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Factorial algebraic group actions and categorical quotients

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ABSTRACT

Given an action of an affine algebraic group with only trivial characters on a factorial variety, we ask for categorical quotients. We characterize existence in the category of algebraic varieties. Moreover, allowing constructible sets as quotients, we obtain a more general existence result, which, for example, settles the case of a finitely generated algebra of invariants. As an application, we provide a combinatorial GIT-type construction of categorical quotients for actions of not necessarily reductive groups on, e.g. complete varieties with finitely generated Cox ring via lifting to the characteristic space.

1. Introduction

Consider the action of an affine algebraic group G on a normal variety X defined over an algebraically closed field $\mathbb{K}$. In most cases, the orbit space X/G does not inherit the structure of a variety and it is the main task of Geometric Invariant Theory to provide reasonable replacements. A common concept with minimal requirements is the *categorical quotient*: this is a G-invariant morphism $\pi : X \to Y$ such that for every other G-invariant morphism $\varphi : X \to Z$, there exists a unique morphism

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$\psi : Y \to Z$ with $\varphi = \psi \circ \pi$. If G is reductive and X is affine, then Hilbert's finiteness theorem guarantees existence of a categorical quotient $\pi : X \to Y$ with $Y := \operatorname{Spec} \Gamma(X, \mathcal{O})^G$ which, in general, is not an orbit space. However, as soon as one of the conditions "G reductive" and "X affine" is not satisfied, even categorical quotients need not exist any more, see Example 4.1 for the first and [2] for the second one. In this article, we investigate existence of categorical quotients in the following setting: we say that the action of G on X is *factorial*, if every invariant hypersurface $D \subseteq X$ is the zero set of an invariant function $f \in \Gamma(X, \mathcal{O})$; compare [12] for a related concept. As we will see, this setting naturally occurs via lifting to the Cox ring whenever a group G with trivial character group $\mathbb{X}(G)$, e.g. a semisimple or a unipotent one, acts on a normal variety X with finitely generated divisor class group.

Similarly to the reductive case, the algebra of invariants plays a central role in the construction of quotients, compare also the work on unipotent group actions [9,14,15] and [7]. In contrast to the reductive case, even for affine X, the algebra of invariants need not be finitely generated. However, in our setting there always exist finitely generated normal subalgebras $A \subseteq \Gamma(X, \mathcal{O})^G$, which are large in the sense that they have $\mathbb{K}(X)^G$ as their field of fractions, see Lemma 3.2. This provides at least candidates $\pi' : X \to Y'$ with $Y' := \operatorname{Spec} A$ for a quotient. An obvious obstruction to being a categorical quotient is non-surjectivity; this even happens if X is affine and the algebra of invariants is finitely generated, i.e., we may take $A = \Gamma(X, \mathcal{O})^G$. In general, the image $Y = \pi'(X)$ is a constructible set. This motivates an excursion to the category of constructible spaces, i.e., spaces locally isomorphic to constructible subsets of affine varieties, see Section 2 for details and [5] for a related concept. We ask whether the map $\pi : X \to Y$ sending $x \in X$ to $\pi'(x) \in Y$ is a *categorical quotient in the category of constructible spaces*, i.e., every G-invariant morphism $X \to Z$ to a constructible space Z factors uniquely through $\pi : X \to Y$; note that a positive answer allows in particular to associate a unique quotient to the action. We say that a categorical quotient $\pi : X \to Y$ is *strong*, if for every open $V \subseteq Y$, the restriction $\pi : \pi^{-1}(V) \to V$ is a categorical quotient. Here comes our first result.

Theorem 1.1. *Consider a normal variety X with a factorial action of an affine algebraic group G. Let $A \subseteq \Gamma(X, \mathcal{O})^G$ be a finitely generated normal subalgebra having $\mathbb{K}(X)^G$ as its field of fractions, $\pi' : X \to Y'$, where $Y' := \operatorname{Spec} A$, the canonical morphism and set $Y := \pi(X)$. Then the following statements are equivalent.*

(i) *The morphism $\pi : X \to Y$, $x \mapsto \pi'(x)$ is a categorical quotient in the category of constructible spaces for the G-action on X.*
(ii) *The pullback $\pi^* : \Gamma(Y, \mathcal{O}) \to \Gamma(X, \mathcal{O})^G$ is an isomorphism.*
(iii) *There is an open subset $Y'' \subseteq Y'$ with $Y \subseteq Y''$ and $Y'' \setminus Y$ is of codimension at least two in Y''.*

Moreover, if one of these statements holds, then $\pi : X \to Y$ is even a strong categorical quotient.

