s \in \mathbf{Z}$. The *graded character* (resp. Hilbert series) of $V \in \operatorname{Ob}\mathcal{I}$ is the element of the space of power series $\mathbf{Z}[P][[q, q^{-1}]]$, given by

$$\operatorname{ch}_{\mathrm{gr}} V = \sum_{r\in\mathbf{Z}} \operatorname{ch}_{\mathfrak{g}}(V[r])q^r, \quad \mathbb{H}(V) = \sum_{r\in\mathbf{Z}} \dim V[r]q^r.$$

Given $V \in \operatorname{Ob}\mathcal{I}$, the restricted dual is

$$V^* = \bigoplus_{r\in\mathbf{Z}} V[r]^*, \quad V^*[r] = V[-r]^*.$$

Then $V^* \in \operatorname{Ob}\mathcal{I}$ with the usual action:

$$(xt^s)v^*(w) = -v^*(xt^s w),$$

and $(V^*)^* \cong V$ as objects of $\mathcal{I}$. Note that if $V \in \operatorname{Ob}\mathcal{I}$, then

$$\operatorname{ch}_{\mathrm{gr}} V^* := \sum_{r\in\mathbf{Z}} \operatorname{ch}_{\mathfrak{g}}(V[r]^*)u^{-r}.$$

3 The main result

3.1. Let $\mathrm{ev}_0 : \mathfrak{g}[t] \to \mathfrak{g}$ be the homomorphism of Lie algebras, which maps $x \otimes f \mapsto f(0)x$. The kernel of this map is a graded ideal in $\mathfrak{g}[t]$, and hence, any $\mathfrak{g}$-module V can be regarded in an obvious way as a graded $\mathfrak{g}[t]$-module denoted $\mathrm{ev}_0 V$. Clearly, $\mathrm{ev}_0 V$ is an object of $\mathcal{I}$ if $\dim V < \infty$.

For each $r \in \mathbf{Z}$ and $\lambda \in P^+$, we abbreviate $V(\lambda, r) := \tau_r \,\mathrm{ev}_0 V_\lambda$. Fix an element $v_{\lambda,r} \in V(\lambda, r)$ corresponding to v_λ. The following elementary result was proved in [5, Proposition 1.3].

Lemma *Any irreducible object in $\mathcal{I}$ is isomorphic to $V(\mu, r)$ for a unique element $(\mu, r) \in P^+ \times \mathbf{Z}$ and $V(\mu, r)^* \cong V(-w_0\mu, -r)$. Further, $V \in \operatorname{Ob}\mathcal{I}$ is semisimple iff*

$$V \cong \oplus_{(\lambda,r)\in P^+\times\mathbf{Z}} V(\lambda, r)^{m(\lambda,r)}, \quad m(\lambda, r) \in \mathbf{Z}_+.$$

□

Suppose that $\dim V < \infty$ and r is minimal such that $V[r] \neq 0$. Then, we have a short exact sequence of $\mathfrak{g}[t]$-modules

$$0 \to \bigoplus_{s>r} V[s] \to V \to \tau_r \,\mathrm{ev}_0 V[r] \to 0.$$

A simple induction on dim V now proves that for all $(\lambda, s) \in P^+ \times \mathbf{Z}$, we have

$$[V : V(\lambda, s)] = \dim \operatorname{Hom}_{\mathfrak{g}}(V(\lambda), V[s]) = \dim \operatorname{Hom}_{\mathfrak{g}}(V[s], V(\lambda)), \tag{3.1}$$

where $[V : V(\lambda, s)]$ is the multiplicity of $V(\lambda, s)$ in a Jordan–Hölder series of V (the Jordan–Hölder theorem obviously holds in $\mathcal{I}$).

3.2. For $\lambda \in P^+$ and $r \in \mathbf{Z}$, the local Weyl module, $W_{\text{loc}}(\lambda, r)$, is the $\mathfrak{g}[t]$-module generated by an element $w_{\lambda,r}$ with relations:

$$\mathfrak{n}^+[t]w_{\lambda,r} = 0, \quad (x_i^-)^{\lambda(h_i)+1}w_{\lambda,r} = 0,$$
$$(h \otimes t^s)w_{\lambda,r} = \delta_{s,0}\lambda(h)w_{\lambda,r},$$

where $i \in I, h \in \mathfrak{h}$ and $s \in \mathbf{Z}_+$. The next proposition summarizes the results on the local Weyl module, which are needed to state our main result. A proof of this proposition can be found in [7].

Proposition *Let* $(\lambda, r) \in P^+ \times \mathbf{Z}$. *The* $\mathfrak{g}[t]$*-module* $W_{\text{loc}}(\lambda, r)$ *is indecomposable and finite-dimensional. Moreover,* $\dim W_{\text{loc}}(\lambda, r)_\lambda = \dim W_{\text{loc}}(\lambda, r)[r]_\lambda = 1$, *and* $V(\lambda, r)$ *is the unique irreducible quotient of* $W_{\text{loc}}(\lambda, r)$. □

3.3. For $(\lambda, r) \in P^+ \times \mathbf{Z}$, the global Weyl module $W(\lambda, r)$ is generated as a $\mathfrak{g}[t]$-module by an element $w_{\lambda,r}$ with relations:

$$\mathfrak{n}^+[t]w_{\lambda,r} = 0, \quad (x_i^-)^{\lambda(h_i)+1}w_{\lambda,r} = 0, \quad hw_{\lambda,r} = \lambda(h)w_{\lambda,r},$$

where $i \in I$ and $h \in \mathfrak{h}$. Parts (i) and (ii) of the following result can be found for loop algebras in [7, Theorem 1, Proposition 2.1]. The same proof works for the current algebras. One can also use Theorems 4.4 and 6.1(ii) in [4]. Part (iii) is proved in Section 3 of [3].

Proposition *For* $(\lambda, r) \in P^+ \times \mathbf{Z}$, *we have that* $W(\lambda, r)$ *is an indecomposable object of* $\mathcal{I}$ *and* $\operatorname{wt} W(\lambda, r) = \operatorname{wt} V(\lambda, r)$. *Further,*

(i) $W_{\text{loc}}(\lambda, r)$ *is a quotient of* $W(\lambda, r)$, *and* $V(\lambda, r)$ *is the unique irreducible quotient of* $W(\lambda, r)$.
(ii) $W(0, r) \cong V(0, r)$ *and if* $\lambda \neq 0$, *the modules* $W(\lambda, r)$ *are infinite-dimensional.*
(iii) We have,

$$\operatorname{ch}_{\text{gr}} W(\lambda, r) = \operatorname{ch}_{\text{gr}} V(\lambda, r) + \sum_{s>r} \sum_{\mu \leq \lambda} \dim \operatorname{Hom}_{\mathfrak{g}}(W\lambda, r)[s] : V(\mu)) \operatorname{ch}_{\text{gr}} V(\mu, s),$$

and $\{\operatorname{ch}_{\text{gr}} W(\lambda, r) : (\lambda, r) \in P^+ \times \mathbf{Z}\}$ *is a linearly independent subset of* $\mathbf{Z}[P][[u, u^{-1}]]$. □

3.4. In the paper, we shall be interested primarily in the subcategory of $\mathcal{I}$ which consists of objects which have only finitely many nonzero pieces with negative grade. It is not hard to see (and we say more about this later) that such objects obviously have a nontrivial maximal semisimple quotient and in fact a decreasing filtration where the successive quotients are semisimple. On the other hand, the objects could have trivial socle. This makes it natural to consider decreasing filtrations for such modules. The dual case was studied in [2] where one has increasing filtrations.

