Abh. Math. Semin. Univ. Hambg. (2013) 83:187–217
DOI 10.1007/s12188-013-0080-4

Modular forms of orthogonal type and Jacobi theta-series

F. Cléry · V. Gritsenko

Received: 1 March 2013 / Published online: 24 May 2013

Abstract In this paper, we study Jacobi forms of half-integral index for any even integral positive definite lattice L (classical Jacobi forms from the book of Eichler and Zagier correspond to the lattice $A_1 = \langle 2\rangle$). We construct Jacobi forms of singular (respectively, critical) weight in all dimensions $n \geq 8$ (respectively, $n \geq 9$). We give the Jacobi lifting for Jacobi forms of half-integral indices and we obtain an additive lifting construction of new reflective modular forms which are natural generalizations to $\mathrm{O}(2, n)$ ($n = 4$, 5 and 6) of the Igusa modular form Δ_5.

Keywords Jacobi forms for lattices and root systems · Theta-series and Weil representations · Modular forms of orthogonal type · Lifting of modular forms

Mathematics Subject Classification 11F50 · 11F27 · 11F55 · 17B22 · 17B67

1 Introduction

The divisor of a strongly reflective modular form with respect to an integral orthogonal group of signature $(2, n)$ is determined by reflections. Such modular forms determine Lorentzian Kac–Moody Lie (super) algebras. The most famous reflective modular form is the Borcherds function Φ_{12} with respect to $\mathrm{O}^+(II_{2,26})$ which determines the Fake Monster Lie algebra (see [1]). One can consider reflective modular forms as automorphic discriminants or multi-dimensional Dedekind η-functions (see [2, 3, 17–20]). Reflective modular forms play also an

Communicated by U. Kühn.

F. Cléry
Korteweg-de Vries Instituut, Universiteit van Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands
e-mail: f.l.d.clery@uva.nl

V. Gritsenko (✉)
Laboratoire Paul Painlevé, University Lille 1, 59655 Villeneuve d'Ascq Cedex, France
e-mail: Valery.Gritsenko@math.univ-lille1.fr

important role in complex algebraic geometry (see [21] and [14]). All of them are Borcherds automorphic products and some of them can be constructed as additive (or Jacobi) lifting. If a reflective modular form can be obtained by a Jacobi lifting then one has a simple formula for its Fourier coefficients which determine the generators and relations of Lorentzian Kac–Moody algebras (see [17]).

In [14], the second author constructed *the Borcherds-Enriques form* Φ_4, the automorphic discriminant of the moduli space of Enriques surfaces (see [2]), as Jacobi lifting, $\mathrm{Lift}(\vartheta(\tau, z_1)\dots\vartheta(\tau, z_8))$, of the tensor product of eight classical Jacobi theta-series (see [11] for the definition of Lift which provides a modular form on orthogonal group by its first Fourier–Jacobi coefficient). This new construction of Φ_4 gives an answer to a problem formulated by K.-I. Yoshikawa ([33]) and to a question of J.A. Harvey and G. Moore ([24]) about the second Lorentzian Kac–Moody super Lie algebra determined by the Borcherds–Enriques form Φ_4 and its quasi-pullbacks.

Another application of reflective modular forms of type $\mathrm{Lift}(\vartheta(\tau, z_1)\dots\vartheta(\tau, z_8))$ is the construction of new examples of modular varieties of orthogonal type of Kodaira dimension 0 (see the beginning of Sect. 2). The first two examples of this type of dimension 3 are related to reflective Siegel cusp forms of weight 3 and Siegel modular three-folds having compact Calabi–Yau models (see [5, 16] and [9]). In the case of dimension 4, the unique cusp form of weight 4 was defined in [14] as a Borcherds product but it can also be constructed as a lifting of a Jacobi form of half-integral index with a character of order 2 of the full modular group $\mathrm{SL}_2(\mathbb{Z})$ (see Example 3.4 in Sect. 2). Jacobi forms of half-integral index in one variable are very important in the theory of Lorentzian Kac–Moody algebras of hyperbolic rank 3 corresponding to Siegel modular forms (see [17–19]). Moreover, they are very natural in the structure theory of classical Jacobi forms (in the sense of Eichler and Zagier [8]) and in applications to topology and string theory (see [7, 13]).

In this paper, we consider Jacobi forms of half-integral index for any positive definite lattice L (Jacobi forms in [8] correspond to the lattice $A_1 = \langle 2\rangle$). Jacobi forms in many variables naturally appeared in the theory of affine Lie algebras (see [25] and [26]). One can consider Jacobi forms as vector-valued modular forms in one variable. Vector-valued modular forms are used in the additive Borcherds lifting (see [3, Sect. 14]) which is a genralization of the Jacobi lifting of [11]. In this paper, we follow the general approach to Jacobi forms proposed in [11]. The first section contains all necessary definitions and basic results on Jacobi forms in many variables. It turns out that the order of the character of the integral Heisenberg group of such Jacobi forms is always at most 2 (see Proposition 2.3). Using the classical Jacobi theta-series, we give examples of Jacobi forms for the root lattices. We show, at the end of the first section (Examples 2.8–2.9), that the natural theta-products give all Jacobi forms of singular weight (or vector valued $\mathrm{SL}_2(\mathbb{Z})$-modular forms of weight 0 related to the Weil representation) for the lattices D_m.

In Sect. 2, we give the Jacobi lifting for Jacobi forms of half-integral index with a possible character. This explicit construction has many advantages: one can see immediately a part of its divisor, the maximal modular group of the lifting, etc. (Compare our construction of Φ_4 with the construction of S. Kondo in [28].)

We build many modular forms of singular, critical and canonical weights on orthogonal groups. In particular, we give in Example 3.4 the Jacobi lifting construction of a new strongly reflective modular form of singular weight on $\mathrm{O}(2, 6)$. This modular forms gives a tower of four reflective modular forms based on the classical Igusa modular form Δ_5.

In Sect. 3, we analyze Jacobi forms of singular (the minimal possible) and critical (singular weight $+\frac{1}{2}$) weights using the theta-products and their pullbacks (see Proposition 4.1–4.2). In this way, we construct Jacobi forms of singular and critical weights in all dimensions

$n \geq 8$ (see Propositions 4.3–4.11). In particular, using this approach, we give a new explanation of theta-quarks in Corollary 4.4, which are the simplest examples of holomorphic theta-blocks (see [23]).

2 Jacobi group and Jacobi modular forms

In this section, we discuss Jacobi forms of orthogonal type. In the definitions below, we follow the papers [10, 11] where Jacobi forms were considered as modular forms with respect to a parabolic subgroup of an orthogonal group of signature $(2, n)$.

By a lattice we mean a free $\mathbb{Z}$-module equipped with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ with values in $\mathbb{Z}$. A lattice is even if (l,l) is even for all its elements. Let L_2 be a lattice of signature $(2, n_0+2)$. All the bilinear forms we deal with can be extended to $L_2 \otimes \mathbb{C}$ (respectively to $L_2 \otimes \mathbb{R}$) by $\mathbb{C}$-linearity (respectively by $\mathbb{R}$-linearity) and we use the same notations for these extensions. Let

$$\mathcal{D}(L_2) = \left\{[\mathcal{Z}] \in \mathbb{P}(L_2 \otimes \mathbb{C}) \mid (\mathcal{Z},\mathcal{Z}) = 0,\ (\mathcal{Z},\overline{\mathcal{Z}}) > 0\right\}^+$$

be the (n_0+2)-dimensional bounded symmetric Hermitian domain of type IV associated to the lattice L_2 (here $+$ denotes one of its two connected components). We denote by $\mathrm{O}^+(L_2 \otimes \mathbb{R})$ the index 2 subgroup of the real orthogonal group preserving $\mathcal{D}(L_2)$. Then $\mathrm{O}^+(L_2)$ is the intersection of the integral orthogonal group $\mathrm{O}(L_2)$ with $\mathrm{O}^+(L_2 \otimes \mathbb{R})$. We use the similar notation $\mathrm{SO}^+(L_2)$ for the special orthogonal group.

In this paper, we assume that L_2 is an even lattice of signature $(2, n_0+2)$ containing two hyperbolic planes

$$L_2 = U \oplus U_1 \oplus L(-1), \qquad U \simeq U_1 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where L is a positive definite even lattice of rank n_0 and $L(-1)$ denotes its rescaling by -1. We fix a basis of the hyperbolic plane $U = \mathbb{Z}e \oplus \mathbb{Z}f$: $e \cdot f = (e, f) = 1$ and $e^2 = f^2 = 0$. Similarly $U_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}f_1$. Let F be the totally isotropic plane spanned by f and f_1 and let P_F be the parabolic subgroup of $\mathrm{SO}^+(L_2)$ that preserves F. This corresponds to a 1-dimensional cusp of the modular variety $\mathrm{SO}^+(L_2)\backslash\mathcal{D}(L_2)$ (see [21]). We choose a basis of L_2 of the form $(e, e_1, \ldots, f_1, f)$ where $\ldots$ denote a basis of $L(-1)$. In this basis, the quadratic form associated to the bilinear form on L_2 has the following Gram matrix

$$S_2 = \left(\begin{array}{cc|ccc|cc} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \hline 0 & 0 & & & & 0 & 0 \\ \vdots & \vdots & & -S & & \vdots & \vdots \\ 0 & 0 & & & & 0 & 0 \\ \hline 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right)$$

where S is a positive definite integral matrix with even entries on the main diagonal. We denote the *positive definite* even integral bilinear form on the lattice L by $(.,.)$ and the bilinear form of signature $(1, n_0+1)$ on the hyperbolic lattice $U_1 \oplus L(-1)$ by $(.,.)_1$. Therefore, for any $v = ne_1 + l + mf_1 \in L_1$, we have $(v, v)_1 = 2nm - (l,l)$.

The subgroup $\Gamma^J(L)$ of P_F of elements acting trivially on the sublattice L is called the *Jacobi group*. The Jacobi group has a subgroup isomorphic to $\mathrm{SL}_2(\mathbb{Z})$. For any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we denote

$$\{A\} := \begin{pmatrix} A^* & 0 & 0 \\ 0 & \mathbf{1}_{n_0} & 0 \\ 0 & 0 & A \end{pmatrix} \in \Gamma^J(L), \quad \text{where } A^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^tA^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \tag{1}$$

The second standard subgroup of $\Gamma^J(L)$ is the Heisenberg group $H(L)$ acting trivially on the totally isotropic plane F. This is the central extension $\mathbb{Z} \rtimes (L \times L)$. More precisely, we define

$$H(L) = \left\{ [x, y; r] : x, y \in L,\ r \in \frac{1}{2}\mathbb{Z} \text{ with } r + \frac{1}{2}(x, y) \in \mathbb{Z} \right\}$$

where

$$[x, y; r] := \begin{pmatrix} 1 & 0 & {}^tyS & (x,y)/2 - r & (y,y)/2 \\ 0 & 1 & {}^txS & (x,x)/2 & (x,y)/2 + r \\ 0 & 0 & \mathbf{1}_{n_0} & x & y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

with S the positive definite Gram matrix of the quadratic form L of rank n_0, $(x, y) = {}^txSy$ and we consider x and y as column vectors. The multiplication in $H(L)$ is given by the following formula

$$[x, y; r] \cdot \left[x', y'; r'\right] = \left[x + x', y + y'; r + r' + \frac{1}{2}\big(\big(x, y'\big) - \big(x', y\big)\big)\right]. \tag{3}$$

In particular, the center of $H(L)$ is equal to $\{[0, 0; r],\ r \in \mathbb{Z}\}$. We introduce a subgroup of $H(L)$

$$H_s(L) = \big\langle [x, 0; 0],\ [0, y; 0] \,|\, x, y \in L \big\rangle$$

with a smaller center and we call it *minimal* integral Heisenberg group of L. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $H(L)$ by conjugation:

$$A.[x, y; r] := \{A\}[x, y; r]\big\{A^{-1}\big\} = [dx - cy, -bx + ay; r]. \tag{4}$$

Using (3) and (4), one can define the integral Jacobi group or its subgroup $\Gamma_s^J(L)$ as the semidirect product of $\mathrm{SL}_2(\mathbb{Z})$ with the Heisenberg group $H(L)$ or $H_s(L)$

$$\Gamma^J(L) \simeq \mathrm{SL}_2(\mathbb{Z}) \rtimes H(L) \quad \text{and} \quad \Gamma_s^J(L) \simeq \mathrm{SL}_2(\mathbb{Z}) \rtimes H_s(L).$$

Extending the coefficients, we can define the real Jacobi group which is a subgroup of the real orthogonal group: $\Gamma^J(L \otimes \mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R}) \rtimes H(L \otimes \mathbb{R})$.

In what follows, we need characters of Jacobi groups. Let $\chi : \Gamma^J(L) \to \mathbb{C}^*$ be a character of finite order. Its restriction to $\mathrm{SL}_2(\mathbb{Z})$, $\chi|_{\mathrm{SL}_2(\mathbb{Z})}$, is an even power v_η^D of the multiplier system of the Dedekind η-function and we have

$$\chi\big(\{A\} \cdot [x, y; r]\big) = v_\eta^D(A) \cdot \nu\big([x, y; r]\big), \quad \text{where } \chi|_{\mathrm{SL}_2(\mathbb{Z})} = v_\eta^D,\ \nu = \chi|_{H(L)}. \tag{5}$$

If D is odd then we obtain a multiplier system of the Jacobi group. The properties of the character of the Heisenberg group are clarified by the next proposition.

Proposition 2.1 1. *Let $s(L)\in\mathbb{N}^*$ (resp. $n(L)$) denote the generator of the integral ideal generated by (x,y) (resp. (x,x)) for x and y in L. Let $[H(L),H(L)]$ be the derivated group of $H(L)$. Then*

$$\big[H(L),H(L)\big]=\big[H_s(L),H_s(L)\big]=\big\{[0,0;r]\,|\,r\in s(L)\mathbb{Z}\big\}.$$

This subgroup is the center of $H_s(L)$.

2. *Let $\nu: H(L)\to\mathbb{C}^*$ be a character of finite order which is invariant with respect to the* $\mathrm{SL}_2(\mathbb{Z})$*-action*: $\nu(A.[x,y;r])=\nu([x,y;r])$. *Then*

$$\nu\big([x,y;r]\big)=e^{\pi it((x,x)+(y,y)-(x,y)+2r)}$$

where $t\in\mathbb{Q}$ such that $t\cdot s(L)\in\mathbb{Z}$. The restriction $\nu|_{H_s(L)}$ is a binary character which is trivial if $t\cdot n(L)\in 2\mathbb{Z}$.

Remark The constants $s(L)$ and $n(L)$ are called *scale* and *norm* of the integral lattice L. The scale $s(L)$ is the greatest common divisor of the entries of the Gram matrix S of L. For any even lattice L the norm $n(L)$ is an even divisor of $s(L)$.

Proof The first property follows from the formula for the commutator of the elements of $H(L)$

$$[x,y;r]\cdot\big[x',y';r'\big]\cdot[x,y;r]^{-1}\cdot\big[x',y';r'\big]^{-1}=\big[0,0;\big(x,y'\big)-\big(x',y\big)\big]\tag{6}$$

because $[x,y;r]^{-1}=[-x,-y;-r]$.

Considering the restriction of the character to the center of $H(L)$, isomorphic to $\mathbb{Z}$, we get $\nu([0,0;r])=\exp(2\pi itr)$ with $t\in\mathbb{Q}$. The formula for the commutator (6) shows that $t\cdot s(L)\in\mathbb{Z}$.

