Multiplicities of degenerations of matrices and mixed volumes of Cayley polyhedra

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1. Introduction.

The local version of D. Bernstein's formula [Ber] expresses the local degree of a germ of a proper analytic map in terms of the Newton polyhedra of its components, provided that the principal parts of its components are in general position (see Theorem 5). We generalize this formula as follows.

Let $A : \mathbb{C}^m \to \mathbb{C}^{n\times k}$ be a germ of an $(n \times k)$-matrix with analytic entries, where $n \leqslant k$ (we denote the space of all $(n \times k)$-matrices by $\mathbb{C}^{n\times k}$). If $\operatorname{rk} A(0) < n$ and $\operatorname{rk} A(x) = n$ for all $x \neq 0$, then $m \leqslant k - n + 1$. Suppose that $m = k - n + 1$ (in particular, if $n = 1$, then this means that $A : \mathbb{C}^m \to \mathbb{C}^m$ is a germ of a proper analytic map). The intersection number $m(A)$ of the germ $A(\mathbb{C}^m)$ and the set of all degenerate matrices in $\mathbb{C}^{n\times k}$ is well defined, because the codimension of degenerate matrices in $\mathbb{C}^{n\times k}$ equals $k - n + 1$. In particular, if $n = 1$, then $m(A)$ equals the local degree of the map $A : \mathbb{C}^m \to \mathbb{C}^m$.

Definition 1. Let $A : \mathbb{C}^m \to \mathbb{C}^{n\times k}$ be a germ of an $(n\times k)$-matrix with analytic entries, such that $m = k-n+1$, $\operatorname{rk} A(0) < n$ and $\operatorname{rk} A(x) = n$ for all $x \neq 0$. Then the intersection number $m(A)$ will be called *the multiplicity (of degeneration)* of the germ A.

We recall the relation of this number to algebraic and topological invariants, motivating our interest to it.

Relation to Buchsbaum-Rim multiplicities. In the notation of Definition 1, the multiplicity of the matrix A is equal to $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^m,0}/\langle\text{maximal minors of } A\rangle$, where $\mathcal{O}_{\mathbb{C}^m,0}$ is the ring of germs of analytic functions on $\mathbb{C}^m$ near the origin. In particular, it equals the Buchsbaum-Rim multiplicity of the submodule of $\mathcal{O}^n_{\mathbb{C}^m,0}$, generated by the columns of A (see, for example, Proposition 2.3 in [G]).

Relation to characteristic classes. Let v_i be a holomorphic section of a vector bundle $\mathcal{I}$ of rank k on a smooth $(k-n+1)$-dimensional complex manifold M for $i = 1, \dots, n$. Suppose that there is a finite number of points $x \in M$ such that the vectors $v_1(x), \dots, v_n(x)$ are linearly dependent. Denote the set of all such points by X. Near each point $x \in X$, choosing a local basis $s_1, \dots, s_k$ in the bundle $\mathcal{I}$, one can represent v_i as a linear combination $v_i = a_{i,1}s_1 + \ldots + a_{i,k}s_k$, where $a_{i,j}$ are the entries of an $(n \times k)$-matrix $A : M \to \mathbb{C}^{n\times k}$ defined near x. Denote the multiplicity of A by m_x. Then the Chern number $c_{k-n+1}(\mathcal{I}_q) \cdot [M]$ is equal to the sum of the multiplicities m_x over all points $x \in X$ (see, for example, [GH]).

The aim of this paper is to present a formula for the multiplicity of a matrix A in terms of the Newton polyhedra of the entries of A, provided that the principal parts

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of the entries are in general position. In [Biv], a similar formula is given under the assumption that all the entries from the same row of the matrix A have the same Newton polyhedron. [E05] contains a general formula (see Theorem 23), which is somewhat indirect in the sense that one has to increase the dimension of polyhedra under consideration in order to formulate the answer. The aim of this paper is to simplify this answer combinatorially (see Theorem 7), so that no higher-dimensional polyhedra are involved.

In Sections 2 and 3, we present the formula for the multiplicity of a matrix and the condition of general position for the principal parts of the entries of a matrix, respectively. In Sections 5, this formula is deduced from Theorem 23, which expresses the multiplicity of a matrix in terms of the mixed volume of pairs of certain polyhedra (this notion is introduced in Section 4). This requires a formula for the mixed volume of Cayley polyhedra (Theorem 24, the proof given in Section 7), which follows from the Oda equality $(A \cap \mathbb{Z}^n) + (B \cap \mathbb{Z}^n) = (A + B) \cap \mathbb{Z}^n$ for some class of bounded lattice polyhedra $A, B \subset \mathbb{R}^n$ (see Section 6).

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2. Multiplicity in terms of Newton polyhedra.

A *polyhedron* in $\mathbb{R}^n$ is the intersection of a finite number of closed half-spaces. A *face* of a polyhedron A is the intersection of A and the boundary of a closed half-space, containing A. Note that the empty set is a face of every polyhedron. *The Minkowski sum* of sets A and B in $\mathbb{R}^n$ is the set $A + B = \{a + b \,|\, a \in A,\, b \in B\}$. Note that $\varnothing + A = \varnothing$ for every A.