We say that $M \in \operatorname{Ob}\mathcal{I}$ admits a filtration by global Weyl modules if there exists a decreasing family of submodules

$$M = M_0 \supset M_1 \supset \cdots, \quad \bigcap_k M_k = \{0\},$$

such that

$$M_k/M_{k+1} \cong \bigoplus_{(\lambda,r)\in P^+\times\mathbf{Z}} W(\lambda,r)^{m_k(\lambda,r)},$$

for some choice of $m_k(\lambda,r) \in \mathbf{Z}_+$. Since $\dim M[r]_\lambda < \infty$ for all $(\lambda,r) \in P^+ \times \mathbf{Z}$, we see that if M has a filtration by global Weyl modules, then $m_k(\lambda,r) = 0$ for all but finitely many k. Further, we have

$$\operatorname{ch}_{\mathrm{gr}} M = \sum_{k\geq 0} \operatorname{ch}_{\mathrm{gr}} M_k/M_{k+1} = \sum_{(\lambda,r)\in\mathbf{Z}} \left(\sum_{k\geq 0} m_k(\lambda,r)\right) \operatorname{ch}_{\mathrm{gr}} W(\lambda,r).$$

Proposition 3.3(iii) now implies that the filtration multiplicity

$$[M : W(\lambda,r)] = \sum_{k\geq 0} m_k(\lambda,r),$$

is well defined and independent of the choice of the filtration.

3.5. The category $\mathcal{I}$ contains the projective cover of a simple object. For $(\lambda,r) \in P^+ \times \mathbf{Z}$, set

$$P(\lambda,r) = \mathbf{U}(\mathfrak{g}[t]) \otimes_{\mathbf{U}(\mathfrak{g})} V(\lambda,r). \tag{3.2}$$

Note that

$$P(\lambda,r)[r] \cong_{\mathfrak{g}} V(\lambda), \quad P(\lambda,r)[s] = 0 \quad s < r.$$

The following was proved in [5, Proposition 2.1].

Proposition *For $(\lambda,r) \in P^+ \times \mathbf{Z}$, the object $P(\lambda,r)$ is generated by the element $p_{\lambda,r} = 1 \otimes v_\lambda$ with defining relations:*

$$\mathfrak{n}^+ p_{\lambda,r} = 0, \quad h p_{\lambda,r} = \lambda(h) p_{\lambda,r}, \quad (x_i^-)^{\lambda(h_i)+1} p_{\lambda,r} = 0,$$

and is the projective cover in $\mathcal{I}$ of $V(\lambda, r)$. Moreover, if $M \in \operatorname{Ob}\mathcal{I}$ then

$$\operatorname{Hom}_{\mathcal{I}}(P(\lambda, r), M) \cong \operatorname{Hom}_{\mathfrak{g}}(V(\lambda), M[r]).$$

□

3.6. The main result of this paper is the following. It was conjectured in [3] for all $\mathfrak{g}$ and proved there in the case of $\mathfrak{sl}_2$.

Theorem *Assume that $\mathfrak{g}$ is of type $\mathfrak{sl}_{n+1}$. For $(\lambda, r) \in P^+ \times \mathbf{Z}$, the module $P(\lambda, r)$ has a filtration by global Weyl modules and*

$$[P(\lambda, r) : W(\mu, s)] = [W_{\mathrm{loc}}(\mu, r) : V(\lambda, s)].$$

Remark If A is any graded commutative associative algebra, then we can set $\mathfrak{g}[A] := \mathfrak{g} \otimes A$ and define the category $\mathcal{I}$ accordingly. The global and local Weyl modules and the projective modules have their analogs, and hence, one could ask if Theorem 3.6 remains true in this case. The graded characters of the local and global Weyl modules which play a crucial role in our paper are not known in this generality. However, the first step of the proof of the theorem (see Proposition 3.8 below) does go through in this level of generality.

3.7. The proof of the theorem is in two steps, but to state these two steps we need some additional notation and an alternative definition of $W(\lambda, r)$. From now on, we fix a linear order (which is a refinement of the standard partial order) on P^+ via $P^+ = \{\lambda_0, \lambda_1, \dots, \lambda_k, \dots\}$ so that:

$$\lambda_r - \lambda_s \in Q^+ \implies r \geq s.$$

We shall need the following result. Versions of this have been proved in the literature (see [4] for instance). But we include a proof here since we need it in this precise form for this paper.

Lemma *For $k \geq 0$, the global Weyl module $W(\lambda_k, r)$ is the quotient of $P(\lambda_k, r)$ obtained by imposing the single additional relation $\mathfrak{n}^+[t]p_{\lambda_k,r} = 0$, and hence,* $\operatorname{wt} W(\lambda_k, r) \subset \lambda_k - Q^+$*. Equivalently, $W(\lambda_k, r)$ is the maximal quotient of $P(\lambda_k, r)$ whose weights lie in $\cup_{s=0}^k \lambda_s - Q^+$.*

Proof The first statement is obvious from the defining relations of $W(\lambda_k, r)$ and $P(\lambda_k, r)$. For the second, let

$$\tilde{W} = P(\lambda_k, r) \Big/ \sum_{s>k} \mathbf{U}(\mathfrak{g}[t]) P(\lambda_k, r)_{\lambda_s}.$$

Clearly

$$\operatorname{wt} \tilde{W} \subset \bigcup_{s=0}^{k} \lambda_s - Q^+,$$

and $\tilde{W}$ is the maximal quotient with this property. Let $\tilde{w} \in \tilde{W}$ be the image of $p_{\lambda_k,r}$. Since wt $W(\lambda_k, r) \subset \lambda_k - Q^+$, it follows that $W(\lambda_k, r)$ is a quotient of $\tilde{W}$ via a morphism which maps $\tilde{w} \to w_{\lambda_k,r}$. The element $w' = (x_i^+ \otimes t^s)\tilde{w}$ has weight $\lambda_k + \alpha_i > \lambda_k$. If it is nonzero in $\tilde{W}$, then it would follow from the representation theory of $\mathfrak{g}$ that $\tilde{W}_{\lambda_s} \neq 0$ for some $s > k$, which is a contradiction. Hence, $\mathfrak{n}^+[t]\tilde{w} = 0$, and there exists a well-defined surjective morphism $W(\lambda_k, r) \to \tilde{W}$ sending $w_{\lambda,r} \to \tilde{w}$ proving that $W(\lambda, r) \cong \tilde{W}$ as required. □

3.8. The first step needed to prove the main result is the following.

Proposition *Let $\mathfrak{g}$ be an arbitrary simple Lie algebra and let $M \in \mathrm{Ob}\,\mathcal{I}$ be such that $M[r] = 0$ for all $r << 0$. There exists a decreasing filtration*

$$M = M_0 \supset M_1 \supset \cdots, \quad \bigcap_k M_k = \{0\},$$

and surjective morphisms

$$\varphi_k : \bigoplus_{r \in \mathbf{Z}_+} W(\lambda_k, r)^{\oplus m(k,r)} \longrightarrow M_k/M_{k+1} \to 0, \quad k \geq 0$$

where $m(k, r) = \dim \mathrm{Hom}_{\mathcal{I}}(M, W_{\mathrm{loc}}(-w_0\lambda_k, -r)^*)$.