The invariance of the character with respect to $A=-\mathbf{1}_2$, $A=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $A=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ gives us $\nu([x,y;r])=\nu([-x,-y;r])=\nu([-y,x;r])=\nu([x,y-x;r])$. Therefore $\nu([x,0;0])=\nu([-x,0;0])=\nu([x,0;0])^{-1}$, $\nu([0,y;0])=\nu([y,0;0])$ and $\nu([x,0;0])=\nu([x,-x;0])$. We have

$$[x,-x;0]=[x,0;0]\cdot[0,-x;0]\cdot\left[0,0;\frac{1}{2}(x,x)\right].$$

Therefore $\nu([x,0;0])=\nu([x,-x;0])=e^{\pi it(x,x)}$ and the final formula follows from the decomposition

$$[x,y;r]=\left[x,y;\frac{1}{2}(x,y)\right]\cdot\left[0,0;r-\frac{1}{2}(x,y)\right]=[x,0;0]\cdot[0,y;0]\cdot\left[0,0;r-\frac{1}{2}(x,y)\right].$$

We see that $t\cdot(x,x)\in\mathbb{Z}$. Therefore the order of $\nu|_{H_s(L)}$ is equal to 1 or 2. □

In order to define Jacobi forms, we have to fix a tube realization of the homogeneous domain $\mathcal{D}(L_2)$ related to the 1-dimensional boundary component of its Baily–Borel compactification corresponding to the Jacobi group related to the isotropic flag $\langle f\rangle\subset\langle f,f_1\rangle$. Let $[\mathcal{Z}]=[\mathcal{X}+i\mathcal{Y}]\in\mathcal{D}(L_2)$. Then

$$(\mathcal{X},\mathcal{Y})=0,\qquad(\mathcal{X},\mathcal{X})=(\mathcal{Y},\mathcal{Y})\quad\text{and}\quad(\mathcal{Z},\overline{\mathcal{Z}})=2(\mathcal{Y},\mathcal{Y})>0.$$

Using the basis $\langle e, f\rangle_{\mathbb{Z}} = U$ we write $\mathcal{Z} = z'e + \widetilde{Z} + zf$ with $\widetilde{Z} \in L_1 \otimes \mathbb{C}$, where

$$L_1 = U_1 \oplus L(-1)$$

is the hyperbolic lattice of signature $(1, n_0 + 1)$ with the bilinear form $(\cdot\,,\cdot)_1$. We note that $z \neq 0$. (If $z = 0$ the real and imaginary parts of $\widetilde{Z}$ form two orthogonal vectors of positive norm in the hyperbolic lattice $L_1 \otimes \mathbb{R}$.) Thus $[{}^t\mathcal{Z}] = [(-\frac{1}{2}(Z, Z)_1, {}^tZ, 1)]$. Using the basis $\langle e_1, f_1\rangle_{\mathbb{Z}} = U_1$ of the second hyperbolic plane in L, we see that $\mathcal{D}(L_2)$ is isomorphic to the tube domain

$$\mathcal{H}(L) = \mathcal{H}_{n_0+2}(L) = \left\{ Z = \begin{pmatrix} \omega \\ \mathfrak{z} \\ \tau \end{pmatrix} \;\middle|\; \tau, \omega \in \mathbb{H}_1,\ \mathfrak{z} \in L \otimes \mathbb{C},\ (\operatorname{Im} Z, \operatorname{Im} Z)_1 > 0 \right\}$$

where

$$(\operatorname{Im} Z, \operatorname{Im} Z)_1 = 2\operatorname{Im}(\omega)\operatorname{Im}(\tau) - \big(\operatorname{Im}(\mathfrak{z}), \operatorname{Im}(\mathfrak{z})\big) > 0.$$

We fix the isomorphism $[\mathrm{pr}] : \mathcal{H}(\mathrm{L}) \to \mathcal{D}(\mathrm{L}_2)$ defined by the 1-dimensional cusp F fixed above

$$Z = \begin{pmatrix} \omega \\ \mathfrak{z} \\ \tau \end{pmatrix} \mapsto \mathrm{pr}(Z) = \begin{pmatrix} -\frac{1}{2}(Z, Z)_1 \\ \omega \\ \mathfrak{z} \\ \tau \\ 1 \end{pmatrix} \mapsto \big[\mathrm{pr}(Z)\big]. \tag{7}$$

The map pr gives us the embedding of $\mathcal{H}(L)$ into the affine cone $\mathcal{D}^\bullet(L_2)$ over $\mathcal{D}(L_2)$. Using the map [pr], we can define a linear-fractional action of $M \in \mathrm{O}^+(L_2 \otimes \mathbb{R})$ on the tube domain

$$M \cdot \mathrm{pr}(Z) = J(M, Z) \cdot \mathrm{pr}\big(M\langle Z\rangle\big)$$

where the automorphic factor $J(M, Z)$ is the last (non-zero) element of the column vector $M \cdot \mathrm{pr}(Z) \in \mathcal{D}^\bullet(L_2)$. In particular, for the standard elements of the Jacobi group, we have the following action

$$\{A\}\langle Z\rangle = {}^t\!\left(\omega - \frac{c(\mathfrak{z}, \mathfrak{z})}{2(c\tau + d)}, \frac{{}^t\mathfrak{z}}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R});$$

$$[x, y; r]\langle Z\rangle = {}^t\!\left(\omega + \frac{1}{2}(x, x)\tau + (x, \mathfrak{z}) + \frac{1}{2}(x, y) + r,\ {}^t(\mathfrak{z} + x\tau + y),\ \tau\right)$$

$x, y \in L \otimes \mathbb{R}$ and $r \in \mathbb{R}$. We note that $J(\{A\}, Z) = c\tau + d$ and $J([x, y; r], Z) = 1$. For a function $\psi : \mathcal{H}(L) \to \mathbb{C}$, we define as usual

$$(\psi|_k M)(Z) := J(M, Z)^{-k}\psi\big(M\langle Z\rangle\big), \quad M \in \mathrm{O}^+(L_2 \otimes \mathbb{R}).$$

In the next definition, Jacobi forms are considered as modular forms with respect to the parabolic subgroup $\Gamma^J(L)$ of $\mathrm{O}^+(L_2)$.

Definition 2.2 Let χ be a character (or a multiplier system) of finite order of $\Gamma^J(L)$, k be integral or half-integral and t be a (positive) rational number. A holomorphic function

$\varphi : \mathbb{H}_1 \times (L \otimes \mathbb{C}) \to \mathbb{C}$ is called a holomorphic *Jacobi form* of weight k and index t with a character (or a multiplier system) χ if the function

$$\widetilde{\varphi}(Z) = \varphi(\tau, \mathfrak{z}) e^{2i\pi t\omega}, \quad Z = \begin{pmatrix} \omega \\ \mathfrak{z} \\ \tau \end{pmatrix} \in \mathcal{H}(L)$$

satisfies the functional equation

$$\widetilde{\varphi}|_k M = \chi(M)\widetilde{\varphi} \quad \text{for any } M \in \Gamma^J(L) \tag{8}$$

and is holomorphic at "infinity" (see the condition (11) below).

Remarks (1) We show below that for any non zero Jacobi form of rational index t we have $t \cdot s(L) \in \mathbb{Z}$ where $s(L)$ is the scale of the lattice (see Proposition 2.1).

(2) One can reduce this definition to the only two cases $t = 1$ and $t = \frac{1}{2}$ (see Proposition 2.3).

(3) One can give another definition of Jacobi forms in more intrinsic terms (see (9), (10) and Definition 2.2′).

In order to precise the condition "to be holomorphic at infinity", we analyze the character χ, the functional equation and the Fourier expansion of Jacobi forms. We decompose the character into two parts

$$\chi = \chi|_{\mathrm{SL}_2(\mathbb{Z})} \times \chi|_{H(L)} = \chi_1 \times \nu, \quad \text{where } \chi_1 = v_\eta^D$$

(see (5)) and ν satisfies the condition of Proposition 2.1. For a central element $[0, 0; (x, y)] \in H_s(L)$ $(x, y \in L)$, the Eq. (8) gives $\nu([0, 0; (x, y)]) = e^{2\pi i t(x,y)} = 1$. Therefore

$$t \cdot s(L) \in \mathbb{Z} \quad \text{if } \varphi \text{ is not identically zero}$$

and

$$\nu\big([x, y; r]\big) = e^{\pi i t((x,x)+(y,y)-(x,y)+2r)}, \quad [x, y; r] \in H(L)$$

as in Proposition 2.1.

The formulae above for the action of the generators of the Jacobi group on the tube domain define also an action, denoted by $M\langle\tau, \mathfrak{z}\rangle$, of the real Jacobi group $\Gamma^J(L\otimes\mathbb{R})$ on the domain $\mathbb{H}_1 \times (L \otimes \mathbb{C})$. We can always add to any $(\tau, \mathfrak{z}) \in \mathbb{H}_1 \times (L \otimes \mathbb{C})$ a complex number $\omega \in \mathbb{H}_1$ such that $Z = \begin{pmatrix} \omega \\ \mathfrak{z} \\ \tau \end{pmatrix}$ belongs to $\mathcal{H}(L)$. If we denote the first component of $M\langle Z\rangle$ (that is the component along the vector e_1 of our basis) by $\omega\{M\langle Z\rangle\}$ for $M \in \Gamma^J(L \otimes \mathbb{R})$ then

$$J_{k,t}(M; \tau, \mathfrak{z}) = J(M, Z)^k e^{-2i\pi t\omega\{M\langle Z\rangle\}} e^{2i\pi t\omega}$$

defines an automorphic factor of weight k and index t for the Jacobi group. For the generators of the Jacobi group, we get

$$J_{k,t}\big(\{A\}; \tau, \mathfrak{z}\big) = (c\tau + d)^k e^{i\pi t \frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

and

$$J_{k,t}\big([x, y; r]; \tau, \mathfrak{z}\big) = e^{-2i\pi t(\frac{1}{2}(x,x)\tau+(x,\mathfrak{z})+\frac{1}{2}(x,y)+r)}, \quad x, y \in L \otimes \mathbb{R},\ r \in \mathbb{R}.$$

We also get an action of the Jacobi group on the space of functions defined on $\mathbb{H}_1 \times (L \otimes \mathbb{C})$:

$$(\varphi|_{k,t} M)(\tau, \mathfrak{z}) := J_{k,t}^{-1}(M; \tau, \mathfrak{z})\varphi\big(M\langle\tau, \mathfrak{z}\rangle\big).$$

Then the Eq. (8) in the definition of Jacobi forms is equivalent to

$$(\varphi|_{k,t} M)(\tau, \mathfrak{z}) = \chi(M)\varphi(\tau, \mathfrak{z}), \quad M \in \Gamma^J(L).$$

For the generators of the Jacobi group we obtain

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}\right) = \chi(A)(c\tau + d)^k e^{i\pi t \frac{c(\mathfrak{z},\mathfrak{z})}{(c\tau+d)}}\varphi(\tau, \mathfrak{z}) \tag{9}$$

for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and

$$\varphi(\tau, \mathfrak{z} + x\tau + y) = \chi\left(\left[x, y; \frac{1}{2}(x, y)\right]\right)e^{-i\pi t((x,x)\tau + 2(x,\mathfrak{z}))}\varphi(\tau, \mathfrak{z}) \tag{10}$$

for all $x, y \in L$.

(Note that $t(x, y) \in \mathbb{Z}$ for any x, y in L if $\varphi \not\equiv 0$.) The variables τ and $\mathfrak{z}$ are called modular and abelian variables. To clarify the last condition of Definition 2.2, we consider the Fourier expansion of a Jacobi form φ.

We see that the function φ have the following periodic properties

$$\varphi(\tau + 1, \mathfrak{z}) = e^{2\pi i \frac{D}{24}}\varphi(\tau, \mathfrak{z}) \quad \text{and} \quad \varphi(\tau, \mathfrak{z} + 2y) = v\big([0, 2y; 0]\big)\varphi(\tau, \mathfrak{z}) = \varphi(\tau, \mathfrak{z}).$$

The function φ is called *holomorphic at infinity* if it has the Fourier expansion of the following type

$$\varphi(\tau, \mathfrak{z}) = \sum_{\substack{n \in \mathbb{Q}_{\geqslant 0},\, n \equiv \frac{D}{24} \bmod \mathbb{Z},\ l \in \frac{1}{2}L^\vee \\ 2nt - (l,l) \geqslant 0}} f(n, l)e^{2i\pi(n\tau + (l,\mathfrak{z}))} \tag{11}$$

where $L^\vee$ is the dual lattice of the even positive definite lattice L. This condition is equivalent to the fact that the function $\widetilde{\varphi}$ is holomorphic at the zero-dimensional cusp defined by the isotropic vector f in the first copy U in the lattice $L_2 = U \oplus U_1 \oplus L(-1)$.

The Definition 2.2 suits well for the applications considered in this paper but we can give another definition which does not depend on the orthogonal realization of the Jacobi group $\Gamma_s^J(L) \simeq \mathrm{SL}_2(\mathbb{Z}) \rtimes H_s(L)$.

Definition 2.2′ A holomorphic function $\varphi : \mathbb{H}_1 \times (L \otimes \mathbb{C}) \to \mathbb{C}$ is called a holomorphic Jacobi form of weight $k \in \frac{1}{2}\mathbb{Z}$ and index $t \in \mathbb{Q}$ with a character (or a multiplier system) of finite order $\chi : \Gamma_s^J(L) \to \mathbb{C}^*$ if φ satisfies the functional equations (9) and (10) and has a Fourier expansion of type (11).

Remarks (1) *The classical Jacobi forms of Eichler and Zagier*. Note that for $n_0 = 1$, the tube domain $\mathcal{H}(L)$ is isomorphic to the classical Siegel upper half-space of genus 2. If $L \simeq A_1 = \langle 2\rangle$ is the lattice $\mathbb{Z}$ with quadratic form $2x^2$ and $\chi = \mathrm{id}$ then the definition above is identical to the definition of Jacobi forms of integral weight k and index t given in the book [8].

(2) The difference between the Definitions 2.2 and 2.2′ is the character of the center of the "orthogonal" Heisenberg group $H(L)$. It is more natural to consider a Jacobi forms $\varphi(\tau,\mathfrak{z})$ as a modular form with respect to the minimal Jacobi group $\Gamma_s^J(L)$ and the *extended* Jacobi form $\widetilde{\varphi}(Z)=\varphi(\tau,\mathfrak{z})e^{2\pi i t\omega}$ with respect to $\Gamma^J(L)$.

We denote the vector space of Jacobi forms from Definition 2.2′ by $J_{k,L;t}(\chi)$ where $\chi=v_\eta^D\times\nu$ is defined by a character (or a multiplier system) v_η^D of $\mathrm{SL}_2(\mathbb{Z})$ and a binary (or trivial) character ν of $H_s(L)$. The space of Jacobi forms from Definition 2.2 is denoted by $\widetilde{J}_{k,L;t}(v_\eta^D\times\widetilde{\nu})$ with evident modification for the central part of $\widetilde{\nu}$. The character χ of $\Gamma^J(L)$ and its restriction $\widetilde{\chi}=\chi|_{\Gamma_s^J(L)}$ determine each other uniquely and we denote both of them by χ, $\widetilde{J}_{k,L;t}(\chi)\simeq J_{k,L;t}(\chi)$ and we will identify these spaces.