Definition **2.** Let B_i be a face of a polyhedron $\Delta_i \subset \mathbb{R}^m$ for $i = 1, \ldots, k$. The collection of faces $(B_1, \ldots, B_k)$ is said to be *compatible*, if the sum $B_1 + \ldots + B_k$ is a non-empty bounded face of the sum $\Delta_1 + \ldots + \Delta_k$.

Denote the positive orthants of $\mathbb{R}^m$ and $\mathbb{Z}^m$ by $\mathbb{R}^m_+$ and $\mathbb{Z}^m_+$ respectively. For each point $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$, denote the monomial $x_1^{a_1} \ldots x_m^{a_m}$ by x^a.

Definition **3.** *The Newton polyhedron* Δ_f of a germ of an analytic function $f = \sum_{a \in \mathbb{Z}^m_+} c_a x^a : \mathbb{C}^m \to \mathbb{C}$ is the convex hull of the union $\bigcup\limits_{a \,|\, c_a \neq 0} (a + \mathbb{R}^m_+)$.

Definition **4.** *The restriction* $f|_B$ of a germ $f = \sum_{a \in \mathbb{Z}^m_+} c_a x^a$ to a bounded subset B of the Newton polyhedron Δ_f is the polynomial $\sum\limits_{a \in \mathbb{Z}^m \cap B} c_a x^a$. The restriction of f to the union of all bounded faces of Δ_f is called *the principal part* of f. The restriction to the empty set equals zero by definition.

The principal parts of the components of a map $f : \mathbb{C}^m \to \mathbb{C}^m$ form *the principal part* of f, and the principal parts of the entries of an $(n \times k)$-matrix $A : \mathbb{C}^m \to \mathbb{C}^{n \times k}$ form *the principal part* of A.

For a polyhedron $\Delta \subset \mathbb{R}^m_+$, denote the number of integer lattice points in the difference $\mathbb{R}^m_+ \setminus \Delta$ by $I(\Delta)$. Recall the local version of D. Bernstein's formula [Ber] (it can be deduced, for example from M. Oka's formula [O90]):

THEOREM **5.** *Let* $f = (f_1, \ldots, f_m) : \mathbb{C}^m \to \mathbb{C}^m$ *be a germ of an analytic map near the origin, and the differences* $\mathbb{R}^m_+ \setminus \Delta_{f_i}$ *are bounded.*
1) The local degree of f *is greater than or equal to*

$$\sum_{0<p\leqslant m} (-1)^{m-p} \sum_{0<i_1<\ldots<i_p\leqslant m} I(\Delta_{f_{i_1}} + \ldots + \Delta_{f_{i_p}}), \tag{$*$}$$

provided that f *is proper.*
2) The germ f *is proper, and its local degree equals* $(*)$, *if and only if, for each compatible collection of faces* $B_1, \ldots, B_m$ *of the polyhedra* $\Delta_1, \ldots, \Delta_m$*, the system of polynomial equations* $f_1|_{B_1} = \ldots = f_m|_{B_m} = 0$ *has no roots in* $(\mathbb{C} \setminus \{0\})^m$.

REMARK. The principal parts, which satisfy the condition from part (2) of this theorem, form a dense algebraic set in the space of principal parts of maps with given Newton polyhedra of components.

The main result of this paper is the following generalization of this fact to multiplicities of matrices.

DEFINITION **6.** The *tropical semiring* P of polyhedra is the set of all convex polyhedra in $\mathbb{R}^n$ (including the empty one) with the additive operation

$$A \vee B = \text{ convex hull of } A \cup B$$

and the Minkowski sum as the multiplicative operation

$$A + B = \{a + b \,|\, a \in A,\, b \in B\}.$$

The name is justified by the fact that the support functions of $A \vee B$ and $A + B$ are equal to the maximum and the sum of the support functions of A and B respectively. All the polyhedra A, satisfying the equation $A + \mathbb{R}^m_+ = A$, form a subring $P_+ \subset P$, and $\mathbb{R}^m_+$ is the unit in this subring. In particular, whenever the sum of polyhedra $A_j \in P_+$ is taken over an empty set of indices $J = \varnothing$, we set $\sum_{j\in J} A_\alpha = \mathbb{R}^m_+$ by definition.

THEOREM **7.** *Let* $A = (a_{i,j}) : \mathbb{C}^m \to \mathbb{C}^{n\times k}$ *be a germ of an* $(n \times k)$*-matrix with analytic entries,* $m = k - n + 1$*, and the differences* $\mathbb{R}^m_+ \setminus \Delta_{a_{i,j}}$ *are bounded.*
1) The multiplicity of the matrix A *is greater than or equal to*

$$\sum_{\substack{J\subset\{1,\ldots,k\}\\ b_1+\ldots+b_n=|J|}} (-1)^{k-|J|} I\Bigl(\bigvee_{\substack{J_1\sqcup\ldots\sqcup J_n=J\\ |J_1|=b_1,\ldots,|J_n|=b_n}} \sum_{\substack{i=1,\ldots,n\\ j\in J_i}} \Delta_{a_{i,j}} \Bigr), \tag{$**$}$$

provided that $\operatorname{rk} A(x) = n$ *for all* $x \neq 0$*. Here the first summation is taken over all non-empty* $J \subset \{1, \ldots, k\}$ *and all collections of non-negative integers* b_i *that sum up to* $|J|$*, and* $\bigvee$ *is taken over all decompositions of* J *into disjoint sets* J_i *of size* b_i.
2) We have $\operatorname{rk} A(x) = n$ *for all* $x \neq 0$*, and the multiplicity of* A *equals* $(**)$*, if and only if the principal part of* A *is in general position in the sense of Definition 17.*

It is a purely combinatorial problem to deduce this fact from Theorem 23, and it will be addressed in Section 5.