The proposition will be proved in the next section.

3.9. The second step in the proof of the theorem is the following.

Proposition *Assume that $\mathfrak{g}$ is of type $\mathfrak{sl}_{n+1}$. We have,*

$$\mathbb{H}(P(\lambda, 0)) = \sum_{k \geq 0} \sum_{r \in \mathbf{Z}_+} [W_{\mathrm{loc}}(\lambda_k, 0) : V(\lambda, r)]\mathbb{H}(W(\lambda_k, r)) \tag{3.3}$$

$$= \sum_{k \geq 0} \left(\sum_{r \geq 0} [W_{\mathrm{loc}}(\lambda_k, 0) : V(\lambda, r)]u^r \right) \mathbb{H}(W(\lambda_k, 0)). \tag{3.4}$$

This proposition is proved in the last two sections of this paper.

3.10. Observe that we can apply Proposition 3.8 to $P(\lambda, r)$. Using the following equalities which follow from Proposition 3.5 and standard properties of duals and grade shift operators, we get

$$\begin{aligned} m(k, r) &= \dim \mathrm{Hom}_{\mathcal{I}}(P(\lambda, 0), W_{\mathrm{loc}}(-w_0\lambda_k, -r)^*) \\ &= \dim \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), W_{\mathrm{loc}}(-w_0\lambda_k, -r)^*[0]) \\ &= \dim \mathrm{Hom}_{\mathfrak{g}}(W_{\mathrm{loc}}(-w_0\lambda_k, -r)[0], V(-w_0\lambda)) \\ &= \dim \mathrm{Hom}_{\mathfrak{g}}(W_{\mathrm{loc}}(\lambda_k, 0)[r], V(\lambda)) \\ &= [W_{\mathrm{loc}}(\lambda_k, 0) : V(\lambda, r)]. \end{aligned}$$

Hence, for $\ell \geq 0$, we have

$$\dim P(\lambda, 0)[\ell] \leq \sum_{k,r \geq 0} [W_{\mathrm{loc}}(\lambda_k, 0) : V(\lambda, r)] \dim W(\lambda_k, r)[\ell].$$

Proposition 3.9 implies that for $\mathfrak{sl}_{n+1}$, equality holds, and hence, the surjective maps φ_k are actually isomorphisms for all $k \geq 0$. This proves Theorem 3.6.

3.11. In the last section, we also establish the analog of Theorem 3.6 in certain subcategories of $\mathcal{I}$. Given $k \geq 0$, let $\mathcal{I}^k_>$ be the full subcategory of $\mathcal{I}_>$ consisting of objects M such that

$$\mathrm{wt}\, M \subset \bigcup_{s=0}^{k} \lambda_s - Q^+.$$

The modules $V(\lambda_s, r)$, $W_{\mathrm{loc}}(\lambda_s, r)$ and $W(\lambda_s, r)$ are objects of $\mathcal{I}^k_>$ for all $s \leq k$ and $r \in \mathbf{Z}$. Let $P^k(\lambda_s, r)$ be the maximal quotient of $P(\lambda_s, r)$, which lies in $\mathcal{I}^k_>$. Then, $P^k(\lambda_s.r)$ is the projective cover of $V(\lambda_s, r)$ in $\mathcal{I}^k_>$.

Theorem *Assume that $\mathfrak{g}$ is of type $\mathfrak{sl}_{n+1}$. Let $s, k \in \mathbf{Z}_+$ with $s \leq k$. The object $P^k(\lambda_s, r)$ has a finite filtration by global Weyl modules, and*

$$[P^k(\lambda_s, r) : W(\lambda_\ell, p)] = [W_{\mathrm{loc}}(\lambda_\ell, r) : V(\lambda_s, p)].$$

4 The o-canonical filtration

Let $\mathcal{I}_>$ be the full subcategory of $\mathcal{I}$ consisting of objects M such that there exists $r \in \mathbf{Z}$ (depending on M) with $M[p] = 0$ for all $p < r$. It follows from Sect. 3.2 that $P(\lambda, r) \in \mathrm{Ob}\, \mathcal{I}_>$ for all $\lambda \in P^+$.

4.1. We begin this section with the following proposition which summarizes the properties of the duals of the projective, global and local Weyl modules.

Proposition *For $(\lambda_k, r) \in P^+ \times \mathbf{Z}$, set*

$$I(\lambda_k, r) = P(-w_0\lambda_k, -r)^*.$$

(i) $I(\lambda_k, r)$ is an injective object of $\mathcal{I}$ with a unique irreducible submodule, which is isomorphic to $V(\lambda_k, r)$.
(ii) The maximal submodule of $I(\lambda, r)$ whose weights are in the union of cones $\lambda_s - Q^+$, $0 \leq s \leq k$ is isomorphic to $W(-w_0\lambda, -r)^*$.
(iii) $W_{\mathrm{loc}}(-w_0\lambda_k, -r)^*$ is isomorphic to the maximal submodule M of $I(\lambda, r)$ satisfying

$$\mathrm{wt}\, M \subset \bigcup_{s=0}^{r} \lambda_s - Q^+, \quad M[s]_{\lambda_k} \neq 0 \implies s = r.$$

□

By abuse of notation, we shall freely identify $V(-w_0\lambda_k, -r)$ with its isomorphic copy in $W_{\mathrm{loc}}(\lambda_k, r)^*$, similar remarks apply for the corresponding submodules of $I(\lambda, r)$.

4.2. Given $M \in \mathrm{Ob}\,\mathcal{I}_>$, let

$$M_k = \sum_{s\geq k} \mathbf{U}(\mathfrak{g}[t])M_{\lambda_s}, \quad s, k \in \mathbf{Z}_+.$$

Clearly $M_0 = M$, $M_k \in \mathrm{Ob}\,\mathcal{I}_>$ for all $k \geq 0$ and

$$M_k/M_{k+1} = \mathbf{U}(\mathfrak{g}[t])(M_k/M_{k+1})_{\lambda_k}, \tag{4.1}$$

$$\mathfrak{n}^+[t]\,(M_k/M_{k+1})_{\lambda_k} = 0, \quad (h - \lambda_k(h))\,(M_k/M_{k+1})_{\lambda_k} = 0, \quad h \in \mathfrak{h}. \tag{4.2}$$

We claim that

$$\bigcap_{k\in\mathbf{Z}_+} M_k = \{0\},$$

and call $M = M_0 \supset M_1 \cdots$ the *o-canonical filtration* of $M \in \mathrm{Ob}\,\mathcal{I}_>$.