A function

$$\varphi(\tau,\mathfrak{z})=\sum_{n,l}f(n,l)e^{2i\pi(n\tau+(l,\mathfrak{z}))}\in J_{k,L;t}(\chi)$$

is called a *Jacobi cusp form* if $f(n,l)\neq 0$ only if the hyperbolic norm of its index is positive: $2nt-(l,l)>0$. We denote this vector space by $J_{k,L;t}^{cusp}(\chi)$. We define the order of φ as follows

$$\mathrm{Ord}(\varphi)=\min_{f(n,l)\neq 0}\big(2nt-(l,l)\big). \tag{12}$$

When the character (or the multiplier system) is trivial, we omit it in the notation of these spaces. If $\chi=v_\eta^D\times\mathrm{id}$, we omit the trivial part. We see from the definition that $\varphi\equiv 0$ if $t<0$. If $t=0$ then $J_{k,L;0}(\chi)=M_k(\mathrm{SL}_2(\mathbb{Z}),\chi|_{\mathrm{SL}_2})$. In fact, a Jacobi form corresponds to a vector-valued modular form of integral or half-integral weight related to the Weil representation of the lattice $L(t)$ and $J_{k,L;t}(\chi)$ is finite dimensional (see Sect. 3).

The notation $L(t)$ stands for the lattice L equipped with bilinear for $t(\cdot,\cdot)$. We proved above that if $J_{k,L;t}(\chi)\neq\{0\}$ then the lattice $L(t)$ is integral. Any Jacobi form with trivial character can be considered as Jacobi form of index 1 (see [11, Lemma 4.6]). In general, we have the following

Proposition 2.3 1. *If $L(t)$ is an even lattice then*

$$J_{k,L;t}(\chi)=J_{k,L(t);1}(\chi).$$

If this space is non-trivial then the Heisenberg part $\nu=\chi|_{H_s(L)}$ of the character is trivial.

2. *If $L(t)$ is integral odd (non-even) lattice then*

$$J_{k,L;t}(\chi)=J_{k,L(2t);\frac{1}{2}}(\chi).$$

In this case, the character $\nu=\chi|_{H_s(L)}$ is of order 2.

3. *If $\varphi(\tau,\mathfrak{z})\in J_{k,L;t}(v_\eta^D\times\nu)$ then $\varphi(\tau,2\mathfrak{z})\in J_{k,L;4t}(v_\eta^D\times\mathrm{id})$.*

Remark This proposition shows that we have to distinguish in fact only between index 1 and $\frac{1}{2}$. In what follows, we denote by $J_{k,L}(\chi)$ the space of Jacobi forms of index 1.

Proof If $L(t)$ is even then $t(x,x)\in 2\mathbb{Z}$ for any $x\in L$. Therefore $\chi([x,0;0])=e^{i\pi t(x,x)}=1$ and the Heisenberg part of χ is trivial. If $L(t)$ is odd then there exists $x\in L$ such that $t(x,x)$ is odd. Therefore the Heisenberg part of χ is non-trivial.

We prove the proposition about the indexes using a map that we will need in Sect. 2. We define an application for $N \in \mathbb{Q}_{>0}$

$$\pi_N : \mathcal{H}\big(L(N)\big) \to \mathcal{H}(L), \qquad \pi_N : \begin{pmatrix} \omega \\ \mathfrak{z} \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} \omega/N \\ \mathfrak{z} \\ \tau \end{pmatrix}. \tag{13}$$

This map corresponds to the multiplication $I_N \cdot \mathrm{pr}(Z)$ with $Z \in \mathcal{H}(L(N))$ and $\mathrm{pr}(Z) \in \mathcal{D}^\bullet(U \oplus U_1 \oplus L(-N))$ where $I_N = \mathrm{diag}(N^{-1}\mathbf{1}_2, \mathbf{1}_{n_0}, \mathbf{1}_2)$. We prove the second claim. (The proof of the first one is similar.)

If $\varphi \in J_{k,L;t}(\chi)$ then

$$\widetilde{\varphi}_{1/2}(Z) = \varphi(\tau, \mathfrak{z})e^{\pi i \omega} = \widetilde{\varphi} \circ \pi_{2t}(Z), \quad Z \in \mathcal{H}\big(L(2t)\big)$$

is a holomorphic function on $\mathcal{H}(L(2t))$. We add index $2t$ to $\{A\}$ and h in order to indicate the elements of the Jacobi groups $\Gamma^J(L(2t))$. First, we see that $I_{2t}\{A\}_{2t}I_{2t}^{-1} = \{A\}$. Therefore

$$\widetilde{\varphi}_{1/2}|_k\{A\}_{2t}(Z) = \chi(A)\widetilde{\varphi}_{1/2}(Z).$$

Secondly, we have

$$I_{2t}[x, y; r]_{2t}I_{2t}^{-1} = \left[x, y; \frac{r}{2t}\right] = \left[x, y; \frac{1}{2}(x, y)\right] \cdot \left[0, 0; \frac{r - t(x, y)}{2t}\right] \quad \text{for } x, y \in L.$$

Therefore

$$\begin{aligned} \widetilde{\varphi}_{1/2}|_k[x, y; r]_{2t}(Z) &= \widetilde{\varphi}|_k\left(\left[x, y; \frac{1}{2}(x, y)\right] \cdot \left[0, 0; \frac{r}{2t} - \frac{1}{2}(x, y)\right]\right)\big(\pi_{2t}(Z)\big) \\ &= \chi\left(\left[x, y; \frac{1}{2}(x, y)\right]\right)e^{\pi i(r - t(x,y))}\widetilde{\varphi}_{1/2}(Z) \quad \text{where } r - t(x, y) \in \mathbb{Z}. \end{aligned}$$

It means that $\widetilde{\varphi}_{1/2}$ is an (extended) Jacobi form of index $\frac{1}{2}$ with the same $\mathrm{SL}_2(\mathbb{Z})$- and Heisenberg characters with respect to the even lattice $L(2t)$. $\square$

In the Definitions 2.2 and 2.2′ and in the proof of the last proposition, we used two interpretations of Jacobi forms as a function on $\mathbb{H}_1 \times (L \otimes \mathbb{C})$ and on the tube domain $\mathcal{H}(L)$. For any $\tau = u + iv \in \mathbb{H}_1$ and $\mathfrak{z} \in L \otimes \mathbb{C}$, we can find $\omega = u_1 + iv_1 \in \mathbb{H}_1$ such that $2v_1v - (\mathrm{Im}(\mathfrak{z}), \mathrm{Im}(\mathfrak{z})) > 0$ or, equivalently, such that ${}^t(\omega, {}^t\mathfrak{z}, \tau) \in \mathcal{H}(L)$. In the next lemma, we fix an independent part of this parameter ω.

Lemma 2.4 *Let $Z = {}^t(\omega, {}^t\mathfrak{z}, \tau) \in \mathcal{H}(L)$. Then the quantity*

$$\widetilde{v}(Z) = v_1 - \frac{(\mathrm{Im}(\mathfrak{z}), \mathrm{Im}(\mathfrak{z}))}{2v} > 0$$

is invariant with respect to the action of the real Jacobi group $\Gamma^J(L \otimes \mathbb{R})$.

Proof For any $Z = X + iY \in \mathcal{H}(L)$ we consider its image $[\mathcal{Z}] = [\mathcal{X} + i\mathcal{Y}] = [\mathrm{pr}(Z)] = [{}^t(-\frac{1}{2}(Z, Z)_1, {}^tZ, 1)]$ in the projective homogeneous domain $\mathcal{D}(L_2)$.

For any $M \in \mathrm{O}^+(L_2 \otimes \mathbb{R})$, we have

$$(M\mathcal{Z}, M\overline{\mathcal{Z}}) = (\mathcal{Z}, \overline{\mathcal{Z}}) = 2(\mathcal{Y}, \mathcal{Y})_1 = 2\big(2v_1 \cdot v - \big(\mathrm{Im}(\mathfrak{z}), \mathrm{Im}(\mathfrak{z})\big)\big) = 4v \cdot \widetilde{v}(Z).$$

By the definition of the action of the group $O^+(L_2 \otimes \mathbb{R})$ on the tube domain, we have

$$\begin{aligned} 4v \cdot \widetilde{v}(\mathcal{Z}) &= 2(\mathcal{Y}, \mathcal{Y})_1 = (M\mathcal{Z}, M\overline{\mathcal{Z}}) = J(M, \mathcal{Z}) \cdot \overline{J(M, \mathcal{Z})}\big(M\langle \mathcal{Z}\rangle, M\langle \overline{\mathcal{Z}}\rangle\big) \\ &= 2\big|J(M, \mathcal{Z})\big|^2 \big(\mathcal{Y}\big(M\langle \mathcal{Z}\rangle\big), \mathcal{Y}\big(M\langle \mathcal{Z}\rangle\big)\big)_1 = 4\big|J(M, \mathcal{Z})\big|^2 v\big(M\langle \mathcal{Z}\rangle\big) \cdot \widetilde{v}\big(M\langle \mathcal{Z}\rangle\big) \\ &= 4v \cdot \widetilde{v}\big(M\langle \mathcal{Z}\rangle\big). \end{aligned}$$

□

Remark The quantity $\widetilde{v}(Z) \in \mathbb{R}_{>0}$ is a free part of the variable Z in the extended Jacobi form $\widetilde{\varphi}(Z)$:

$$\big(\tau, {}^t\mathfrak{z}\big) \mapsto {}^t\left(x_1 + i\left(\widetilde{v} + \frac{(\mathrm{Im}(\mathfrak{z}), \mathrm{Im}(\mathfrak{z}))}{2v}\right), {}^t\mathfrak{z},\ \tau\right) \in \mathcal{H}(L).$$

The Jacobi forms with respect to a lattice L form a bigraded ring $J_{*,L;*}$ with respect to weights and indexes. In the next proposition, we define a direct (or tensor) product of two Jacobi forms. Its proof follows directly from the definition.

Proposition 2.5 *Let $\varphi_1 \in J_{k_1,L_1;t}(\chi_1 \times \nu_1)$ and $\varphi_2 \in J_{k_2,L_2;t}(\chi_2 \times \nu_2)$ two Jacobi forms of the same index t. Then*

$$\varphi_1 \otimes \varphi_2(\tau, \mathfrak{z}_1, \mathfrak{z}_2) := \varphi_1(\tau, \mathfrak{z}_1) \cdot \varphi_2(\tau, \mathfrak{z}_2) \in J_{k_1+k_2,L_1\oplus L_2;t}(\chi_1\chi_2 \times \nu_1\nu_2)$$

where $\nu_1\nu_2$ is a character of the group $H(L_1 \oplus L_2)$ defined by

$$(\nu_1\nu_2)\big([x, y; r]\big) = \nu_1\left(\left[x_1, y_1; \frac{1}{2}(x_1, y_1)\right]\right)\nu_2\left(\left[x_2, y_2; \frac{1}{2}(x_2, y_2)\right]\right)e^{i\pi t((x_1,y_1)+(x_2,y_2)+2r)}$$

for $[x, y; r] = [x_1 \oplus x_2, y_1 \oplus y_2; r] \in H(L_1 \oplus L_2)$. The tensor product of two Jacobi forms is a cusp form if at least one of them is a cusp form.

It is known (see [10]) that the space $J_{k,L;t}(\chi)$ is trivial if $k < \frac{n_0}{2}$ where $\operatorname{rank} L = n_0$. The minimal possible weight $k = \frac{n_0}{2}$ is called *singular weight*. For any Jacobi form φ of singular weight the hyperbolic norm of the index of a non-zero Fourier coefficient $f(n, l)$ (see (11)) is equal to zero: $2nt - (l, l) = 0$, i.e. $\mathrm{Ord}(\varphi) = 0$.

Example 2.6 (The Jacobi theta-series) The Jacobi theta-series of characteristic $(\frac{1}{2}, \frac{1}{2})$ (see [29]) is defined by

$$\vartheta(\tau, z) = \sum_{n\in\mathbb{Z}} \left(\frac{-4}{n}\right) q^{\frac{n^2}{8}} r^{\frac{n}{2}} = -q^{1/8} r^{-1/2} \prod_{n\geqslant 1} \big(1 - q^{n-1}r\big)\big(1 - q^n r^{-1}\big)\big(1 - q^n\big) \tag{14}$$

where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}_1$ and $r = e^{2\pi iz}$, $z \in \mathbb{C}$. This is the simplest example of Jacobi form of half-integral index. The theta-series ϑ satisfies two functional equations

$$\vartheta(\tau, z + x\tau + y) = (-1)^{x+y} e^{-\pi i(x^2\tau+2xz)} \vartheta(\tau, z), \quad (x, y) \in \mathbb{Z}^2$$

and

$$\vartheta\left(A\langle\tau\rangle, \frac{z}{c\tau + d}\right) = v_\eta^3(A)(c\tau + d)^{\frac{1}{2}} e^{\pi i \frac{cz^2}{c\tau+d}} \vartheta(\tau, z), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

where v_η is the multiplier system of the Dedekind η-function. Using our notations, $A_1 = \langle 2\rangle$, we have

$$\vartheta \in J_{\frac{1}{2},A_1;\frac{1}{2}}\big(v_\eta^3 \times v_H\big)$$

where, for short, $v_{H(A_1)} = v_H$ is defined by:

$$v_H\big([x,y;r]\big) = (-1)^{x+y+xy+r}, \quad [x,y;r] \in H(A_1) = H(\mathbb{Z}).$$

The Jacobi theta-series ϑ is the Jacobi form of singular weight $\frac{1}{2}$ with a non-trivial character of the Heisenberg group. This Jacobi form was not mentioned in [8] but it plays an important role in the construction of the basic Jacobi forms and reflective modular forms (see [13, 14, 19]). We remind that

$$\operatorname{div} \vartheta = \big\{h\langle z=0\rangle \mid h \in H(\mathbb{Z})\big\} = \{z = x\tau + y \mid x, y \in \mathbb{Z}\}.$$

The Jacobi theta-series ϑ having the triple product formula (14) will be the first main function in our construction of Jacobi forms for orthogonal lattices.

Example 2.7 (Jacobi forms of singular weight for mA_1) Using the Jacobi theta-series, we can construct Jacobi forms of singular weight for $mA_1 = \langle 2\rangle \oplus \cdots \oplus \langle 2\rangle$. The tensor product of m Jacobi theta-series is a Jacobi form of singular weight and index $\frac{1}{2}$ for mA_1:

$$\vartheta_{mA_1}(\tau, z_1, \ldots, z_m) = \prod_{1\leqslant j\leqslant m} \vartheta(\tau, z_j) \in J_{\frac{m}{2},mA_1;\frac{1}{2}}\big(v_\eta^{3m} \times v_H^{\otimes m}\big) \tag{15}$$

where

$$v_H^{\otimes m}\big([x,y;r]\big) = v_{H(mA_1)}\big([x,y;r]\big) = (-1)^{r+\sum_{j=1}^m x_j+y_j+x_jy_j}$$

for any x_j, y_j and r in $\mathbb{Z}$. For even m, we can construct Jacobi forms of singular weight and index 1 because

$$\vartheta_{2A_1}^{(1)}(\tau, z_1, z_2) = \vartheta(\tau, z_1+z_2)\cdot\vartheta(\tau, z_1-z_2) \in J_{1,2A_1}\big(v_\eta^6\big). \tag{16}$$

Taking different orthogonal decompositions of the lattice $8A_1$, we obtain 105 Jacobi forms of weight 4 and index 1 with trivial character.