Example **8.** Theorem 7 appears to be more convenient than Theorem 23 in many important special cases. For instance, in the classical case of homogeneous $a_{i,j}$, Theorem 7 unlike Theorem 23 gives a closed formula for the multiplicity in terms of the degrees $d_{i,j}$ of the components $a_{i,j}$. For $J \subset \{1,\ldots,k\}$ and a decomposition $|J| = b_1 + \ldots + b_n$ into non-negative integers, introduce the number

$$d^J_{b_1,\ldots,b_n} = \min_{\substack{J_1\sqcup\ldots\sqcup J_n=J\\ |J_1|=b_1,\ldots,|J_n|=b_n}} \sum_{\substack{i=1,\ldots,n\\ j\in J_i}} d_{i,j}.$$

Corollary 9. *In the setting of Theorem 7, assume that the components $a_{i,j}$ are homogeneous polynomials of degree $d_{i,j}$.*

1) The multiplicity of the matrix A is greater or equal to

$$\sum_{\substack{J\subset\{1,\ldots,k\}\\ b_1+\ldots+b_n=|J|}} (-1)^{k-|J|}\binom{m+d^J_{b_1,\ldots,b_n}-1}{m}.$$

2) The multiplicity is strictly greater than this number or is infinite, if and only if the entries are not in general position in the following sense: there exist integer numbers $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_k$ and non-zero $x \in \mathbb{C}^m$ such that $d_{i,j} \geqslant \alpha_i + \beta_j$ for every i and j, and the matrix of the entries $\delta^{\alpha_i+\beta_j}_{d_{i,j}} a_{i,j}(x)$ is effectively degenerate (as usual, δ^q_p is 1 if $p = q$ and 0 otherwise).

Example **10.** Note that, unlike in the complete intersection case $n = 1$, the multiplicity of such a homogeneous matrix can be strictly greater than expected, but still finite. For example, if $(m,n,k) = (2,2,3)$, then the matrix

$$\begin{pmatrix} x+y & (x+y)^2+y^2 & x+y \\ x+y & x+y & (x+y)^2+2y^2 \end{pmatrix}$$

has multiplicity 6, which is strictly greater than the answer 3, given by Part 1 for a generic matrix of degree $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. This is because the matrix above is not in general position (consider $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = \beta_3 = 0$ in the notation of Part 2).

Example **11.** Let us expand the answer given by Theorem 7 in the simplest case $(m,n,k) = (2,2,3)$. Denote $\Delta_{a_{i,j}}$ by $\Delta_{i,j}$, then $(**)$ equals

$$I(\Delta_{1,1}+\Delta_{1,2}+\Delta_{1,3})+I\Big((\Delta_{2,1}+\Delta_{1,2}+\Delta_{1,3})\vee(\Delta_{1,1}+\Delta_{2,2}+\Delta_{1,3})\vee(\Delta_{1,1}+\Delta_{1,2}+\Delta_{2,3})\Big)+$$
$$+I\Big((\Delta_{1,1}+\Delta_{2,2}+\Delta_{2,3})\vee(\Delta_{2,1}+\Delta_{1,2}+\Delta_{2,3})\vee(\Delta_{2,1}+\Delta_{2,2}+\Delta_{1,3})\Big)+I(\Delta_{2,1}+\Delta_{2,2}+\Delta_{2,3})-$$
$$-I(\Delta_{1,1}+\Delta_{1,2})-I(\Delta_{1,1}+\Delta_{1,3})-I(\Delta_{1,2}+\Delta_{1,3})-I(\Delta_{2,1}+\Delta_{2,2})-I(\Delta_{2,1}+\Delta_{2,3})-I(\Delta_{2,2}+\Delta_{2,3})-$$
$$-I((\Delta_{1,1}+\Delta_{2,2})\vee(\Delta_{1,2}+\Delta_{2,1}))-I((\Delta_{1,1}+\Delta_{2,3})\vee(\Delta_{1,3}+\Delta_{2,1}))-I((\Delta_{1,3}+\Delta_{2,2})\vee(\Delta_{1,2}+\Delta_{2,3}))-$$
$$+I(\Delta_{1,1})+I(\Delta_{1,2})+I(\Delta_{1,3})+I(\Delta_{2,1})+I(\Delta_{2,2})+I(\Delta_{2,3}).$$

Example **12.** If $\Delta_{i,j} = \Delta_i$ does not depend on the column j, then the answer, given by Theorems 7 and 23, admits a much simpler form $\sum_{1\leqslant i_1\leqslant\ldots\leqslant i_m\leqslant k} \mathrm{MV}(\Delta_{i_1}, \ldots, \Delta_{i_m})$. If $\Delta_{i,j} = \Delta_j$ does not depend on the row i, then the answer, given by Theorems 7 and 23, admits a much simpler form $\sum_{1\leqslant j_1<\ldots<j_m\leqslant k} \mathrm{MV}(\Delta_{j_1}, \ldots, \Delta_{j_m})$. Both of these facts can be easily deduced from Theorem 23 (see [E06] and [E09] for details). The latter one was discovered earlier in a much more general setting by Bivià-Ausina ([Biv]).