For the claim, note that since $M[s] = 0$ for all $s << 0$, we have

$$\#\left\{\left(\bigcup_{p\leq r} \mathrm{wt}\, M[p]\right)\bigcap P^+\right\} < \infty.$$

Hence, we can choose $k_0 \in \mathbf{Z}_+$ such that

$$M_{\lambda_s} \subset \bigoplus_{p>r} M[p], \quad s \geq k_0.$$

Using the definition of M_{k_0} we now get that $M_{k_0}[r] = \{0\}$ which establishes the claim.

4.3. We now prove,

Lemma *For $M \in \mathrm{Ob}\,\mathcal{I}_>$, and $k, r \in \mathbf{Z}_+$, we have an isomorphism of vector spaces,*

$$\mathrm{Hom}_{\mathfrak{g}}(M_k/M_{k+1}, V(\lambda_k, r)) \cong \mathrm{Hom}_{\mathcal{I}}(M, W_{\mathrm{loc}}(-w_0\lambda_k, -r)^*). \tag{4.3}$$

Proof Let $\iota_k : M_k/M_{k+1} \to M/M_{k+1}$ and $\iota_{\lambda_k,r} : V(\lambda_k, r) \to I(\lambda_k, r)$ be the canonical injective morphisms. Given $\psi \in \mathrm{Hom}_{\mathfrak{g}}(M_k/M_{k+1}, V(\lambda_k, r))$, let $\tilde{\psi} : M/M_{k+1} \to I(\lambda_k, r)$ be the map such that

$$\tilde{\psi}\iota_k = \iota_{\lambda,r}\psi.$$

Since

$$\mathrm{wt}\, M/M_{k+1} \subset \bigcup_{s=0}^{k} \lambda_s - Q^+, \quad (M/M_{k+1})_{\lambda_k} \cong (M_k/M_{k+1})_{\lambda_k}$$

and

$$\psi((M_k/M_{k+1})[p]_{\lambda_k}) = 0, \quad p \neq r,$$

it follows from Proposition 4.1(iii) that

$$\operatorname{Im} \tilde{\psi} \subset W_{\rm loc}(-w_0\lambda_k, -r)^*.$$

If $\pi : M \to M/M_{k+1}$ is the canonical projection, we see now that the assignment $\psi \to \pi.\tilde{\psi}$ is an injective linear map $\operatorname{Hom}_{\mathfrak{g}}(M_k/M_{k+1}, V(\lambda_k, r)) \longrightarrow \operatorname{Hom}_{\mathcal{I}}(M, W_{\rm loc}(-w_0\lambda_k, -r)^*)$.

For the opposite map, let $\psi \in \operatorname{Hom}_{\mathcal{I}}(M, W_{\rm loc}(-w_0\lambda_k, -r)^*)$. Observe that $\psi(M_{k+1}) = 0$, since wt $W_{\rm loc}(-w_0\lambda_k, -r)^*) \subset \lambda_k - Q^+$. Hence, we have a nonzero map $M/M_{k+1} \to W_{\rm loc}(-w_0\lambda_k, -r)^*$, and we let $\tilde{\psi}$ be the restriction of this map to M_k/M_{k+1}. Since $V(\lambda_k, r)$ is the unique irreducible submodule of $W_{\rm loc}(-w_0\lambda_k, -r)^*$, we have

$$\operatorname{Im} \psi \cap V(\lambda_k, r) \neq 0 \quad \operatorname{Im}(M/M_{k+1}) \cap V(\lambda_k, r) \neq 0.$$

Since $M_{\lambda_k} = (M_k)_{\lambda_k}$, it now follows that $\operatorname{Im} \tilde{\psi} = V(\lambda_k, r)$, and the assignment

$$\psi \to \tilde{\psi}, \quad \operatorname{Hom}_{\mathcal{I}}(M, W_{\rm loc}(-w_0\lambda_k, -r)^*) \to \operatorname{Hom}_{\mathfrak{g}}(M_k/M_{k+1}, V(\lambda_k, r)),$$

is an injective map, and the Lemma is proved. □

4.4. Let head(M) be the maximal semisimple quotient of M and $\mathbf{h} : M \to \operatorname{head}(M)$ be the corresponding map. If r is minimal such that $M[r] \neq 0$, then $\oplus_{p>r} M[p]$ is a proper submodule of M. The corresponding quotient is semisimple and isomorphic to $\operatorname{ev}_0 M[r]$ and hence head$(M) \neq 0$. Moreover, by Lemma 3.1, we have

$$\operatorname{head}(M) = \bigoplus_{(\lambda,\ell)\in P^+\times\mathbf{Z}} V(\lambda, \ell)^{\oplus \dim \operatorname{Hom}_{\mathcal{I}}(M, V(\lambda,\ell))}.$$

The map $\mathbf{h}$ lifts to a surjective map

$$\tilde{\mathbf{h}} : \bigoplus_{(\lambda,\ell)\in P^+\times\mathbf{Z}} P(\lambda, \ell)^{\oplus \dim \operatorname{Hom}_{\mathcal{I}}(M, V(\lambda,\ell))} \longrightarrow M \to 0. \tag{4.4}$$

The fact that a lift $\tilde{\mathbf{h}}$ of $\mathbf{h}$ exists that is obvious since the $P(\lambda, \ell)$ are projective and we have surjective maps $P(\lambda, \ell) \to V(\lambda, \ell) \to 0$ sending $p_{\lambda,\ell} \to 1 \otimes v_{\lambda,\ell}$. If $M' = M/\operatorname{Im} \tilde{\mathbf{h}}$ is nonzero, then head$(M') \neq 0$ and hence is a semisimple quotient of M as well. But this contradicts the fact that head(M) is the maximal semisimple quotient of M and the fact that

$$\operatorname{Im} \tilde{\mathbf{h}} \supset \operatorname{head}(M).$$

4.5. For $r \in \mathbf{Z}$, set $m(k,r) = \dim \operatorname{Hom}_{\mathfrak{g}}(M_k/M_{k+1}, V(\lambda_k, r))$ and notice that

$$\operatorname{head}(M_k/M_{k+1}) \cong \bigoplus_{r \in \mathbf{Z}} V(\lambda_k, r)^{\oplus m(k,r)}.$$

Using Corollary 3.5, we see that the map

$$\bigoplus_{r \in \mathbf{Z}} P(\lambda_k, r)^{\oplus m(k,r)} \to M_k/M_{k+1} \to 0,$$

defined in (4.4) factors through to

$$\bigoplus_{r \in \mathbf{Z}} W(\lambda_k, r)^{\oplus m(k,r)} \longrightarrow M_k/M_{k+1} \to 0.$$

Proposition 3.8 now follows by using Lemma 4.3.

