Topology of locally conformally Kähler manifolds with potential

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Abstract
Locally conformally Kähler (LCK) manifolds with potential are those which admit a Kähler covering with a proper, automorphic, global potential. Existence of a potential can be characterized cohomologically as vanishing of a certain cohomology class, called the Bott-Chern class. Compact LCK manifolds with potential are stable at small deformations and admit holomorphic embeddings into Hopf manifolds. This class strictly includes the Vaisman manifolds. We show that every compact LCK manifold with potential can be deformed into a Vaisman manifold. Therefore, every such manifold is diffeomorphic to a smooth elliptic fibration over a Kähler orbifold. We show that the pluricanonical condition on LCK manifolds introduced by G. Kokarev is equivalent to vanishing of the Bott-Chern class. This gives a simple proof of some of the results on topology of pluricanonical LCK-manifolds, discovered by Kokarev and Kotschick.

1 Introduction

The main object of the present paper is the following notion.

Definition 1.1: A **locally conformally Kähler (LCK)** manifold is a complex Hermitian manifold, with a Hermitian form ω satisfying $d\omega = \omega\wedge\theta$, where θ is a closed 1-form, called **the Lee form** of M.

Sometimes an LCK manifold is defined as a complex manifold which has a Kähler covering $\widetilde{M}$, with the deck transform group acting on $\widetilde{M}$ by conformal homotheties. This definition is equivalent to the first one, up to conformal equivalence.

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A compact LCK manifold never admits a Kähler structure, unless the cohomology class $\theta \in H^1(M)$ vanishes (see [Va]). Further on, we shall usually assume that θ is non-exact, and $\dim_{\mathbb{C}} M \geqslant 3$.

LCK manifolds form an interesting class of complex non-Kähler manifolds, including all non-Kähler surfaces which are not class VII. In many situations, the LCK structure becomes useful for the study of topology and complex geometry of an LCK-manifold.

We would like to investigate the LCK geometry along the same lines as used to study the Kähler manifolds. Existence of a Kähler structure gives all kinds of constraints on the topology of M (even-dimensionality if $H^{odd}(M)$, strong Lefschetz, homotopy formality). It is thus natural to ask what can we say about the topology of a compact LCK manifold.

Not much is known in the general case. We list several known facts:

(1) The LCK manifolds are not necessarily homotopy formal: the Kodaira surfaces are not homotopy formal (it has non-vanishing Massey products), but they are LCK ([B1]).

(2) In [Va], I. Vaisman conjectured that for a compact LCK manifold, $h^1(M)$ odd. This was disproven by Oeljeklaus and Toma in [OT]. In the same paper he also conjectured that no compact LCK manifold can be homotopy equivalent to a Kähler manifold. This is still unknown.

(3) All compact Vaisman manifolds (see below) have odd b_1.

(4) All non-Kähler complex surfaces admit an LCK structure ([B1]), except some of Kodaira class VII surfaces. For Kodaira class VII with $b_2 = 0$, LCK structures are known to exist on two types of Inoue surfaces (Tricerri, [Tr]), and do not exist on all of the third type (Belgun, [B1]). For $b_2 > 0$, all Kodaira class VII surfaces are conjectured to admit a spherical shell (Ma. Kato, I. Nakamura; see e.g. [N]). The known examples of minimal Kodaira class VII surfaces are either hyperbolic or parabolic Inoue surfaces. LCK structures on hyperbolic Inoue surfacse were recently constructed by A. Fujiki and M. Pontecorvo ([FP]). There are no known examples of non-LCK, non-Kähler surfaces, except the example of Inoue surface of class S^+ considered by Belgun.

(5) In complex dimension > 2, only known examples of non-Kähler, non-LCK manifolds with infinite fundamental group are products of a non-Kähler manifold with finite π_1 and a Kaehler manifold with infinite π_1.

Among the LCK manifolds, a distinguished class is the following:

Definition 1.2: An LCK manifold (M, ω, θ) is called **Vaisman** if $\nabla\theta = 0$, where ∇ is the Levi-Civita connection of the metric $g(\cdot, \cdot) = \omega(I\cdot, \cdot)$.

The universal cover of a Vaisman manifold can be precisely described. We need the following

Definition 1.3: A **conical Kähler manifold** is a Kähler manifold (C, ω) equipped with a free, proper holomorpic flow $\rho: \mathbb{R} \times C \longrightarrow C$, with ρ acting by homotheties as follows: $\rho(t)^*\omega = e^t\omega$. The space of orbits of ρ is called **a Sasakian manifold**.

