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Gaiotto–Witten superpotential and Whittaker D-modules on monopoles

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Dedicated to the memory of Andrei Zelevinsky

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ABSTRACT

Let G be an almost simple simply connected group over $\mathbb{C}$. For a positive element α of the coroot lattice of G let $\overset{\circ}{Z}{}^{\alpha}$ denote the space of maps from $\mathbb{P}^1$ to the flag variety $\mathcal{B}$ of G sending $\infty \in \mathbb{P}^1$ to a fixed point in $\mathcal{B}$ of degree α. This space is known to be isomorphic to the space of framed G-monopoles on $\mathbb{R}^3$ with maximal symmetry breaking at infinity of charge α.

In [6] a system of (étale, rational) coordinates on $\overset{\circ}{Z}{}^{\alpha}$ is introduced. In this note we compute various known structures on $\overset{\circ}{Z}{}^{\alpha}$ in terms of the above coordinates. As a byproduct we give a natural interpretation of the Gaiotto–Witten superpotential studied in [8] and relate it to the theory of Whittaker D-modules discussed in [9].

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1. Introduction

1.1. Zastava spaces

Let G be an almost simple simply connected algebraic group over $\mathbb{C}$. We denote by $\mathcal{B}$ the flag variety of G. Let us also fix a pair of opposite Borel subgroups B, B_- whose intersection is a maximal torus T (thus we have $\mathcal{B} = G/B = G/B_-$).

Let Λ denote the cocharacter lattice of T; since G is assumed to be simply connected, this is also the coroot lattice of G. We denote by $\Lambda_+ \subset \Lambda$ the sub-semigroup spanned by positive coroots. We say that $\alpha \geq \beta$ (for $\alpha, \beta \in \Lambda$) if $\alpha - \beta \in \Lambda_+$.

It is well-known that $H_2(\mathcal{B}, \mathbb{Z}) = \Lambda$ and that an element $\alpha \in H_2(\mathcal{B}, \mathbb{Z})$ is representable by an algebraic curve if and only if $\alpha \in \Lambda_+$. Let $\overset{\circ}{Z}{}^{\alpha}$ denote the space of maps $\mathbb{P}^1 \to \mathcal{B}$ of degree α sending $\infty \in \mathbb{P}^1$ to $B \in \mathcal{B}$. It is known [6] that this is a smooth symplectic affine algebraic variety, which can be identified with the space of framed G-monopoles on $\mathbb{R}^3$ with maximal symmetry breaking at infinity of charge α [10,11].

The scheme $\overset{\circ}{Z}{}^{\alpha}$ is endowed with a number of remarkable structures (listed below). On the other hand in [6] the authors introduce a system of (birational, étale) coordinates on $\overset{\circ}{Z}{}^{\alpha}$. The purpose of the present note is to compute how these structures look like in the above coordinates. In particular, it turns out that the Gaiotto–Witten superpotential [8] admits a natural interpretation in terms of Whittaker D-modules of [9].

1.2. Quasi-maps

The scheme $\overset{\circ}{Z}{}^{\alpha}$ has a natural partial compactification Z^{α}. It can be realized as the space of based *quasi-maps* of degree α; set-theoretically it can be described in the following way:

$$Z^{\alpha} = \bigsqcup_{0 \leq \beta \leq \alpha} \overset{\circ}{Z}{}^{\beta} \times \mathbb{A}^{\alpha - \beta},$$

where for $\gamma \in \Lambda_+$ we denote by $\mathbb{A}^{\gamma}$ the space of all colored divisors $\sum \gamma_i x_i$ with $x_i \in \mathbb{A}^1$, $\gamma_i \in \Lambda_+$ such that $\sum \gamma_i = \gamma$.

1.3. A "symmetric" definition of the zastava space

Fix $\lambda, \mu \in \Lambda$. Let us denote by $\overset{\circ}{Z}{}^{\lambda,\mu}$ the scheme classifying the following data:

1) A G-bundle $\mathcal{F}$ on $\mathbb{P}^1$ with a trivialization at $\infty \in \mathbb{P}^1$.

2) A B-structure $\mathcal{F}_B$ on $\mathcal{F}$ such that the induced T-bundle $\mathcal{F}_{T,+}$ is of degree λ. We require that $\mathcal{F}_B$ is equal to B at ∞.

3) A B_--structure $\mathcal{F}_{B_-}$ on $\mathcal{F}$ such that the induced T-bundle $\mathcal{F}_{T,-}$ is of degree μ. We require that $\mathcal{F}_{B_-}$ is equal to B_- at ∞.

It is easy to see that this is indeed a scheme. Moreover, we claim that $\overset{\circ}{Z}{}^{\lambda,\mu}$ is naturally isomorphic to $\overset{\circ}{Z}{}^{\lambda-\mu}$ (Section 6).

1.4. Structures on the zastava space

It is easy to see that the space $\overset{\circ}{Z}{}^{\alpha}$ is endowed with the following structures (the precise constructions are given in the main body of the paper):

(1) The scheme $\overset{\circ}{Z}{}^{\alpha}$ possesses a natural symplectic structure [6].

(2) There is a natural morphism $\pi_\alpha : \overset{\circ}{Z}{}^{\alpha} \to \mathbb{A}^\alpha$. Moreover, given $\beta, \gamma \in \Lambda_+$ let $(\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}}$ denote the space of pairs of colored divisors of degrees β and γ which are mutually disjoint. If $\alpha = \beta + \gamma$ then we have a natural étale map $(\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \to \mathbb{A}^\alpha$. The *factorization* is a canonical isomorphism

$$\mathfrak{f}_{\beta,\gamma} : (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\alpha} Z^\alpha \xrightarrow{\sim} (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\beta \times \mathbb{A}^\gamma} (Z^\beta \times Z^\gamma).$$

We shall refer to the latter as the *factorization property* of Zastava.

(3) The Cartan involution on G (which interchanges B and B_- and induces the map $t \mapsto t^{-1}$ on T) induces an involution ι on $\overset{\circ}{Z}{}^{\alpha}$ (this is clear from the point of view of the definition of $\overset{\circ}{Z}{}^{\alpha}$ given in Section 1.3).

(4) Let $\partial Z^\alpha = Z^\alpha \backslash \overset{\circ}{Z}{}^{\alpha}$. Then ∂Z^α is a Cartier divisor and moreover it is the divisor of zeros of some function F_α on Z^α which is invertible on $\overset{\circ}{Z}{}^{\alpha}$ (this function is unique up to a multiplicative scalar).

(5) Fix $\lambda, \mu \in \Lambda$ such that $\lambda - \mu = \alpha$. Then for every simple root $\check{\alpha}_i$ of G we have canonical maps $\mathfrak{E}^\alpha_{\lambda,+,i} : \overset{\circ}{Z}{}^{\alpha} \to H^1(\mathbb{P}^1, \mathcal{O}(\langle -\check{\alpha}_i, \lambda\rangle))$, $\mathfrak{E}^\alpha_{\mu,-,i} : \overset{\circ}{Z}{}^{\alpha} \to H^1(\mathbb{P}^1, \mathcal{O}(\langle -\check{\alpha}_i, \mu\rangle))$. The precise definition is given in Section 6, so let us just explain the definition for $G = SL(2)$ here. In this case $Z^{\lambda,\mu} \simeq Z^\alpha$ just classifies rank 2 vector bundles $\mathcal{F}$ on $\mathbb{P}^1$ with trivialized determinant together with two short exact sequences $0 \to \mathcal{L}_+ \to \mathcal{F} \to \mathcal{L}_+^{-1} \to 0$ and $0 \to \mathcal{L}_- \to \mathcal{F} \to \mathcal{L}_-^{-1} \to 0$ with $\deg \mathcal{L}_+ = -\lambda$, $\deg \mathcal{L}_- = -\mu$, where we identify the lattice Λ with $\mathbb{Z}$ in a natural way. In addition $\mathcal{F}$ is endowed with a trivialization at ∞, which is compatible with $\mathcal{L}_+$ and $\mathcal{L}_-$; in particular $\mathcal{L}_+$ and $\mathcal{L}_-$ also get a trivialization at ∞ which allows us to identify them canonically with $\mathcal{O}(-\lambda)$ and $\mathcal{O}(-\mu)$ (here we use a notation $\mathcal{O}(n)$, $n \in \mathbb{Z}$, for a line bundle on $\mathbb{P}^1$ trivialized at $\infty \in \mathbb{P}^1$). Hence the above short exact sequences define elements in $H^1(\mathbb{P}^1, \mathcal{O}(-2\lambda))$ and $H^1(\mathbb{P}^1, \mathcal{O}(-2\mu))$.

Let $\chi^\lambda_{i,+} : \overset{\circ}{Z}{}^{\lambda,\mu} \times H^0(\mathbb{P}^1, \mathcal{O}(\langle \lambda, \check{\alpha}_i\rangle - 2)) \to \mathbb{C}$ be the composition of $\mathfrak{E}^\alpha_{\lambda,+,i}$ and the natural pairing $H^0(\mathbb{P}^1, \mathcal{O}(\langle \lambda, \check{\alpha}_i\rangle - 2)) \times H^1(\mathbb{P}^1, \mathcal{O}(-\langle \lambda, \check{\alpha}_i\rangle)) \to \mathbb{C}$. Note that an element of $H^0(\mathbb{P}^1, \mathcal{O}(\langle \lambda, \check{\alpha}_i\rangle - 2)$ can be regarded as a polynomial K_i of one variable z of degree $\leq \langle \lambda, \check{\alpha}_i\rangle - 2$. Similarly, we let $\chi^\mu_{i,-} : \overset{\circ}{Z}{}^{\lambda,\mu} \times H^0(\mathbb{P}^1, \mathcal{O}(\langle \mu, \check{\alpha}_i\rangle - 2)) \to \mathbb{C}$ be the corresponding function (obtained by replacing $\mathfrak{E}^\alpha_{\lambda,+,i}$ with $\mathfrak{E}^\alpha_{\mu,-,i}$). We set $\mathfrak{E}^\alpha_{\lambda,+}$

to be the direct sum of all the $\mathfrak{E}^{\alpha}_{\lambda,+,i}$ and similarly for $\mathfrak{E}^{\alpha}_{\mu,-}$ (sometimes we shall drop the indices λ, μ and α when it does not lead to a confusion). Obviously the maps $\mathfrak{E}_+$ and $\mathfrak{E}_-$ are interchanged by the involution ι.

1.5. Coordinates on zastava

A system of étale birational coordinates on $\overset{\circ}{Z}{}^{\alpha}$ is introduced in Section 2.2. Let us recall the definition for $G = SL(2)$. In this case $\overset{\circ}{Z}{}^{\alpha}$ consists of all maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree α which send ∞ to 0. We can represent such a map by a rational function $\frac{R}{Q}$ where Q is a monic polynomial of degree α and R is a polynomial of degree $< \alpha$. Let $w_1, \dots, w_\alpha$ be the zeros of Q. Set $y_r = R(w_r)$. Then the functions $(y_1, \dots, y_\alpha, w_1, \dots, w_\alpha)$ form a system of étale birational coordinates on $\overset{\circ}{Z}{}^{\alpha}$.

For general G the definition of the above coordinates is quite similar. In this case given a point in $\overset{\circ}{Z}{}^{\alpha}$ we can define polynomials R_i, Q_i where i runs through the set of vertices of the Dynkin diagram of G and

(1) Q_i is a monic polynomial of degree $\langle \alpha, \check{\omega}_i \rangle$

(2) R_i is a polynomial of degree $< \langle \alpha, \check{\omega}_i \rangle$.

Hence, we can define (étale, birational) coordinates $(y_{i,r}, w_{i,r})$ where i is as above and $r = 1, \dots, \langle \alpha, \check{\omega}_i \rangle$. It will be convenient for us to use slightly modified coordinates $\mathbf{y}_{i,r} := y_{i,r} \prod_{j \neq i} Q_j^{\langle \alpha_j, \check{\alpha}_i \rangle / 2}(w_{i,r})$. Then the main result of this note is the following

Theorem 1.6.