5 A combinatorial interlude

5.1. Let $\{x_j : 1 \le j \le r\}$ be a set of indeterminates. The symmetric group S_r acts naturally on the polynomial ring $\mathbf{Z}[x_1, \ldots, x_r]$, and we let Λ_r be the corresponding ring of invariants. Set $|x| = x_1 \cdots x_r$ and denote by Λ_r' the localization of Λ_r at $|x|$. Equivalently, Λ_r' is the ring of invariants for the action of S_r on the integer valued ring of Laurent polynomials in the x_j, $1 \le j \le r$. Let $\hat{\Lambda}_r$ be the ring of all symmetric power series in $x_1, \ldots, x_r$; thus, elements of $\hat{\Lambda}_r$ are of the form $\sum_{\ell \ge 0} p_\ell$ where $p_\ell \in \Lambda_r$ is homogeneous of degree ℓ.

5.2. Given any $f \in \mathbf{Z}[x_1^{\pm}, \ldots, x_r^{\pm 1}]$, let $[f]_0$ be the constant term of f and let f^* be given by $f^*(x_1, \ldots, x_r) = f(x_1^{-1}, \ldots, x_r^{-1})$. The Macdonald inner product $(\cdot, \cdot) : \Lambda_r' \times \Lambda_r' \to \mathbf{Z}$ is defined by

$$(f, g) = \frac{1}{r!} \left[f g^* \prod_{1 \le i < j \le r} \left(1 - \frac{x_i}{x_j}\right)\left(1 - \frac{x_j}{x_i}\right) \right]_0, \quad f, g \in \Lambda_r'.$$

Since $(f, g) = 0$ for homogeneous $f, g \in \Lambda_r$ of different degrees, one can extend $(\cdot, \cdot)$ naturally to a pairing $\Lambda_r \times \hat{\Lambda}_r \to \mathbf{Z}$. The following is not hard to prove and can be found in [12].

Lemma *(i) For any $f, g, h \in \Lambda_r'$ one has*

$$(fg\,,\, h) = (f\,,\, g^* h).$$

(ii) For any $f \in \Lambda_r$ and $\ell_1, \ldots, \ell_r \in \mathbf{Z}$, one has:

$$\left(f, \frac{1}{\prod\limits_{1 \le i,j \le r} (1 - x_i \ell_j)} \right) = f(\ell_1, \ldots, \ell_r).$$

□

5.3. Let Par(r) be the set of all partitions $\xi = (\xi_1 \ge \cdots \ge \xi_r \ge 0)$ with at most r parts. Given $\xi = (\xi_1 \ge \xi_2 \ge \cdots \xi_r \ge 0) \in \text{Par}(r)$ let

$$m_\xi = \sum_{\sigma \in S_r} x_{\sigma(1)}^{\xi_1} \cdots x_{\sigma(r)}^{\xi_r} \in \Lambda_r.$$

The set $\{m_\xi : \xi \in \text{Par}(r)\}$ is an integral basis of Λ_r, called the *symmetrized monomial basis*.

The basis of Λ_r consisting of Newton polynomials is given as follows. For $0 \le j \le r$ and a partition $\xi = (\xi_1 \ge \cdots \ge \xi_r \ge 0)$, set

$$p_j = x_1^j + x_2^j + \cdots + x_r^j, \quad p_\xi = p_{\xi_1} \ldots p_{\xi_r}.$$

The basis of Schur functions s_ξ is defined as follows. Given $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbf{N}^r$, let

$$d_{\mathbf{m}} = \det \begin{pmatrix} x_1^{m_1} & \ldots & x_r^{m_1} \\ \vdots & \vdots & \vdots \\ x_1^{m_r} & \ldots & x_r^{m_r} \end{pmatrix}$$

Then, it can be shown that for a partition ξ, the polynomial $d_{(\xi_1+r-1,\xi_2+r-2,\ldots,\xi_r)}$ is divisible by $d_{(r-1,r-2,\ldots,1,0)}$, and the ratio is the Schur function s_ξ. If $\xi = (\xi_1 \ge \cdots \ge \xi_r)$, we have

$$|x| s_\xi(x_1, \ldots, x_n) = s_{\tilde{\xi}}(x_1, \ldots, x_n), \quad \tilde{\xi} = (\xi_1 + 1 \ge \cdots \ge \xi_r + 1).$$

In particular, this means that if $\xi \in \text{Par}(r)$ and $\lambda \in \text{Par}(r-1)$ is such that $\lambda_s = \xi_s - \xi_r$, $1 \le s \le r-1$, then

$$s_\xi(x_1, \ldots, x_r) = |x|^{\xi_r} s_\lambda(x_1, \ldots, x_r). \tag{5.1}$$

Moreover, it is well known that the elements

$$s_{\lambda,\ell} = |x|^\ell s_\lambda(x_1, \ldots, x_r), \quad \lambda \in \text{Par}(r-1), \ \ell \in \mathbf{Z},$$

form an orthonormal (with respect to) $\mathbf{Z}$-linear basis of Λ'_r, and

$$\Lambda'_r = \bigoplus_{\ell \in \mathbf{Z}} \Lambda_r^0 \cdot |x|^\ell, \quad \Lambda_r = \bigoplus_{\ell \ge 0} \Lambda_r^0 \cdot |x|^\ell,$$

where Λ_r^0 is the $\mathbf{Z}$-linear span of $\{s_\lambda(x_1, \ldots, x_r) : \lambda \in \mathrm{Par}(r-1)\}$.

5.4. Define elements $R_r, R_r' \in \hat{\Lambda}_r$ by

$$R_r = \frac{1}{(1-x_1)^r \cdots (1-x_r)^r}, \quad R_r' = \frac{1 - x_1 \cdots x_r}{(1-x_1)^r \cdots (1-x_r)^r}.$$

Lemma *We have*

$$R_r = \sum_{\xi \in \mathrm{Par}(r)} s_\xi(1, \ldots, 1) s_\xi(x_1, \ldots, x_r),$$
$$R_r' = \sum_{\lambda \in \mathrm{Par}(r-1)} s_\lambda(1, \ldots, 1) s_\lambda(x_1, \ldots, x_r).$$

Proof Let $y_1, \ldots, y_r$ be another set of indeterminates. Then, setting $y_1 = \cdots = y_r = 1$ in the Cauchy identity [12, I, (4.3)]:

$$\frac{1}{\prod_{1 \le i,j \le r} (1 - x_i y_j)} = \sum_{\xi \in \mathrm{Par}(r)} s_\xi(x_1, \ldots, x_r) s_\xi(y_1, \ldots, y_r)$$

gives the first identity of the Lemma. To prove the second one, we use (5.1) to get

$$\begin{aligned}
\frac{1}{\prod_{1 \le i,j \le r} (1 - x_i y_j)} &= \sum_{\xi \in \mathrm{Par}(r)} s_\xi(x_1, \ldots, x_r) s_\xi(y_1, \ldots, y_r) \\
&= \sum_{\lambda \in \mathrm{Par}(r-1)} s_\lambda(x_1, \ldots, x_r) s_\lambda(y_1, \ldots, y_r) \sum_{\ell \in \mathbf{Z}_+} |x|^\ell |y|^\ell \\
&= \sum_{\lambda \in \mathrm{Par}(r-1)} \frac{s_\lambda(x_1, \ldots, x_r) s_\lambda(y_1, \ldots, y_r)}{1 - |x||y|}.
\end{aligned}$$