Example 2.8 (Jacobi forms of singular weight for D_m) We recall the definition of the even quadratic lattice D_m. (We denote by A_m, D_m, E_m the lattices generated by the corresponding root systems.) We use the standard Euclidian basis $\langle e_i\rangle_{i=1}^m$ ($(e_i, e_j) = \delta_{ij}$) in $\mathbb{Z}^m$. Then

$$D_m = \big\{(x_1, \ldots, x_m) \in \mathbb{Z}^m \mid x_1 + \cdots + x_m \in 2\mathbb{Z}\big\} \quad (m \geqslant 1)$$

is the maximal even sublattice in $\mathbb{Z}^m$. The theta-product (15) is a Jacobi form of index 1 for D_m with trivial Heisenberg character because the quadratic form $x_1^2 + \cdots + x_m^2$ is even on D_m

$$\vartheta_{D_m}(\tau, \mathfrak{z}_m) = \vartheta(\tau, z_1)\cdot\cdots\cdot\vartheta(\tau, z_m) \in J_{\frac{m}{2},D_m}\big(v_\eta^{3m}\big). \tag{17}$$

We note that $D_2 \cong 2A_1$ and $\vartheta_{D_2} = \vartheta_{2A_1}^{(1)}$. For the lattice D_4 we can give two more examples:

$$\vartheta_{D_4}^{(2)}(\tau, \mathfrak{z}_4) = \vartheta\left(\tau, \frac{-z_1+z_2+z_3+z_4}{2}\right)\vartheta\left(\tau, \frac{z_1-z_2+z_3+z_4}{2}\right)$$

$$\times\, \vartheta\left(\tau, \frac{z_1+z_2-z_3+z_4}{2}\right)\vartheta\left(\tau, \frac{z_1+z_2+z_3-z_4}{2}\right) \in J_{2,D_4}\big(v_\eta^{12}\big) \qquad (18)$$

and

$$\vartheta_{D_4}^{(3)}(\tau, \mathfrak{z}_4) = \vartheta\left(\tau, \frac{z_1+z_2+z_3+z_4}{2}\right)\vartheta\left(\tau, \frac{z_1+z_2-z_3-z_4}{2}\right)$$
$$\times\, \vartheta\left(\tau, \frac{z_1-z_2-z_3+z_4}{2}\right)\vartheta\left(\tau, \frac{z_1-z_2+z_3-z_4}{2}\right) \in J_{2,D_4}\big(v_\eta^{12}\big). \qquad (19)$$

Analyzing the divisors of the Jacobi forms, we obtain the relation

$$\vartheta_{D_4}(\tau, \mathfrak{z}_4) = \vartheta_{D_4}^{(2)}(\tau, \mathfrak{z}_4) + \vartheta_{D_4}^{(3)}(\tau, \mathfrak{z}_4).$$

Jacobi forms and Weil representation The Jacobi forms can be considered as vector-valued $\mathrm{SL}_2(\mathbb{Z})$-modular forms (see [3, 11, 26, 30, 32]) related to the Weil representation. To compare the examples considered above with vector-valued modular forms, we recall the definitions from [11] for Jacobi forms of index one.

Let L an even positive definite lattice of rank n_0 and

$$\varphi(\tau, \mathfrak{z}) = \sum_{\substack{n\in\mathbb{Z},\ l\in L^\vee\\ 2n-(l,l)\geqslant 0}} f(n,l)e^{2i\pi(n\tau+(l,\mathfrak{z}))} \in J_{k,L}$$

a Jacobi form of weight k and index one. By $q = q(L)$ we denote the level of the lattice L, i.e. the smallest integer such that $L^\vee(q)$ is an even lattice. Then we have the following representation (see [11, Lemma 2.3] with $m = 1$)

$$\varphi(\tau, \mathfrak{z}) = \sum_{\mu\in D(L)} \phi_\mu(\tau)\theta_\mu^L(\tau, \mathfrak{z})$$

where $D(L) = L^\vee/L$ is the discriminant group of L,

$$\phi_\mu(\tau) = \sum_{\substack{r\geqslant 0\\ \frac{2r}{q}\equiv-(\mu,\mu) \bmod 2\mathbb{Z}}} f_h(r)\exp\left(2\pi i\,\frac{r}{q}\tau\right), \qquad f_\mu(r) = f\left(\frac{2r+(\mu,\mu)}{2q}, \mu\right)$$

and

$$\theta_\mu^L(\tau, \mathfrak{z}) = \sum_{l\in\mu+L} e^{i\pi((l,l)\tau+2(l,\mathfrak{z}))}$$

is the theta-series with characteristic μ. For any matrix $M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the theta-vector $\Theta_L(\tau, \mathfrak{z}) = (\theta_\mu^L(\tau, \mathfrak{z}))_{\mu\in D(L)}$ has the following transformation property

$$\Theta_L\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right) = (c\tau+d)^{\frac{n_0}{2}} U(M)\exp\left(\frac{\pi i c(\mathfrak{z},\mathfrak{z})}{c\tau+d}\right)\Theta_L(\tau, \mathfrak{z})$$

where $U(M)$ is a unitary matrix. In particular, for $T = \left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)$ we have

$$U(T) = \mathrm{diag}\big(e^{i\pi(\mu,\mu)}\big)_{\mu\in D(L)}, \qquad U(S) = (-i)^{\frac{n_0}{2}}\Big(\sqrt{|D(L)|}\Big)^{-\frac{1}{2}}\big(e^{-2i\pi(\mu,\nu)}\big)_{\mu,\nu\in D(L)}.$$

Therefore $\Phi(\tau) = (\phi_\mu(\tau))_{\mu\in D(L)}$ is a holomorphic vector-valued modular form of weight $k - \frac{n_0}{2}$ for the conjugated representation $\overline{U}(M)$ of $SL_2(\mathbb{Z})$. In particular, the weight of any holomorphic Jacobi form is greater or equal to $\frac{n_0}{2}$ (see [10]). We note also that the theta-series θ^L_μ are linearly independent and Θ_L is invariant with respect to the action of the stable orthogonal group $\widetilde{\mathrm{O}}(L)$ (see Theorem 3.2 below) and, in particular, with respect to the Weyl group of the lattice L, $W_2(L)$, which is a subgroup of $\widetilde{\mathrm{O}}(L)$ generated by 2-reflections in the lattice L.

We get the simplest example for an even unimodular lattice N of rank n_0 (see [10] and [11, Lemma 4.1])

$$\Theta_N(\tau,\mathfrak{z}) = \sum_{l\in N} e^{\pi i(l,l)\tau+2\pi i(l,\mathfrak{z})} \in J_{\frac{n_0}{2},N}. \tag{20}$$

Moreover we have that two linear spaces of Jacobi forms are isomorphic

$$J_{k_1,L_1} \cong J_{k_1+\frac{n_2-n_1}{2},L_2}$$

if L_1 (rank $L_1 = n_1$) and L_2 (rank $L_2 = n_2$) are two lattices with isomorphic discriminant forms (see [11, Lemma 2.4]).

Example 2.9 (Weil representation for D_m) Recall that $|D_m^\vee/D_m| = 4$ and

$$D_m^\vee/D_m = \{\mu_i,\ i \bmod 4\} = \left\{0,\ \frac{1}{2}(e_1+\cdots+e_m),\ e_1,\ \frac{1}{2}(e_1+\cdots+e_{m-1}-e_m) \bmod D_m\right\}$$

is the cyclic group of order 4 generated by $\mu_1 = \frac{1}{2}(e_1+\cdots+e_m) \bmod D_m$, if m is odd, and the product of two groups of order 2, if m is even. We have the following matrix of inner products in the discriminant group of D_m of the non-trivial classes modulo D_m

$$\big((\mu_i,\mu_j)\big)_{i,j\neq 0} = \begin{pmatrix} \frac{m}{4} & \frac{1}{2} & \frac{m-2}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{m-2}{4} & \frac{1}{2} & \frac{m}{4} \end{pmatrix} \quad \big(\mu_i \in D_m^\vee/D_m\big)$$

where the diagonal elements are taken modulo $2\mathbb{Z}$ and the non-diagonal elements are taken modulo $\mathbb{Z}$. We note that the discriminant group of D_m depends only on $m \bmod 8$. This gives the formulae for $U(T)$ and $U(S)$.

(1) For $m \equiv 4 \bmod 8$, we have

$$U(T) = \mathrm{diag}(1,\ -1,\ -1,\ -1), \quad U(S) = -\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

We put $\theta_i^{D_m}(\tau,\mathfrak{z}_{\mathrm{m}}) := \theta_{\mu_i}^{D_m}(\tau,\mathfrak{z}_{\mathrm{m}})$ for $i \bmod 4$. The matrices $U(T)$ and $U(S)$ have the three common eigenvectors:

$$\theta_1^{D_m} - \theta_3^{D_m}, \quad \theta_1^{D_m} - \theta_2^{D_m}, \quad \theta_2^{D_m} - \theta_3^{D_m} \quad (m \equiv 4 \bmod 8).$$

If $m = 4$ we get the Jacobi forms ϑ_{D_4}, $\vartheta_{D_4}^{(1)}$ and $\vartheta_{D_4}^{(2)}$ obtained above as theta-products.

(2) For $m \equiv 0 \bmod 8$, we have

$$U(T) = \mathrm{diag}(1, 1, -1, 1), \qquad U(S) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

These lattices have again two linearly independent common eigenvectors. The first one is the theta-product $\vartheta_{D_m} = \theta_1^{D_m} - \theta_3^{D_m}$. The second eigenvector $\vartheta_{D_m^+} = \theta_0^{D_m} + \theta_1^{D_m}$ is equal to the Jacobi theta-series of the unimodular lattice $D_m^+ = \langle D_m, \mu_1 \rangle$. In particular, $D_8^+ = E_8$ and $\vartheta_{D_m^+} = \Theta_{E_8}$ (see (20)). To understand better the role of the Jacobi theta-series ϑ, we consider one more case.

(3) For $m \equiv 1 \bmod 8$, we have

$$U(T) = \mathrm{diag}\big(1, e^{i\frac{\pi}{4}}, -1, e^{i\frac{\pi}{4}}\big), \qquad U(S) = \frac{1}{2}e^{-i\frac{\pi}{4}}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

and $\vartheta_{D_m} = \theta_1^{D_m} - \theta_3^{D_m}$ is the only Jacobi form of singular weight. Moreover, for $m = 1$ we get

$$\vartheta_{D_1}(\tau, z) \in J_{\frac{1}{2},\langle 4\rangle;1}\big(v_\eta^3\big) = J_{\frac{1}{2},A_1;2}\big(v_\eta^3\big) = J_{\frac{1}{2},2}\big(v_\eta^3\big).$$

The last space is the space of classical Jacobi forms of weight $\frac{1}{2}$, index 2 with the multiplier system v_η^3. It is easy to check that $\vartheta_{D_1}(\tau, 0) = \vartheta_{D_1}(\tau, \frac{1}{2}) = 0$. Therefore

$$\vartheta_{D_1} = \vartheta(\tau, 2z).$$

(4) Analyzing $U(T)$ and $U(S)$ for all other m modulo 8, we get only one common eigenvector corresponding to the theta-product $\vartheta_{D_m} = \theta_1^{D_m} - \theta_3^{D_m}$. Therefore Example 1.8 contains all possible Jacobi forms of singular weight (and index one) for D_m.

Example 2.10 (The lattice E_6) Let $E_6^\vee$ be the dual lattice of E_6 and $D(E_6)$ its discriminant group. We have

$$D(E_6) = E_6^\vee / E_6 \simeq \mathbb{Z}/3\mathbb{Z} \quad \text{and} \quad q_{D(E_6)} = -q_{D(A_2)}.$$

The discriminant group has the following system of representatives (see [4], Planche V): $D(E_6) = \{0, \mu, 2\mu\}$ where $\mu^2 \equiv \frac{4}{3} \mod 2\mathbb{Z}$. We have

$$U(T) = \mathrm{diag}\big(1, \rho^2, \rho^2\big), \qquad U(S) = \frac{i}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \rho^2 & \rho \\ 1 & \rho & \rho^2 \end{pmatrix}$$

with $\rho = e^{\frac{2i\pi}{3}}$. We get

$$\theta_{E_6}(\tau, \mathfrak{z}_6) = (\theta_1 - \theta_2)(\tau, \mathfrak{z}_6) \in J_{3,E_6}\big(v_\eta^{16}\big).$$

This Jacobi form is invariant with respect to the Weyl group $W(E_6)$.

The simple construction of Jacobi forms using products of Jacobi theta-series has a lot of advantages. First, we get Jacobi forms of singular weight with a very simple divisor. Second, we can easily determine the maximal group of symmetries with respect to the abelian variable. This fact is important in the next section in which we construct modular forms of singular weight with respect to orthogonal groups.

Example 2.11 (The Jacobi theta-series $\vartheta_{3/2}$) We can get more examples using the second theta-series of weight $1/2$ and index $3/2$ with respect to the full modular group $\mathrm{SL}_2(\mathbb{Z})$. This function is related to twisted affine Lie algebras and is important in the construction of basic reflective Siegel modular forms (see [19])

$$\vartheta_{3/2}(\tau,z)=\frac{\eta(\tau)\vartheta(\tau,2z)}{\vartheta(\tau,z)}\in J_{\frac{1}{2},A_1;\frac{3}{2}}(v_\eta\times v_H)=J_{\frac{1}{2},\langle 6\rangle;\frac{1}{2}}(v_\eta\times v_H) \tag{21}$$

which is given by the quintiple product formula

$$\begin{aligned}\vartheta_{3/2}(\tau,z)&=\sum_{n\in\mathbb{Z}}\left(\frac{12}{n}\right)q^{\frac{n^2}{24}}r^{\frac{n}{2}}\\&=q^{\frac{1}{24}}r^{-\frac{1}{2}}\prod_{n\geqslant 1}\big(1+q^{n-1}r\big)\big(1+q^nr^{-1}\big)\big(1-q^{2n-1}r^2\big)\big(1-q^{2n-1}r^{-2}\big)\big(1-q^n\big).\end{aligned}$$

We have

$$\vartheta_{mA_1(3)}(\tau,z_1,\dots,z_m)=\prod_{1\leqslant j\leqslant m}\vartheta_{3/2}(\tau,z_j)\in J_{\frac{m}{2},m\langle 6\rangle;\frac{1}{2}}\big(v_\eta^m\times v_H^{\otimes m}\big) \tag{22}$$

(we recall that $m\langle 6\rangle=mA_1(3)$ denotes the orthogonal sum of m copies of the lattice $\langle 6\rangle$ of rank one). The same theta-product can be considered as a Jacobi form of index 1 for the lattice $D_m(3)$ and

$$\vartheta_{D_m(3)}(\tau,\mathfrak{z}_m)=\vartheta_{3/2}(\tau,z_1)\cdot\dots\cdot\vartheta_{3/2}(\tau,z_m)\in J_{\frac{m}{2},D_m(3)}\big(v_\eta^m\big) \tag{23}$$

where $D_m(3)$ is the lattice D_m renormalized by 3. In this simple way, we construct examples of Jacobi forms of singular weight with trivial character for even $n_0\geqslant 8$: D_8, $D_7\oplus D_3(3)$, $D_6\oplus D_6(3)$, $D_5\oplus D_9(3)$ and so on (see Proposition 4.6).

