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Ordinary Semicascades and Their Ergodic Properties

A. V. Romanov

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ABSTRACT. A relationship is considered between ergodic properties of a discrete dynamical system on a compact metric space Ω and characteristics of companion algebro-topological objects, namely, the Ellis enveloping semigroup E, the Köhler enveloping operator semigroup Γ, and the semigroup G being the closure of the convex hull of Γ in the weak-star topology on the operator space $\operatorname{End} C^*(\Omega)$. The main results are formulated for ordinary (having metrizable semigroup E) semicascades and for tame dynamical systems determined by the condition $\operatorname{card} E \leqslant \mathfrak{c}$. A classification of compact semicascades in terms of topological properties of the semigroups specified above is given.

KEY WORDS: semicascade, ergodic properties, nonchaotic dynamics, tame dynamical system, enveloping semigroup, Choquet simplex.

Introduction. This paper describes ergodic properties of the discrete dynamical system (Ω, φ) generated by a continuous transformation of a compact metric space Ω in terms of three algebro-topological objects associated with (Ω, φ). These objects are two enveloping semigroups described in [1] and [2] (the Ellis enveloping semigroup $E(\Omega, \varphi)$ and the Köhler enveloping operator semigroup $\Gamma(\Omega, \varphi)$) and the operator semigroup $G(\Omega, \varphi)$ introduced in the author's previous paper [3], which is related to $\Gamma(\Omega, \varphi)$. We also study the dependence between the three semigroups mentioned above, which is of independent interest. As shown in [3], topological characteristics of the semigroups $E(\Omega, \varphi)$, $\Gamma(\Omega, \varphi)$, and $G(\Omega, \varphi)$, which are compact in appropriate topologies, are closely related to ergodic properties of the semicascade (Ω, φ). It turns out that this dependence is especially pronounced for the class of discrete dynamical systems with metrizable enveloping semigroup $E(\Omega, \varphi)$, which we call *ordinary systems*. A series of results on the Ellis semigroups of such systems (including nondiscrete ones) can be found in the survey [4]. It is known that all weakly almost periodic dynamical system are ordinary.

In this paper, we propose an alternative definition of an ordinary semicascade (Ω, φ) (see Theorem 4), which consists in the requirement that the induced semigroup $G(\Omega, \varphi)$ be metrizable. We also consider the class of so-called tame dynamical systems (Ω, φ), which was first introduced by Köhler in [2] and studied in [4]–[6]. Section 2 presents an hierarchical summary of known and new properties of semicascades (Ω, φ), which we formulate largely in terms of the semigroups $E(\Omega, \varphi)$, $\Gamma(\Omega, \varphi)$, and $G(\Omega, \varphi)$.

At the same time, the main results of the paper refer to the weak-star convergence of operator ergodic means for ordinary and tame semicascades. The idea of considering this convergence in ergodic theory essentially goes back to the classical paper [7] by Krylov and Bogolyubov.

1. Preliminaries. Let φ be a continuous (not necessarily invertible) transformation of a compact metric space Ω, and let (Ω, φ) be the corresponding semicascade on Ω. The map $\varphi\colon \Omega \to \Omega$ generates linear shifts U and $V = U^*$ on the space $X = C(\Omega)$ of continuous scalar functions on Ω and on the dual space X^* of Borel measures on Ω. We denote the normed space of bounded linear operators on X^* by $\operatorname{End} X^*$. The enveloping semigroup $E(\Omega, \varphi)$ of the semicascade (Ω, φ) is the closure of the family $\{\varphi^n,\ n \geqslant 0\}$ in the Hausdorff topology of pointwise convergence on the space of all maps $\Omega \to \Omega$ [1]. Note that, although only invertible dynamical systems were considered in [1], most of the results obtained there remain valid in the noninvertible case. The semigroup $\Gamma(\Omega, \varphi)$ is the closure of the set $\Gamma_0 = \{V^n,\ n \geqslant 0\}$ in the weak-star operator topology W^*O on the operator space $\operatorname{End} X^*$ [2]. The operator semigroup $G(\Omega, \varphi)$ was defined in [3] as the W^*O-closed convex