Hence,

$$\frac{1 - x_1 \cdots x_r y_1 \cdots y_r}{\prod_{1 \le i,j \le r} (1 - x_i y_j)} = \sum_{\lambda \in \mathrm{Par}(r-1)} s_\lambda(x_1, \ldots, x_r) s_\lambda(y_1, \ldots, y_r).$$

Now, setting $y_1 = \cdots = y_r = 1$ completes the proof of the second identity. □

5.5. We now prove,

Proposition *Let* $\lambda \in \mathrm{Par}(r-1)$, $\ell \in \mathbf{Z}_+$. *If* $f \in \Lambda_r$ *is such that* $\ell \ge \deg(f)$, *then*

$$(s_{\lambda,\ell}\,,\ f R_r) = f(1, \ldots, 1) s_\lambda(1, \ldots, 1)$$

Proof By Lemma 5.2 (i), we have

$$(s_{\lambda,\ell}\ ,\ fR_r) = (s_{\lambda,\ell} f^*\ ,\ R_r).$$

Since $\ell \geq \deg f$, we have $s_{\lambda,\ell} f^* = s_{\lambda,\ell-\deg(f)}(|x|^{\deg(f)} f^*) \in \Lambda_r$ and we can now use Lemma 5.2(ii) with $\ell_1 = \cdots = \ell_r = 1$ to get

$$(s_{\lambda,\ell} f^*\ ,\ R_r) = (s_{\lambda,\ell} f^*)(1, \ldots, 1) = s_\lambda(1, \ldots, 1) f(1, \ldots, 1),$$

as required. □

5.6. Define the element $Q \in \mathbf{Z}[[q]] \otimes \hat{\Lambda}_r$ by

$$Q := \frac{(qx_1 \cdots x_r; q)_\infty}{(x_1; q)_\infty^r \cdots (x_r; q)_\infty^r}$$

where $(a; q)_\infty := \prod_{i=0}^{\infty}(1 - aq^i)$.

Our goal is to find an asymptotic expansion of Q based on the expansion

$$Q = \sum_{\lambda \in \mathrm{Par}(r-1), \ell \geq 0} \phi_{\lambda,\ell}(q) s_{\lambda,\ell},$$

where

$$\phi_{\lambda,\ell}(q) = (s_{\lambda,\ell}\ ,\ Q) = \sum_{s \geq 0} a^s_{\lambda,\ell} q^s \in \mathbf{Z}[[q]]. \tag{5.2}$$

Define integers c_m by requiring,

$$\frac{1}{(q; q)_\infty^{r^2-1}} = \sum_{m \geq 0} c_m q^m.$$

Proposition *For* $\lambda \in \mathrm{Par}(r-1)$ *and* $\ell \geq 0$ *we have*

$$a^m_{\lambda,\ell} = c_m s_\lambda(1, \ldots, 1), \ \text{ if } \ \ell \geq rm.$$

Proof Setting,

$$Q' = \frac{(qx_1 \ldots x_r; q)_\infty}{(qx_1; q)_\infty^r \ldots (qx_r; q)_\infty^r},$$

it is clear that

$$Q = \frac{(qx_1 \cdots x_r; q)_\infty}{(x_1; q)_\infty^r \cdots (x_r; q)_\infty^r} = R_r \cdot Q'.$$

Expanding $Q' = \sum_{m\geq 0} Q'_m q^m$ with all $Q_m \Lambda_r$, we obtain from (5.2):

$$\phi_{\lambda,\ell}(q) = (s_{\lambda,\ell}, Q) = \left(s_{\lambda,\ell}, R_r \sum_{m\geq 0} Q'_m q^m \right) = \sum_{m\geq 0} a^m_{\lambda,\ell} q^m \ ,$$

that is, $a^m_{\lambda,\ell} = (s_{\lambda,\ell}, R_r Q'_m)$.

Since $\deg(Q'_m) \leq rm$, Proposition 5.5 implies that

$$a^m_{\lambda,\ell} = Q'_m(1, \ldots, 1) s_\lambda(1, \ldots, 1), \quad \text{if } \ \ell \geq mr.$$

The proposition follows by noticing that if we set $x_1 = \cdots = x_r = 1$, we have

$$\frac{1}{(q;q)_\infty^{r^2-1}} = Q'(1, \ldots, 1) = \sum_{m\geq 0} Q'_m(1, \ldots, 1) q^m = \sum_{m\geq 0} c_m q^m.$$

□

5.7. For $\lambda \in \mathrm{Par}(r-1)$ and $\ell \geq 0$ define power series $\psi_{\lambda,\ell} = \sum_{m\geq 0} b^m_{\lambda,\ell} q^m \in \mathbf{Z}[[q]]$ by:

$$\psi_{\lambda,\ell}(q) = \phi_{\lambda,\ell}(q) - \phi_{\lambda,\ell-1}(q), \quad b^m_{\lambda,\ell} = a^m_{\lambda,\ell} - a^m_{\lambda,\ell-1},$$

where we adopt the convention that $\phi_{\lambda,-1} = 0$. Clearly,

$$\phi_{\lambda,\ell}(q) = \sum_{k=0}^{\ell} \psi_{\lambda,k}(q),$$

and moreover, we see from Proposition 5.6 that $b^m_{\lambda,\ell} = 0$ when $\ell > mr$. This means that

$$\sum_{\ell\geq 0} b^m_{\lambda,\ell} < \infty,$$

and hence

$$\sum_{\ell\geq 0} \psi_{\lambda,\ell}(q) = \sum_{\ell\geq 0} \sum_{m\geq 0} b^m_{\lambda,\ell} q^m = \sum_{m\geq 0} \left(\sum_{\ell\geq 0} b^m_{\lambda,\ell} \right) q^m,$$

is a well-defined element of $\mathbf{Z}[[q]]$. Using Proposition 4.6 again, we see that

$$\sum_{k\geq 0} \psi_{\lambda,k} = \lim_{\ell\to\infty} \phi_{\lambda,\ell} = \sum_{m\geq 0} c_m s_\lambda(1, \ldots, 1) q^m = \frac{s_\lambda(1, \ldots, 1)}{(q;q)_\infty^{r^2-1}}.$$

Together with the fact that

$$\sum_{\lambda\in\mathrm{Par}(r-1)}\sum_{\ell\geq 0}\psi_{\lambda,\ell}(q)s_{\lambda,\ell} = \sum_{\lambda\in\mathrm{Par}(r-1)}\sum_{\ell\geq 0}\phi_{\lambda,\ell}(q)(s_{\lambda,\ell}-s_{\lambda,\ell+1})$$
$$= \sum_{\lambda\in\mathrm{Par}(r-1)}\sum_{\ell\geq 0}\phi_{\lambda,\ell}(q)(1-|x|)s_{\lambda,\ell},$$

we have now proved,

Proposition *We have an equality of symmetric power series,*

$$\frac{(x_1\cdots x_r;q)_\infty}{(x_1;q)_\infty^r\cdots(x_r;q)_\infty^r} = \sum_{\lambda\in\mathrm{Par}(r-1)}\sum_{\ell\geq 0}\psi_{\lambda,\ell}(q)s_{\lambda,\ell},$$

where $\psi_{\lambda,\ell}(q)\in q^{\lfloor\frac{\ell}{r}\rfloor}\cdot\mathbf{Z}[[q]]$ *and*

$$\sum_{\ell\geq 0}\psi_{\lambda,\ell}(q) = \frac{s_\lambda(1,\ldots,1)}{(q;q)_\infty^{r^2-1}}.$$