Note that the quotient Y in the theorem inherits its structure from Y' and thus is a quasiaffine constructible space, see Section 2 for the precise definition. If the algebra of invariants is finitely generated, then it obviously fulfills the second condition of Theorem 1.1, and thus we obtain the following.

Corollary 1.2. *Consider a normal variety X with a factorial action of an affine algebraic group G and suppose that $\Gamma(X, \mathcal{O})^G$ is finitely generated. Then $\pi : X \to Y$, $x \mapsto \pi'(x)$, where $\pi' : X \to \operatorname{Spec} \Gamma(X, \mathcal{O})^G$ is the canonical map and $Y = \pi'(X)$ is a strong categorical quotient in the category of constructible spaces for the action of G on X.*

In general, if (strong) categorical quotient exists, its fibers may contain more than one orbit, even if the action is free; see Example 4.2, where we discuss a free action of the additive group $\mathbb{C}$.

We come back to the problem of existence of quotients in the category of varieties. An important observation is that $\Gamma(X, \mathcal{O})^G$ admits a *separating subalgebra* in the sense of Derksen and Kemper [6], i.e., a finitely generated subalgebra that separates any pair of points, which can be separated by invariant functions, see Proposition 3.1. Combining this with our first result, we obtain the following characterization of existence of categorical quotients.

Theorem 1.3. *Let X be a normal variety with a factorial action of an affine algebraic group G. Then the following statements are equivalent.*

(i) *There exists a categorical quotient $\pi : X \to Y$ in the category of varieties for the action of G on X.*
(ii) *There is a finitely generated normal subalgebra $A \subseteq \Gamma(X, \mathcal{O})^G$ with quotient field $\mathbb{K}(X)^G$ such that the canonical map $X \to \operatorname{Spec} A$ has an open image.*
(ii) *For every normal separating subalgebra $A \subseteq \Gamma(X, \mathcal{O})^G$ with quotient field $\mathbb{K}(X)^G$, the canonical map $X \to \operatorname{Spec} A$ has an open image.*

Moreover, if one of the statements holds, then the categorical quotient $\pi : X \to Y$ is even a strong categorical quotient.

In the case of a finitely generated ring of invariants, we obtain as an immediate consequence the following characterization for existence of a categorical quotient.

Corollary 1.4. *Let X be a normal variety with a factorial action of an affine algebraic group G and suppose that $\Gamma(X, \mathcal{O})^G$ is finitely generated. Then the following statements are equivalent.*

(i) *The G-action on X has a categorical quotient in the category of varieties.*
(ii) *The canonical morphism $\pi : X \to \operatorname{Spec} \Gamma(X, \mathcal{O})^G$ has an open image.*

Moreover, if one of these statements holds, then $\pi : X \to \pi(X)$ is a categorical quotient, and it is even a strong one.

For representations of unipotent groups on finite dimensional vector spaces, the following result shows that existence of a categorical quotient in the category of varieties is equivalent to surjectivity of the morphism $V \to \operatorname{Spec} \Gamma(V, \mathcal{O})^G$; see Example 4.1 for an application.

Theorem 1.5. *Let a unipotent group G act linearly on a finite dimensional vector space V. Then the following statements are equivalent.*

(i) *There exists a categorical quotient in the category of varieties for the action of G on V.*
(ii) *The algebra $\Gamma(V, \mathcal{O})^G$ of invariants is finitely generated and the canonical morphism $\pi : V \to \operatorname{Spec} \Gamma(V, \mathcal{O})^G$ is surjective.*

Moreover, if one of these statements holds, then $\pi : V \to \operatorname{Spec} \Gamma(V, \mathcal{O})^G$ is a strong categorical quotient in the category of varieties for the action of G on V.