(1) *The Poisson brackets of the modified coordinates (with respect to the symplectic structure defined in [6]) are as follows:*

$$\{w_{i,r}, w_{j,s}\} = 0, \ \{w_{i,r}, \mathbf{y}_{j,s}\} = \check{d}_i \delta_{ij} \delta_{rs} \mathbf{y}_{j,s}, \ \{\mathbf{y}_{i,r}, \mathbf{y}_{j,s}\} = 0.$$

(2) *(Recall that the boundary equation F_α is defined up to a multiplicative constant.) We have $F_\alpha = \prod_{i,r} \mathbf{y}_{i,r}^{d_i} = \prod_{i,r} y_{i,r}^{d_i} \prod_{j \neq i} Q_j^{\alpha_j \cdot \alpha_i / 2}(w_{i,r})$.*

(3) *Let us introduce yet another modified system of rational étale coordinates on $\overset{\circ}{Z}{}^{\alpha}$: we define*

$$\mathfrak{y}_{i,r} := \frac{y_{i,r}}{Q_i'(w_{i,r})}, \tag{1.1}$$

where Q_i' stands for the derivative of the polynomial $Q_i(z)$. Then we have

$$\mathfrak{f}_{\beta,\gamma}(w_{i,r}, \mathfrak{y}_{i,r})_{i \in I}^{1 \leq r \leq a_i} = \left((w_{i,r}, \mathfrak{y}_{i,r})_{i \in I}^{1 \leq r \leq b_i}, (w_{i,r}, \mathfrak{y}_{i,r})_{i \in I}^{b_i + 1 \leq r \leq a_i} \right). \tag{1.2}$$

(4) *The involution ι sends $(w_{i,r}, \mathbf{y}_{i,r})$ to $(w_{i,r}, \mathbf{y}_{i,r}^{-1})$.*

(5) *We have*

$$\chi^{\lambda}_{i,+}(\underline{w},\underline{y},\underline{z}) = \sum_{r=1}^{a_i} y_{i,r}^{-1} \frac{\prod_{j\neq i} Q_j^{-\langle \alpha_j,\check{\alpha}_i\rangle}(w_{i,r})}{Q_i'(w_{i,r})} K_i(w_{i,r}) = \sum_{r=1}^{a_i} \mathbf{y}_{i,r}^{-1} \frac{\prod_{j\neq i} Q_j^{-\langle \alpha_j,\check{\alpha}_i\rangle/2}(w_{i,r})}{Q_i'(w_{i,r})} K_i(w_{i,r}). \tag{1.3}$$

Similarly,

$$\chi^{\mu}_{i,-}(\underline{w},\underline{y},\underline{z}) = \sum_{r=1}^{a_i} \mathbf{y}_{i,r} \frac{\prod_{j\neq i} Q_j^{-\langle \alpha_j,\check{\alpha}_i\rangle/2}(w_{i,r})}{Q_i'(w_{i,r})} K_i(w_{i,r}). \tag{1.4}$$

Remark 1.7. The set of irreducible components Irr^{α} of the central factorization fiber $\pi_{\alpha}^{-1}(\alpha\cdot 0)\subset Z^{\alpha}$ is in a natural bijection with the weight α component of the Kashiwara crystal $\mathsf{B}_{\check{\mathfrak{g}}}(\infty)$, [2, Section 14]. The involution induced by ι on $\bigsqcup_{\alpha}\mathrm{Irr}^{\alpha}$ is nothing but the involution $*:\ \mathsf{B}_{\check{\mathfrak{g}}}(\infty)\to\mathsf{B}_{\check{\mathfrak{g}}}(\infty)$ of [12, 8.3].

1.8. Relation with the works of Gaiotto–Witten and Gaitsgory

We keep the notation from Theorem 1.6. Let us observe that a monic polynomial $K(z)$ of degree d is the same as a point in $\mathbb{A}^{(d)}$. Thus if all K_i are monic, together they form an point in $\mathbb{A}^{\lambda-2\rho}$. Thus we may regard $\chi^{\lambda}_{\pm}:=\sum_{i\in I}\chi^{\lambda}_{i,\pm}$ as functions on $\overset{\circ}{Z}{}^{\alpha}\times\mathbb{A}^{\lambda-2\rho}$.

Let $\Lambda=(\lambda_1,\dots,\lambda_n)$ be an unordered collection of dominant coweights whose sum is equal to $\lambda-2\rho$. Then Λ defines a locally closed subvariety $\overset{\circ}{\mathbb{A}}{}^{\Lambda}$ in $\mathbb{A}^{\lambda-2\rho}$ (namely, the moduli space of configurations of *distinct* colored points $\underline{z}=(z_1,\dots,z_n)$ so that the color of z_i is λ_i) and we denote by $\chi^{\Lambda}_{\pm}$ the restriction of $\chi^{\lambda}_{\pm}$ to $\overset{\circ}{Z}{}^{\alpha}\times\overset{\circ}{\mathbb{A}}{}^{\Lambda}$. We now define the (multivalued) superpotentials $\mathcal{W}^{\Lambda,\alpha}_{\pm}:\mathfrak{h}^{\vee}\times\overset{\circ}{Z}{}^{\alpha}\times\overset{\circ}{\mathbb{A}}{}^{\Lambda}\to\mathbb{A}^1$ by setting

$$\mathcal{W}^{\Lambda,\alpha}_{\pm} = \sum_{1\leq n\leq N}\langle\lambda_n,h^*\rangle z_n - \sum_{(i,r)}\langle\alpha_i,h^*\rangle w_{i,r} \pm \log F_{\alpha} + \chi^{\Lambda}_{\pm} + \sum_{1\leq m<n\leq N}\lambda_m\cdot\lambda_n\log(z_m-z_n). \tag{1.5}$$

Note that all the summands except the 3rd and the 4th are pulled back from $\overset{\circ}{\mathbb{A}}{}^{\alpha}\times\mathbb{A}^{\Lambda}$. Also, it is clear from the above definition that the exponential of $\mathcal{W}^{\Lambda}_{+}$ is well defined as a regular function on $\mathfrak{h}^{\vee}\times\overset{\circ}{Z}{}^{\alpha}\times\overset{\circ}{\mathbb{A}}{}^{\Lambda}$. In addition the involution ι transforms $\mathcal{W}^{\Lambda,\alpha}_{+}$ to $\mathcal{W}^{\Lambda,\alpha}_{-}$.

Let us now assume that $G=SL(2)$. Then it follows from Theorem 1.6 that the function $\mathcal{W}^{\Lambda,\alpha}_{-}$ is exactly the Gaiotto–Witten superpotential studied in [8]. We shall from now on

use this name for $\mathcal{W}_-^{\Lambda,\alpha}$ for any G. We see that the exponential of the Gaiotto–Witten super-potential is well-defined on $\mathfrak{h}^\vee \times \overset{\circ}{Z}{}^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda$ (this is not immediately clear from the coordinate description).

On the other hand, let $\varkappa \in \mathbb{C}$ be an irrational number. Then the work of Gaitsgory [9] easily implies the following result:

Theorem 1.9. *Let $M_-^{\varkappa,\alpha,\Lambda}$ denote the D-module on $\overset{\circ}{Z}{}^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda$ generated by the function $\exp(\varkappa \mathcal{W}_-^{\Lambda,\alpha})$. Let $\pi^{\alpha,\Lambda}: \mathfrak{h}^\vee \times \overset{\circ}{Z}{}^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda \to \mathfrak{h}^\vee \times \mathbb{A}^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda$ be the corresponding morphism. Then we have $\pi^{\alpha,\Lambda}_!(M_-^{\varkappa,\alpha,\Lambda}) = \pi^{\alpha,\Lambda}_*(M_-^{\varkappa,\alpha,\Lambda})$ and it is isomorphic to the minimal extension of the D-module on the open stratum generated by the function*

$$\prod_{1\le n\le N} \exp(\langle \lambda_n, \varkappa h^*\rangle z_n) \times \prod_{(i,r)} \exp(-\langle \alpha_i, \varkappa h^*\rangle w_{i,r}) \times$$
$$\times \prod_{(i,r)\neq(j,s)} (w_{i,r}-w_{j,s})^{\varkappa \alpha_i\cdot\alpha_j/2} \times \prod_{(i,r),1\le n\le N} (z_n - w_{i,r})^{-\varkappa\alpha_i\cdot\lambda_n} \times$$
$$\times \prod_{1\le m<n\le N} (z_m - z_n)^{\varkappa\lambda_m\cdot\lambda_n}.$$

1.10. Remark

The above theorem is essentially due to Gaiotto and Witten when restricted to the open stratum (in this case it is not difficult to deduce it from the coordinate description of the superpotential). Interpreting the superpotential in terms of (1.5) allows one to extend this statement to all of $\mathbb{A}^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda$ using the work of Gaitsgory. It would be interesting to find an interpretation of this refined statement in terms of the Landau–Ginzburg model studied by Gaiotto and Witten.

2. Recollections about zastava

2.1. Notations

G is an almost simple simplyconnected complex algebraic Lie group. We fix its Cartan and Borel subalgebras $T \subset B \subset G$ with the Lie algebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. The set of simple roots is denoted I; the simple roots (resp. coroots) are denoted $\check{\alpha}_i$ (resp. α_i), $i \in I$. We fix a Weyl group invariant symmetric bilinear form $?\cdot?$ on the Cartan Lie algebra $\mathfrak{h}$ such that the square length of a *short* coroot is $\alpha_i \cdot \alpha_i = 2$. This bilinear form gives rise to an isomorphism $\mathfrak{h}^\vee \xrightarrow{\sim} \mathfrak{h}$ so that the root lattice X generated by $\{\check{\alpha}_i\}_{i\in I}$ embeds into $\mathfrak{h}$. We then have $\check{\alpha}_i \cdot \check{\alpha}_i \in \{2, 1, \frac{2}{3}\}$, and $\alpha_i \cdot \alpha_i \in \{2, 4, 6\}$. We set $d_i = \frac{\alpha_i\cdot\alpha_i}{2}$. Let d be the ratio of the square lengths of the long and short coroots, so that $d \in \{1, 2, 3\}$. We set $\check{d}_i = d/d_i$. Then $\langle \alpha_i, \check{\alpha}_j\rangle = \frac{\alpha_i\cdot\alpha_j}{d_j} = d_i\check{\alpha}_i \cdot \check{\alpha}_j = d\frac{\check{\alpha}_i\cdot\check{\alpha}_j}{\check{d}_i}$.

For $\alpha = \sum_{i\in I} a_i\alpha_i,\ a_i \in \mathbb{N}$, we consider the corresponding zastava space Z^α (see e.g. [1]) with an open smooth subvariety $Z^\alpha \supset \overset{\circ}{Z}{}^\alpha$: the moduli space of degree α based maps from $C = \mathbb{P}^1$ to the flag variety $\mathcal{B} = G/B$ (also known as the moduli space of framed G-monopoles on $\mathbb{R}^3$ of topological charge α with the maximal symmetry breaking at infinity). The complementary boundary divisor is denoted $\partial Z^\alpha := Z^\alpha \setminus \overset{\circ}{Z}{}^\alpha$.

2.2. Coordinates on zastava

Let z be a coordinate on $C = \mathbb{P}^1$. We think of the zastava space Z^α in its Plücker embedding as of collections of degree $\langle\alpha,\check\lambda\rangle$ $V_{\check\lambda}$-valued polynomials (here $\check\lambda$ is a dominant weight, and $V_{\check\lambda}$ is the corresponding irreducible representation) such that the highest weight component is of the form $z^{\langle\alpha,\check\lambda\rangle} + \dots$ (the smaller powers of z), and all the other weight components are of degree strictly smaller than $\langle\alpha,\check\lambda\rangle$. In particular, if $\check\lambda = \check\omega_i$, a fundamental weight, then the highest weight component is denoted Q_i (a monic polynomial of degree $a_i = \langle\alpha,\check\omega_i\rangle$), and the prehighest weight ($= \check\omega_i - \check\alpha_i$) component is denoted R_i (a polynomial of degree $< a_i$). The polynomial Q_i is determined uniquely by the (unordered) set of its roots $w_{i,r},\ 1 \le r \le a_i$. The ramified cover $\varpi :\ \widehat{Z}^\alpha \to Z^\alpha$ is formed by all the orderings of the roots of all the polynomials $Q_i,\ i \in I$. We have regular functions $y_{i,r} := R_i(w_{i,r})$ on $\widehat{Z}^\alpha$. According to [6, Remark 2], on the open subset where all the roots $w_{i,r},\ i \in I$ are distinct (and ϖ is unramified), $\{w_{i,r}, y_{i,r}\}$ form a coordinate system (an open embedding into $\mathbb{A}^{\langle\alpha,2\check\rho\rangle}$).

2.3. A symplectic form and modified coordinates

The main result of [6] is a construction of a symplectic form on $\overset{\circ}{Z}{}^\alpha$ which extends as a Poisson structure to Z^α. According to [6, Proposition 2], the Poisson brackets of the coordinates of Section 2.2 are as follows: $\{w_{i,r}, w_{j,s}\} = 0$, $\{w_{i,r}, y_{j,s}\} = \check d_i\delta_{ij}\delta_{rs}y_{j,s}$, $\{y_{i,r}, y_{j,s}\} = d\check\alpha_i \cdot \check\alpha_j \frac{y_{i,r}y_{j,s}}{w_{i,r}-w_{j,s}}$ for $i \ne j$, and finally $\{y_{i,r}, y_{i,s}\} = 0$.

Following the private communications of S. Oblezin and L. Rybnikov, we consider the modified rational étale coordinates $\mathbf{y}_{i,r} := y_{i,r}\prod_{j\ne i} Q_j^{\langle\alpha_j,\check\alpha_i\rangle/2}(w_{i,r})$ (they are regular only on the open subset where all the roots $w_{i,r},\ i \in I$ are distinct).