Theorem 1.4: ([OV1]) A compact Vaisman manifold is conformally equivalent to a quotient of a conical Kähler manifold by $\mathbb{Z}$ freely acting on (C, ω) by non-isometric homotheties. Moreover, M admits a smooth Riemannian submersion $\sigma: M \longrightarrow S^1$, with Sasakian fibers. ■

We shall be interested in LCK manifolds whose Kähler metric on the universal cover have global potential. Recall first the following:

Definition 1.5: Let (M, I, ω) be a Kähler manifold. Let $d^c := -IdI$. A **Kähler potential** is a function satisfying $dd^c\psi = \omega$. Locally, a Kähler potential always exists, and it is unique up to adding real parts of holomorphic functions.

Claim 1.6: (see e.g. [Ve1]) Let (M, ω, θ) be a Vaisman manifold, and

$$(\widetilde{M}, \widetilde{\omega}) \xrightarrow{\pi} M$$

be its Kähler covering, with $\Gamma \cong \mathbb{Z}$ the deck transform group: $M = \widetilde{M}/\Gamma$. Then $\pi^*\theta$ is exact on $\widetilde{M}$: $\pi^*\theta = d\nu$. Moreover, the function $\psi := e^{-\nu}$ is a Kähler potential: $dd^c\psi = \widetilde{\omega}$. ■

Remark 1.7: In these notations, let $\gamma \in \Gamma$. Since Γ preserves θ, we have $\gamma^*\nu = \nu + c_\gamma$, where c_γ is a constant. Then $\gamma^*\psi = e^{-c_\gamma}\psi$. A function which satisfies such a property for any $\gamma \in \Gamma$ is called **automorphic**.

Definition 1.8: Let (M, ω, θ) be an LCK manifold, $(\widetilde{M}, \widetilde{\omega})$ its Kähler covering, Γ the deck transform group, $M = \widetilde{M}/\Gamma$, and $\psi \in C^\infty(\widetilde{M})$ a Kähler potential, $\psi > 0$. Assume that for any $\gamma \in \Gamma$, $\gamma^*\psi = c_\gamma\psi$, for some constant c_γ. Then ψ is called **an automorphic potential** of M.

To go on, we need to introduce the following:

Definition 1.9: Let (M, ω, θ) be an LCK manifold, and L a trivial line bundle, associated to the representation $\mathrm{GL}(2n, \mathbb{R}) \ni A \mapsto |\det A|^{\frac{1}{n}}$, with flat connection defined as $D := \nabla_0 + \theta$, where ∇_0 is the trivial connection. Then L is called **the weight bundle** of M. Being flat, its holonomy defines a map $\chi : \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ whose image Γ is called **the monodromy group of** M.

We shall denote with the same letter, D, the corresponding Weyl covariant derivative on M.

Proposition 1.10: ([OV4]) Let (M, ω, θ) be an LCK manifold with an automorphic potential. Then there exists another LCK metric on M with automorphic potential and monodromy $\mathbb{Z}$. ■

Remark 1.11: A note on the terminology. In [OV3], we introduced the **LCK manifolds with potential**. An LCK manifold is called **LCK with potential** if it has an automorphic potential and monodromy $\mathbb{Z}$. This terminology can be confusing, because there exist LCK manifolds with automorphic potential, without being "LCK with potential", in this sense.

The main example of a LCK manifold with potential is the Hopf manifold $H_A = (\mathbb{C}^n \setminus \{0\})/\langle A \rangle$, where A is a linear operator with subunitary absolute value of all eigenvalues, see [OV3] where we proved:

Theorem 1.12: ([OV3]) A compact LCK manifold M, $\dim_{\mathbb{C}} M \geqslant 3$ is LCK with potential if and only if it admits a holomorphic embedding to a Hopf manifold H_A. M is Vaisman if and only if A is diagonal. ■

The class of compact LCK manifolds is not stable under small deformations, and the same is true for Vaisman manifolds ([B1]). On the contrary:

Theorem 1.13: ([OV3]) Let (M, I) be a compact complex manifold admitting an LCK metric with LCK potential. Then any small deformation of the complex structure I also admits an LCK metric with LCK potential. ■

2 Deforming a compact LCK manifold with potential to a Vaisman manifold

Theorem 2.1: Let (M, ω, θ), $\dim_{\mathbb{C}} M \geqslant 3$, be an LCK manifold with potential. Then there exists a small deformation of M which admits a Vaisman metric.

Proof: The idea is to embed M into a Hopf manifold defined by a linear operator A, then, using the Jordan-Chevalley decomposition of A to show that its semisimple part preserves some subvariety of $\mathbb{C}^n \setminus \{0\}$, thus yielding an embedding of a small deformation of M into a diagonal Hopf manifold, the Vaisman metric of which can be pulled back on M. We now provide the details.