For example, if the germs $a_{i1} \in \langle x^2, y\rangle$, $a_{i2} \in \langle x, y^3\rangle$, $a_{i3} \in \langle x^2, y^3\rangle$, $i = 1, 2$, are in general position, then the multiplicity of A equals $4I(\Delta_1 + \Delta_2 + \Delta_3) - 3I(\Delta_1 + \Delta_2) - 3I(\Delta_1 + \Delta_3) - 3I(\Delta_2 + \Delta_2) + 2I(\Delta_1) + 2I(\Delta_2) + 2I(\Delta_3) = 4\cdot 16 - 3\cdot(6+9+11) + 2\cdot(2+3+5) = 6$ according to Theorem 7 and $\mathrm{MV}(\Delta_1, \Delta_2) + \mathrm{MV}(\Delta_1, \Delta_3) + \mathrm{MV}(\Delta_2, \Delta_3) = 1 + 2 + 3 = 6$ according to [Biv].

3. General position of principal parts of matrices.

By convention, each polyhedron has the empty face. In particular, some faces $B_{i,j}$ in the following definition may be empty.

Definition **13.** Let $B_{i,j}$ be a bounded face of a polyhedron $\Delta_{i,j} \subset \mathbb{R}^m$ for $i = 1, \ldots, n$, $j = 1, \ldots, k$. The collection of faces $B_{i,j}$ is said to be *matrix-compatible*, if there exist vectors $c_1, \ldots, c_n \in \mathbb{Z}^m$ and compatible faces $B_1, \ldots, B_k$ of the convex hulls $\bigvee_i(\Delta_{i,1} + c_i), \ldots, \bigvee_i(\Delta_{i,k} + c_i)$, such that $B_{i,j} = (B_j - c_i) \cap \Delta_{i,j}$ for each $i = 1, \ldots, n$, $j = 1, \ldots, k$.

Example **14.** Let $\Delta_{i,j} \subset \mathbb{R}^1$ be the rays

$$\begin{pmatrix} [1,\infty) & [1,\infty) & [1,\infty) \\ [1,\infty) & [2,\infty) & [2,\infty) \\ [1,\infty) & [2,\infty) & [2,\infty) \end{pmatrix},$$

then every face $B_{i,j}$ is either the origin of $\Delta_{i,j}$ (denoted by $*$), or empty (denoted by $\varnothing$). In this case, the matrix-compatible collection of faces are

$$\mathcal{B}_1 = \begin{pmatrix} * & * & * \\ * & \varnothing & \varnothing \\ * & \varnothing & \varnothing \end{pmatrix}, \quad \mathcal{B}_2 = \begin{pmatrix} \varnothing & * & * \\ * & * & * \\ * & * & * \end{pmatrix},$$

and 11 more collections with fewer non-empty faces.

Definition **15.** A matrix $M \in \mathbb{C}^{n\times k}$, $n \leqslant k$, is said to be *effectively non-degenerate*, if $(t_1, \ldots, t_n) \cdot M \neq (0, \ldots, 0)$ for all $(t_1, \ldots, t_n) \in (\mathbb{C} \setminus \{0\})^n$.

Example **16.** The complex matrix

$$\begin{pmatrix} a & b & c \\ d & 0 & 0 \\ e & 0 & 0 \end{pmatrix}$$

is effectively non-degenerate if and only if $b = c = 0$ (although it is degenerate for all complex numbers a, b, c, d, e).

For an $(n \times k)$-matrix A with analytic entries $a_{i,j} : \mathbb{C}^m \to \mathbb{C}$ and a collection $\mathcal{B}$ of faces $B_{i,j}$ of the Newton polyhedra $\Delta_{a_{i,j}}$, we denote the matrix with entries $a_{i,j}|_{B_{i,j}}$ by $A|_{\mathcal{B}}$.

Definition **17.** The principal part of an $(n \times k)$-matrix A with analytic entries $a_{i,j} : \mathbb{C}^m \to \mathbb{C}$ is said to be *in general position*, if, for each matrix-compatible collection $\mathcal{B}$ of faces of the Newton polyhedra $\Delta_{a_{i,j}}$ and for each $x \in (\mathbb{C} \setminus \{0\})^m$, the matrix $A|_{\mathcal{B}}(x)$ is effectively non-degenerate.

Remark. Principal parts in general position form a dense algebraic set in the space of principal parts of matrices with given Newton polyhedra of entries. However, this is not true, if we replace the effective non-degeneracy of matrices with the conventional one in Definition 17. For instance, if $(m, n, k) = (1, 3, 3)$, and the Newton polyhedra $\Delta_{a_{i,j}}$ are as in the example to Definition 13, then the only non-trivial condition, imposed by Definition 17, corresponds to the second matrix-compatible collection of faces shown in the example:

$$\det(A|_{\mathcal{B}_2}) = \det \begin{pmatrix} 0 & a^0_{0,1} & a^0_{0,2} \\ a^0_{1,0} & a^0_{1,1} & a^0_{1,2} \\ a^0_{2,0} & a^0_{2,1} & a^0_{2,2} \end{pmatrix} \neq 0,$$

where $a^0_{i,j}$ is the leading coefficient of the series $a_{i,j}$. However, if we replace effective nondegeneracy with nondegeneracy in Definition 17, then no matrix A will satisfy it, because the matrix $A|_{\mathcal{B}_1}$ is always degenerate (see the example to Definition 15).