Lemma 2.4. *The Poisson brackets of the modified coordinates are as follows:* $\{w_{i,r}, w_{j,s}\} = 0$, $\{w_{i,r}, \mathbf{y}_{j,s}\} = \check d_i\delta_{ij}\delta_{rs}\mathbf{y}_{j,s}$, $\{\mathbf{y}_{i,r}, \mathbf{y}_{j,s}\} = 0$.

Proof. Straightforward. □

Note that this is exactly the statement of Theorem 1.6(1).

Definition 2.5. We define the logarithmic coordinates $\mathbf{y}_{i,r} := \log \mathbf{y}_{i,r}$ on an appropriate $\mathbb{Z}^{|\alpha|}$-cover of the open subset of $\widehat{Z}^\alpha$ where all the roots $w_{i,r},\ i \in I$ are distinct, and $y_{i,r} \ne 0$.

2.6. A version of zastava

Given $\lambda, \mu \in X_*(T)$, we consider the moduli stack $\overset{\circ}{Z}{}^{\lambda,\mu}$ of the following data: (a) a G-bundle $\mathcal{F}_G$ on C trivialized at $\infty \in C$; (b) a reduction of $\mathcal{F}_G$ to a B-bundle (a B-structure on $\mathcal{F}_G$) such that the induced T-bundle has degree λ, and the fiber of the B-structure at $\infty \in C$ is $B \subset G$; (c) a reduction of $\mathcal{F}_G$ to a B_--bundle (a B_--structure on $\mathcal{F}_G$) such that the induced T-bundle has degree μ, and the fiber of the B_--structure at $\infty \in C$ is $B_- \subset G$.

According to [3, Section 2], $\overset{\circ}{Z}{}^{\lambda,\mu}$ is representable by a scheme. More precisely, $\alpha := \lambda - \mu$ is automatically a nonnegative combination of positive coroots, and $\overset{\circ}{Z}{}^{\lambda,\mu}$ is isomorphic to the zastava scheme $\overset{\circ}{Z}{}^{\alpha}$.

The Cartan involution of G interchanging B and B_- and acting on T as $t \mapsto t^{-1}$ induces an isomorphism $\iota: \overset{\circ}{Z}{}^{\lambda,\mu} \xrightarrow{\sim} \overset{\circ}{Z}{}^{-\mu,-\lambda}$. The composition $\overset{\circ}{Z}{}^{\alpha} \simeq \overset{\circ}{Z}{}^{\lambda,\mu} \xrightarrow{\iota} \overset{\circ}{Z}{}^{-\mu,-\lambda} \simeq \overset{\circ}{Z}{}^{\alpha}$ is a well defined involution $\iota: \overset{\circ}{Z}{}^{\alpha} \xrightarrow{\sim} \overset{\circ}{Z}{}^{\alpha}$ (independent of the choice of a presentation $\alpha = \lambda - \mu$: the independence is clear from the description of the identification $\overset{\circ}{Z}{}^{\lambda,\mu} \xrightarrow{\sim} \overset{\circ}{Z}{}^{\alpha}$ of [3, Section 2]).

3. Factorization (proof of Theorem 1.6(3))

3.1. Factorization in coordinates

Recall the fundamental *factorization* property of zastava spaces. For $\alpha = \beta + \gamma$ we have a natural morphism $a: \mathbb{A}^\beta \times \mathbb{A}^\gamma \to \mathbb{A}^\alpha$. An open subset $(\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \subset (\mathbb{A}^\beta \times \mathbb{A}^\gamma)$ is formed by the pairs (D_β, D_γ) of disjoint divisors $D_\beta, D_\gamma \in \mathbb{A}^1$. The *factorization* is a canonical isomorphism

$$\mathfrak{f}_{\beta,\gamma}: (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\alpha} Z^\alpha \xrightarrow{\sim} (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\beta \times \mathbb{A}^\gamma} (Z^\beta \times Z^\gamma).$$

We introduce yet another modified system of rational étale coordinates on Z^α: we define

$$\mathfrak{y}_{i,r} := \frac{y_{i,r}}{Q_i'(w_{i,r})}. \tag{3.1}$$

Let $\beta = \sum_{i\in I} b_i\alpha_i$, $\gamma = \sum_{i\in I} c_i\alpha_i$, so that $a_i = b_i + c_i$.

Proposition 3.2. $\mathfrak{f}_{\beta,\gamma}(w_{i,r}, \mathfrak{y}_{i,r})_{i\in I}^{1\le r\le a_i} = \left((w_{i,r}, \mathfrak{y}_{i,r})_{i\in I}^{1\le r\le b_i}, (w_{i,r}, \mathfrak{y}_{i,r})_{i\in I}^{b_i+1\le r\le a_i}\right)$.

Proof. We recall the construction of the factorization isomorphism. Let U stand for the unipotent radical of the Borel B, and let U_- be the unipotent radical of the opposite Borel (with the same Cartan torus T) B_-. Let $\overline{G/U}$ stand for the affinization of the base affine space. The quotient stack $U_-\backslash\overline{G/U}/T$ has an open dense point; and the

complement is a Cartier (Schubert) divisor D. Now Z^α is the moduli space of degree α maps $C \to U_-\backslash\overline{G/U}/T$ (i.e. such that the induced T-bundle on C has degree α) such that $\infty \in C$ goes to the complement of the Schubert divisor, see e.g. [3].

For $\phi \in Z^\alpha$, the pullback of the Schubert divisor $\phi^* D$ is nothing but $\pi_\alpha(\phi) \in (C\backslash\infty)^\alpha$. Given $\phi_1 \in Z^\beta$, $\phi_2 \in Z^\gamma$ with disjoint $\pi_\beta(\phi_1), \pi_\gamma(\phi_2)$, we construct the corresponding $\phi \in Z^\alpha$ as follows. Note that the disjointness condition guarantees that $\mathcal{U}_1 := C \setminus \phi_1^* D$ and $\mathcal{U}_2 := C \setminus \phi_2^* D$ cover C, and $\phi_1|_{\mathcal{U}_1 \cap \mathcal{U}_2} = \phi_2|_{\mathcal{U}_1 \cap \mathcal{U}_2}$ (the constant map to the point). So we define ϕ by gluing ϕ_1 and ϕ_2 over $\mathcal{U}_1 \cap \mathcal{U}_2$.

Now let us replace G, U, U_-, T by SL_2^i, U^i, U_-^i, T^i corresponding to the i-th root. Then $\overline{SL_2^i/U^i}$ is isomorphic to a 2-dimensional vector space V_i; the right action of T^i is isomorphic to the scalar action of $\mathbb{C}^*$; the left action of U_-^i is isomorphic to the one coming from the natural left action of SL_2^i. We have the canonical homomorphisms $\chi_i : U_- \twoheadrightarrow U_-^i$, and $\check{\alpha}_i : T \to T^i$. We also have a natural projection $\mathrm{pr}_i : \overline{G/U} \twoheadrightarrow \overline{SL_2^i/U^i}$. In effect, $\overline{G/U}$ in Plücker realization consists of collections of vectors in the irreducible G-modules. In particular, each collection contains a vector $v_{\check{\omega}_i} \in V_{\check{\omega}_i}$. So we set $\mathrm{pr}_i(v_{\check{\lambda}})_{\check{\lambda} \in X^*(T)^+} := \mathrm{pr}_i(v_{\check{\omega}_i}) \in V_{\check{\omega}_i}^{\mathrm{Rad}\, P_i} = V_i$. It is straightforward to check that pr_i is $\chi_i : U_- \twoheadrightarrow U_-^i$-equivariant, and $\check{\alpha}_i : T \to T^i$-equivariant. In other words, we have a morphism of stacks $\mathrm{pr}_i : U_-\backslash\overline{G/U}/T \to U_-^i\backslash\overline{SL_2^i/U^i}/T^i$, and the inverse image of the Schubert divisor $D_i \subset U_-^i\backslash\overline{SL_2^i/U^i}/T^i$ lies inside the Schubert divisor $D \subset U_-\backslash\overline{G/U}/T$ (in fact, this inverse image coincides with the corresponding irreducible component of D). Hence we obtain the same named projection $\mathrm{pr}_i : Z_{\mathfrak{g}}^\alpha \to Z_{\mathfrak{sl}_2^i}^{a_i}$, and the following diagram commutes:

$$\begin{array}{ccc} Z_{\mathfrak{g}}^\alpha & \xrightarrow{\mathrm{pr}_i} & Z_{\mathfrak{sl}_2^i}^{a_i} \\ {\scriptstyle \pi_\alpha}\downarrow & & \downarrow{\scriptstyle \pi_{a_i}} \\ \mathbb{A}^\alpha & \xrightarrow{\mathrm{pr}_i} & \mathbb{A}^{(a_i)} \end{array} \tag{3.2}$$

Moreover, the following diagram commutes as well:

$$\begin{array}{ccc} (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\alpha} Z_{\mathfrak{g}}^\alpha & \xrightarrow{\mathfrak{f}_{\beta,\gamma}} & (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\beta \times \mathbb{A}^\gamma} (Z_{\mathfrak{g}}^\beta \times Z_{\mathfrak{g}}^\gamma) \\ {\scriptstyle \mathrm{pr}_i}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_i} \\ (\mathbb{A}^{(b_i)} \times \mathbb{A}^{(c_i)})_{\mathrm{disj}} \times_{\mathbb{A}^{(a_i)}} Z_{\mathfrak{sl}_2^i}^{a_i} & \xrightarrow{\mathfrak{f}_{b_i,c_i}} & (\mathbb{A}^{(b_i)} \times \mathbb{A}^{(c_i)})_{\mathrm{disj}} \times_{\mathbb{A}^{(b_i)} \times \mathbb{A}^{(c_i)}} (Z_{\mathfrak{sl}_2^i}^{b_i} \times Z_{\mathfrak{sl}_2^i}^{c_i}) \end{array} \tag{3.3}$$

Hence the proposition is reduced to the case of $\mathfrak{g} = \mathfrak{sl}_2^i$ that will be dealt with in the next section.

3.3. Factorization for SL_2

In this section $G = SL_2^i$, and to unburden the notations we will write G, U, U_-, T for SL_2^i, U^i, U_-^i, T^i. We will use another point of view on the factorization. Namely, we

will think of $Z^a \ni \phi : \ C \to U_-\backslash\overline{G/U}/T$ as of a G-bundle $\mathcal{F}$ on C with a generalized B-structure, and a U_--structure transversal to the B-structure at $\infty \in C$. These generically transversal structures define a generic trivialization of $\mathcal{F}$, i.e. a point of the Beilinson–Drinfeld Grassmannian Gr_{BD}. Moreover, since any U_--bundle over C is trivial, $\mathcal{F}$ is trivial too, and its trivialization at $\infty \in C$ extends to a canonical global trivialization. Thus the above trivialization (coming from two transversal structures) may be viewed as a rational function $C \to G$; more precisely, as a rational function $f : \ C \to U_-$ (because of the reduction to U_-) sending $\infty \in C$ to the neutral element of U_-. Now recall that $G = SL_2^i$, and $U_- = \mathbb{G}_a = \mathbb{A}^1$. Then in the elementary terms f is nothing but $\frac{R_i}{Q_i}$.

Back to factorization, it arises from the factorization of the Beilinson–Drinfeld Grassmannian. Given G-bundles $\mathcal{F}_1, \mathcal{F}_2$ with trivializations σ_1, σ_2 defined on the open subsets $\mathcal{U}_1, \mathcal{U}_2 \subset C$ such that $\mathcal{U}_1 \cup \mathcal{U}_2 = C$ we construct a new bundle $\mathcal{F}$ with trivialization σ on $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ by gluing $\mathcal{F}_1|_{\mathcal{U}_2}$ and $\mathcal{F}_2|_{\mathcal{U}_1}$ over $\mathcal{U}$ where they are both trivialized.

Given $Z^b \ni \phi_1$ (resp. $Z^c \ni \phi_2$) corresponding to $(\mathcal{F}_1, \mathcal{U}_1, \sigma_1)$ (resp. $(\mathcal{F}_2, \mathcal{U}_2, \sigma_2)$) and $f_1 = \frac{R_1}{Q_1}$ (resp. $f_2 = \frac{R_2}{Q_2}$) we want to compute the result of gluing $Z^a \ni \phi$ corresponding to $(\mathcal{F}, \mathcal{U}, \sigma)$ and $f = \frac{R}{Q}$. Note that by the construction, the principal part of f at $C \setminus \mathcal{U}_1$ (resp. $C \setminus \mathcal{U}_2$) coincides with the principal part of f_1 at $C \setminus \mathcal{U}_1$ (resp. with that of f_2 at $C \setminus \mathcal{U}_2$). On the other hand, the rational function f of degree a vanishing at $\infty \in C$ is uniquely determined by its principal parts at $(C\setminus\mathcal{U}_1)\cup(C\setminus\mathcal{U}_2)$. We conclude $f = f_1 + f_2$. This is equivalent to the desired formula of (3.1) and Proposition 3.2 (since the principal part of f at $w_{i,r}$, i.e. the residue of fdz at $w_{i,r}$, is given by the formula (3.1)).