3 The lifting of Jacobi forms of half-integral index

The lifting of the Jacobi form ϑ_{D_8} (see (17)) is a reflective modular form with respect to the orthogonal group $\mathrm{O}^+(2U\oplus D_8(-1))$ (see [14]) which is equal to the Borcherds–Enriques automorphic discriminant Φ_4 of the moduli space of the Enriques surfaces introduced in [2]. The lifting of the Jacobi form

$$\eta^9(\tau)\vartheta_{D_5}(\tau,\mathfrak{z}_5)\in J_{7,D_5}$$

determined the unique canonical differential form on the modular variety of the orthogonal group $\widetilde{\mathrm{SO}}^+(2U\oplus D_5(-1))$ having Kodaira dimension 0. In [14], there were found three such modular varieties of dimension 4, 6 and 7. The cusp form of the modular variety of dimension 4 is defined by a Jacobi form of half-integral index with a character of order 2 (see Example 3.4 below). In this section, we give a variant of the lifting of Jacobi forms of half-integral index with a character. This theorem is a generalization of Theorem 3.1 in [11] (the case of Jacobi forms of orthogonal type with trivial character) and Theorem 1.12 in [19] (the case of Siegel modular forms with respect to a paramodular group of genus 2). All these constructions are particular cases of Borcherds additive lifting (see [3, Sect. 14]) of vector-valued modular forms. Nevertheless, our approach related to Jacobi forms gives in a natural way many new important examples of reflective modular forms for orthogonal groups. The Theorem 3.2 is a necessary tool for this purpose.

We can define a Hecke operator which multiplies the index of Jacobi forms. This operator is similar to the operator V_m of [8] or to the 'minus'-Hecke operator introduced in [11, 12] in the case of Siegel modular forms of arbitrary genus or for the modular forms for orthogonal groups. We apply such operators to elements of $J_{k,L;t}(v_\eta^D \times \nu)$ where ν is a binary character of the minimal integral Heisenberg group $H_s(L)$.

Proposition 3.1 *Let $\varphi \in J_{k,L;t}(v_\eta^D \times \nu)$ not identically zero. We assume that k is integral, t is rational and D is an even divisor of 24. If $Q = \frac{24}{D}$ is odd, we assume that the character of the minimal integral Heisenberg group $\nu : H_s(L) \to \{\pm 1\}$ is trivial. Then, for any natural m coprime to Q, the function*

$$\varphi|_{k,t} T_-^{(Q)}(m)(\tau, \mathfrak{z}) = \sum_{\substack{ad=m,\ a>0 \\ b \bmod d}} a^k v_\eta^D(\sigma_a) \varphi\left(\frac{a\tau + bQ}{d}, a\mathfrak{z}\right),$$

where $\sigma_a \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_a \equiv \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \bmod Q$, belongs to $J_{k,L;mt}(v_{\eta,m}^D \times \nu)$. The new $\mathrm{SL}_2(\mathbb{Z})$-character is defined as follows:

$$v_{\eta,m}^D(A) = v_\eta^D(A_m) \quad \textit{for all } A \in \mathrm{SL}_2(\mathbb{Z})$$

with $A_m \left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right) A \bmod Q$. The character $v_{\eta,m}^D$ depends only on $m \bmod Q$.

Proof It is known that $\mathrm{Ker}\, v_\eta^D$ contains the principle congruence subgroup $\Gamma(Q) < \mathrm{SL}_2(\mathbb{Z})$ (see [19, Lemma 1.2]). We consider the following subgroup $\Gamma^J(Q) \simeq \Gamma(Q) \rtimes \mathrm{Ker}(\nu)$ of the Jacobi group. We identify it with the corresponding parabolic subgroup in the orthogonal group $\mathrm{SO}^+(2U \oplus L(-1))$. For $(m, Q) = 1$, let

$$T^{(Q)}(m) = \Gamma(Q) \sum_{\substack{ad=m,\ a>0 \\ b \bmod d}} \sigma_a \begin{pmatrix} a & bQ \\ 0 & d \end{pmatrix}$$

be the usual Hecke operator for $\Gamma(Q)$. To the element $T^{(Q)}(m)$, we associate the element $T_-^{(Q)}(m)$ of the Hecke ring of the parabolic subgroup (see [11] and [19])

$$T_-^{(Q)}(m) = \Gamma^J(Q) \sum_{\substack{ad=m,\ a>0 \\ b \bmod d}} \{\sigma_a\} M_{a,b,d}$$

where $M_{a,b,d} = \mathrm{diag}\left(\left(\begin{smallmatrix} a & -bQ \\ 0 & d \end{smallmatrix}\right),\ \mathbf{1}_{n_0},\ m^{-1}\left(\begin{smallmatrix} a & bQ \\ 0 & d \end{smallmatrix}\right)\right)$. This is a sum of some double cosets with respect to $\Gamma^J(Q)$. We consider the extended Jacobi form $\widetilde{\varphi}(Z) = \varphi(\tau, \mathfrak{z}) e^{2i\pi t\omega}$ with $Z =$

${}^t(\omega, {}^t\mathfrak{z}, \tau) \in \mathcal{H}(L)$ which is modular with respect to the parabolic subgroup. Then we have

$$\widetilde{\psi}(Z) = \big(\widetilde{\varphi}|_k T_-^{(Q)}(m)\big)(Z) = \sum_{\substack{ad=m,\ a>0\\ b \bmod d}} \big(\widetilde{\varphi}|_k\{\sigma_a\}M_{a,b,d}\big)(Z).$$

By definition, we have

$$\big(\widetilde{\varphi}|_k\{\sigma_a\}M_{a,b,d}\big)(Z) = a^k v_\eta^D(\sigma_a)\varphi\left(\frac{a\tau + bQ}{d}, a\mathfrak{z}\right)e^{2i\pi mt\omega}.$$

Therefore, the Hecke operator of the proposition corresponds to the Hecke operator $T_-^{(Q)}(m)$ of the parabolic subgroup $\Gamma^J(Q)$ acting on the modular forms with respect to the parabolic subgroup $\Gamma^J(Q)$. We remark that the new index of the extended function on $\mathcal{H}(L)$ is equal to mt. The case of modular transformations is similar to the theory of usual Hecke operators (see [31]). If $A \in \mathrm{SL}_2(\mathbb{Z})$, then somewhat lenghty but easy calculations give us

$$\widetilde{\psi}|_k\{A\} = \sum_{\substack{a'd'=m,\ a'>0\\ b' \bmod d'}} \big(\widetilde{\varphi}\big|_k\{A_m\}\big)|_k\{\sigma_{a'}\}M_{a',b',d'} = v_\eta^D(A_m)\widetilde{\psi}.$$

This is due to the fact that the group $\Gamma(Q)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$ and then

$$\Gamma^J(Q)\{A_m\}^{-1}\{\sigma_a\}M_{a,b,d}\{A\} \neq \Gamma^J(Q)\{A_m\}^{-1}\{\sigma_{a'}\}M_{a',b',d'}\{A\}$$

for distinct a and a' prime to Q. Secondly, we consider the abelian transformations. Let $h = [x, y; r] \in H_s(L)$, then

$$\widetilde{\psi}|_k h = \sum_{\substack{ad=m,\ a>0\\ b \bmod d}} \nu\big(h'_{a,b,d}\big)\widetilde{\varphi}|_k\{\sigma_a\}M_{a,b,d}$$

where $h'_{a,b,d} = \{\sigma_a\}(M_{a,b,d}\cdot h)\{\sigma_a^{-1}\} = [x', y'; r']$ and

$$\big[x', y'; r'\big] = \big[(\delta d + \gamma bQ)x - a\gamma y, -(\beta d + \alpha bQ)x + \alpha a y; mr\big] \in H(L)$$

with $\sigma_a = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \equiv \left(\begin{smallmatrix}a^{-1} & 0\\ 0 & a\end{smallmatrix}\right) \bmod Q$. We note that

$$\big(x', y'\big) \equiv \big(m(\alpha\delta + \beta\gamma) + 2\alpha\gamma abQ\big)(x, y) \equiv m(x, y) \bmod 2s(L).$$

Therefore, if $\nu = \mathrm{id}$ then

$$\nu\big(\big[x', y'; r'\big]\big) = e^{2\pi it(mr - \frac{1}{2}m(x,y))}$$

because $t\cdot s(L) \in \mathbb{Z}$. This proves the formula for odd Q. If Q is even, we have $[x', y'; r'] = [mx + Q\widetilde{x}, y + Q\widetilde{y}; mr]$. Then

$$\begin{aligned}&\left[-Q\widetilde{x}, -Q\widetilde{y}; -\frac{Q^2}{2}(\widetilde{x}, \widetilde{y})\right]\cdot\big[x', y'; r'\big]\\ &\quad = \left[mx, y; mr + \frac{Q}{2}\big(-\big(\widetilde{x}, y'\big) + m\big(\widetilde{y}, x'\big) - Q(\widetilde{x}, \widetilde{y})\big)\right].\end{aligned}$$

As Q is even, we have $\nu([-Q\widetilde{x}, -Q\widetilde{y}; -\frac{Q^2}{2}(\widetilde{x}, \widetilde{y})]) = 1$. But, in this case, $m = 2m_0 + 1$ is odd so

$$\nu\big([x', y'; r']\big) = \nu\big([mx, y; mr]\big) = \nu\big([x, y; mr - m_0(x, y)]\big) = \nu\big([x, y; r]\big).$$

We calculate the Fourier expansion of $\varphi|_{k,t}T_-^{(Q)}(m)$ in the proof of Theorem 3.2 (see below). It shows that it is a holomorphic Jacobi form. □

Let L be an even lattice. *The stable orthogonal group* $\widetilde{\mathrm{O}}^+(L)$ is the subgroup of $\mathrm{O}^+(L)$ whose elements induce the identity on the discriminant group $D(L) = L^\vee/L$

$$\widetilde{\mathrm{O}}^+(L) = \big\{g \in \mathrm{O}^+(L) \mid \forall\, l \in L^\vee :\ g(l) - l \in L\big\}.$$

Theorem 3.2 *Let $\varphi \in J_{k,L;t}(v_\eta^D \times \nu)$, k be integral, t be rational and D be an even divisor of 24. If the conductor $Q = \frac{24}{D}$ is odd, we assume that ν is trivial. Fix $\mu \in (\mathbb{Z}/Q\mathbb{Z})^*$. Then the function*

$$\mathrm{Lift}_\mu(\varphi)(Z) = f(0,0)E_k(\tau) + \sum_{\substack{m \equiv \mu \bmod Q \\ m \geqslant 1}} m^{-1}\big(\widetilde{\varphi}|_k T_-^{(Q)}(m)\big) \circ \pi_{Qt}(Z),$$

is a modular form of weight k with respect to the stable orthogonal group $\widetilde{\mathrm{O}}^+(2U \oplus L(Qt))$ of the even lattice $L(Qt)$ with a character of order Q induced by $v_{\eta,\mu}^D$, the binary Heisenberg character ν of $H_s(L(Qt))$ and the character $e^{2i\pi\frac{\mu}{Q}}$ of the center of $H(L(Qt))$. In the formula above, $f(0,0)$ is the zeroth Fourier coefficient of φ, E_k is the Eisenstein series of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$ and the map π_{Qt} was defined in (13).

Proof *The Eisenstein series E_k.* First we note that $f(0,0)$ could be non-zero only for the trivial character $v_\eta^D = \mathrm{id}$. In this case, $\varphi(\tau, 0) = f(0,0) + \cdots$ is a non-zero modular form of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$. Therefore $k \geqslant 4$ and E_k is well defined. We note that E_k is a Jacobi form of index 0.

The lattice $L(Qt)$. The lattice $L(t)$ is integral for a non zero Jacobi form φ. If Q is odd then $L(t)$ is even because the character ν is trivial in this case (see Proposition 2.3). Therefore, for all Q, the lattice $L(Qt)$ is even.

The character of $\Gamma^J(L(Qt))$. According to Proposition 3.1

$$\varphi_m(\tau, \mathfrak{z}) = \big(\varphi|_{k,t}T_-^{(Q)}(m)\big)(\tau, \mathfrak{z}) \in J_{k,L;mt}\big(v_{\eta,\mu}^D \times \nu\big).$$

We can defined an extended Jacobi form using the map π_{Qt} (see (13)). According to Proposition 2.3

$$\varphi_m(\tau, \mathfrak{z})e^{2i\pi\frac{m}{Q}\omega} \in \widetilde{J}_{k,L(Qt);\frac{m}{Q}}\big(v_{\eta,\mu}^D \times \nu\big)$$

is a modular form of weight k with respect to the parabolic subgroup $\Gamma^J(L(Qt))$ of the orthogonal group $\widetilde{\mathrm{SO}}^+(2U \oplus L(-Qt))$. We note that the character ν of the minimal integral Heisenberg group $H_s(L(Qt))$ is extended to the center of $H(L(Qt))$ by the formula

$$\nu\big([0, 0; r]\big) = e^{2\pi i\frac{m}{Q}r} = e^{2\pi i\frac{\mu}{Q}r}.$$

If $f(0,0)\neq 0$ then $v_\eta^D=\mathrm{id}$, i.e. $D=24$, $Q=1$ and $\nu=\mathrm{id}$. Therefore all terms in the sum defining the lifting $\mathrm{Lift}_\mu(\varphi)$ have the same character with respect to $\Gamma^J(L(Qt)) < \widetilde{\mathrm{SO}}^+(2U\oplus L(-Qt))$.

Convergence. Let $Z={}^t(\omega,{}^t\mathfrak{z},\tau)\in\mathcal{H}(L(Qt))$. The extended Jacobi form $\widetilde{\varphi}(Z)=\varphi(\tau,\mathfrak{z})\exp(2\pi i\frac{\omega}{Q})$ of index $\frac{1}{Q}$ is holomorphic at "infinity" ($\mathrm{Im}\,\omega\to+\infty$). Therefore $|\widetilde{\varphi}|$ is bounded in any neighborhood of infinity (see [5] and [27]). We can rewrite this fact using the free parameter $\widetilde{v}=\widetilde{v}(Z)>0$ from Lemma 2.4. Then we have

$$\big|\varphi(\tau,\mathfrak{z})\big|\exp\left(-\frac{2\pi}{Q}\,\frac{(\mathrm{Im}(\mathfrak{z}),\mathrm{Im}(\mathfrak{z}))}{2v}\right)<C$$

is bounded for $v=\mathrm{Im}(\tau)>\varepsilon$ and the exponential term does not depend on the action of $\Gamma^J(L(Qt))$. Using the action of $\mathrm{SL}_2(\mathbb{Z})<\Gamma^J(L(Qt))$, we obtain that

$$\big|\varphi(\tau,\mathfrak{z})\big|\exp\left(-\frac{2\pi}{Q}\frac{(\mathrm{Im}(\mathfrak{z}),\mathrm{Im}(\mathfrak{z}))}{2v}\right)<Cv^{-k}$$

if $v\leqslant\varepsilon$ (see [5, Sect. 2] for similar considerations). Now we can get an estimation of all terms in the sum for $\mathrm{Lift}_\mu(\varphi)$ for $v>\varepsilon$. We have

$$\left|a^k\varphi\left(\frac{a\tau+bQ}{d},a\mathfrak{z}\right)\exp\left(-2\pi\frac{1}{Q}\left(\frac{(\mathrm{Im}(a\mathfrak{z}),\mathrm{Im}(a\mathfrak{z}))}{2va/d}\right)\right)\right|<Cd^kv^{-k}$$

if $\frac{a}{d}v\leqslant\varepsilon$. If $\frac{a}{d}v>\varepsilon$ then we have $<Ca^k$. In the both cases we see that the term above depending on (a,b,d) is smaller than $C_\varepsilon m^k$. It gives us

$$\left|m^{-1}\varphi_m(\tau,\mathfrak{z})e^{2i\pi\frac{m}{Q}\omega}\right|<C_\varepsilon m^k\sigma_0(m)\exp\left(-\frac{2\pi m}{Q}\widetilde{v}\right)<C_\varepsilon m^{k+1}\exp\left(-2\pi\frac{m}{Q}\widetilde{v}\right)$$

where $\widetilde{v}(Z)>0$. Therefore the function $\mathrm{Lift}_\mu(\varphi)$ is well defined and it transforms like a modular form of weight k and character $v_{\eta,\mu}^D\times\nu\times e^{2\pi i\frac{\mu}{Q}r}$ with respect to the parabolic subgroup $\Gamma^J(L(Qt))$.