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hull of Γ_0. The sets $E(\Omega,\varphi)$, $\Gamma(\Omega,\varphi)$, and $G(\Omega,\varphi)$ are compact in the topologies specified above. If $A = A(\Omega)$ and $K = K(\Omega)$ are sets of Borel probability measures and Dirac measures $\delta(\omega)$ on Ω and they are compact in the weak-star topology on $X^* = C^*(\Omega)$, then the semigroup $G(\Omega,\varphi)$ can be interpreted as the enveloping semigroup of the action $W \times A \xrightarrow{V} A$ of the Abelian semigroup W of nonnegative finitely supported number sequences with unit sum under convolution as multiplication. We have $V\delta(\omega) = \delta(\varphi\omega)$, $\omega \in \Omega$, so that $PK \subset K$ for all operators $P \in \Gamma(\Omega,\varphi)$. The correspondence $V \to \varphi$ induces a continuous algebraic homomorphism $\pi\colon P \to p$ of the semigroup $\Gamma(\Omega,\varphi)$ to the semigroup $E(\Omega,\varphi)$, which is determined in an obvious way by the relation

$$P\delta(\omega) = \delta(p\omega), \qquad \omega \in \Omega, \tag{1}$$

and turns out to be an epimorphism.

Definition 1. A semicascade (Ω,φ) is said to be *ordinary* if its enveloping semigroup $E(\Omega,\varphi)$ is metrizable.

Definition 2 (see [4]). We say that a dynamical system (Ω,φ) is *nonchaotic* if any closed semi-invariant set $\Theta \subset \Omega$ ("semi-invariant" means that $\varphi\Theta \subset \Theta$) contains a trajectory $o(\omega)$, $\omega \in \Theta$, stable in the sense of Lyapunov with respect to the semicascade (Θ,φ).

Here and in what follows, $o(\omega) = \{\varphi^n\omega,\ n \geqslant 0\}$ for $\omega \in \Omega$. The notion of a tame dynamical system was initially introduced in [2] as follows.

Definition 3. A semicascade (Ω,φ) is said to be *tame* if, for any continuous function $x \in X$ and any increasing sequence $\{n(k)\}$ of positive integers,

$$\inf_a \Big\| \sum_{k=1}^{\infty} a_k x_{n(k)} \Big\|_X = 0,$$

where $x_m(\omega) = x(\varphi^m\omega)$ $(\omega \in \Omega)$ and the infimum is over all finitely supported real sequences $a = \{a_k\}$ with $\sum_{k=1}^{\infty} |a_k| = 1$.

Let Π_1 denote the set of transformations of the compact space Ω belonging to the first Baire class in Baire's increasing hierarchy of classes of functions, and let Π_b denote the set of all Borel maps $p\colon \Omega \to \Omega$.

Let G_0 be the convex hull of the set Γ_0 in $\operatorname{End} X^*$. A net of operators $T_\alpha \in G_0$ is said to be *ergodic* if $T_\alpha(I - V) \xrightarrow{W^*O} 0$, where $I = \mathrm{id}$. Ergodic sequences of operators $T_n \in G_0$ are defined accordingly. The sequence of Cesàro means $V_n = \frac{1}{n}(I + V + \dots + V^{n-1})$ is ergodic, because $V_n(I - V) = n^{-1}(I - V^n)$ and $\|V^n\| = 1$. Note that the W^*O-convergence of an ergodic net of operators T_α is equivalent to the convergence of the ergodic means $T_\alpha^* x \to z$, $z \in X^{**}$, in the weak-star topology of the space $X^{**} = C^{**}(\Omega)$ for any function $x \in X$. For ergodic sequences, such a convergence is equivalent to pointwise convergence on the compact set Ω, but in the general case, it may be stronger.

2. Classification of properties of compact semicascades. Consider the following classes D1–D6 of properties of a semicascade (Ω,φ); they are largely related to the problem on the weak-star convergence of ergodic means, which is considered in this paper.

D1: (a1) The compact space $E(\Omega,\varphi)$ is metrizable;
(b1) the system (Ω,φ) is nonchaotic;
(c1) the compact space $G(\Omega,\varphi)$ is metrizable.