This completes the proof of the proposition. □

3.4. Another factorization

Recall from Section 3.3 that the factorization isomorphism

$$\mathfrak{f}_{\beta,\gamma} : \ (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\alpha} \overset{\circ}{Z}{}^\alpha \xrightarrow{\sim} (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\beta \times \mathbb{A}^\gamma} (\overset{\circ}{Z}{}^\beta \times \overset{\circ}{Z}{}^\gamma)$$

(Section 3.1) is induced by the embedding $\overset{\circ}{Z}{}^\alpha \hookrightarrow \mathrm{Gr}_{BD}(U_-) \hookrightarrow \mathrm{Gr}_{BD}(G)$. Given $\underline{x} = \sum_m \alpha_m \cdot x_m \in \mathbb{A}^\alpha$ the fiber $\pi_\alpha^{-1}(\underline{x})$ goes under this embedding to $\prod_m (\mathfrak{T}_0 \cap \mathfrak{S}_{\alpha_m}) \subset \mathrm{Gr}_{BD}(G)$. Here $\mathfrak{T}_0 \subset \mathrm{Gr}_{G,x_m}$ (resp. $\mathfrak{S}_{\alpha_m} \subset \mathrm{Gr}_{G,x_m}$) is the semiinfinite orbit $U_-(\mathcal{K}_{x_m}) \cdot 0$ (resp. $U(\mathcal{K}_{x_m}) \cdot \alpha_m$), and $\mathcal{K}_{x_m} \supset \mathcal{O}_{x_m}$ is the local field (resp. ring) around the point $x_m \in C$, and $\alpha_m \in \mathrm{Gr}_G$ is a T-fixed point. Note that $\mathfrak{T}_0 \subset \mathrm{Gr}_G$ is canonically isomorphic to $\mathrm{Gr}_{U_-} \subset \mathrm{Gr}_G$.

We also have a natural embedding $\overset{\circ}{Z}{}^{0,-\alpha} \hookrightarrow \mathrm{Gr}_{BD}(G)$ sending the fiber over $\underline{x}$ to $\prod_m (\mathfrak{T}_{-\alpha_m} \cap \mathfrak{S}_0) \subset \mathrm{Gr}_{BD}(G)$. Note that $\mathfrak{S}_0 \subset \mathrm{Gr}_G$ is canonically isomorphic to $\mathrm{Gr}_U \subset \mathrm{Gr}_G$. Under the identification $\overset{\circ}{Z}{}^\alpha \simeq \overset{\circ}{Z}{}^{0,-\alpha}$ the factorization of Gr_U induces the factorization

$$\mathfrak{f}^+_{\beta,\gamma} : \ (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\alpha} \overset{\circ}{Z}{}^\alpha \xrightarrow{\sim} (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\beta \times \mathbb{A}^\gamma} (\overset{\circ}{Z}{}^\beta \times \overset{\circ}{Z}{}^\gamma).$$

Recall the Cartan involution $\overset{\circ}{Z}{}^{\alpha} \simeq \overset{\circ}{Z}{}^{\alpha,0} \xrightarrow{\iota} \overset{\circ}{Z}{}^{0,-\alpha} \simeq \overset{\circ}{Z}{}^{\alpha}$ of Section 2.6. The following lemma is used in the next Section 4.

Lemma 3.5. *The following diagram commutes:*

$$\begin{array}{ccc} (\mathbb{A}^{\beta} \times \mathbb{A}^{\gamma})_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha}} \overset{\circ}{Z}{}^{\alpha} & \xrightarrow{\mathfrak{f}_{\beta,\gamma}} & (\mathbb{A}^{\beta} \times \mathbb{A}^{\gamma})_{\mathrm{disj}} \times_{\mathbb{A}^{\beta} \times \mathbb{A}^{\gamma}} (\overset{\circ}{Z}{}^{\beta} \times \overset{\circ}{Z}{}^{\gamma}) \\ \downarrow{\scriptstyle \mathrm{Id} \times \iota} & & \downarrow{\scriptstyle \mathrm{Id} \times \iota \times \iota} \\ (\mathbb{A}^{\beta} \times \mathbb{A}^{\gamma})_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha}} \overset{\circ}{Z}{}^{\alpha} & \xrightarrow{\mathfrak{f}^{+}_{\beta,\gamma}} & (\mathbb{A}^{\beta} \times \mathbb{A}^{\gamma})_{\mathrm{disj}} \times_{\mathbb{A}^{\beta} \times \mathbb{A}^{\gamma}} (\overset{\circ}{Z}{}^{\beta} \times \overset{\circ}{Z}{}^{\gamma}) \end{array} \tag{3.4}$$

Proof. Obvious. □

4. Cartan involution (proof of Theorem 1.6(4))

4.1. Involution in coordinates

Recall the modified coordinates $\mathbf{y}_{i,r}$ of Section 2.3, and the Cartan involution $\iota : \overset{\circ}{Z}{}^{\alpha} \to \overset{\circ}{Z}{}^{\alpha}$ of Section 2.6.

Proposition 4.2. *The involution* $\iota : \overset{\circ}{Z}{}^{\alpha} \to \overset{\circ}{Z}{}^{\alpha}$ *in coordinates acts as follows:* $\iota : (w_{i,r}, \mathbf{y}_{i,r}) \mapsto (w_{i,r}, \mathbf{y}_{i,r}^{-1})$ *(equivalently,* $(w_{i,r}, y_{i,r}) \mapsto (w_{i,r}, y_{i,r}^{-1} \prod_{j \neq i} Q_j^{-\langle \alpha_j, \check{\alpha}_i \rangle}(w_{i,r}))$*).*

Proof. Recall that a B-structure on $\mathcal{F}_G$ is encoded in a collection $\kappa_{\check{\lambda}} : \mathcal{L}_{\check{\lambda}} \hookrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$ of line subbundles satisfying the Plücker relations. Equivalently, we can consider a collection $\kappa^{*}_{-w_0\check{\lambda}} : \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G} \twoheadrightarrow {}'\mathcal{L}_{\check{\lambda}}$ of the quotient line bundles satisfying the Plücker relations (we have ${}'\mathcal{L}_{\check{\lambda}} = \mathcal{L}^{*}_{-w_0\check{\lambda}}$). Similarly, a B_--structure on $\mathcal{F}_G$ is encoded in a collection of line subbundles $\kappa^{-}_{\check{\lambda}} : \mathcal{L}^{-}_{\check{\lambda}} \hookrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$ or equivalently, a collection of the quotient line bundles $\kappa^{-*}_{-w_0\check{\lambda}} : \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G} \twoheadrightarrow {}'\mathcal{L}^{-}_{\check{\lambda}}$. Let P_i (resp. P_i^-) be the i-type subminimal parabolic subgroup containing B (resp. B_-). Then a B-structure on $\mathcal{F}_G$ induces a P_i-structure on $\mathcal{F}_G$ that gives rise to a 2-dimensional subbundle $\mathcal{V}_i \hookrightarrow \mathcal{V}^{\check{\omega}_i}_{\mathcal{F}_G}$ (associated to the 2-dimensional subspace of invariants $V^{\mathrm{Rad}\, P_i}_{\check{\omega}_i} \subset V_{\check{\omega}_i}$). Similarly, a B_--structure on $\mathcal{F}_G$ induces a P_i^--structure on $\mathcal{F}_G$ that gives rise to a 2-dimensional quotient bundle $\mathcal{V}^{\check{\omega}_i}_{\mathcal{F}_G} \twoheadrightarrow {}'\mathcal{V}^{-}_{i}$.

We have the natural embedding $\mathcal{L}_{\check{\omega}_i} \hookrightarrow \mathcal{V}_i$ and the natural projection ${}'\mathcal{V}^{-}_{i} \twoheadrightarrow {}'\mathcal{L}_{\check{\omega}_i}$. We define the line bundle $\mathcal{M}_i := \mathcal{V}_i / \mathcal{L}_{\check{\omega}_i}$ so that we have a short exact sequence $0 \to \mathcal{L}_{\check{\omega}_i} \to \mathcal{V}_i \to \mathcal{M}_i \to 0$. We define the line bundle ${}'\mathcal{M}_i$ as the kernel of ${}'\mathcal{V}^{-}_{i} \twoheadrightarrow {}'\mathcal{L}_{\check{\omega}_i}$ so that we have a short exact sequence $0 \to {}'\mathcal{M}_i \to {}'\mathcal{V}^{-}_{i} \to {}'\mathcal{L}_{\check{\omega}_i} \to 0$. We also consider the composition $\mathcal{L}_{\check{\omega}_i} \hookrightarrow \mathcal{V}^{\check{\omega}_i}_{\mathcal{F}_G} \twoheadrightarrow {}'\mathcal{V}^{-}_{i}$. We define $\mathcal{N}_i$ as the cokernel of this composed map. Note that generically over $\overset{\circ}{Z}{}^{\lambda,\mu}$ this composed map is an embedding of the line bundle $\mathcal{L}_{\check{\omega}_i}$, so that $\mathcal{N}_i$ is a line bundle as well, and we have a short exact sequence $0 \to \mathcal{L}_{\check{\omega}_i} \to {}'\mathcal{V}^{-}_{i} \to \mathcal{N}_i \to 0$. Given a general $(\mathcal{F}_G, \kappa_{\check{\lambda}}, \kappa^{-*}_{\check{\lambda}}) \in \overset{\circ}{Z}{}^{\lambda,\mu}$ such that $\mathcal{N}_i$ is a line bundle, we consider the following diagram:

$$\begin{array}{ccccc} \mathcal{L}_{\check{\omega}_i} & \longrightarrow & \mathcal{V}_i & \longrightarrow & \mathcal{M}_i \\ \| & & \downarrow & & \mathfrak{a}\downarrow \\ \mathcal{L}_{\check{\omega}_i} & \longrightarrow & {}'\mathcal{V}_i^- & \longrightarrow & \mathcal{N}_i \end{array} \tag{4.1}$$

Here the rows are the above short exact sequences, the middle vertical map is defined as the composition $\mathcal{V}_i \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\omega}_i} \twoheadrightarrow {}'\mathcal{V}_i^-$, and the right vertical map $\mathfrak{Q}$ is defined as follows. Note that the trivialization of $\mathcal{F}_G$ at $\infty \in C$ compatible with the B, B_--structures gives rise to the trivializations of $\mathcal{L}_{\check{\omega}_i}, \mathcal{M}_i, \mathcal{N}_i$ at $\infty \in C$. For degree reasons, $\mathcal{M}_i$ is canonically isomorphic to $\mathcal{O}_C(\langle \lambda, -\check{\omega} + \check{\alpha}_i \rangle)$, and $\mathcal{N}_i$ is canonically isomorphic to $\mathcal{O}_C(\langle \alpha, \check{\omega}_i \rangle + \langle \mu, -\check{\omega}_i + \check{\alpha}_i \rangle)$. Finally $\mathfrak{Q} \in \mathrm{Hom}(\mathcal{M}_i, \mathcal{N}_i) = \Gamma(C, \mathcal{O}(\langle \alpha, 2\check{\omega}_i - \check{\alpha}_i \rangle))$ is defined as $\prod_{j \neq i} Q_j^{-\langle \alpha_j, \check{\alpha}_i \rangle}$.

Lemma 4.3. *The diagram (4.1) commutes.*

Proof. Straightforward. □

Now given a general $(\mathcal{F}_G, \kappa_{\check{\lambda}}, \kappa_{\check{\lambda}}^{-*}) \in \overset{\circ}{Z}{}^{\lambda,\mu} \simeq \overset{\circ}{Z}{}^{\alpha}$, the coordinates $w_{i,r}$ are nothing but the points of C where the line subbundles $\mathcal{L}_{\check{\omega}_i} \hookrightarrow {}'\mathcal{V}_i^-$ and ${}'\mathcal{M}_i \hookrightarrow {}'\mathcal{V}_i^-$ are not transversal. The trivialization of $\mathcal{L}_{\check{\omega}_i}$, ${}'\mathcal{M}_i$ at $\infty \in C$ gives rise to a canonical trivialization of these line bundles restricted to $\mathbb{A}^1 = C \setminus \{\infty\}$. Hence at a nontransversality point $w_{i,r} \in \mathbb{A}^1$ we have two collinear vectors in the fiber ${}'\mathcal{V}_i^-|_{w_{i,r}}$, and the coordinate $y_{i,r}$ is nothing but their ratio.