It would be thus interesting to describe a collection of minors of the matrices $A|_{\mathcal{B}}$, such that
1) If the principal part of A is in general position, then these minors vanish.
2) The principal parts for which these minors vanish form a (closed algebraic) set of positive codimension in the space of all principal parts of matrices with given Newton polyhedra of entries.

This reduces to the following problem: given $K \subset \mathbb{N}^2$, assume that $a_{i,j}$ are independent variables for $(i, j) \in K$, and the entries of the matrix A are $a_{i,j}$ for $(i, j) \in K$ and equal 0 for $(i, j) \notin K$. Find a collection of minors $\mathcal{A}$ of the matrix A, such that
1) If A is effectively nondegenerate, then $\mathcal{A} = 0$.
2) We have $\mathcal{A} \neq 0$ for generic $a_{i,j}, (i, j) \in K$.

4. Mixed volumes of pairs of polyhedra.

Definition **18.** Polyhedra Δ_1 and Δ_2 in $\mathbb{R}^n$ are said to be *parallel* if $a + \Delta_1 \subseteq \Delta_1 \Leftrightarrow a + \Delta_2 \subseteq \Delta_2$ for every point $a \in \mathbb{R}^n$.

Definition **19.** ([E05], [E06]) 1) A pair of polyhedra Δ_1, Δ_2 in $\mathbb{R}^n$ is called *bounded* if both $\Delta_1 \setminus \Delta_2$ and $\Delta_2 \setminus \Delta_1$ are bounded. The set of all bounded pairs of polyhedra parallel to a given convex cone $C \subset \mathbb{R}^n$ is denoted by BP_C.

2) *The Minkowski sum* $(\Delta_1, \Delta_2) + (\Gamma_1, \Gamma_2)$ of two pairs from BP_C is the pair $(\Delta_1 + \Gamma_1, \Delta_2 + \Gamma_2) \in \mathrm{BP}_C$.

3) *The volume* $\mathrm{Vol}(\Delta_1, \Delta_2)$ of a bounded pair (Δ_1, Δ_2) is the difference $\mathrm{Vol}(\Delta_1 \setminus \Delta_2) - \mathrm{Vol}(\Delta_2 \setminus \Delta_1)$.

4) *The mixed volume* is the symmetric multilinear (with respect to Minkowski summation) function $\mathrm{MV} : \underbrace{\mathrm{BP}_C \times \ldots \times \mathrm{BP}_C}_{n} \to \mathbb{R}$ such that $\mathrm{MV}(A, \ldots, A) = \mathrm{Vol}(A)$ for every pair $A \in \mathrm{BP}_C$.

There exists a unique such function MV (see [E06], Section 4, Lemma 3 for existance, uniqueness and all other basic facts about the mixed volume of pairs, mentioned below). Recall that a polyhedron is said to be *lattice* if its vertices are integer lattice points. The mixed volume of pairs of n-dimensional lattice polyhedra is a rational number with denominator $n!$.

EXAMPLE. If C consists of one point, then BP_C consists of pairs of bounded polyhedra, and

$$\mathrm{MV}\big((\Delta_1, \Gamma_1), \ldots, (\Delta_n, \Gamma_n)\big) = \mathrm{MV}(\Delta_1, \ldots, \Delta_n) - \mathrm{MV}(\Gamma_1, \ldots, \Gamma_n),$$

where MV in the right hand side is the classical mixed volume of bounded polyhedra. If C is not bounded, then both terms in the right hand side are infinite, but "their difference makes sense".

One can use the following formula to express the mixed volume of pairs in terms of mixed volumes of polyhedra ([E06], Section 4, Lemma 3).

LEMMA **20.** *For bounded pairs $(\Delta_i, \Gamma_i) \in \mathrm{BP}_C$, $i = 1, \ldots, n$, let $H \subset \mathbb{R}^n$ be a half-space such that $C \cap H$ is bounded and $\Delta_i \setminus H = \Gamma_i \setminus H$. Then*

$$\mathrm{MV}\big((\Delta_1, \Gamma_1), \ldots, (\Delta_n, \Gamma_n)\big) = \mathrm{MV}(\Delta_1 \cap H, \ldots, \Delta_n \cap H) - \mathrm{MV}(\Gamma_1 \cap H, \ldots, \Gamma_n \cap H),$$

where MV *in the right hand side is the classical mixed volume of bounded polyhedra.*

For a bounded pair of (closed) polyhedra $(\Delta, \Gamma) \in \mathrm{BP}_C$, define $I(\Delta, \Gamma)$ as the number of integer lattice points in the difference $\Delta \setminus \Gamma$ minus the number of integer lattice points in the difference $\Gamma \setminus \Delta$.

LEMMA **21.** *For bounded pairs of lattice polyhedra $A_i \in \mathrm{BP}_C$, we have*

$$n!\,\mathrm{MV}(A_1, \ldots, A_n) = \sum_{0<p\leqslant m} (-1)^{n-p} \sum_{0<i_1<\ldots<i_p\leqslant n} I(A_{i_1} + \ldots + A_{i_p}).$$

PROOF. For the classical mixed volume of bounded polyhedra, this equality is well known (see, for example, [Kh]). The general case can be deduced to the case of bounded polyhedra by the previous lemma. □

5. Proof of Theorem 7.

The following theorem is a special case of Theorem 5 from [E06].