Fourier expansion of $\mathrm{Lift}_\mu(\varphi)$. In the summation of the Fourier expansion of $\varphi\in J_{k,L;t}(v_\eta^D\times\nu)$, we have $n\equiv\frac{D}{24}\bmod\mathbb{Z}$ (see (11)). Rewriting n in terms of the conductor $Q=\frac{24}{D}$, the Fourier expansion of the function φ has the following form

$$\varphi(\tau,\mathfrak{z})=\sum_{\substack{n\equiv 1\bmod Q,\,n\geqslant 0\\ l\in\frac{1}{2}L^\vee\\ 2\frac{nt}{Q}-(l,l)\geqslant 0}} f\left(\frac{n}{Q},l\right)e^{2i\pi(\frac{n}{Q}\tau+(l,\mathfrak{z}))}.$$

After the summation over b mod d in the action of the Hecke operator, we get

$$\begin{aligned}&m^{-1}\big(\widetilde{\varphi}|_kT_-^{(Q)}(m)\big)\circ\pi_{Qt}(Z)\\&\quad=\sum_{\substack{ad=m\\a>0}}a^{k-1}v_\eta^D(\sigma_a)\sum_{\substack{nd\equiv 1\bmod Q,n\geqslant 0\\ l\in\frac{1}{2}L^\vee\\ 2\frac{nd}{Q}t-(l,l)\geqslant 0}} f\left(\frac{nd}{Q},l\right)e^{2i\pi(\frac{na}{Q}\tau+a(l,\mathfrak{z})+\frac{ad}{Q}\omega)}.\end{aligned}$$

So we have

$$\operatorname{Lift}_\mu(\varphi)(Z)=\sum_{\substack{m\equiv\mu \bmod Q\\ m\geqslant 1}}\ \sum_{\substack{ad=m\\ a>0}} a^{k-1}v_\eta^D(\sigma_a)$$
$$\times\sum_{\substack{nd\equiv 1 \bmod Q\\ l\in\frac12 L^\vee\\ 2\frac{nd}{Q}t-(l,l)\geqslant 0}} f\left(\frac{nd}{Q},l\right)e^{2i\pi(\frac{na}{Q}\tau+a(l,\mathfrak{z})+\frac{ad}{Q}\omega)}.$$

But $nd\equiv 1 \bmod Q \Leftrightarrow an\equiv\mu \bmod Q$ because for any $(\mu,24)=1$ we have $\mu^2\equiv 1 \bmod 24$. (We note that 24 is the maximal natural number with this property.) Using this property, we obtain the Fourier expansion of the lifting

$$\operatorname{Lift}_\mu(\varphi)(Z)=\sum_{\substack{m,n\equiv\mu \bmod Q\\ m,n\geqslant 1\\ l\in\frac12 L^\vee\\ 2\frac{nm}{Q}t-(l,l)\geqslant 0}}\left(\sum_{a\mid(n,l,m)} a^{k-1}v_\eta^D(\sigma_a)f\left(\frac{nm}{Qa^2},\frac{l}{a}\right)\right)e^{2i\pi(\frac{n}{Q}\tau+(al,\mathfrak{z})+\frac{m}{Q}\omega)}.$$

We can reformulate the condition on the hyperbolic norm of the index (n,l,m) of the Fourier coefficient in terms of the lattice $L(Qt)$: $2\frac{nm}{Q^2}-\frac{1}{Qt}(l,l)\geqslant 0$.

The formula for the Fourier expansion is symmetric with respect to τ and ω. The involution V which permutes the isotropic vectors e_1 and f_1 in the second copy of the hyperbolic plane of the lattice $U\oplus U_1\oplus L(-Qt)$ realizes the transformation $\tau\leftrightarrow\omega$ and $\mathfrak{z}\leftrightarrow\mathfrak{z}$. We see that $V\in\widetilde{\mathrm{O}}^+(2U\oplus L(-Qt))$, $\det(V)=-1$, $J(V,Z)=1$ and

$$\operatorname{Lift}_\mu(\varphi)|_k V=\operatorname{Lift}_\mu(\varphi).$$

It is known (see [11, p. 1194] or [22, Proposition 3.4]) that

$$\widetilde{\mathrm{O}}^+\big(2U\oplus L(-Qt)\big)=\big\langle\Gamma^J\big(L(Qt)\big),\ V\big\rangle.$$

Therefore, the function $\operatorname{Lift}_\mu(\varphi)$ is a modular form of weight k with a character of order Q with respect to $\widetilde{\mathrm{O}}^+(2U\oplus L(-Qt))$. □

Remark to Theorem 3.2 If $\mu=1$ then $\operatorname{Lift}(\varphi)=\operatorname{Lift}_1(\varphi)\not\equiv 0$ because its first Fourier–Jacobi coefficient $\widetilde{\varphi}$ is not zero. For $\mu\neq 1$ the function $\operatorname{Lift}_\mu(\varphi)$ might be identically zero. See [19, Example 1.15] for a non-zero μ-lifting in the case of signature $(2,3)$.

At the end of the section we give the first application of Theorem 3.2.

Example 3.3 (Modular forms of singular weight) The first example of such modular forms was given in [10]:

$$\operatorname{Lift}(\Theta_{E_8})=\frac{1}{240}+\sum_{\substack{n,m\geq 0,\ \ell\in E_8\\ 2nm=(\ell,\ell)\\ (n,m)\neq(0,0)}}\sigma_3\big((n,\ell,m)\big)e^{2\pi i(n\tau+(\ell,\mathfrak{z})+m\omega)}\in M_4\big(\mathrm{O}^+(II_{2,10})\big)$$

where $\sigma_3((n,\ell,m))$ is the sum of the cubes of all divisors of the greatest common divisor of n, m and $\ell\in E_8$. This function is sometimes called the simplest modular form. Using the theta-products (15)–(23), we can define modular forms of singular weight on orthogonal groups with a character induced by v_η^D-character for all even divisors of 24. We give some examples below in order to illustrate different cases:

$$\mathrm{Lift}(\vartheta_{D_8}) \in M_4\big(\widetilde{\mathrm{O}}^+\big(2U \oplus D_8(-1)\big)\big), \qquad \mathrm{Lift}(\vartheta_{D_8(3)}) \in M_4\big(\widetilde{\mathrm{O}}^+\big(2U \oplus D_8(-9)\big), \chi_3\big),$$
$$\mathrm{Lift}(\vartheta_{4A_1}) \in M_2\big(\widetilde{\mathrm{O}}^+\big(2U \oplus 4A_1(-1)\big), \chi_2\big), \qquad \mathrm{Lift}(\vartheta_{D_{24}(3)}) \in M_{12}\big(\widetilde{\mathrm{O}}^+\big(2U \oplus D_{24}(-3)\big)\big),$$
$$\mathrm{Lift}(\vartheta_{2A_1}) \in M_1\big(\widetilde{\mathrm{O}}^+\big(2U \oplus 2\langle -4\rangle\big), \chi_4\big), \qquad \mathrm{Lift}(\vartheta_{D_2(3)}) \in M_1\big(\widetilde{\mathrm{O}}^+\big(2U \oplus 2\langle -36\rangle\big), \chi_{12}\big)$$

where χ_n denotes a character of order $n = \frac{24}{D}$ of the corresponding orthogonal group.

We note that in many cases *the maximal modular group* of the lifting is larger than the stable orthogonal group $\widetilde{\mathrm{O}}^+(2U \oplus L(-1))$. For example, the maximal modular group of $\mathrm{Lift}(\eta^d \vartheta_{D_m})$ for any d and m such that $d + 3m \equiv 0 \bmod 24$ is the full orthogonal group $\mathrm{O}^+(2U \oplus D_m(-1))$ if $m \neq 4$. The form $\mathrm{Lift}(\eta^d \vartheta_{D_m})$ is anti-invariant with respect to the involution of the Dynkin diagram (the reflection with respect to a vector with square 4). If $m = 4$ then

$$\mathrm{O}^+\big(2U \oplus D_4(-1)\big) / \widetilde{\mathrm{O}}^+\big(2U \oplus D_4(-1)\big) \cong S_3.$$

The liftings of ϑ_{D_4}, $\vartheta_{D_4}^{(2)}$, $\vartheta_{D_4}^{(3)}$ (see Example 1.8) are modular with respect to three different subgroups of order 3 in $\mathrm{O}^+(2U \oplus D_4(-1))$.

The lifting of any theta-products vanishes along the divisors of the corresponding Jacobi forms. In particular $\mathrm{Lift}(\vartheta_{4A_1})$ vanishes with order one along $z_i = 0$. It is known that the full divisor of this modular form is equal to the union of all modular transformations of $z_i = 0$, i.e. this is a singular reflective modular form with the simplest possible divisor (see [14]). The same is true for $\mathrm{Lift}(\vartheta_{D_8})$. The Fourier expansion of $\mathrm{Lift}(\vartheta_{4A_1})$ (or $\mathrm{Lift}(\vartheta_{D_8})$) written in a fixed Weyl chamber of the corresponding orthogonal group will define generators and relations of Lorentzian Kac–Moody algebras (see [17–19] and a forthcoming paper of Gritsenko and Nikulin about reflective groups of rank $\geqslant 4$). Here we consider the formula for $4A_1$ which was given without proof in [14].

Example 3.4 (Jacobi lifting, the modular tower $4A_1$ and modular forms of "Calabi–Yau type") We consider the following theta-product as a Jacobi form of index $\frac{1}{2}$

$$\vartheta_{4A_1}(\tau, \mathfrak{z}_4) = \vartheta(\tau, z_1) \ldots \vartheta(\tau, z_4) \in J_{2,4A_1;\frac{1}{2}}\big(v_\eta^{12} \times v_H^{\otimes 4}\big).$$

According to Theorem 3.2, we get

$$\Phi_2(Z) := \mathrm{Lift}(\vartheta_{4A_1})(\tau, \mathfrak{z}_4, \omega) \in M_2\big(\mathrm{O}^+\big(2U \oplus 4A_1(-1)\big), \chi_2\big)$$

where χ_2 is a character of order 2 of the full orthogonal group. The modular form Φ_2 is reflective with the simplest possible divisor (see [14]). The Fourier expansion of this fundamental reflective form of singular weight is the following

$$\Phi_2(Z) = \sum_{\ell=(l_1,\ldots,l_4),\ l_i \equiv \frac{1}{2} \bmod \mathbb{Z}} \times \sum_{\substack{n,m \in \mathbb{Z}_{>0} \\ n \equiv m \equiv 1 \bmod \mathbb{Z} \\ nm-(\ell,\ell)=0}} \sigma_1\big((n, \ell, m)\big) \left(\frac{-4}{2l_1}\right) \ldots \left(\frac{-4}{2l_4}\right) e^{\pi i (n\tau + (\ell, \mathfrak{z}_4) + m\omega)}$$

where $\sigma_1(n) = \sum_{d|n} d$. The quasi-pullbacks (see [21]) of Φ_2 along the divisors are again reflective (see [14]). In this way we obtain the $4A_1$-tower of reflective modular forms in six, five, four and three variables with respect to $\mathrm{O}^+(2U \oplus nA_1(-1))$ for $n = 4, 3, 2$ and 1:

$$\Phi_2 = \mathrm{Lift}(\vartheta_{4A_1}), \qquad \mathrm{Lift}\big(\eta^3\vartheta_{3A_1}\big),$$
$$K_4(\tau, z_1, z_2, \omega) := \mathrm{Lift}\big(\eta^6(\tau)\vartheta(\tau, z_1)\vartheta(\tau, z_2)\big), \qquad \Delta_5 = \mathrm{Lift}\big(\eta^9(\tau)\vartheta(\tau, z)\big)$$

where $\Delta_5 \in S_5(\mathrm{Sp}_2(\mathbb{Z}), \chi_2)$ is the Igusa modular form (a square root of the first Siegel cusp form of weight 10). The modular form Δ_5 determines one of the most fundamental Lorentzian Kac–Moody algebras related to the second quantized elliptic genus of $K3$ surfaces (see [7, 17] and [13]). The modular form

$$K_4 \in S_4\big(\widetilde{\mathrm{SO}}^+\big(2U \oplus 2A_1(-1)\big), \chi_2\big)$$

is the second member of the modular $4A_1$-tower based on Δ_5. This form defines an (elliptic) Lorentzian Kac–Moody algebra of signature $(1, 3)$ (see a forthcoming paper of Gritsenko and Nikulin). Moreover $K_4(Z)dZ$ is the only canonical differential form on the orthogonal modular variety

$$M_{\chi_2}(2A_1) = \Gamma_{\chi_2} \setminus \mathcal{D}\big(2U \oplus 2A_1(-1)\big)$$

of complex dimension 4 and of Kodaira dimension 0 where $\Gamma_{\chi_2} = \ker(\chi_2)$ (see [14]). The first example of cusp forms of this type was considered in [16] where it was shown that the modular form

$$\Delta_1 = \mathrm{Lift}\big(\eta(\tau)\vartheta(\tau, z)\big) \in S_1\big(\widetilde{\mathrm{O}}^+\big(2U \oplus \langle -6\rangle\big), \chi_6\big)$$

determines the unique, up to a constant, canonical differential form $\Delta_1^3(Z)dZ$ on the Barth-Nieto modular Calabi–Yau three-fold. The second example of Siegel cusp forms of canonical weight with the simplest possible divisor was constructed in [5]:

$$\nabla_3 = \mathrm{Lift}\big(\eta(\tau)\eta(2\tau)^4\vartheta(\tau, z)\big) \in S_3\big(\Gamma_0^{(2)}(2), \chi_2\big)$$

where $\Gamma_0^{(2)}(2) < \mathrm{Sp}_2(\mathbb{Z})$ and χ_2 is its character of order 2. A Calabi-Yau model of the Siegel modular three-fold $\Gamma_0^{(2)}(2)_{\chi_2} \setminus \mathbb{H}_2$ was found in [9]. The modular form in four variables K_4 is the next example of a cusp form of "Calabi–Yau type" similar to the Siegel modular forms Δ_1^3 and ∇_3.

Question 3.5 We can ask a question about the existence of a compact model of Calabi–Yau type of the modular variety $M_{\chi_2}(2A_1)$ of dimension 4 defined above.

4 Modular forms of singular and critical weights

The minimal possible weight (singular weight) of holomorphic Jacobi form for L is $\frac{n_0}{2}$ where $n_0 = \mathrm{rank}\, L$. The first weight for which Jacobi cusp forms might appear is equal to $\frac{n_0+1}{2}$. This weight is called *critical*. In the case of classical modular forms in one variable, the critical weight is equal to 1. The simplest possible example of modular forms of critical weight in our context is the cusp form $\Delta_1 = \mathrm{Lift}(\eta\,\vartheta)$ of weight 1 with a character of order 6 for the lattice $2U \oplus \langle -6\rangle$ of signature $(2, 3)$. We mentioned in Example 3.4 that this function determines one of the basic Lorentzian Kac–Moody algebras in the Gritsenko–Nikulin classification (see [18, 19]) and it induces the unique canonical differential form on a special Calabi–Yau three-folds, the Barth–Nieto quintic. We can construct a simple example of modular form of critical weight with trivial character using Theorem 3.2. This is

$$\mathrm{Lift}(\eta\,\vartheta_{D_{23}(3)}) \in M_{12}\big(\widetilde{\mathrm{O}}^+\big(2U \oplus D_{23}(-3)\big)\big)$$

which is a modular form with trivial character with respect to the orthogonal group of signature $(2, 25)$. In this section, we construct examples of Jacobi cusp forms of critical weight for all even ranks. For this aim, we use the pullback of Jacobi forms of singular weight such that its Fourier coefficient $f(0,0)=0$. This is exactly the case of ϑ_{D_m}.