Since the Cartan involution $\overset{\circ}{Z}{}^{\alpha} \simeq \overset{\circ}{Z}{}^{\lambda,\mu} \xrightarrow{\iota} \overset{\circ}{Z}{}^{-\mu,-\lambda} \simeq \overset{\circ}{Z}{}^{\alpha}$ takes $(\mathcal{F}_G, \kappa_{\check{\lambda}}, \kappa_{\check{\lambda}}^{-*})$ to $(\mathcal{F}_G, \kappa_{\check{\lambda}}^{-*}, \kappa_{\check{\lambda}})$, and interchanges the line bundles $\mathcal{L}_{\check{\omega}_i}, \mathcal{M}_i$ with and without primes, the proposition follows. □

5. An equation of the boundary (proof of Theorem 1.6(2))

5.1. An equation in modified coordinates

A regular function F_α on Z^α was constructed in [1, Section 4] such that the divisor of F_α is the boundary divisor ∂Z^α (the multiplicities of various irreducible components of the boundary are 1 or d), see [1, Lemma 4.2]. Recall the modified coordinates $\mathbf{y}_{i,r}$ of Section 2.3.

Theorem 5.2. *There is $c_\alpha \in \mathbb{C}^*$ such that $c_\alpha F_\alpha = \prod_{i,r} \mathbf{y}_{i,r}^{d_i} = \prod_{i,r} y_{i,r}^{d_i} \prod_{j \neq i} Q_j^{\alpha_j \cdot \alpha_i / 2}(w_{i,r})$.*

The rest of the section is devoted to the proof of the theorem.

5.3. Invertible functions on zastava

Let us denote the RHS of Theorem 5.2 by $\mathfrak{F}_\alpha$. If we can prove that Y_α is a regular function on Z^α invertible on $\overset{\circ}{Z}{}^\alpha$ with a correct order of vanishing at ∂Z^α, then $\mathfrak{F}_\alpha/F_\alpha$ is a rational function on Z^α regular and nonvanishing at $\overset{\circ}{Z}{}^\alpha$ and at the generic points of the irreducible components of the divisor ∂Z^α. Due to normality of Z^α [1, Corollary 2.10], the ratio $\mathfrak{F}_\alpha/F_\alpha$ is a regular invertible function on Z^α. Then according to the following lemma, the ratio $\mathfrak{F}_\alpha/F_\alpha$ is a nonzero constant c_α.

Lemma 5.4. $\Gamma(Z^\alpha, \mathcal{O}^*_{Z^\alpha}) = \mathbb{C}^*$.

Proof. Recall the factorization morphism $\pi_\alpha : Z^\alpha \to \mathbb{A}^\alpha$. Let $\Delta \subset \mathbb{A}^\alpha$ be the diagonal divisor. For an off-diagonal configuration $D \in \mathbb{A}^\alpha$ the fiber $\pi_\alpha^{-1}(D)$ is isomorphic to the $\langle \alpha, \check{\rho}\rangle$-dimensional affine space. Hence for $f \in \Gamma(Z^\alpha, \mathcal{O}^*_{Z^\alpha})$ the restriction of f to any off-diagonal fiber of π_α is constant. Hence $f = \bar{f} \circ \pi_\alpha$ for a certain (invertible) function $\bar{f}$ on $\mathbb{A}^\alpha$. Such $\bar{f}$ is necessarily constant. □

5.5. Codimension one: A_1 and $A_1 \times A_1$

The order of vanishing of $\mathfrak{F}_\alpha$ at the generic points of the irreducible components of ∂Z^α clearly coincides with that of F_α: see [1, Lemma 4.2]. We prove the regularity of $\mathfrak{F}_\alpha$. Due to normality of Z^α it suffices to check the regularity at the generic points of divisors $w_{i,r} = w_{j,s}$. By the factorization property, it suffices to consider the case $\alpha = \alpha_i + \alpha_j$. The case when $\alpha_i \cdot \alpha_j = 0$ being evident, we start with $i = j$. Then we can assume $\mathfrak{g} = \mathfrak{sl}_2$, so that $Z^2_{\mathfrak{sl}_2} \simeq \mathbb{A}^4 = \{(Q_i = z^2 + \mathsf{a}_1 z + \mathsf{a}_2, R_i = \mathsf{b}_0 z + \mathsf{b}_1)\}$. We have $Q_i = (z - w_1)(z - w_2)$, $R_i = (y_1(z - w_2) - y_2(z - w_1))/(w_1 - w_2)$, so that $y_1 y_2$ is the resultant $R(Q_i, R_i)$: a regular function on $Z^2_{\mathfrak{sl}_2}$, an equation of the boundary.

5.6. Codimension one: A_2

Next assume $i \neq j$, and $\alpha_i \cdot \alpha_j \neq 0$, and $d_i = d_j$. Then we can assume $\mathfrak{g} = \mathfrak{sl}_3$. Both fundamental representations $V_{\check{\omega}_i}, V_{\check{\omega}_j}$ of $\mathfrak{sl}_3$ are 3-dimensional. The zastava space $Z^{\alpha_i+\alpha_j}_{\mathfrak{sl}_3}$ is formed by the polynomials with values in $V_{\check{\omega}_i}, V_{\check{\omega}_j}$ of the form $(z - w_i, y_i, u)$, $(z - w_j, y_j, -u)$ such that $y_i y_j + (w_i - w_j)u = 0$. We have $\frac{y_i}{\sqrt{w_j - w_i}} \frac{y_j}{\sqrt{w_i - w_j}} = \sqrt{-1}u$: a regular function on $Z^{\alpha_i+\alpha_j}_{\mathfrak{sl}_3}$, an equation of the boundary.

5.7. Codimension one: B_2

Next assume $i \neq j$, and $\alpha_i \cdot \alpha_j \neq 0$, and $d_i = 2$, $d_j = 1$. Then we can assume $\mathfrak{g} = \mathfrak{sp}_4$. The fundamental representation $V_{\check{\omega}_i}$ (resp. $V_{\check{\omega}_j}$) is 4-dimensional (resp. 5-dimensional). The Plücker coordinates for $Z^{\alpha_i+\alpha_j}_{\mathfrak{sp}_4}$ are as follows:

$$\begin{array}{ccc} c_{03} & \boxed{b_{01}} & z+A_1 \\ \boxed{b_{03}} & c_{02} & \boxed{z+A_2} \\ c & \boxed{b_{02}} & b_{12} \end{array}.$$

Here the boxed coordinates are the weight components of $V_{\check\omega_i}$, and the remaining ones are the weight components of $V_{\check\omega_j}$. They are placed in the weight lattice of $Sp(4)$. The origin of the weird notation is in [7, Example 2.3.2]. The Plücker equations are as follows. First, we have the natural pairing $V_{\check\omega_j}\otimes V_{\check\omega_j}\to\mathbb{C}$ coming from $V_{\check\omega_j}\subset\Lambda^2 V_{\check\omega_i}$, and $\Lambda^4 V_{\check\omega_i}=\mathbb{C}$. The $V_{\check\omega_j}$-valued polynomial must be selforthogonal. The vanishing of the leading coefficient of the selfpairing is $c=0$. The vanishing of the degree zero coefficient of the selfpairing is $2c_{03}b_{12}-2c_{02}^2=0$. Second, we have the projection $V_{\check\omega_i}\otimes V_{\check\omega_j}\to\Lambda^3 V_{\check\omega_i}$ which must vanish on our polynomials. The vanishing of the leading coefficient of the projection is $c_{02}=b_{02}$, $c_{03}=b_{03}$. The vanishing of the degree zero coefficient of the projection is $-c_{02}b_{02}+cA_2+b_{12}b_{03}=0$, $A_1b_{02}-A_2c_{02}-b_{01}b_{12}=0$, $b_{02}c_{03}-b_{01}c-b_{03}c_{02}=0$, $A_2c_{03}-A_1b_{03}+b_{01}c_{02}=0$. Note that the former quadratic equation is equivalent to the *first* quadratic Plücker equation. All in all, we can take $A_1,A_2,b_{12},b_{01},b_{02},b_{03}$ as independent coordinates, and we will have three quadratic equations: $b_{02}(A_1-A_2)=b_{01}b_{12}$, $b_{03}(A_1-A_2)=b_{01}b_{02}$, $b_{02}^2=b_{12}b_{03}$ (noncomplete intersection of three quadrics). To compare with the coordinates of Section 2.2: $w_i=-A_1$, $w_j=-A_2$, $y_i=b_{01}$, $y_j=b_{12}$.

We have $(\frac{y_i}{\sqrt{w_j-w_i}})^2\frac{y_j}{w_i-w_j}=-\frac{b_{01}^2b_{12}}{(w_i-w_j)^2}=-\frac{b_{01}b_{02}}{w_i-w_j}=-b_{03}$: a regular function on $Z^{\alpha_i+\alpha_j}_{\mathfrak{sp}_4}$, an equation of the boundary.

5.8. Codimension one: G_2

Next assume $i\neq j$, and $\alpha_i\cdot\alpha_j\neq 0$, and $d_i=3$, $d_j=1$. Then $\mathfrak{g}$ is of type G_2. We have the regular functions w_i, w_j, y_i, y_j on $Z^{\alpha_i+\alpha_j}_{\mathfrak{g}}$. We have to show that $(\frac{y_i}{\sqrt{w_j-w_i}})^3\frac{y_j}{\sqrt{(w_i-w_j)^3}}=\sqrt{-1}\frac{y_i^3y_j}{(w_i-w_j)^3}$ is a regular function on $Z^{\alpha_i+\alpha_j}_{\mathfrak{g}}$. According to the formulas of Section 2.3, the Poisson bracket $\{y_i,y_j\}=-3\frac{y_iy_j}{w_i-w_j}$ is a regular function. Furthermore, $\{y_i,\frac{y_iy_j}{w_i-w_j}\}=y_i\frac{\{y_i,y_j\}}{w_i-w_j}+y_iy_j\{y_i,\frac{1}{w_i-w_j}\}=-3\frac{y_i^2y_j}{(w_i-w_j)^2}+\frac{y_i^2y_j}{(w_i-w_j)^2}=-2\frac{y_i^2y_j}{(w_i-w_j)^2}$ is a regular function. Finally, $\{y_i,\frac{y_i^2y_j}{(w_i-w_j)^2}\}=-\frac{y_i^3y_j}{(w_i-w_j)^3}$ is a regular function on $Z^{\alpha_i+\alpha_j}_{\mathfrak{g}}$.

5.9. Invertibility

The last thing to check is the invertibility of $\mathfrak{F}_\alpha$ on $\overset{\circ}{Z}{}^\alpha$. To this end recall the Cartan involution $\iota:\overset{\circ}{Z}{}^\alpha\to\overset{\circ}{Z}{}^\alpha$ of Section 2.6 and note that according to Proposition 4.2 we have $\mathfrak{F}_\alpha\circ\iota=\mathfrak{F}_\alpha^{-1}$.

The theorem is proved. □

Remark 5.10. Here is an alternative way to prove the invertibility of $\mathfrak{F}_\alpha$ on $\overset{\circ}{Z}{}^\alpha$: much shorter but less elementary. According to [1, Proposition 4.4], the weight of F_α with respect to the loop rotations in all the examples of Section 5.6, Section 5.7, Section 5.8 is equal to one. If $\mathfrak{F}_\alpha$ were not regular on Z^α, for certain $m > 0$ the function $(w_i - w_j)^m \mathfrak{F}_\alpha$ would be regular on Z^α and invertible on $\overset{\circ}{Z}{}^\alpha$. Thus, the ratio $(w_i - w_j)^m \mathfrak{F}_\alpha / F_\alpha$ would be invertible on Z^α (since the numerator and denominator have the same order of vanishing at the boundary ∂Z^α) and hence constant by Lemma 5.4. Since the weight of both $(w_i - w_j)$ and $\mathfrak{F}_\alpha$ with respect to the loop rotations is also equal to one, we conclude $m = 0$: a contradiction with our assumption $m > 0$. Hence we have proved the regularity of $\mathfrak{F}_\alpha$ on Z^α and simultaneously the invertibility of $\mathfrak{F}_\alpha$ on $\overset{\circ}{Z}{}^\alpha$. The general case is reduced to the above examples by factorization.

This completes the proof of Theorem 5.2. □

6. An Ext calculation (proof of Theorem 1.6(5))

6.1. PGL_2-bundles

A PGL_2-bundle with a flag on $C = \mathbb{P}^1$ can be viewed as a short exact sequence $0 \to \mathcal{L} \to \mathcal{V} \to \mathcal{M} \to 0$ ($\mathcal{L}$ and $\mathcal{M}$ are the line bundles, and $\mathcal{V}$ is a rank two vector bundle) modulo the twistings by the line bundles. In particular, the line bundle $\mathcal{M}^{-1} \otimes \mathcal{L} = \mathcal{H}om(\mathcal{M}, \mathcal{L})$ is well defined: this is nothing but the induction of the Borel bundle to the Cartan bundle. We consider the moduli stack $\mathcal{F}_2$ of PGL_2-bundles with a flag on C equipped with a trivialization at $\infty \in C$ of the corresponding line bundle $\mathcal{M}^{-1} \otimes \mathcal{L}$. The connected components of $\mathcal{F}_2$ are numbered by the integers $\deg \mathcal{M} - \deg \mathcal{L}$. On a connected component $\mathcal{F}_2^n$, we have a canonical isomorphism $\mathcal{H}om(\mathcal{M}, \mathcal{L}) = \mathcal{O}_C(-n)$, and so $\mathrm{Ext}^1(\mathcal{M}, \mathcal{L}) = H^1(C, \mathcal{O}(-n)) = H^0(C, \mathcal{O}(n-2))^\vee$. Thus we have a morphism $E : \mathcal{F}_2^n \to H^0(C, \mathcal{O}(n-2))^\vee$.