D2: (a2) The compact space $G(\Omega,\varphi)$ is Fréchet–Urysohn;
(b2) $\operatorname{card} G(\Omega,\varphi) = \mathfrak{c}$.

D3: (a3) The semicascade (Ω,φ) is a tame system;
(b3) the compact space $E(\Omega,\varphi)$ is Fréchet–Urysohn;
(c3) $\operatorname{card} E(\Omega,\varphi) \leqslant \mathfrak{c}$;
(d3) $E(\Omega,\varphi) \subset \Pi_1$;
(e3) $E(\Omega,\varphi) \subset \Pi_b$.

D4: (a4) The operators $T \in G(\Omega, \varphi)$ are determined by their values at the Dirac measures $\delta \in K(\Omega)$.

D5: (a5) The convex set $G(\Omega, \varphi)$ of operators is a Choquet simplex.

D6: (a6) The operators $P \in \Gamma(\Omega, \varphi)$ are determined by their values at the Dirac measures;

(b6) the function $\pi\colon \Gamma(\Omega, \varphi) \to E(\Omega, \varphi)$ defined by (1) is injective;

(c6) $\operatorname{ex} G = \Gamma$;

(d6) $\operatorname{ex} \Gamma = \Gamma$.

Here $\Gamma = \Gamma(\Omega, \varphi)$, $G = G(\Omega, \varphi)$, and $\operatorname{ex}(\,\cdot\,)$ denotes the set of extreme points of a set in $\operatorname{End} X^*$. Some properties of a dynamical system (Ω, φ) included in D1, D3, and D6 were considered earlier in [4]–[6]. Although these papers are devoted to invertible dynamical systems, the passage to the noninvertible case turns out to be purely technical in the situations under consideration. Properties (c6) and (d6) were discussed in [3]. A compact topological space is Fréchet–Urysohn if the closure of sets in this space is determined by convergent sequences. Property (a5) means that the cone with base $G(\Omega, \varphi)$ in the space $\operatorname{End} X^*$ is a vector structure [8]. If condition (b6) holds, then the function $\pi\colon \Gamma(\Omega, \varphi) \to E(\Omega, \varphi)$ defined by (1) is simultaneously a homeomorphism and an algebraic isomorphism, which allows us to identify the semigroups $\Gamma(\Omega, \varphi)$ and $E(\Omega, \varphi)$.

Theorem 4. *The properties of a semicascade* (Ω, φ) *in each of the classes* D1, D2, D3, *and* D6 *are pairwise equivalent.*

Thus, each of properties (b1) and (c1) can be taken for an alternative definition of an ordinary system, and each of properties (b3), (c3), (d3), and (e3) completely characterizes a tame system (Ω, φ). The equivalences (a1) $\Longleftrightarrow$ (b1) and (a3) $\Longleftrightarrow$ (b3) $\Longleftrightarrow$ (c3) $\Longleftrightarrow$ (d3) were proved in [4]–[6], and (b6) $\Longleftrightarrow$ (c6) $\Longleftrightarrow$ (d6) were obtained in [3]. As we see, a semicascade (Ω, φ) is tame if and only if its enveloping semigroup $E(\Omega, \varphi)$ consists of Borel transformations. Therefore, if this semigroup contains transformations not of the first Baire class, then it contains also transformation not being Borel.

Thus, we can consider classes $\mathscr{D}1$–$\mathscr{D}6$ of semicascades (Ω, φ) having any of the (equivalent) properties in D1–D6. It is known (see [4]) that the class $\mathscr{D}3$ of tame dynamical systems does not coincide with the class $\mathscr{D}1$ of ordinary systems.