Definition **22.** For polyhedra $\Delta_1, \ldots, \Delta_n \subset \mathbb{R}^m$, define the *Cayley polyhedron* $\Delta_1 * \ldots * \Delta_n$ as the convex hull of the union

$$\bigcup_i \{b_i\} \times \Delta_i \subset \mathbb{R}^{n-1} \oplus \mathbb{R}^m,$$

where $b_1, \ldots, b_n$ are the points $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$ and $(0, 0, \ldots, 0)$ in $\mathbb{R}^{n-1}$. Denote $\mathbb{R}^m_+ * \ldots * \mathbb{R}^m_+$ by D.

For germs of analytic functions $a_1, \ldots, a_n$ on $\mathbb{C}^m$ near the origin, denote the sum $t_1 a_1 + \ldots + t_{n-1} a_{n-1} + a_n$ by $a_1 * \ldots * a_n$, where $t_1, \ldots, t_{n-1}$ are the standard coordinates on $\mathbb{C}^{n-1}$.

Theorem **23.** *([E05], [E06], [E09]) Let A be an $(n \times k)$-matrix with entries $a_{i,j} : \mathbb{C}^m \to \mathbb{C}$ which are germs of analytic functions near the origin. Suppose that the Newton polyhedra $\Delta_{i,j}$ of the germs $a_{i,j}$ intersect all coordinate axes in $\mathbb{R}^m$.*
1) The multiplicity of A is greater than or equal to

$$(m + n - 1)! \operatorname{MV}\big((D, \Delta_{1,1} * \ldots * \Delta_{n,1}), \ldots, (D, \Delta_{1,k} * \ldots * \Delta_{n,k})\big). \qquad (***)$$

2) We have $\operatorname{rk} A(x) = n$ for all $x \neq 0$, and the multiplicity of A equals $(***)$, if and only if, for each compatible collection of faces $B_1, \ldots, B_k$ of the polyhedra $\Delta_{1,1} * \ldots * \Delta_{n,1}$, $\ldots$, $\Delta_{1,k} * \ldots * \Delta_{n,k}$, the polynomials $(a_{1,1} * \ldots * a_{n,1})|_{B_1}$, $\ldots$, $(a_{1,k} * \ldots * a_{n,k})|_{B_k}$ have no common zeroes in $(\mathbb{C} \setminus \{0\})^{n-1} \times (\mathbb{C} \setminus \{0\})^m$.

The "only if" part of (2) is actually proved in [E06], but is explicitly formulated and discussed only in [E09], Theorem 1.21.

Recall that $|S \cap \mathbb{Z}^m|$ is denoted by $I(S)$ for a bounded set $S \in \mathbb{R}^m$. If the symmetric difference of (closed) lattice polyhedra Γ and Δ in $\mathbb{R}^m$ is bounded, denote the difference $I(\Gamma \setminus \Delta) - I(\Delta \setminus \Gamma)$ by $I(\Gamma, \Delta)$. For pairs of polyhedra (Γ_i, Δ_i) in $\mathbb{R}^m$, denote the pair $(\bigvee_i \Gamma_i, \bigvee_i \Delta_i)$ by $\bigvee_i (\Gamma_i, \Delta_i)$ and the pair $(\Gamma_1 * \ldots * \Gamma_n, \Delta_1 * \ldots * \Delta_n)$ by $(\Gamma_1, \Delta_1) * \ldots * (\Gamma_n, \Delta_n)$.

Theorem **24.** *If $B_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, k$, are bounded lattice polyhedra in $\mathbb{R}^m$ or pairs of lattice polyhedra in BP_C, and $m = k - n + 1$, then the mixed volume of $B_{1,j} * \ldots * B_{n,j}$, $j = 1, \ldots, k$, equals*

$$\frac{1}{k!} \sum_{\substack{J \subset \{1,\ldots,k\} \\ b_1 + \ldots + b_n = |J|}} (-1)^{k-|J|} I\Big(\bigvee_{\substack{J_1 \sqcup \ldots \sqcup J_n = J \\ |J_1| = b_1, \ldots, |J_n| = b_n}} \sum_{\substack{i=1,\ldots,n \\ j \in J_i}} B_{i,j} \Big).$$

Note that some of $B_{i,j}$ may be empty. The proof is given in Section 7. Theorem 7 follows from Theorems 23 and 24 (one can easily check that the condition of general position in Theorem 23(2) coincides with the one given by Definition 17).

6. Fans and lattice points of polyhedra.

Here we prove the equality

$$(A \cap \mathbb{Z}^q) + (B \cap \mathbb{Z}^q) = (A + B) \cap \mathbb{Z}^q$$

for some class of bounded lattice polyhedra $A, B \subset \mathbb{R}^q$ (see [O97] for a conjecture in the general case).

Definition **25.** *A (rational) cone* in $\mathbb{R}^q$ generated by (rational) vectors $v_1, \ldots, v_m$ is the set of all linear combinations of $v_1, \ldots, v_m$ with positive coefficients.

Note that, according to this definition, a cone is not a closed set unless it is a vector subspace of $\mathbb{R}^q$, and is not an open set unless it is q-dimensional.