6.2. A map to Ext

By Plücker, we may view a B-structure on $\mathcal{F}_G$ as a collection of line subbundles $\mathcal{L}_{\check\lambda} \subset \mathcal{V}^{\check\lambda}_{\mathcal{F}_G}$ satisfying the Plücker relations (here $\check\lambda$ runs through the cone of dominant weights of G, and $\mathcal{V}^{\check\lambda}_{\mathcal{F}_G}$ is the vector bundle associated to the irreducible G-module $V_{\check\lambda}$ with the highest weight $\check\lambda$). For a B-structure coming from a point of $\overset{\circ}{Z}{}^{\lambda,\mu}$ we have $\deg \mathcal{L}_{\check\lambda} = -\langle \lambda, \check\lambda \rangle$. The trivialization at $\infty \in C$ extends to a canonical isomorphism $\mathcal{L}_{\check\lambda} = \mathcal{O}_C(-\langle \lambda, \check\lambda \rangle)$. Since the assignment $\check\lambda \mapsto \mathcal{L}_{\check\lambda}$ is multiplicative in $\check\lambda$: $\mathcal{L}_{\check\mu + \check\nu} = \mathcal{L}_{\check\mu} \otimes \mathcal{L}_{\check\nu}$, we can extend the notion of $\mathcal{L}_{\check\lambda}$ for arbitrary weights $\check\gamma \in X^*(T)$. We have a canonical isomorphism $\mathcal{L}_{\check\gamma} = \mathcal{O}_C(-\langle \lambda, \check\gamma \rangle)$.

For $i \in I$ we have a morphism $\mathfrak{P}_i : \overset{\circ}{Z}{}^{\lambda,\mu} \to \mathcal{F}_2^{\langle \lambda, \check\alpha_i \rangle}$ defined as follows. Let $G \supset P_i \supset B$ be the i-type subminimal parabolic subgroup. We have the projection $P_i \to L_i$ to the

corresponding Levi, also we have the projection $L_i \to PGL_2$, and so the composed projection $B \hookrightarrow P_i \twoheadrightarrow L_i \twoheadrightarrow PGL_2$. Given a B-reduction of a G-bundle on $\mathcal{F}_G$ trivialized at $\infty \in C$, we consider the induced PGL_2-bundle. It comes equipped with a flag and trivialization at $\infty \in C$. This is our desired $\mathfrak{P}_i$ applied to $\mathcal{F}_G$ with the B-structure (note that we have not used the B_--structure).

If we specialize to the case $(\lambda, \mu) = (\alpha, 0)$, the assumption $\mu = 0$ guarantees that the B_--structure further reduces to the U_--structure (where U_- is the radical of B_-). Since any U_--bundle on $C = \mathbb{P}^1$ is automatically trivial, the ambient G-bundle $\mathcal{F}_G$ is trivial too, and its trivialization at $\infty \in C$ extends to a canonical trivialization over C. Thus we arrive at an identification $\overset{\circ}{Z}{}^{\alpha,0} \simeq \overset{\circ}{Z}{}^{\alpha}$ with the usual zastava space, i.e. the moduli space of degree α based maps from (C, ∞) to $(\mathcal{B}, B)$. For $\phi \in \overset{\circ}{Z}{}^{\alpha}$ let us describe explicitly two particular representatives (2-dimensional bundles with a flag) of $\mathfrak{P}_i(\phi)$.

We have a projection $p_i : \mathcal{B} = G/B \to G/P_i =: \mathcal{B}_i$. We define $\mathcal{B}^i := \mathcal{B} \times_{\mathcal{B}_i} \mathcal{B}$, and $p^i : \mathcal{B}^i \to \mathcal{B}$ (the first projection). By construction, p^i is a $\mathbb{P}^1$-bundle over $\mathcal{B}$ equipped with a canonical (diagonal) section $\Delta^i : \mathcal{B} \to \mathcal{B}^i$. We define $\mathcal{V}'_i := p^i_* \mathcal{O}_{\mathcal{B}^i}(\Delta^i) \supset \mathcal{L}'_i := p^i_* \mathcal{O}_{\mathcal{B}^i} = \mathcal{O}_{\mathcal{B}}$. Thus we get a short exact sequence $0 \to \mathcal{L}'_i \to \mathcal{V}'_i \to \mathcal{M}'_i \to 0$ trivialized at $B \in \mathcal{B}$; here $\mathcal{M}'_i = \mathcal{O}_{\mathcal{B}}(\check{\alpha}_i)$. Finally, $\mathfrak{P}_i(\phi) = \{0 \to \phi^* \mathcal{L}'_i \to \phi^* \mathcal{V}'_i \to \phi^* \mathcal{M}'_i \to 0\}$.

Alternatively, let $\mathcal{V}_{\check{\omega}_i}$ be the trivial vector bundle over $\mathcal{B}$ associated with the fundamental G-module $V_{\check{\omega}_i}$. It has a line subbundle $\mathcal{L}_i$: the fiber $\mathcal{L}_i|_{B'}$ is the B'-highest line $V^{\mathrm{Rad}\, B'}_{\check{\omega}_i}$. If P'_i is the i-type subminimal parabolic containing B', then the invariants $V^{\mathrm{Rad}\, P'_i}_{\check{\omega}_i}$ are 2-dimensional (the highest and next highest lines), and as B' varies in $\mathcal{B}$, we obtain a 2-dimensional subbundle $\mathcal{V}_i \subset \mathcal{V}_{\check{\omega}_i}$. Thus we have a short exact sequence $0 \to \mathcal{L}_i \to \mathcal{V}_i \to \mathcal{M}_i \to 0$ trivialized at $B \in \mathcal{B}$; here $\mathcal{L}_i = \mathcal{O}_{\mathcal{B}}(-\check{\omega}_i)$, and $\mathcal{M}_i = \mathcal{O}_{\mathcal{B}}(-\check{\omega}_i + \check{\alpha}_i)$. Again, we have $\mathfrak{P}_i(\phi) = \{0 \to \phi^* \mathcal{L}_i \to \phi^* \mathcal{V}_i \to \phi^* \mathcal{M}_i \to 0\}$.

Finally, we define $\mathfrak{E}_i : \overset{\circ}{Z}{}^{\lambda,\mu} \to \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{L}_{\check{\alpha}_i}) = \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C(\langle \mu - \lambda, \check{\alpha}_i \rangle))$ as the composition $\overset{\circ}{Z}{}^{\lambda,\mu} \simeq \overset{\circ}{Z}{}^{\lambda-\mu,0} \xrightarrow{\mathfrak{P}_i} \mathcal{F}_2^{\langle \lambda-\mu, \check{\alpha}_i \rangle} \xrightarrow{E} \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C(\langle \mu - \lambda, \check{\alpha}_i \rangle))$.

6.3. Recollections of [9]

We recall some of the constructions of [9] in the particular case of a curve of genus 0 (projective line C). In this case the canonical bundle ω_C is isomorphic to $\mathcal{O}_C(-2)$, and we choose a square root $\omega_C^{1/2} \simeq \mathcal{O}_C(-1)$. Let $\Lambda = (\lambda_1, \dots, \lambda_N)$ be an ordered collection of dominant coweights. Let $\overset{\circ}{\mathbb{A}}{}^{\Lambda}$ be the moduli space of ordered configurations of distinct points $(z_1, \dots, z_N \in \mathbb{A}^1)$. Let $\overset{\circ}{\mathbb{A}}{}^{\alpha,\Lambda} \subset \mathbb{A}^{\alpha} \times \overset{\circ}{\mathbb{A}}{}^{\Lambda}$ be the open subspace formed by the configurations of pairwise distinct points. For $i \in I$ and $\underline{z} \in \overset{\circ}{\mathbb{A}}{}^{\Lambda}$ we define a monic polynomial $K_i(z) := \prod_{1 \le n \le N} (z - z_n)^{\langle \lambda_n, \check{\alpha}_i \rangle}$.

Given a point $\underline{z} \in \overset{\circ}{\mathbb{A}}{}^{\Lambda}$, we consider a moduli stack $\mathfrak{W}_{\underline{z},\Lambda}$ classifying the following data: (a) A G-bundle $\mathcal{F}_G$ on C; (b) For each dominant weight $\check{\lambda}$ a nonzero map $\kappa^{\check{\lambda}} : \omega_C^{\langle \rho, \check{\lambda} \rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$ having the poles of order exactly λ_i at z_i, and regular nonvanishing at $C \setminus \{\underline{z}\}$. Here

$\mathcal{V}^{\check\lambda}_{\mathcal{F}_G}$ is the vector bundle associated to $\mathcal{F}_G$ and the irreducible G-module $V^{\check\lambda}$, and $\omega_C^{\langle\rho,\check\lambda\rangle}$ stands for $(\omega_C^{1/2})^{\otimes\langle 2\rho,\check\lambda\rangle}$. The collection of maps $\kappa^{\check\lambda}$ must satisfy the Plücker relations (cf. [9, 2.1, 2.6]).

Alternatively, note that $\kappa^{\check\lambda}|_{\omega_C^{\langle\rho,\check\lambda\rangle}(-\sum_{n=1}^N\langle\lambda_n,\check\lambda\rangle\cdot z_n)}$ is a regular embedding, and the image is a line subbundle $\mathcal{L}_{\check\lambda}\subset\mathcal{V}^{\check\lambda}_{\mathcal{F}_G}$. So $\mathfrak{W}_{\underline{z},\Lambda}$ is the moduli stack of the collections of line subbundles $\mathcal{L}_{\check\lambda}\subset\mathcal{V}^{\check\lambda}_{\mathcal{F}_G}$ satisfying the Plücker relations plus the identifications $\mathcal{L}_{\check\lambda}=\omega_C^{\langle\rho,\check\lambda\rangle}(-\sum_{n=1}^N\langle\lambda_n,\check\lambda\rangle\cdot z_n)$.

Since the assignment $\check\lambda\mapsto\mathcal{L}_{\check\lambda}$ is multiplicative in $\check\lambda$: $\mathcal{L}_{\check\mu+\check\nu}=\mathcal{L}_{\check\mu}\otimes\mathcal{L}_{\check\nu}$, we can extend the notion of $\mathcal{L}_{\check\lambda}$ for arbitrary weights $\check\gamma\in X^*(T)$. The construction of Section 6.2 defines a morphism $\mathfrak{E}_i:\ \mathfrak{W}_{\underline{z},\Lambda}\to\mathrm{Ext}^1(\mathcal{O}_C,\mathcal{L}_{\check\alpha_i})$. The canonical embedding $\mathcal{L}_{\check\alpha_i}\hookrightarrow\omega_C^{\langle\rho,\check\alpha_i\rangle}=\omega_C$ gives rise to the projection $\mathrm{Ext}^1(\mathcal{O}_C,\mathcal{L}_{\check\alpha_i})\to\mathrm{Ext}^1(\mathcal{O}_C,\omega_C)=\mathbb{A}^1$. Composing it with $\mathfrak{E}_i$ we obtain a function $\chi_i:\ \mathfrak{W}_{\underline{z},\Lambda}\to\mathbb{A}^1$.

Following [9, 4.3] we consider the moduli stack $\overset{\circ}{\mathcal{Z}}{}^\alpha\to\mathfrak{W}_{\underline{z},\Lambda}$ classifying the same data as $\mathfrak{W}_{\underline{z},\Lambda}$ plus (a) a trivialization of $\mathcal{F}_G$ at $\infty\in C$ such that the B-structure (given by the collection $\{\kappa^{\check\lambda}\}$) at ∞ coincides with $B\subset G$; (b) an additional B_--structure on $\mathcal{F}_G$ of degree $-2\rho-\alpha$ equal at $\infty\in C$ to $B_-\subset G$. By an abuse of notation we preserve the notation $\chi_i:\ \overset{\circ}{\mathcal{Z}}{}^\alpha\to\mathbb{A}^1$ for the composition of $\chi_i:\ \mathfrak{W}_{\underline{z},\Lambda}\to\mathbb{A}^1$ and the projection $\overset{\circ}{\mathcal{Z}}{}^\alpha\to\mathfrak{W}_{\underline{z},\Lambda}$. According to [9, 4.5, 4.6], the stack $\overset{\circ}{\mathcal{Z}}{}^\alpha$ is actually a scheme; moreover, we have a canonical isomorphism $\overset{\circ}{\mathcal{Z}}{}^\alpha=\overset{\circ}{Z}{}^{-2\rho,-2\rho-\alpha}=\overset{\circ}{Z}{}^\alpha$. Thus we obtain a function $\chi_i:\ \overset{\circ}{Z}{}^\alpha\to\mathbb{A}^1$. If we allow $\underline{z}$ to vary in $\overset{\circ}{\mathbb{A}}{}^\Lambda$, we obtain the same named function χ_i on $\overset{\circ}{Z}{}^\alpha\times\overset{\circ}{\mathbb{A}}{}^\Lambda$.