Let $M < L$ be an even sublattice of L. We can consider the Heisenberg group of M as a subgroup of $H(L)$. Therefore if $\mathrm{rank}(M) = \mathrm{rank}(L)$ then the Jacobi forms with respect to L can be considered as Jacobi forms with respect to M. In the next proposition, we consider the operation of pullback.

Proposition 4.1 *Let $M < L$ be a sublattice of L and* $\mathrm{rank}(M) < \mathrm{rank}(L)$

$$M \oplus M^\perp < L, \qquad \mathfrak{z} = \mathfrak{z}_m \oplus \mathfrak{z}_\perp \in L \otimes \mathbb{C} = \big(M \oplus M^\perp\big) \otimes \mathbb{C}.$$

For any $\varphi(\tau, \mathfrak{z}) \in J_{k,L;t}(\chi \times \nu)$ its pullback is also a Jacobi form

$$\varphi|_M := \phi(\tau, \mathfrak{z}_m) = \varphi(\tau, \mathfrak{z})|_{(\mathfrak{z}_\perp = 0)} \in J_{k,M;t}(\chi \times \nu|_{\Gamma^J(M)}).$$

The pullback of a Jacobi cusp form is a cusp form or 0.

Proof We note that the pullback of Jacobi form might be the zero-function. What is more interesting is that the pullback might be a cusp form although the original function is not.

The functional equations (9)–(10) are evidently true for $\varphi|_M$. To calculate its Fourier expansion, we consider the embedding of the lattices

$$M \oplus M^\perp < L < L^\vee < M^\vee \oplus \big(M^\perp\big)^\vee.$$

We have to analyze the M-projection of any vector l in $\frac{1}{2}L^\vee = L(2)^\vee$ in the Fourier expansion (11). If the character ν of the minimal Heisenberg group is trivial then we do not need the coefficient $\frac{1}{2}$ before the lattices dual to L and M in the calculation below. For any $l \in \frac{1}{2}L^\vee = L(2)^\vee$, we have the following decomposition

$$l = l_m \oplus l_\perp = \mathrm{pr}_{M(2)^\vee}(l) \oplus \mathrm{pr}_{(M(2)^\perp)^\vee}(l) \in M(2)^\vee \oplus \big(M(2)^\perp\big)^\vee.$$

In the coordinates $\mathfrak{z} = \mathfrak{z}_m \oplus \mathfrak{z}_\perp$, we have

$$\varphi(\tau, \mathfrak{z}) = \sum_{n \geqslant 0,\, l = l_m \oplus l_\perp} f(n, l) e^{2\pi i (n\tau + (l_m, \mathfrak{z}_m) + (l_\perp, \mathfrak{z}_\perp))}.$$

Therefore

$$\varphi|_M(\tau, \mathfrak{z}_m) = \sum_{n \geqslant 0,\, l_m \in M(2)^\vee} \Bigg(\sum_{\substack{l_\perp \in (M(2)^\perp)^\vee \\ l_m \oplus l_\perp \in L(2)^\vee}} f(n, l_m \oplus l_\perp) \Bigg) e^{2\pi i (n\tau + (l_m, \mathfrak{z}_m))}.$$

We note that $2nt - (l_m, l_m) \geqslant (l_\perp, l_\perp) \geqslant 0$. The last inequality is strict if φ is a cusp form. □

Using the operation of pullback, we can construct Jacobi *cusp* forms of critical weight starting from Jacobi forms of singular weight if the constant term $f(0,0)$ of the last one is equal to zero. The estimation on $2nt-(l_m,l_m)$ at the end of the proof of the last proposition gives us the following estimation of the order at infinity (see (12)) of the pullback.

Corollary 4.2 *In the conditions of Proposition 4.1 we have*

$$\operatorname{Ord}\bigl(\varphi|_M(\tau,\mathfrak{z}_m)\bigr)\geqslant \min\bigl\{(l_\perp,l_\perp)\mid l_\perp=\operatorname{pr}_{(M^\perp)^\vee}(l)\text{ such that } f(n,l)\neq 0\bigr\}.$$

In particular, if $\operatorname{pr}_{(M^\perp)^\vee}(l)\neq 0$ *for all* $f(n,l)\neq 0$ *then the pullback* $\varphi|_M$ *is a cusp form or the zero-function.*

Using the last corollary, we can construct new important examples of Jacobi forms of singular and critical weights. We recall that by $J_{k,L}$ we denote the space of Jacobi forms of index one.

We define the root lattice A_m as a sublattice of D_{m+1}

$$A_m=\bigl\{(x_1,\ldots,x_{m+1})\in\mathbb{Z}^{m+1}\mid x_1+\cdots+x_{m+1}=0\bigr\}<D_{m+1}.$$

We note that $A_1\cong\langle 2\rangle$, $A_1\oplus A_1\cong D_2$ and $A_3\cong D_3$.

Proposition 4.3 (1) *Let* $v=2(b_1,\ldots,b_m)\in\mathbb{Z}^m$ *be an element in* D_m *with at least two non-zero coordinates* b_i *such that* $(b_1+\cdots+b_m)\equiv 1 \mod 2$ *and g.c.d.* $(b_1,\ldots,b_m)=1$. *Then*

$$\vartheta_{D_m}|_{v^\perp}\in J^{cusp}_{\frac{m}{2},v^\perp_{D_m}}\bigl(v_\eta^{3m}\bigr)\quad \textit{and}\quad \vartheta_{D_m(3)}|_{v^\perp}\in J^{cusp}_{\frac{m}{2},v^\perp_{D_m(3)}}\bigl(v_\eta^{m}\bigr)$$

is a non-zero Jacobi cusp form of critical weight such that

$$\operatorname{Ord}(\vartheta_{D_m}|_{v^\perp})=\frac{1}{(v,v)}>0\quad\textit{and}\quad \operatorname{Ord}(\vartheta_{D_m(3)}|_{v^\perp})=\frac{1}{3(v,v)}>0.$$

(2) *The theta-product*

$$\vartheta_{A_m}(\tau,z_1,\ldots,z_m)=\vartheta(\tau,z_1)\cdot\cdots\cdot\vartheta(\tau,z_m)\cdot\vartheta(\tau,z_1+\cdots+z_m)\in J_{\frac{m+1}{2},A_m}\bigl(v_\eta^{3m+3}\bigr)$$

is a Jacobi form of critical weight. If m is even then ϑ_{A_m} *is a Jacobi cusp form and*

$$\operatorname{Ord}(\vartheta_{A_m})=\frac{1}{4(m+1)}>0.$$

(3) *For the renormalized lattice* $A_m(3)$, *the Jacobi form*

$$\vartheta_{A_m(3)}(\tau,z_1,\ldots,z_m)=\vartheta_{3/2}(\tau,z_1)\cdot\cdots\cdot\vartheta_{3/2}(\tau,z_m)\cdot\vartheta_{3/2}(\tau,z_1+\cdots+z_m)$$

belongs to $J_{\frac{m+1}{2},A_m(3)}(v_\eta^{m+1})$. *For even* m,

$$\operatorname{Ord}(\vartheta_{A_m(3)})=\frac{1}{12(m+1)}>0.$$

Proof (1) If in v only one $b_i \neq 0$ then $\vartheta_{D_m}|_{v^\perp} \equiv 0$. To prove the lemma, we calculate the Fourier expansion of the pullback function. The discriminant group of D_m was given in Example 2.9. The Fourier expansion of ϑ_{D_m} has the following form

$$\vartheta_{D_m}(\tau, \mathfrak{z}_m) = \sum_{\substack{n \in \mathbb{Q}_{>0},\ l \in \frac{1}{2}\mathbb{Z}^m \\ 2n-(l,l)=0}} f(n,l)\, e^{2\pi i (n\tau + (l, \mathfrak{z}_m))}$$

where

$$f(n,l) = \left(\frac{-4}{2l}\right) = \left(\frac{-4}{2l_1}\right) \cdot \cdots \cdot \left(\frac{-4}{2l_m}\right)$$

is the product of the generalized Kronecker symbols modulo 4. In particular, all coordinates $2l_i$ are odd. If v is a vector which satisfies the condition of the proposition then $(l, v) = 2(l_1 b_1 + \cdots + l_m b_m) \equiv 1 \mod 2$. The lattice $\langle v \rangle^\vee$ is generated by $\frac{v}{(v,v)}$. We get that

$$l_\perp = \mathrm{pr}_{\langle v \rangle^\vee}(l) = (l, v)\frac{v}{(v, v)} \neq 0$$

is always non trivial. Moreover, there exists a vector $2l = (2l_i)$ with odd coordinates such that $(l, v) = 1$. According to Corollary 4.2

$$\mathrm{Ord}(\vartheta_{D_m}|_{v^\perp}) = \left| \left(\frac{v}{(v, v)}, \frac{v}{(v, v)} \right) \right| = \frac{1}{|(v, v)|} > 0$$

and $\vartheta_{D_m}|_{v^\perp}$ is a Jacobi cusp form. The proof for $D_m(3)$ is quite similar.

(2) We have $A_m = v^\perp_{D_{m+1}}$ where $v = 2(1, \ldots, 1) \in D_{m+1}$. In particular $z_{m+1} = -(z_1 + \cdots + z_m)$ and $\vartheta_{A_m} = -\vartheta_{D_{m+1}}|_{v^\perp}$. If m is even then v satisfies the condition in (1) and $\mathrm{Ord}(\vartheta_{A_m}) = \frac{1}{4(m+1)}$. The proof of (3) is similar. □

Example Since $A_3 \cong D_3$, there exist a Jacobi form of singular and two Jacobi forms (cusp and non-cusp) of critical weight for this lattice

$$\vartheta_{D_3} \in J_{\frac{3}{2}, D_3}(v_\eta^9), \qquad \eta\,\vartheta_{D_3} \in J^{cusp}_{2, D_3}(v_\eta^{10}), \qquad \vartheta_{A_3} \in J_{2, A_3}(v_\eta^{12}).$$

We can construct many Jacobi forms of singular, critical and other small weights using the equalities of the previous proposition. For any $\varphi \in J_{k, L; t}(\chi)$, we denote by $\varphi^{[n]}$ the direct (tensor) product of n-copies of φ, i.e. the Jacobi form for the lattice nL

$$\varphi^{[n]}\big(\tau, (\mathfrak{z}_1, \ldots, \mathfrak{z}_n)\big) = \varphi(\tau, \mathfrak{z}_1) \cdot \cdots \cdot \varphi(\tau, \mathfrak{z}_n) \in J_{nk, nL; t}\big(\chi^n\big).$$

The next example is very important.

Corollary 4.4 *There exists a Jacobi form of singular weight for* A_2

$$\sigma_{A_2}(\tau, z_1, z_2) = \frac{\vartheta(\tau, z_1)\vartheta(\tau, z_2)\vartheta(\tau, z_1 + z_2)}{\eta(\tau)} \in J_{1, A_2}\big(v_\eta^8\big). \tag{24}$$

In particular, the Jacobi form of singular weight $\sigma_{3A_2} = \sigma_{A_2}^{[3]} \in J_{3, 3A_2}$ *has trivial character.*

Proof We note that $\eta(\tau)^{-1} = q^{-1/24}(1 + q(\dots))$. Therefore

$$\operatorname{Ord}\left(\frac{\varphi(\tau, \mathfrak{z})}{\eta(\tau)}\right) = \operatorname{Ord}\big(\varphi(\tau, \mathfrak{z})\big) - \frac{1}{12}.$$

Thus σ_{A_2} is holomorphic Jacobi form of singular weight for A_2. □

Remarks (1) The Jacobi form σ_{A_2} is equal to the denominator function of the affine Lie algebra A_2 (see [26] and [6]). We consider the Jacobi forms related to the denominator functions of all affine Kac–Moody Lie algebras in a forthcoming paper of V. Gritsenko and K.-I. Iohara.

(2) The lifting of σ_{3A_2} is a reflective modular form of singular weight. The lifting of $\eta^8\sigma_{2A_2}$ determined the unique canonical differential form on a modular variety of Kodaira dimension 0 (see [14]).

(3) The form σ_{A_2} is the first example of Jacobi form obtained as theta/eta-quotients. Using such Jacobi form, we can produce important classical Jacobi forms in one variable called *theta-blocks*, see Corollary 4.9 and [23].

Using the same principle, we obtain

Corollary 4.5 *The Jacobi forms given below are cusp forms of critical weight.*

$$\kappa_{2A_4} = \frac{\vartheta_{A_4} \otimes \vartheta_{A_4}}{\eta} \in J^{cusp}_{\frac{9}{2},2A_4}\big(v_\eta^5\big), \qquad \kappa_{A_4\oplus A_6} = \frac{\vartheta_{A_4} \otimes \vartheta_{A_6}}{\eta} \in J^{cusp}_{\frac{11}{2},A_4\oplus A_6}\big(v_\eta^{11}\big).$$

Let $v_5 = 2(2, 1, 0, \dots, 0) \in D_m$ $(m \geqslant 2)$ *and* $v_7 = 2(2, 1, 1, 1, 0, \dots, 0) \in D_n$ $(n \geqslant 4)$. *Then*

$$\frac{\vartheta_{D_m}|_{v_5^\perp} \otimes \vartheta_{D_n}|_{v_a^\perp}}{\eta} \in J^{cusp}_{\frac{m+n-1}{2}, D_m|_{v_5^\perp} \oplus D_n|_{v_a^\perp}}\big(v_\eta^{3(n+m)-1}\big)$$

where $a = 5$ *or* 7.

Proof According to Proposition 4.3

$$\operatorname{Ord}(\kappa_{2A_4}) = \frac{1}{60}, \qquad \operatorname{Ord}(\kappa_{A_4\oplus A_6}) = \frac{1}{420}, \qquad \operatorname{Ord}(\vartheta_{D_m}|_{v_5^\perp}) = \frac{1}{20},$$
$$\operatorname{Ord}(\vartheta_{D_m}|_{v_7^\perp}) = \frac{1}{28}.$$
□

Remark In the same way, we get *non-cusp* Jacobi forms of weight $\frac{n_0}{2} + 1$ (singular weight $+1$):

$$\frac{\vartheta^{[5]}_{A_4}}{\eta^3} \in J_{11,5A_4}, \qquad \frac{\vartheta^{[7]}_{A_6}}{\eta^3} \in J_{23,7A_6}, \qquad \frac{\vartheta^{[3]}_{A_8}}{\eta} \in J_{13,3A_8}\big(v_\eta^8\big).$$

The first two functions have trivial character. It might be that these functions are interesting Eisenstein series. We can also mention the non-cusp form

$$\vartheta^{[3]}_{A_2(3)}/\eta \in J_{4,3A_2(3)}\big(v_\eta^8\big).$$

Proposition 4.6 *Theta-products give examples of Jacobi forms of singular weight with trivial character for some lattices of all even ranks* $\geqslant 6$.