□

5.8. Let q, t be indeterminates, and let $\mathbf{Q}(q,t)$ be the field of rational functions in q and t over the field $\mathbf{Q}$ of rational numbers. The *Macdonald scalar product* on $\Lambda_r(q,t)=\Lambda_r\otimes\mathbf{Q}(q,t)$ is defined on the Newton polynomials by

$$\langle p_\xi, p_\psi\rangle_{q,t} = \delta_{\xi,\psi}\prod_{i=1}^{r} i^{n_i}n_i!\prod_{s=1}^{\ell(\xi)}\frac{1-q^{\xi_s}}{1-t^{\xi_s}},$$

for $\xi,\psi\in\mathrm{Par}(r)$, where $n_i=|\{k:\xi_k=i\}|$ and $\ell(\xi)$ is the number of nonzero parts of ξ. The Macdonald polynomial $P_\xi(x;q,t)$ in $x=(x_1,\ldots,x_r)$ is the orthonormal basis of $\Lambda_r(q,t)$ obtained by applying the Gram–Schmidt process to the lexicographically ordered basis of monomial symmetric functions. Thus,

$$P_\xi(x;q,t)=m_\xi+\sum_{\psi<\xi}u_{\xi,\psi}m_\xi(x),\quad u_{\xi,\psi}\in\mathbf{Q}(q,t).$$

Proposition *For* $r\geq 1$ *one has:*

$$\frac{1}{\prod_{1\leq i,j\leq r}(x_iy_j;q)_\infty} = \sum_{\xi\in\mathrm{Par}(r)}\frac{P_\xi(x;q,0)P_\xi(y;q,0)}{(q;q)_{\xi_1-\xi_2}\cdots(q;q)_{\xi_{r-1}-\xi_r}(q;q)_{\xi_r}}, \qquad (5.3)$$

where $(a;q)_m:=\prod_{i=0}^m(1-aq^i)$ *denotes the (shifted) q-Pochhammer symbol.*

Proof It is shown in [12, VI,(4.19)] that

$$\prod_{i,j=1}^{r} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \sum_{\xi \in \mathrm{Par}(r)} b_\xi(q,t) P_\xi(x;q,t) P_\xi(y;q,t), \tag{5.4}$$

where $b_\xi(q,t) = \langle P_\xi(x,q,t), P_\xi(x,q,t)\rangle_{q,t}^{-1}$ is computed by the following recursive formula in [12, VI (6.19)],

$$b_0(q,t) = 1, \quad b_\xi(q,t) = b_{\xi'}(q,t) \prod_{i=1}^{r} \frac{1 - q^{\xi_i - 1} t^{r+1-i}}{1 - q^{\xi_i} t^{r-i}} \tag{5.5}$$

for $\xi = (\xi_1 \geq \cdots \geq \xi_r) \in \mathrm{Par}(r)$ with $\xi_r > 0$, and $\xi' = (\xi_1 - 1 \geq \cdots \geq \xi_r - 1)$.

If we set $t = 0$ in the above formulas, we see that the left-hand side of (5.4) is precisely the left-hand side of (5.3). Also, the recursion (5.5) simplifies:

$$b_\xi(q,0) = b_{\xi'}(q,0) \frac{1}{1 - q^{\xi_i}}$$

and gives by induction:

$$b_\xi(q,0) = (q;q)_{\xi_1 - \xi_2}^{-1} \cdots (q;q)_{\xi_{r-1} - \xi_r}^{-1} (q;q)_{\xi_r}^{-1}.$$

The proposition is proved. □

5.9. We now prove the following result.

Proposition *We have,*

$$\frac{(x_1 \cdots x_r; q)_\infty}{\prod_{1 \leq j \leq r} (x_j; q)_\infty} = \sum_{\lambda \in \mathrm{Par}(r-1)} \frac{P_\lambda(1, \ldots, 1; q, 0) P_\lambda(x_1, \ldots, x_r; q, 0)}{(q;q)_{\lambda_1 - \lambda_2} \ldots (q;q)_{\lambda_{r-2} - \lambda_{r-1}} (q;q)_{\lambda_{r-1}}}.$$

Proof Using the fact that

$$P_\xi(x;q,t) = |x|^{\xi_r} P_\lambda(x;q,t), \quad \lambda = (\xi_1 - \xi_r \geq \cdots \geq \xi_{r-1} - \xi_r \geq 0),$$

we get,

$$\begin{aligned}
&\sum_{\xi \in \mathrm{Par}(r)} \frac{P_\xi(x;q,0) P_\xi(y;q,0)}{(q;q)_{\xi_1 - \xi_2} \cdots (q;q)_{\xi_{r-1} - \xi_r} (q;q)_{\xi_r}} \\
&= \sum_{\ell \geq 0} \frac{|xy|^\ell}{(q;q)_\ell} \sum_{\lambda \in \mathrm{Par}(r-1)} \frac{P_\lambda(x;q,0) P_\lambda(y;q,0)}{(q;q)_{\lambda_1} \cdots (q;q)_{\lambda_{r-1}}} \\
&= \frac{1}{(|xy|;q)_\infty} \sum_{\lambda \in \mathrm{Par}(r-1)} \frac{P_\lambda(x;q,0) P_\lambda(y;q,0)}{(q;q)_{\lambda_1} \cdots (q;q)_{\lambda_{r-1}}},
\end{aligned}$$

where we have used the fact that for any indeterminate a, we have

$$\sum_{k=0}^{\infty} \frac{a^k}{(q;q)_k} = \frac{1}{(a;q)_\infty}.$$

Setting $y_1 = \cdots y_r = 1$ and using Proposition 5.8 complete the proof. □

5.10. The following is now an immediate consequence of Propositions 5.7 and 5.8.

Lemma *For* $\mu \in \mathrm{Par}(r-1)$*, write*

$$P_\mu(x_1, \dots, x_r; q, 0) = \sum_{\ell \geq 0} \sum_{\lambda \in \mathrm{Par}(r-1)} \eta^{\mu}_{\lambda,\ell}(q) s_{\lambda,\ell}(x_1, \dots, x_r).$$

Then,

$$\frac{s_\lambda(1,\dots,1)}{(q;q)_\infty^{r^2-1}} = \sum_{\mu \in \mathrm{Par}(r-1)} \sum_{\ell \geq 0} \frac{\eta^{\mu}_{\lambda,\ell}(q) P_\mu(1,\dots,1;q,0)}{(q;q)_{\mu_1-\mu_2} \cdots (q;q)_{\mu_{r-2}-\mu_{r-1}} (q;q)_{\mu_{r-1}}}.$$

6 Proof of Proposition 3.9

The proof involves putting together known results on the Hilbert series of the projective, global Weyl and local Weyl modules with the combinatorial identities, which were established in the previous section. We will use the notation of the previous sections freely.