Theorem 6.4. *The function χ_i on $\overset{\circ}{Z}{}^\alpha\times\overset{\circ}{\mathbb{A}}{}^\Lambda$ in the coordinates $(\underline{w},\underline{y},\underline{z})$ is given by*

$$\chi_i(\underline{w},\underline{y},\underline{z})=\sum_{r=1}^{a_i}y_{i,r}^{-1}\frac{\prod_{j\neq i}Q_j^{-\langle\alpha_j,\check\alpha_i\rangle}(w_{i,r})}{Q_i'(w_{i,r})}K_i(w_{i,r})$$
$$=\sum_{r=1}^{a_i}\mathbf{y}_{i,r}^{-1}\frac{\prod_{j\neq i}Q_j^{-\langle\alpha_j,\check\alpha_i\rangle/2}(w_{i,r})}{Q_i'(w_{i,r})}K_i(w_{i,r}).$$

Proof. Recall the involution $\iota:\ \overset{\circ}{Z}{}^\alpha\xrightarrow{\sim}\overset{\circ}{Z}{}^\alpha$ of Section 2.6. We have to prove that $\chi_i\circ\iota(\underline{w},\underline{y},\underline{z})=\sum_{r=1}^{a_i}y_{i,r}K_i(w_{i,r})/Q_i'(w_{i,r})$. Recall also the modified coordinates $\mathfrak{y}_{i,r}:=\frac{y_{i,r}}{Q_i'(w_{i,r})}$ of (3.1). Thus we have to prove

$$\chi_i\circ\iota(\underline{w},\underline{\mathfrak{y}},\underline{z})=\sum_{r=1}^{a_i}\mathfrak{y}_{i,r}K_i(w_{i,r}).\tag{6.1}$$

Recall the map $\mathfrak{E}_i:\ \overset{\circ}{Z}{}^{\lambda,\mu}\to\mathrm{Ext}^1(\mathcal{O}_C,\mathcal{L}_{\check\alpha_i})$ of Section 6.2. We have to prove its factorization property, i.e. the commutativity of the following diagram:

$$\begin{array}{ccc} (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\alpha} \overset{\circ}{Z}{}^\alpha & \xrightarrow{\mathfrak{f}^+_{\beta,\gamma}} & (\mathbb{A}^\beta \times \mathbb{A}^\gamma)_{\mathrm{disj}} \times_{\mathbb{A}^\beta \times \mathbb{A}^\gamma} (\overset{\circ}{Z}{}^\beta \times \overset{\circ}{Z}{}^\gamma) \\ \mathfrak{E}_i \big\downarrow & & \mathfrak{E}_i \times \mathfrak{E}_i \big\downarrow \\ \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{L}_{\check{\alpha}_i}) & \xleftarrow{+} & \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{L}_{\check{\alpha}_i}) \times \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{L}_{\check{\alpha}_i}) \end{array} \tag{6.2}$$

Unraveling the definition of $\mathfrak{E}_i$ and using the compatibility of factorizations (3.3) we reduce the problem to $G = SL_2^i$. This problem is formulated as follows. For a point $x \in C$ and a line bundle $\mathcal{E}$ on C we have a natural map $\varphi_x : (\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{K}_x)/(\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{O}_x) \to H^1(C, \mathcal{E})$ arising from the identification $(\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{K}_x)/(\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{O}_x) = j_*\mathcal{E}/\mathcal{E}$ and the boundary map in the long exact cohomology sequence coming from $0 \to \mathcal{E} \to j_*\mathcal{E} \to j_*\mathcal{E}/\mathcal{E} \to 0$ (here j is the open embedding $C \setminus \{x\} \hookrightarrow C$). Given a short exact sequence $0 \to \mathcal{L} \to \mathcal{V} \to \mathcal{M} \to 0$ as in Section 6.1, its local splittings form a torsor $\mathcal{T}$ over $\mathcal{E} = \mathcal{H}om(\mathcal{M}, \mathcal{L})$ with the class $e(\mathcal{T}) \in H^1(C, \mathcal{E})$. Given a generic splitting of this exact sequence we obtain a generic section s of $\mathcal{T}$. Choosing a local splitting around x we obtain $s_x \in \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{K}_x$ whose principal part in $(\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{K}_x)/(\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{O}_x)$ is well defined, i.e. independent of the choice of a local splitting. Then clearly $e(\mathcal{T}) = \sum_{x \in C} \varphi_x(s_x)$.

This completes the proof of Theorem 6.4. □

6.5. *More recollections (proof of Theorem 1.9)*

Recall the sequence of morphisms $\overset{\circ}{Z}{}^\alpha \simeq \overset{\circ}{Z}{}^{-2\rho,-2\rho-\alpha} = \overset{\circ}{\mathcal{Z}}{}^\alpha \to \mathfrak{W}_{\underline{z},\Lambda} \to \mathrm{Bun}_G(C)$ of Section 6.3. The line bundle $\mathcal{P}$ on $\overset{\circ}{Z}{}^\alpha$ is defined as the inverse image of the determinant line bundle on $\mathrm{Bun}_G(C)$, cf. [9, 2.2]. In fact, $\mathcal{P}$ is the restriction of the same named line bundle on Z^α with the canonical section F_α, see [1, 4.9]. Hence F_α gives rise to a canonical trivialization of $\mathcal{P}$ on $\overset{\circ}{Z}{}^\alpha$. Given a level $\varkappa \in \mathbb{C}$, Gaitsgory constructs a certain $\mathcal{P}^\varkappa$-twisted D-module $\mathcal{F}^\varkappa_{\underline{z},\Lambda}$ on $\overset{\circ}{Z}{}^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda$ (as a lift from $\mathfrak{W}_{\underline{z},\Lambda} \times \overset{\circ}{\mathbb{A}}{}^\Lambda$) [9, 2.7]. It is smooth of rank 1 on $\overset{\circ}{Z}{}^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda$ but has irregular singularities at $\partial Z^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda$. In case $\varkappa$ is irrational, $\mathcal{F}^\varkappa_{\underline{z},\Lambda}$ is clean. The trivialization of $\mathcal{P}$ on $\overset{\circ}{Z}{}^\alpha$ gives rise to the identification of $\mathcal{P}^\varkappa$-twisted D-modules with the usual D-modules, and then the corresponding D-module $\mathcal{F}^{\varkappa,\mathrm{triv}}_{\underline{z},\Lambda}$ on $\overset{\circ}{Z}{}^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda$ is generated by the function $F^\varkappa_\alpha \cdot \exp(\chi_i) \cdot \prod_{1 \le m < n \le N} (z_m - z_n)^{\varkappa \lambda_m \cdot \lambda_n}$.

According to [9, Lemma 4.9] there is a canonical isomorphism $\mathcal{P} = \pi_\alpha^* \mathcal{P}_{\mathbb{A}^\alpha}$ for a certain line bundle $\mathcal{P}_{\mathbb{A}^\alpha}$ on $\mathbb{A}^\alpha$ [9, 3.2]. For irrational $\varkappa$ [9, Theorem 6.2] identifies $\pi_{\alpha!}\mathcal{F}^\varkappa_{\underline{z},\Lambda} \xrightarrow{\sim} \pi_{\alpha *}\mathcal{F}^\varkappa_{\underline{z},\Lambda}$ as the minimal extension $\mathcal{L}^\varkappa_{\underline{z},\Lambda}$ of a smooth rank 1 $\mathcal{P}^\varkappa_{\mathbb{A}^\alpha}$-twisted D-module from the open diagonal stratum of $\mathbb{A}^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda$. The trivialization [9, 3.12] of $\mathcal{P}_{\mathbb{A}^\alpha}$ on $\mathbb{A}^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda$ gives rise to the identification of $\mathcal{P}^\varkappa_{\mathbb{A}^\alpha}$-twisted D-modules with the usual D-modules, and then the corresponding D-module $\mathcal{L}^{\varkappa,\mathrm{triv}}_{\underline{z},\Lambda}$ is the minimal extension of the D-module on the open stratum generated by the function

$$\prod_{(i,r) \ne (j,s)} (w_{i,r} - w_{j,s})^{\varkappa \alpha_i \cdot \alpha_j/2} \times \prod_{(i,r), 1 \le n \le N} (z_n - w_{i,r})^{-\varkappa \alpha_i \cdot \lambda_n} \times \prod_{1 \le m < n \le N} (z_m - z_n)^{\varkappa \lambda_m \cdot \lambda_n}.$$

More generally, we consider the D-module $M^{\varkappa,\alpha,\Lambda}$ on $\mathfrak{h}^\vee \times \overset{\circ}{Z}{}^\alpha \times \overset{\circ}{\mathbb{A}}{}^\Lambda$ generated by the function $\prod_{1\le n\le N}\exp(\langle\lambda_n,\varkappa h^*\rangle z_n)\cdot\prod_{(i,r)}\exp(-\langle\alpha_i,\varkappa h^*\rangle w_{i,r})\cdot F_\alpha^\varkappa\cdot\exp(\chi_i)\cdot\prod_{1\le m<n\le N}(z_m-z_n)^{\varkappa\lambda_m\cdot\lambda_n}$. Then for irrational $\varkappa$ the direct image $\pi_{\alpha!}M^{\varkappa,\alpha,\Lambda}\xrightarrow{\sim}\pi_{\alpha*}M^{\varkappa,\alpha,\Lambda}$ is isomorphic to the minimal extension $\mathcal{M}^{\varkappa,\mathrm{triv}}_{\mathfrak{h}^\vee,\underline{z},\Lambda}$ of the D-module on the open diagonal stratum of $\mathfrak{h}^\vee\times\mathbb{A}^\alpha\times\overset{\circ}{\mathbb{A}}{}^\Lambda$ generated by the function

$$\prod_{1\le n\le N}\exp(\langle\lambda_n,\varkappa h^*\rangle z_n)\times\prod_{(i,r)}\exp(-\langle\alpha_i,\varkappa h^*\rangle w_{i,r})\times$$
$$\times\prod_{(i,r)\ne(j,s)}(w_{i,r}-w_{j,s})^{\varkappa\alpha_i\cdot\alpha_j/2}\times\prod_{(i,r),1\le n\le N}(z_n-w_{i,r})^{-\varkappa\alpha_i\cdot\lambda_n}\times$$
$$\times\prod_{1\le m<n\le N}(z_m-z_n)^{\varkappa\lambda_m\cdot\lambda_n}.$$

In effect, the isomorphism $\pi_{\alpha!}M^{\varkappa,\alpha,\Lambda}\xrightarrow{\sim}\pi_{\alpha*}M^{\varkappa,\alpha,\Lambda}\xrightarrow{\sim}\mathcal{M}^{\varkappa,\mathrm{triv}}_{\mathfrak{h}^\vee,\underline{z},\Lambda}$ follows from the isomorphism $\pi_{\alpha!}\mathcal{F}^{\varkappa,\mathrm{triv}}_{\underline{z},\Lambda}\xrightarrow{\sim}\pi_{\alpha*}\mathcal{F}^{\varkappa,\mathrm{triv}}_{\underline{z},\Lambda}\xrightarrow{\sim}\mathcal{L}^{\varkappa,\mathrm{triv}}_{\underline{z},\Lambda}$ and the projection formula. The latter isomorphism is proved in [9] for any *fixed* value of $\underline{z}$. To prove it for variable $\underline{z}$ it remains to identify the monodromy of the one-dimensional local system $\pi_{\alpha!}M^{\varkappa,\alpha,\Lambda}$ on the open diagonal stratum. This follows from the computation of the Proposition 6.7 below.