Proof The corresponding Jacobi forms are tensor products of the following Jacobi theta-products

$$\begin{array}{llll} \sigma_{A_2} & \text{with character } v_\eta^8, & \vartheta_{D_m} & \text{with character } v_\eta^{3m}, \\ \vartheta_{D_4}^{(i)} & \text{with character } v_\eta^{3m} \ (i=2,3), & \vartheta_{2A_1}^{(1)} & \text{with character } v_\eta^6, \\ \vartheta_{D_m(3)} & \text{with character } v_\eta^m. & & \end{array}$$

See (24), (17), (18), (19), (16) and (23). Below, we give a list of lattices of rank smaller or equal to 24 since for larger ranks one can use the periodicity of the characters:

$$n=6,\ 3A_2; \qquad n=8,\ D_8,\ 2D_4,\ 8A_1; \qquad n=10,\ D_7\oplus D_3(3),\ A_2\oplus D_4\oplus D_4(3);$$
$$n=12,\ D_6\oplus D_6(3),\ 2A_2\oplus D_8(3),\ 6A_2; \qquad n=14,\ D_5\oplus D_9(3);$$
$$n=16,\ D_{16},\ D_4\oplus D_{12}(3); \qquad n=18,\ D_3\oplus D_{15}(3); \qquad n=20,\ D_2\oplus D_{20}(3);$$
$$n=22,\ D_1\oplus D_{21}(3); \qquad n=24,\ D_{24},\ 12A_2,\ D_{24}(3).$$

We note that we consider $8A_1$ as $4(A_1\oplus A_1)$. The corresponding Jacobi form is the product of four functions of type $\vartheta_{2A_1}^{(1)}$. Moreover, instead of any D_m in this list, we can put a direct sum $D_{m_1}\oplus\cdots\oplus D_{m_k}$ with $m_1+\cdots+m_k=m$. □

Jacobi forms of singular weight with respect to the full Jacobi group of a lattice L have a $\mathrm{SL}_2(\mathbb{Z})$-character of type v_η^{2m} if the rank of L is even (the singular weight is integral) or a multiplier system of type v_η^{2m+1} if the rank is odd (the singular weight is half-integral). Analyzing the examples of theta-products given above, we get the following table of possible characters v_η^m

rank n	d : **character of type** v_η^d
1	1, 3
2	2, 4, 6, 8
3	3, 5, 7, 9, 11
4	4, 6, 8, 10, 12, 14, 16
5	5, 7, 9, 11, 13, 15, 17, 19
6	6, 8, 10, 12, 14, 16, 18, 20, 22, 24
7	7, 9, 11, 13, 15, 17, 19, 21, 23, 1, 3
8	8, 10, 12, 14, 16, 18, 20, 22, 24, 2, 4, 6.

As corollary, we obtain

Proposition 4.7 *If $n\geqslant 8$ is even (respectively, $n\geqslant 9$ is odd) and $d\equiv n \bmod 2$ then there exists a lattice L of rank n such that the space of Jacobi forms of singular weight $J_{\frac{n}{2},L}(v_\eta^d)$ is not empty.*

Remark For some n, we can prove that this table contains all possible characters. We are planning to come to this question in another publication.

Now we would like to analyze Jacobi forms of critical weight. First, note that the multiplication by η gives us the simplest such Jacobi form

$$\eta\,\vartheta_{D_{23}(3)} \in J_{12,D_{23}(3)}. \tag{25}$$

The tensor product of a Jacobi form of singular weight and Jacobi form of critical weight has critical weight for the corresponding lattice. In particular, there exist two simple series of Jacobi cusp forms with trivial character for the lattices $A_m \oplus D_n$ where m is even and $m+n \equiv 7 \mod 8$

$$\vartheta_{A_m\oplus D_n} = \vartheta_{A_m}(\tau, \mathfrak{z}_m) \otimes \vartheta_{D_n}(\tau, \mathfrak{z}_n) \in J^{cusp}_{(m+n+1)/2,\, A_m\oplus D_n}, \tag{26}$$

$$\vartheta_{A_m\oplus D_{3n}(3)} = \vartheta_{A_m}(\tau, \mathfrak{z}_m) \otimes \vartheta_{D_{3n}(3)}(\tau, \mathfrak{z}_{3n}) \in J^{cusp}_{(m+3n+1)/2,\, A_m\oplus D_{3n}(3)}. \tag{27}$$

In particular, we get examples of Jacobi cusp forms of weight one with character in one abelian variable. The simplest examples of such forms can be found in [18] (see also [15] where many different cusp theta-products of small weights were considered):

$$\eta(\tau)\vartheta_{3/2}(\tau, 2z) \in J_{1,D_1(3)}\big(v_\eta^2\big), \qquad \eta(\tau)\vartheta(\tau, 2z) \in J_{1,D_1}\big(v_\eta^4\big).$$

To get more interesting examples, we take the pullback of σ_{A_2}. We consider A_2 as the sublattice $v^{\perp}_{D_3}$ where $v = 2(1,1,1)$. Let $u = 2(u_1, u_2, u_3) \in D_3$ and u_a be the projection of u on $A_2^\vee$, i.e. $u = u_a + u_v$ where $u_a \in \langle v^{\perp}\rangle^\vee = A_2^\vee$ and $u_v \in \langle v^\vee\rangle = \langle \frac{v}{12}\rangle$. We set $\sigma_{A_2}|_u := \sigma_{A_2}|_{(u_a)^{\perp}_{A_2}}$.

Proposition 4.8 *Let $u = 2(u_1, u_2, u_3) \in D_3$ such that $u_i \neq u_j$ and $u_1 + u_2 + u_3 \not\equiv 0 \mod 3$. Then $\sigma_{A_2}|_u$ is a Jacobi cusp form of critical weight 1 with character v_η^8.*

Proof We note first that if $u_i \neq u_j$ then the pullback $\sigma_{A_2}|_u$ is not identically zero. According to the proof of Proposition 4.1 and Corollary 4.4, the Fourier expansion of the Jacobi form σ_{A_2} of singular weight 1 has the following form

$$\sigma_{A_2}(\tau, \mathfrak{z}_2) = \sum_{\substack{n>0,\, l_a \in A_2^\vee \\ 2n-(l_a,l_a)=0 \\ l_a \pm v^\vee \in \frac{1}{2}\mathbb{Z}^3}} f(n, l_a)\, e^{2\pi i (n\tau + (l_a, \mathfrak{z}_2))}.$$

More exactly, in the last summation, we have $l^\vee = l_a \pm \frac{v}{12} = \frac{1}{2}(l_1, l_2, l_3)$ with odd l_i because the division by η does not change the $\mathfrak{z}_2$-part of ϑ_{A_2}. Let u_a be the projection of u on $A_2^\vee$, i.e. $u = u_a + u_v$ where $u_a \in \langle v^{\perp}\rangle^\vee = A_2^\vee$ and $u_v \in \langle v^\vee\rangle = \langle \frac{v}{12}\rangle$. We have to analyze the Fourier expansion of $\sigma_{A_2}|_u$. As in the proof of Proposition 4.1, we set $l_a = l_u \oplus l_\perp$ where $(l_u, u_a) = 0$. If the hyperbolic norm of the index of a Fourier coefficient $f_u(n, l_u)$ of $\sigma_{A_2}|_u$ is equal to zero then $l_\perp = 0$. Therefore $(l_u, u_a) = (l_a, u_a) = (l_a, u) = 0$ and

$$\big(l^\vee, u\big) = \pm\frac{(u, v)}{12} = \pm\frac{u_1 + u_2 + u_3}{3} \in \mathbb{Z}.$$

The last inclusion is not possible. Thus the pullback $\sigma_{A_2}|_u$ is a cusp form. □

We note that the Jacobi form in Proposition 3.8 is a classical Jacobi form of type [8]. We give its more explicit form in the next corollary

Corollary 4.9 *Let $a, b \in \mathbb{Z}_{>0}$. The following function, called theta-quark,*

$$\theta_{a,b}(\tau, z) = \frac{\vartheta(\tau, az)\vartheta(\tau, bz)\vartheta(\tau, (a+b)z)}{\eta(\tau)} \in J_{1,A_1;a^2+ab+b^2}\left(v_\eta^8\right)$$

is holomorphic Jacobi form of Eichler–Zagier type of weight 1, *index* (a^2+ab+b^2) *and character v_η^8. This is a Jacobi cusp form if $a \not\equiv b \mod 3$.*

Proof We can assume that a and b are coprime. We obtain this function as $\sigma_{A_2}|_u$ for $u = 2(b, -a, 0)$. □

Remark The Jacobi form $\theta_{a,b}$ was proposed by the second author many years ago in his talks on canonical differential forms on Siegel modular three-folds. The Jacobi forms of similar types, called *theta-blocks*, are studied in the paper [23] where the Fourier expansion of theta-quark $\theta_{a,b}$ is found explicitly. The method of the proof of Proposition 4.8 can be used for other Jacobi forms when one takes a pullback on a sublattice of co-rank 2.

Propositions 4.3 and 4.8 give a method to pass from Jacobi forms of singular weight to Jacobi forms of critical weight. We have noticed that the tensor product of Jacobi forms of singular and critical weights is a form of critical weight. In some cases, we can divide some products of two forms of critical weight by η (see Corollary 4.5). We can control that the obtained Jacobi form is a cusp (or non-cusp) form. Analyzing the table of characters before Proposition 4.7, we obtain

Proposition 4.10 *If $n \geqslant 7$ is odd (respectively, $n \geqslant 8$ is even) and $d \equiv n+1 \mod 2$ then there exists a lattice L of rank n such that the space of Jacobi forms of critical weight $J^{(cusp)}_{\frac{n+1}{2},L}(v_\eta^d)$ is not empty.*

The analogue of Proposition 4.6 is the following

Proposition 4.11 *Theta-products give examples of Jacobi cusp forms of critical weight with trivial character for some lattices of all odd ranks $\geqslant 5$.*

Proof The corresponding Jacobi forms of critical weight are pullbacks (see Proposition 4.3) of Jacobi forms of singular weight of Proposition 4.6. One can also use $\vartheta_{A_m(3)}$ instead of ϑ_{A_m} in theta-products. □

Acknowledgements This work was supported by the grant ANR-09-BLAN-0104-01 and Labex CEMPI (Lille). The authors are grateful to the Max-Planck-Institut für Mathematik in Bonn for support and for providing excellent working conditions.

References

1. Borcherds, R.E.: Automorphic forms on $O_{s+2,2}(R)$ and infinite products. Invent. Math. **120**, 161–213 (1995)
2. Borcherds, R.E.: The moduli space of Enriques surfaces and the fake monster Lie superalgebra. Topology **35**, 699–710 (1996)
3. Borcherds, R.E.: Automorphic forms with singularities on Grassmannians. Invent. Math. **132**, 491–562 (1998)

4. Bourbaki, N.: Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: Systèmes de racines. Éléments de mathématique. Fasc. XXXIV. Actualités Scientifiques et Industrielles, vol. 1337. Hermann, Paris (1968)
5. Cléry, F., Gritsenko, V.: Siegel modular forms of genus 2 with the simplest divisor. Proc. Lond. Math. Soc. **102**, 1024–1052 (2011)
6. Desreumaux, C.: Construction de formes automorphes réflectives sur un espace de dimension 4. J. Théor. Nr. Bordx. **18**, 89–111 (2006)
7. Dijkgraaf, R., Moore, G., Verlinde, E., Verlinde, H.: Elliptic genera of symmetric products and second quantized strings. Commun. Math. Phys. **185**, 197–209 (1997)
8. Eichler, M., Zagier, D.: The Theory of Jacobi Forms. Progress in Mathematics, vol. 55. Birkhäuser, Boston (1985)
9. Freitag, E., Salvati Manni, R.: Some Siegel threefolds with a Calabi-Yau model. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **9**(4), 833–850 (2010)
10. Gritsenko, V.: Jacobi functions of n-variables. Zap. Nauč. Semin. POMI **168**, 32–45 (1988). English transl. in J. Soviet Math. **53**, 243–252 (1991)
11. Gritsenko, V.: Modular forms and moduli spaces of abelian and K3 surfaces. Algebra Anal. **6**, 65–102 (1994). English translation in St. Petersburg Math. J. **6**, 1179–1208 (1995)
12. Gritsenko, V.: Irrationality of the moduli spaces of polarized Abelian surfaces. Int. Math. Res. Not. **6**, 235–243 (1994)
13. Gritsenko, V.: Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms. St. Petersburg Math. J. **11**, 100–125 (1999)
14. Gritsenko, V.: Reflective modular forms and algebraic geometry. arXiv:1005.3753, 28 pp.
15. Gritsenko, V., Hulek, K.: Commutator covering of Siegel threefolds. Duke Math. J. **94**, 509–542 (1998)
16. Gritsenko, V., Hulek, K.: The modular form of the Barth–Nieto quintic. Int. Math. Res. Not. **17**, 915–938 (1999)
17. Gritsenko, V., Nikulin, V.: Siegel automorphic form correction of some Lorentzian Kac–Moody Lie algebras. Am. J. Math. **119**, 181–224 (1997)
18. Gritsenko, V., Nikulin, V.: Automorphic forms and Lorentzian Kac-Moody algebras. I. Int. J. Math. **9**, 153–200 (1998)
19. Gritsenko, V., Nikulin, V.: Automorphic forms and Lorentzian Kac-Moody algebras. II. Int. J. Math. **9**, 200–275 (1998)
20. Gritsenko, V., Nikulin, V.: The arithmetic mirror symmetry and Calabi–Yau manifolds. Commun. Math. Phys. **200**, 1–11 (2000)
21. Gritsenko, V., Hulek, K., Sankaran, G.: The Kodaira dimension of the moduli of K3 surfaces. Invent. Math. **169**, 215–241 (2007)
22. Gritsenko, V., Hulek, K., Sankaran, G.: Abelianisation of orthogonal groups and the fundamental group of modular varieties. J. Algebra **322**, 463–478 (2009)
23. Gritsenko, V., Skoruppa, N.-P., Zagier, D.: Theta-blocks. Preprint
24. Harvey, J.A., Moore, G.: Exact gravitational threshold correction in the Ferrara-Harvey-Strominger-Vafa model. Phys. Rev. D **57**, 2329–2336 (1998)
25. Kac, V.: Infinite Dimensional Lie Algebras. Cambridge Univ. Press, Cambridge (1990)
26. Kac, V., Peterson, D.H.: Infinite-dimensional Lie algebras, theta functions and modular forms. Adv. Math. **53**, 125–264 (1984)
27. Klingen, H.: Uber Kernfunktionen für Jacobiformen und Siegelsche Modulformen. Math. Ann. **285**, 405–416 (1989)
28. Kondo, S.: The moduli space of Enriques surfaces and Borcherds products. J. Algebr. Geom. **11**, 601–627 (2002)
29. Mumford, D.: Tata Lectures on Theta I. Progress in Mathem., vol. 28. Birkhäuser, Boston (1983)
30. Scheithauer, N.: On the classification of automorphic products and generalized Kac-Moody algebras. Invent. Math. **164**, 641–678 (2006)
31. Shimura, G.: Introduction to the Arithmetic Theory of Automorphic Functions. Princeton Univ. Press, Princeton (1971)
32. Skoruppa, N.-P.: Jacobi forms of critical weight and Weil representations. In: Modular Forms on Schiermonnikoog, pp. 239–266. Cambridge Univ. Press, Cambridge (2008)
33. Yoshikawa, K.-I.: Calabi-Yau threefolds of Borcea-Voisin, analytic torsion, and Borcherds products. Astérisque **327**, 351–389 (2009)