Theorem 5. *The inclusions* $\mathscr{D}1 \subset \mathscr{D}2 \subset \mathscr{D}3 \subset \mathscr{D}6$ *and* $\mathscr{D}1 \subset \mathscr{D}4 \subset \mathscr{D}5 \subset \mathscr{D}6$ *hold.*

Thus, in the ordinary case, a compact convex subset $G(\Omega, \varphi)$ of the linear space $\operatorname{End} X^*$ is a Choquet simplex, and any operator $T \in G(\Omega, \varphi)$ is determined by its values at the Dirac measures $\delta(\omega)$, $\omega \in \Omega$. In the case of a tame semicascade (Ω, φ), the set of extreme points of $G(\Omega, \varphi)$ coincides with $\Gamma(\Omega, \varphi)$. The implication (a3) $\Longrightarrow$ (b6) was also proved in [2]. The class of ordinary semicascades $\mathscr{D}1$ is fairly large, because it consists of all discrete compact dynamical systems nonchaotic in the sense of Definition 2. On the other hand, in [4] there is an example of a minimal distal cascade on the 2-torus which is not contained even in the class $\mathscr{D}6$, which is largest among those considered above.

3. Ergodic properties of ordinary and tame semicascades. In this section we state assertions on ergodic properties of semicascades under consideration. Their proofs are based on the constructions of Section 2 and on results obtained in [3]. The convergence of an ergodic net of operator T_α in the space $\operatorname{End} X^*$ is understood in the sense of the W^*O topology. We denote the set of ergodic measures of the semicascade (Ω, φ) by $\Lambda(\Omega)$. A pair of points $\omega_1, \omega_2 \in \Omega$ is said to be proximal if $\inf_{n \geqslant 0} \rho(\varphi^n \omega_1, \varphi^n \omega_2) = 0$ for the metric ρ on the compact space Ω. By L we denote the kernel (the intersection of all two-sided ideals) of the semigroup $G(\Omega, \varphi)$; it consists of those projections $Q \in G(\Omega, \varphi)$ for which $VQ = Q$.

Theorem 6. *If a semicascade* (Ω, φ) *is ordinary, then the following assertions hold.*

(i) *Each ergodic net of operators* T_α *contains a convergent subsequence* $T_{\alpha(k)}$. *All ergodic nets of operators* T_α *converge if and only if, for any* $\omega \in \Omega$, *the closure of the trajectory* $o(\omega)$ *contains a unique minimal set. The last assertion remains true when "nets" is replaced by "sequences."*

(ii) *There exists a convergent ergodic sequence of operators T_n such that, for any $\omega \in \Omega$, the measure $T_n\delta(\omega)$ weakly-star converges in X^* to an ergodic measure $\mu_\omega \in \Lambda(\Omega)$, or, in other words, the asymptotic (with respect to $\{T_n\}$) distribution of all orbits of the dynamical system (Ω, φ) is determined by ergodic measures.*

(iii) *If the proximality relation on Ω is transitive, then all ergodic nets of operators T_α converge.*

(iv) *For each ergodic net of operators T_α, the pointwise convergence of the nets of functions $T_\alpha^* x$ on the compact space Ω for all continuous functions $x \in X$ implies the W^*O-convergence $T_\alpha \to Q$, where $Q \in L$.*

In fact, assertions (i)–(iii) are valid in the class $\mathscr{D}2$ of semicascades, and for (iv) to hold, property (a4) is sufficient. As shown in [3], for any semicascade (Ω, φ), the convergence of all ergodic operator nets is equivalent to the condition $\operatorname{card} L = 1$.

Theorem 7. *In the case of a tame semicascade (Ω, φ), the following assertions hold.*

(i) *The minimal center of attraction $Z(\Omega, \varphi)$ coincides with the closure of the union of all minimal sets.*

(ii) *If the closure of any trajectory $o(\omega)$, $\omega \in \Omega$, contains a unique minimal set, then all ergodic sequences of operators T_n converge, and the support of each ergodic measure $\mu \in \Lambda(\Omega)$ is a minimal set.*

Recall that the minimal center of attraction $Z(\Omega, \varphi)$ of a dynamical system (Ω, φ) is defined as the closure of the union of the supports of all ergodic measures. In this connection, we mention that the strict ergodicity of minimal tame systems was recently proved in [4], [9], and [10]. We emphasize that the results presented in this paper can easily be generalized to the case of dynamical systems with continuous time.

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National Research University Higher School of Economics
e-mail: vitkar48@inbox.ru

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