Definition **26.** A collection of rational cones $C_1, \ldots, C_p$ in $\mathbb{R}^q$ is said to be $\mathbb{Z}$-*transversal*, if $\sum \dim C_i = q$ and the set $\mathbb{Z}^q \cap \bigcup_i C_i$ generates the lattice $\mathbb{Z}^q$.

Definition **27.** *A (rational) fan* Φ in $\mathbb{R}^q$ is a non-empty finite set of nonoverlapping (rational) cones in $\mathbb{R}^q$ such that
1) Each face of each cone from Φ is in Φ,
2) Each cone from Φ is a face of a q-dimensional cone from Φ.

Definition **28.** A collection of fans $\Phi_1, \ldots, \Phi_p$ in $\mathbb{R}^q$ is said to be $\mathbb{Z}$-*transversal with respect to shifts* $c_1 \in \mathbb{R}^q, \ldots, c_p \in \mathbb{R}^q$, if each collection of cones $C_1 \in \Phi_1, \ldots, C_p \in \Phi_p$, such that the intersection $(C_1 + c_1) \cap \ldots \cap (C_p + c_p)$ consists of one point, is $\mathbb{Z}$-transversal.

Definition **29.** *The dual cone* of a face B of a polyhedron $A \subset \mathbb{R}^q$ is the set of all covectors $\gamma \in (\mathbb{R}^q)^*$ such that $\{a \in A \,|\, \gamma(a) = \min \gamma(A)\} = B$. *The dual fan* of a polyhedron is the set of dual cones of all its faces.

Theorem **30.** *If the dual fans of bounded lattice polyhedra* $A_1, \ldots, A_p \subset \mathbb{R}^q$ *are* $\mathbb{Z}$-*transversal with respect to some shifts* $c_1 \in (\mathbb{R}^q)^*, \ldots, c_p \in (\mathbb{R}^q)^*$ *and* $\dim(A_1 + \ldots + A_p) = q$*, then*

$$(A_1 \cap \mathbb{Z}^q) + \ldots + (A_p \cap \mathbb{Z}^q) = (A_1 + \ldots + A_p) \cap \mathbb{Z}^q.$$

Proof. Consider covectors $c_1 \in (\mathbb{R}^q)^*, \ldots, c_p \in (\mathbb{R}^q)^*$ as linear functions on the polyhedra $A_1 \subset \mathbb{R}^q, \ldots, A_p \subset \mathbb{R}^q$ respectively, and denote their graphs in $\mathbb{R}^q \oplus \mathbb{R}^1$ by $\Gamma_1, \ldots, \Gamma_p$. Denote the projection $\mathbb{R}^q \oplus \mathbb{R}^1 \to \mathbb{R}^q$ by π, and denote the ray $\{(0, \ldots, 0, t) \mid t < 0\} \subset \mathbb{R}^q \oplus \mathbb{R}^1$ by L_-.

Each bounded q-dimensional face B of the sum $\Gamma_1 + \ldots + \Gamma_p + L_-$ is the sum of some faces $B_1, \ldots, B_p$ of polyhedra $\Gamma_1 + L_-, \ldots, \Gamma_p + L_-$. $\mathbb{Z}$-transversality with respect to shifts $c_1 \in (\mathbb{R}^q)^*, \ldots, c_p \in (\mathbb{R}^q)^*$ implies that

$$\big(\pi(B_1) \cap \mathbb{Z}^q\big) + \ldots + \big(\pi(B_p) \cap \mathbb{Z}^q\big) = \pi(B_1 + \ldots + B_p) \cap \mathbb{Z}^q.$$

Since the projections of bounded q-dimensional faces of the sum $\Gamma_1 + \ldots + \Gamma_p + L_-$ cover the sum $A_1 + \ldots + A_p$, it satisfies the same equality:

$$(A_1 \cap \mathbb{Z}^q) + \ldots + (A_p \cap \mathbb{Z}^q) = (A_1 + \ldots + A_p) \cap \mathbb{Z}^q. \qquad \square$$

Corollary 31. *Let* $S \subset \mathbb{R}^q$ *be the standard* q*-dimensional simplex, let* $l_1, \ldots, l_p$ *be linear functions on* S *with graphs* $\Gamma_1, \ldots, \Gamma_p$*, and let* l *be the maximal piecewise-linear function on* pS*, such that its graph* Γ *is contained in the sum* $\Gamma_1 + \ldots + \Gamma_p$*. Then, for each integer lattice point* $a \in pS$*, the value* $l(a)$ *equals the maximum of sums* $l_1(c_1) + \ldots + l_p(c_p)$*, where* $(c_1, \ldots, c_p)$ *runs over all* p*-tuples of vertices of* S *such that* $c_1 + \ldots + c_p = a$*.*

Proof. Denote the projection $\mathbb{R}^q \oplus \mathbb{R}^1 \to \mathbb{R}^q$ by π. A q-dimensional face B of Γ, which contains the point $\bigl(a, l(a)\bigr) \in \mathbb{R}^q \oplus \mathbb{R}^1$, can be represented as a sum of faces B_i of simplices Γ_i. Since $\pi(B_1), \ldots, \pi(B_p)$ are faces of the standard simplex, their dual fans are $\mathbb{Z}$-transversal with respect to a generic collection of shifts, and, by Theorem 30,

$$\bigl(\pi(B_1) \cap \mathbb{Z}^q\bigr) + \ldots + \bigl(\pi(B_p) \cap \mathbb{Z}^q\bigr) = \pi(B) \cap \mathbb{Z}^q.$$

In particular, $a = c_1 + \ldots + c_p$ for some integer lattice points $c_i \in \pi(B_i)$, which implies $l(a) = l_1(c_1) + \ldots + l_p(c_p)$. $\square$

Remark. In particular, if the functions $l_1, \ldots, l_p$ are in general position, then all C^q_{p+q} integer lattice points in the simplex pS are projections of vertices of Γ. Translating this into the tropical language, one can prove again the following well-known fact: p generic tropical hyperplanes in the space $\mathbb{R}^q$ subdivide it into C^q_{p+q} pieces.