6.1. The Hilbert series of $P(\lambda, 0)$ is easily calculated by using the Poincare Birkhoff Witt theorem. Thus, we have

$$P(\lambda, 0) = \mathbf{U}(\mathfrak{g}[t]) \otimes_{\mathbf{U}(\mathfrak{g})} V(\lambda) \cong \mathbf{U}(\mathfrak{g}[t]_+) \otimes V(\lambda),$$

where $\mathfrak{g}[t]_+ = \mathfrak{g} \otimes t\mathbf{C}[t]$ and the isomorphism is one of vector spaces. In particular,

$$P(\lambda, 0)[s] = \mathbf{U}(\mathfrak{g}[t]_+)[s] \otimes V(\lambda), \quad \text{and} \quad \mathbf{U}(\mathfrak{g}[t]_+)[s] \cong S(\mathfrak{g}[t]_+)[s],$$

where $S(\mathfrak{g}[t]_+)$ is the symmetric algebra of $\mathfrak{g}[t]_+$, and the isomorphism is again one graded of vector spaces. It follows that

$$\mathbb{H}(P(\lambda, 0)) = \dim V(\lambda) \mathbb{H}(S(\mathfrak{g}[t]_+)) = \frac{\dim V(\lambda)}{(q;q)_\infty^{\dim \mathfrak{g}}}.$$

In the special case when $\mathfrak{g}$ is of type $\mathfrak{sl}_r$, it is well known that $\dim V(\lambda) = s_\lambda(1,\dots,1)$ where we identify P^+ with the set $\mathrm{Par}(r-1)$ by sending $\lambda = \sum_{i=1}^{r-1} \lambda_i \omega_i$ to the partition whose jth part is $\sum_{i=j}^{r-1} \lambda_j \omega_j$. Hence, we have

Lemma *Suppose that* $\mathfrak{g}$ *is of type* $\mathfrak{sl}_r$*. Then,*

$$\mathbb{H}(P(\lambda, 0)) = \frac{s_\lambda(1, \ldots, 1)}{(q; q)_\infty^{r^2-1}}.$$

□

6.2. We now recall the relationship between the graded characters of global and local Weyl modules. This was established in [3, Proposition 3.7], using results proved in [6,7,10,13].

Proposition *For* $\lambda = \sum_{i=1}^n \lambda_i \omega_i \in P^+$*, we have*

$$\mathrm{ch}_{\mathrm{gr}} W(\lambda, 0) = \frac{\mathrm{ch}_{\mathrm{gr}} W_{\mathrm{loc}}(\lambda, 0)}{\prod_{i=1}^n (q; q)_{\lambda_i}}, \quad \mathbb{H}(W(\lambda, 0)) = \frac{\mathbb{H}(W_{\mathrm{loc}}(\lambda, 0))}{\prod_{i=1}^n (q; q)_{\lambda_i}}.$$

□

6.3. The final piece of information we need is on the graded character of the local Weyl modules. We restrict our attention to the case of $\mathfrak{sl}_r$ (but refer the interested reader to [13] for the general case). There is a well-known family of $\mathbf{Z}_+$-graded modules for the subalgebra $\mathfrak{n}^+ \otimes \mathbf{C}[t] \oplus (\mathfrak{n}^- \oplus \mathfrak{h}) \otimes t\mathbf{C}[t]$ of $\mathfrak{g}[t]$ called the Demazure modules (see [11] or [14] for a quick introduction). In the case of $\mathfrak{sl}_r$, it was proved in [14] that the characters of certain Demazure modules which are indexed by P^+ are given by specializing the Macdonald polynomials at $t = 0$. Moreover, these Demazure modules actually admit the structure of a $\mathfrak{g}[t]$-module. The main result of [6] establishes that $W_{\mathrm{loc}}(\lambda)$ is graded isomorphic to such a Demazure module and can be summarized as follows.

Theorem *Assume that* $\mathfrak{g}$ *is of type* $\mathfrak{sl}_r$ *and let* $\lambda = \sum_{i=1}^s \lambda_i \omega_i \in P^+$*. Then*

$$\sum_{k \geq 0} [(W_{\mathrm{loc}}(\lambda, 0)[k] : V(\mu)] q^k = \sum_{\ell \geq 0} \eta_{\mu,\ell}^\lambda(q),$$
$$\mathrm{ch}_{\mathrm{gr}}\, W_{\mathrm{loc}}(\lambda, 0) = \sum_{\mu \in P^+} \sum_{\ell \geq 0} \eta_{\mu,\ell}^\lambda(q)\, \mathrm{ch}_{\mathfrak{g}}\, V(\mu),$$

where the $\eta_{\mu,\ell}^\lambda$ *are as defined in Lemma* 5.10. □

Using Lemma 5.10, we have the following corollary.

Corollary

$$\mathbb{H}(W_{\mathrm{loc}}(\lambda, 0)) = P_\lambda(1, \ldots, 1; q, 0).$$

□

6.4. We now have

$$\sum_{k\geq 0}\sum_{\mu\in P^+} \mathbb{H}(W(\mu,0))[W_{\rm loc}(\mu,0):V(\lambda,k)]q^k$$
$$= \sum_{\mu\in P^+}\left(\sum_{\ell\geq 0}\eta^{\mu}_{\lambda,\ell}(q)\right)\frac{\mathbb{H}(W_{\rm loc}(\mu,0))}{\prod_{i=1}^n (q;q)_{\mu_i}}$$
$$= \sum_{\mu\in P^+}\left(\sum_{\ell\geq 0}\eta^{\mu}_{\lambda,\ell}(q)\right)\frac{P_\mu(1,\dots,1;q,0)}{\prod_{i=1}^n (q;q)_{\mu_i}} = \frac{s_\lambda(1,\dots,1)}{(q;q)_\infty^{r^2-1}},$$

where the last equality is by using Lemma 5.10. Together with Lemma 5.1, we have now established that

$$\mathbb{H}(P(\lambda,0)) = \sum_{k\geq 0}\sum_{\mu\in P^+} \mathbb{H}(W(\mu,0))[W_{\rm loc}(\mu,0):V(\lambda,k)]u^k,$$

which is precisely the statement of Proposition 3.9.

6.5. It remains to prove Theorem 3.11. Set $M = P(\lambda_s, r)$ and let M_ℓ, $\ell \in \mathbf{Z}_+$ be the o-canonical filtration of M. Then, it is clear that

$$M^k = P^k(\lambda_s, r) \cong M/M_{k+1},$$

which proves that the o-canonical filtration of M^k is finite and in fact is given by the submodules M_ℓ/M_{k+1} where $0 \leq \ell \leq k$. Moreover

$$(M_\ell/M_{k+1})/(M_{\ell+1}M_{k+1}) \cong M_\ell/M_{\ell+1},$$

and Theorem 3.11 is proved.

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