Step 1. Let $V = \mathbb{C}^n$, $A \in \operatorname{End}(V)$ be an invertible linear operator with all eigenvalues $|\alpha_i| < 1$, and $H = (V \setminus \{0\})/\langle A \rangle$ be the corresponding Hopf manifold, as constructed in [OV3]. One may see that the complex submanifolds of H are identified with complex subvarieties Z of V, which are smooth outside of $\{0\}$ and are fixed by A. Indeed, by Remmert-Stein theorem ([D], chapter II, ¶8.2), for every complex subvariety $X \subset H$, the closure of $\pi^{-1}(X)$ is complex analytic in $V = \mathbb{C}^n$, where $\pi : V\backslash 0 \longrightarrow H$ is the natural projection.

Step 2. We are going to prove that any such Z is fixed by the flow $G_A := e^{t \log A}$, $t \in \mathbb{R}$, acting on V. Let I_Z be the ideal of Z, and $\hat{I}_Z$ the corresponding ideal in the completion of the structural ring $\mathcal{O}_V$ in $\{0\}$. To prove that I_Z is fixed by G_A, it is enough to show that $\hat{I}_Z$ is fixed by G_A. However, by definition (see [AM, Ch. 10]), $\hat{I}_Z$ is the inverse limit of the projective system $\frac{I_Z}{I_Z \cap \mathfrak{m}^k}$, where $\mathfrak{m}$ is the maximal ideal of $\{0\}$:

$$\hat{I}_Z = \lim_{\leftarrow} \frac{I_Z}{I_Z \cap \mathfrak{m}^k}.$$

To prove that $\hat{I}_Z$ is fixed by G_A it only remains to show that $\frac{I_Z}{I_Z \cap \mathfrak{m}^k}$ is fixed by G_A. But $\frac{I_Z}{I_Z \cap \mathfrak{m}^k}$ is a subspace in the vector space $\mathcal{O}_V/\mathfrak{m}^k$, finite-dimensional by [AM], Corollary 6.11 and Exercise 8.3, and such a subspace, if fixed by A, is automatically fixed by G_A.

Step 3. Now, for any linear operator there exists a unique decomposition $A := SU$ in a product of commuting operators, with S semisimple (diagonal), and U unipotent, i.e. its spectrum contains only the number 1 (this is called the Jordan-Chevalley decomposition and, in this particular situation, for operators acting on a finite dimensional vector space over $\mathbb{C}$, follows easily from the Jordan canonical form). Consequently, for any finite-dimensional representation of $GL(n)$, any vector which is fixed by A, is also fixed by S. By the argument in Step 2, this proves that S fixes the ideal $\hat{I}_Z$, and the subvariety $Z \subset V$.

Step 4. The diagonal Hopf variety $H_S := (V \setminus \{0\})/\langle S\rangle$ contains a Vaisman submanifold $M_1 := (Z \setminus \{0\})/\langle S\rangle$. Since S is contained in a closure of a $GL(V)$-orbit of A, we have also shown that M_1 can be obtained as an arbitrary small deformation of M. ∎

It is well known that every compact Vaisman manifold is diffeomorphic to a quasiregular Vaisman manifold, [OV2]. By definition, the latter is an elliptic fibration over a Kähler orbifold. As a consequence, the above result gives us the possibility to obtain topological information about LCK manifolds with potential from the topology of projective orbifolds. But, once a compact LCK manifold admits an automorphic potential on a covering, it can be deformed to a LCK manifold with a proper potential ([OV4], Corollary 5.3). Hence we have:

Corollary 2.2: The fundamental group of a compact LCK manifold M with an automorphic potential admits an exact sequence

$$0 \longrightarrow G \longrightarrow \pi_1(M) \longrightarrow \pi_1(X) \longrightarrow 0$$

where $\pi_1(X)$ is a fundamental group of a Kähler orbifold, and G a quotient of $\mathbb{Z}^2$ by a subgroup of rank $\leqslant 1$.

Proof: Replacing M with a diffeomorphic Vaisman manifold, we may assume that M is a quasiregular Vaisman manifold, elliptically fibered over a base X. The long exact sequence of homotopy gives

$$\pi_2(X) \xrightarrow{\delta} \pi_1(T^2) \longrightarrow \pi_1(M) \longrightarrow \pi_1(X) \longrightarrow 0$$

The boundary operator δ can be described as follows. Let

$$\gamma :\ \mathbb{Z}^2 \longrightarrow H^2(X)$$

be the map representing the Chern classes of the corresponding $S^1 \times S^1$-fibration. We may interpret this map as a differential of the corresponding Leray spectral sequence, which gives us an exact sequence

$$0 \longrightarrow H^1(X) \longrightarrow H^1(M) \longrightarrow H^1(T^2) \xrightarrow{\gamma} H^2(X).$$