Example **32.** If S in the formulation of Corollary 31 is not the standard simplex, then the statement is not always true. For example, consider

$$S = \operatorname{conv}\bigl\{(1,1),(1,-1),(-1,1),(-1,-1)\bigr\},\ l_1(x,y) = x+y,\ l_2(x,y) = x-y,\ a = (1,0).$$

If, in addition, we allow functions l_j to be concave piecewise linear with integer domains of linearity, then the statement is not true unless S is the standard simplex. That is why we cannot use computations below to simplify the formula in the statement of Theorem 5 from [E06] in general.

7. Proof of Theorem 24.

Rewriting the mixed volume of the pairs $B_{1,i} * \ldots * B_{n,i}$, $i = 1, \ldots, k$, as

$$\sum_{0<p\leqslant m} (-1)^{n-p} \sum_{0<i_1<\ldots<i_p\leqslant n} I\bigl((B_{1,i_1} * \ldots * B_{n,i_1}) + \ldots + (B_{1,i_p} * \ldots * B_{n,i_p})\bigr)$$

by Lemma 21, and applying the following Lemma 33 to every term in this sum, we obtain the statement of Theorem 24.

Lemma **33.** *For bounded pairs of polyhedra $A_{i,j} = (\Delta_{i,j}, \Phi_{i,j}) \in \mathrm{BP}_C$, $i = 1, \ldots, n$, $j = 1, \ldots, p$,*

$$I\bigl((A_{1,1} * \ldots * A_{n,1}) + \ldots + (A_{1,p} * \ldots * A_{n,p})\bigr) =$$

$$= \sum_{\substack{a_1+\ldots+a_n=p\\ a_1\geqslant 0,\ldots,a_n\geqslant 0}} I\Bigl(\bigvee_{\substack{J_1\sqcup\ldots\sqcup J_n=\{1,\ldots,p\}\\ |J_1|=a_1,\ldots,|J_n|=a_n}} \sum_{\substack{i=1,\ldots,n\\ j\in J_i}} \Delta_{i,j},\ \bigvee_{\substack{J_1\sqcup\ldots\sqcup J_n=\{1,\ldots,p\}\\ |J_1|=a_1,\ldots,|J_n|=a_n}} \sum_{\substack{i=1,\ldots,n\\ j\in J_i}} \Phi_{i,j}\Bigr).$$

Proof. Every integer lattice point, participating in the left hand side, is contained in the plane $\{(a_1, \ldots, a_{n-1})\} \times \mathbb{R}^m \subset \mathbb{R}^{n-1} \oplus \mathbb{R}^m$ for some non-negative integer numbers $a_1, \ldots, a_n$, which sum up to p. Thus, it is enough to describe the intersection of the pair $\bigl((A_{1,1} * \ldots * A_{n,1}) + \ldots + (A_{1,p} * \ldots * A_{n,p})\bigr)$ with each of these planes, using the following fact. $\square$

LEMMA **34.** *Suppose that polyhedra $\Delta_{i,j} \subset \mathbb{R}^m$ are parallel to each other for $i = 1, \dots, n$, $j = 1, \dots, p$. Then, for each n-tuple of non-negative integer numbers $a_1, \dots, a_n$ which sum up to p,*

$$\Big(\{(a_1, \dots, a_{n-1})\} \times \mathbb{R}^m\Big) \cap \big((\Delta_{1,1} * \dots * \Delta_{n,1}) + \dots + (\Delta_{1,p} * \dots * \Delta_{n,p})\big) =$$

$$= \{(a_1, \dots, a_{n-1})\} \times \Big(\bigvee_{\substack{J_1 \sqcup \dots \sqcup J_n = \{1,\dots,p\} \\ |J_1| = a_1, \dots, |J_n| = a_n}} \sum_{\substack{i=1,\dots,n \\ j \in J_i}} \Delta_{i,j} \Big) \subset \mathbb{R}^{n-1} \oplus \mathbb{R}^m.$$

PROOF. For each hyperplane $L \subset \mathbb{R}^m$, denote the projection $\mathbb{R}^{n-1} \oplus \mathbb{R}^m \to \mathbb{R}^{n-1} \oplus \mathbb{R}$ along $\{0\} \oplus L$ by π_L. It is enough to prove that the images of the left hand side and the right hand side under π_L coincide for each L. To prove it, apply Corollary 31, setting q to $n-1$, a to $(a_1, \dots, a_{n-1})$, and Γ_j to the maximal bounded face of the projection $\pi_L\big(\Delta_{1,j} * \dots * \Delta_{n,j}\big)$ for every $j = 1, \dots, p$. $\Box$

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