6.6. *The master function and the Gaiotto–Witten superpotential*

The (multivalued) Master function [5, Section 3] on $\mathfrak{h}^\vee\times\overset{\circ}{\mathbb{A}}{}^{\alpha,\Lambda}$ is defined as follows:

$$\Phi(h^*,\underline{w},\underline{z}):=\sum_{1\le n\le N}\langle\lambda_n,h^*\rangle z_n-\sum_{(i,r)}\langle\alpha_i,h^*\rangle w_{i,r}+\sum_{(i,r)\ne(j,s)}\frac{\alpha_i\cdot\alpha_j}{2}\log(w_{i,r}-w_{j,s})-$$
$$-\sum_{(i,r),1\le n\le N}\alpha_i\cdot\lambda_n\log(z_n-w_{i,r})+\sum_{1\le m<n\le N}\lambda_m\cdot\lambda_n\log(z_m-z_n). \tag{6.3}$$

We define an open subvariety $\overset{\circ}{Z}{}^{\alpha,\Lambda}\subset\overset{\circ}{Z}{}^\alpha\times\overset{\circ}{\mathbb{A}}{}^\Lambda$ as the preimage of $\overset{\circ}{\mathbb{A}}{}^{\alpha,\Lambda}$ under the factorization morphism $\pi_\alpha:\ \overset{\circ}{Z}{}^\alpha\times\overset{\circ}{\mathbb{A}}{}^\Lambda\to\mathbb{A}^\alpha\times\overset{\circ}{\mathbb{A}}{}^\Lambda$. Recall the logarithmic coordinates $\mathsf{y}_{i,r}$ of Definition 2.5. The multivalued superpotential $\mathcal{W}^{\Lambda,\alpha}$ on $\mathfrak{h}^\vee\times\overset{\circ}{Z}{}^{\alpha,\Lambda}$ is defined as follows (cf. [8]):

$$\mathcal{W}^{\Lambda,\alpha}(h^*,\underline{w},\underline{\mathsf{y}},\underline{z}):=\sum_{1\le n\le N}\langle\lambda_n,h^*\rangle z_n-\sum_{(i,r)}\langle\alpha_i,h^*\rangle w_{i,r}+\sum_{(i,r)}d_i\mathsf{y}_{i,r}+$$
$$+\sum_{(i,r)}\exp(-\mathsf{y}_{i,r})\frac{\prod_{j\ne i}Q_j^{-\langle\alpha_j,\check{\alpha}_i\rangle/2}(w_{i,r})}{Q_i'(w_{i,r})}K_i(w_{i,r})+$$
$$+\sum_{1\le m<n\le N}\lambda_m\cdot\lambda_n\log(z_m-z_n). \tag{6.4}$$

Proposition 6.7. *a) The restriction of* $\mathcal{W}^{\Lambda,\alpha}$ *to each fiber of the factorization projection* $\mathfrak{h}^\vee \times \overset{\circ}{Z}{}^{\alpha,\Lambda} \to \mathfrak{h}^\vee \times \overset{\circ}{\mathbb{A}}{}^{\alpha,\Lambda}$ *has a unique singular point (with all the derivatives vanishing).*

b) For any $h^* \in \mathfrak{h}^\vee$, $\underline{z} \in \overset{\circ}{\mathbb{A}}{}^{\Lambda}$, *the resulting section* $s_{h^*,\underline{z}} : \overset{\circ}{\mathbb{A}}{}^{\alpha} \hookrightarrow \overset{\circ}{Z}{}^{\alpha}$ *is Lagrangian.*

c) The restriction of $\mathcal{W}^{\Lambda,\alpha}$ *to the section in a) equals the Master function* Φ.

Proof. Straightforward. □

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Appendix A

In this appendix we give another (elementary) derivation of a particular case of Theorem 6.4 for $G = SL(2)$.

A.1. Zastava for SL(2)

For $G = SL(2)$ the coroot lattice is just $\mathbb{Z}$, and given $a \in \mathbb{N}$ the moduli space of based maps $\overset{\circ}{Z}{}^a$ is identified with the moduli space of extensions $0 \to \mathcal{O}_C(-a) \to \mathcal{O}_C \oplus \mathcal{O}_C \to \mathcal{O}_C(a) \to 0$ trivialized at $\infty \in C$. So we have a map $\mathfrak{E} : \overset{\circ}{Z}{}^a \to \mathrm{Ext}^1(\mathcal{O}_C(a), \mathcal{O}_C(-a)) = \Gamma(C, \mathcal{O}_C(2a-2))^\vee$.

Proposition A.2. *For a polynomial* $K \in H^0(C, \mathcal{O}_C(2a-2))$ *we have* $\langle K, \mathfrak{E}\rangle = \sum_{r=1}^{a} y_r^{-1} \frac{K(w_r)}{Q'(w_r)}$.

Proof. We denote $H^0(C, \mathcal{O}_C(1))^\vee$ by V (a 2-dimensional vector space with a base formed by the highest vector x and the lowest vector t). We have $\mathrm{Ext}^1(\mathcal{O}(a), \mathcal{O}(-a)) = \Gamma(C, \mathcal{O}(2a-2))^\vee = \mathrm{Sym}^{2a-2}V$. We will write down an element of $\mathrm{Sym}^{2a-2}V$ in the basis of products of divided powers of x, t : $c_0 x^{(2n-2)} + \ldots + c_k x^{(2a-2-k)} t^{(k)} + \ldots + c_{2n-2} t^{(2a-2)}$.

For a point $\phi \in \overset{\circ}{Z}{}^a$, the first map in the corresponding exact sequence $0 \to \mathcal{O}_C(-a) \to \mathcal{O}_C x \oplus \mathcal{O}_C t \to \mathcal{O}_C(a) \to 0$ is given by a pair of polynomials (Q, R), and the second one is given by $(-R, Q)$. In the corresponding long exact sequence $0 = H^0(\mathcal{O}_C(-a)) \to H^0(\mathcal{O}_C \oplus \mathcal{O}_C) \to H^0(\mathcal{O}_C(a)) \to H^1(\mathcal{O}_C(-a)) \to \ldots$ the boundary map is given by the cup product with our desired Ext^1-class $\mathfrak{E}(\phi)$ in $\mathrm{Sym}^{2a-2} V$. Note that $H^0(\mathcal{O}_C(a)) =$

$\mathrm{Sym}^a V^\vee$, and $H^1(\mathcal{O}_C(-a)) = \mathrm{Sym}^{a-2} V$. So the boundary map is the contraction $\mathrm{Sym}^a V^\vee \to \mathrm{Sym}^{a-2} V$ with the desired class $\mathfrak{E}(\phi) \in \mathrm{Sym}^{2a-2} V$. Since the composition $H^0(\mathcal{O}_C \oplus \mathcal{O}_C) \to H^0(\mathcal{O}_C(a)) \to H^1(\mathcal{O}(-a))$ is 0, and the first map is given by $(-R, Q)$, we conclude that the contraction of $\mathfrak{E}(\phi)$ and Q equals 0, as well as the contraction of $\mathfrak{E}(\phi)$ and R. This is a system of linear equations on $\mathfrak{E}(\phi)$ which defines it up to proportionality. To write down the formula for contraction, we think of Q, R as of differential operators $Q = \partial_x^a + \ldots + \mathsf{a}_k \partial_x^{a-k} \partial_t^k + \ldots + \mathsf{a}_a \partial_t^a$, $R = \mathsf{b}_0 \partial_x^{a-1} + \ldots + \mathsf{b}_k \partial_x^{a-1-k} \partial_t^k + \ldots + \mathsf{b}_a \partial_t^{a-1}$, and then the contraction is nothing but the application of differential operators Q, R to the polynomial $\mathfrak{E} := c_0 x^{(2a-2)} + \ldots + c_k x^{(2a-2-k)} t^{(k)} + \ldots + c_{2a-2} t^{(2a-2)}$.

Note that the matrix of this system of equations is (up to proportionality) exactly the Sylvester matrix $S = \begin{pmatrix} 1 & \mathsf{a}_1 & \ldots \mathsf{a}_a & 0 & \ldots & 0 & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & \mathsf{a}_1 & \ldots & \mathsf{a}_a \\ \mathsf{b}_0 & \mathsf{b}_1 & \ldots & \mathsf{b}_{a-1} & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \mathsf{b}_0 & b_1 & \ldots & \mathsf{b}_{a-1} \end{pmatrix}$ with the middle row (the first one with b's) removed. Solving it via the Cramer rule we obtain $c_k = (-1)^k \det S^{-1}$ times the $(2a-2) \times (2a-2)$-minor of the Sylvester matrix obtained by removing the middle row and the k-th column. Note also that the resultant $\mathsf{R}(Q, R)$ is nothing but $\det S$, and $\mathsf{R}(Q, R) \neq 0$ under our assumptions: $(\mathcal{O}_C \oplus \mathcal{O}_C)/\mathcal{O}_C(-a)$ torsionless.

Equivalently, if we think of Q, R as of two (relatively prime) polynomials in $z = \partial_x/\partial_t$ (as in Section 2.2), then the equation $RD - QF = 1$ has a unique solution such that D is a polynomial in z of degree $a-1$, and F is a polynomial in z of degree $a-2$. The principal part at $\infty \in C$ of the ratio $\frac{D(z)}{Q(z)}$ is nothing but $\frac{c_0}{z} + \frac{c_1}{z^2} + \ldots + \frac{c_{2a-2}}{z^{2a-1}} + \ldots$ (c_k from the previous paragraph). By the Lagrange interpolation we find $c_k = \sum_{r=1}^a \frac{w_r^k D(w_r)}{Q'(w_r)} = \sum_{r=1}^a \frac{w_r^k R(w_r)^{-1}}{Q'(w_r)} = \sum_{r=1}^a \frac{w_r^k y_r^{-1}}{Q'(w_r)}$. The desired formula for $\langle K, \mathfrak{E} \rangle$ follows. □

Remark A.3. We keep the notations introduced in the proof of Proposition A.2. Let us define $\tilde{c}_0, \ldots, \tilde{c}_{2a-2}$ by $\frac{R(z)}{Q(z)} = \frac{\tilde{c}_0}{z} + \frac{\tilde{c}_1}{z^2} + \ldots + \frac{\tilde{c}_{2a-2}}{z^{2a-1}} + \ldots$ Then by the Lagrange interpolation $\tilde{c}_k = \sum_{r=1}^a \frac{w_r^k y_r}{Q'(w_r)}$. According to L. Kronecker [13], the resultant $\mathsf{R}(Q, R) = \det \tilde{L}$ where $\tilde{L}$ is a Hankel matrix $\tilde{L} := \begin{pmatrix} \tilde{c}_0 & \tilde{c}_1 & \tilde{c}_2 & \ldots & \tilde{c}_{a-1} \\ \tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 & \ldots & \tilde{c}_a \\ \tilde{c}_2 & \tilde{c}_3 & \tilde{c}_4 & \ldots & \tilde{c}_{a+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{a-1} & \tilde{c}_a & \tilde{c}_{a+1} & \ldots & \tilde{c}_{2a-2} \end{pmatrix}$. We obtain $\mathsf{R}(Q, R) =$

$\mathsf{R}(D,R)^{-1} = \det L^{-1}$ where L is a Hankel matrix $L := \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{a-1} \\ c_1 & c_2 & c_3 & \dots & c_a \\ c_2 & c_3 & c_4 & \dots & c_{a+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{a-1} & c_a & c_{a+1} & \dots & c_{2a-2} \end{pmatrix}$.

This identity $\mathsf{R}(Q,R) = \det L^{-1}$ was independently obtained by A. Uteshev (private communication). Note that the equation $\det L = 0$ is the equation of the locus in $\mathrm{Ext}^1(\mathcal{O}_C(a), \mathcal{O}_C(-a))$ formed by the extensions with the middle term a *nontrivial* 2-dimensional vector bundle on C [4].

References

[1] A. Braverman, M. Finkelberg, Semi-infinite Schubert varieties and quantum K-theory of flag manifolds, J. Amer. Math. Soc. 27 (2014) 1147–1168.
[2] A. Braverman, M. Finkelberg, D. Gaistgory, Uhlenbeck spaces via affine Lie algebras, Progr. Math. 244 (2006) 17–135.
[3] A. Braverman, M. Finkelberg, D. Gaitsgory, I. Mirković, Intersection cohomology of Drinfeld's compactifications, Selecta Math. (N.S.) 8 (3) (2002) 381–418.
[4] G. Comas, M. Seiguer, On the rank of a binary form, Found. Comput. Math. 11 (1) (2011) 65–78.
[5] G. Felder, Y. Markov, V. Tarasov, A. Varchenko, Differential equations compatible with KZ equations, Math. Phys. Anal. Geom. 3 (2000) 139–177.
[6] M. Finkelberg, A. Kuznetsov, N. Markarian, I. Mirković, A note on a symplectic structure on the space of G-monopoles, Comm. Math. Phys. 201 (1999) 411–421, arXiv:math/9803124; Comm. Math. Phys. 334 (2015) 1153–1155 (Erratum).
[7] M. Finkelberg, L. Rybnikov, Quantization of Drinfeld zastava in type C, J. Algebraic Geom. 1 (2) (2014) 166–180.
[8] D. Gaiotto, E. Witten, Knot invariants from four-dimensional gauge theory, Adv. Theor. Math. Phys. 16 (3) (2012) 935–1086.
[9] D. Gaitsgory, Twisted Whittaker model and factorizable sheaves, Selecta Math. (N.S.) 13 (4) (2008) 617–659.
[10] S. Jarvis, Euclidean monopoles and rational maps, Proc. Lond. Math. Soc. (3) 77 (1) (1998) 170–192.
[11] S. Jarvis, Construction of Euclidean monopoles, Proc. Lond. Math. Soc. (3) 77 (1) (1998) 193–214.
[12] M. Kashiwara, On crystal bases, CMS Conf. Proc. 16 (1995) 155–197.
[13] L. Kronecker, Zur Theorie der Elimination einer Variabeln aus zwei algebraischen Gleichungen, Werke, Bd. 2, Teubner, Leipzig, 1897, pp. 113–192.