Dualizing and using the Hurewicz theorem, we obtain that the boundary map $\pi_2(X) \xrightarrow{\delta} \pi_1(T^2)$ is obtained as a composition of γ^* and the Hurewicz homomorphism $\pi_2(X) \longrightarrow H^2(X)$. The Chern classes of this $S^1 \times S^1$-fibration are easy to compute: one of them is trivial (because M is fibered over a circle), and the other one is non-trivial, because M is non-Kähler, and a total space of an isotrivial elliptic fibration with trivial Chern classes is Kähler. Therefore, the image of δ has rank $\leqslant 1$ in $\pi_1(T^2)$. ■

3 Pluricanonical LCK manifolds are diffeomorphic to Vaisman manifolds

In [K], G. Kokarev introduced the following notion:

Definition 3.1: ([K]) Let (M, ω, θ) be an LCK manifold. Then M is called **pluricanonical** if $(\nabla\theta)^{1,1} = 0$, where $(\cdot)^{1,1}$ denotes the I-invariant part of the tensor.

For this class of LCK manifolds, Kokarev and Kotschick generalized an important result by Siu and Beauville [Si] from Kähler geometry:

Theorem 3.2: ([KK]) Let M be a compact pluricanonical LCK manifold, such that $\pi_1(M)$ admits a surjective homomorphism to a non-abelian free group. Then M admits a surjective holomorphic map with connected fibers to a compact Riemannian surface.

Clearly, the pluricanonical condition is weaker than the Vaisman one. But in the cited papers, no other examples of pluricanonical LCK manifolds are provided but Vaisman ones.

We now prove that the pluricanonical condition is equivalent with the existence of an automorphic potential on a Kähler covering.

Indeed, the Levi-Civita connection ∇ and the Weyl connection D on M are related by the formula ([DO]):

$$\nabla - D = \frac{1}{2}(\theta \otimes \mathrm{id} + \mathrm{id} \otimes \theta - g \otimes \theta^\sharp).$$

Applied on θ, this gives:

$$\nabla\theta - D\theta = -\theta \otimes \theta + \frac{1}{2} g.$$

Hence, the pluricanonical condition $(\nabla\theta)^{1,1} = 0$ is translated into

$$(D\theta)^{1,1} = (\theta \otimes \theta)^{1,1} - \frac{1}{2}g.$$

Since D is torsion-free, this is equivalent to

$$d(I\theta) = \omega - \theta \wedge I\theta.$$

But we can prove the following:

Claim 3.3: Let (M, ω, θ) be an LCK manifold, and $\theta^c := I(\theta)$ the complex conjugate of the Lee form. Then the condition $d\theta^c = \omega - \theta \wedge \theta^c$ is equivalent to the existence of an automorphic potential on a Kähler covering on which the pull-back of θ is exact.

Proof: Let $\widetilde{M}$ be a covering of M on which the pull-back of θ is exact. Denote, for convenience, with the same letters the pull-backs to $\widetilde{M}$ of θ, ω and D. Observe that the Levi-Civita connection of the Kähler metric on $\widetilde{M}$ globally conformal with ω is precisely D. Let $\psi := e^{-\nu}$, where $d\nu = \theta$. Then

$$dd^c\psi = -e^{-\nu} dd^c\nu + e^{-\nu} d\nu \wedge d^c\nu = e^{-\nu}(d^c\theta + \theta \wedge I\theta) = \psi\omega,$$

and hence the pluricanonical condition implies that ψ is an automorphic potential for the Kähler metric $\psi\omega$. The converse is true by the same argument. ■

As we know that the existence of an automorphic potential on a covering allows the deformation to a LCK manifold with potential, taking into account Theorem 2.1, we obtain the following

Corollary 3.4: Any compact pluricanonical LCK manifold is diffeomorphic to a Vaisman manifold.

In view of Corollary 2.2, the restrictions on the fundamental group of a pluricanonical LCK manifold obtained in [KK] using a generalization of Siu's arguments for harmonic maps can be directly obtained for LCK manifolds which admit an automorphic potential on a Kähler covering, by using Corollary 2.2 and the corresponding results for Kähler manifolds. For instance, the following result can be proven (Corollary 4 of [KK]).

Claim 3.5: A non-abelian free group cannot be the fundamental group of an LCK manifold M admitting an LCK potential.

Proof: As follows from Corollary 2.2, $\pi_1(M)$ fits into an exact sequence

$$0 \longrightarrow G \longrightarrow \pi_1(M) \longrightarrow \pi_1(X) \longrightarrow 0,$$

where G is an abelian group of rank $\geqslant 1$. Since all non-trivial normal subgroups of infinite index in a free groups are free, by Nielsen-Schreier theorem, and infinitely generated (see e.g. [G]), G cannot be a normal subgroup of a free group, unless G is trivial. ■

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