The results presented so far are proven in Sections 2 and 3. In Section 4, we discuss examples. An application is given in Section 5. There, we consider the action of an affine algebraic group G with trivial character group $\mathbb{X}(G)$ on an, e.g. complete variety X and assume that the Cox ring $\mathcal{R}(X)$ as well as the subring $\mathcal{R}(X)^G$ are finitely generated. Since Cox rings are (graded) factorial [3], we can apply our results to the lifted action of G on $\operatorname{Spec} \mathcal{R}(X)$. Via a Gel'fand–MacPherson type correspondence, we obtain in Construction 5.1 open G-invariant subsets $U \subseteq X$ with a strong categorical quotient from geometric quotients of a certain torus action on the factorial affine variety $\operatorname{Spec} \mathcal{R}(X)^G$. Among the resulting sets $U \subseteq X$, there are many sets of *finitely generated semistable points* as introduced by Doran and Kirwan in [7]. They fit into a combinatorial picture given by the GIT-fan of a torus action on $\operatorname{Spec} \mathcal{R}(X)^G$.

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2. Constructible quotients

In this section, we prove Theorem 1.1. We begin with presenting the basic concepts concerning constructible spaces.

By a space with functions we mean a topological space X together with a sheaf $\mathcal{O}_X$ of $\mathbb{K}$-valued functions. A morphism of spaces X and Y with functions is a continuous map $\varphi : X \to Y$ such that for every open subset $V \subseteq Y$ and every $g \in \mathcal{O}_Y(V)$, we have $g \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$. If $Y \subseteq X$ is a subset of a space X with functions, then Y is in a natural manner a subspace with functions: firstly, it inherits the subspace topology from X and, secondly, it inherits the sheaf $\mathcal{O}_Y$ of functions that are locally represented as restrictions of functions of $\mathcal{O}_X$. A subset $Y \subseteq X$ is called constructible if it is a union of finitely many locally closed subsets. By a constructible subspace $Y \subseteq X$, we mean a constructible subset $Y \subseteq X$ together with the subspace structure. We are ready to introduce the category of constructible spaces.

- A *quasiaffine constructible space* is a space with functions isomorphic to a constructible subspace of an affine $\mathbb{K}$-variety.
- A *constructible space* is a space with functions admitting a finite cover by open quasiaffine constructible subspaces.
- A *morphism of constructible spaces* is a morphism of the underlying spaces with functions.

Note that the prevarieties form a full subcategory of the category of constructible spaces. Moreover, every constructible subset of a constructible space inherits the structure of a constructible space. We will need the following basic observation.

Lemma 2.1. *Let X' be a normal affine variety and $X \subseteq X'$ a dense constructible subspace. If every closed hypersurface $D \subseteq X'$ meets X, then the restriction $\Gamma(X', \mathcal{O}_{X'}) \to \Gamma(X, \mathcal{O}_X)$ is an isomorphism.*

Proof. Locally every $f \in \Gamma(X, \mathcal{O}_X)$ extends to X'. Since $X \subseteq X'$ is dense, the local extensions can be glued together and thus f extends to an open neighborhood $X'' \subseteq X'$ of X. Normality then gives the claim. □

Similarly one obtains that, given two constructible subspaces $X \subseteq X'$ and $Y \subseteq Y'$ of varieties X' and Y', every morphism $X \to Y$ extends to a morphism $U' \to Y'$ with an open neighborhood $U' \subseteq X'$ of X. This shows in particular that the category of dc-subsets defined by A. Białynicki-Birula [5] is a full subcategory of the category of constructible spaces.

Proof of Theorem 1.1. In order to obtain "(i) ⇒ (ii)", apply the universal property of the categorical quotient to G-invariant functions.

We verify "(ii) ⇒ (iii)". Consider $C := Y' \setminus Y$, let $C_1, \dots, C_r \subseteq C$ denote the irreducible components, which are closed in Y', and set $Y'' := Y' \setminus (C_1 \cup \dots \cup C_r)$. By Lemma 2.1, we have

$$\Gamma\big(Y'', \mathcal{O}\big) = \Gamma(Y, \mathcal{O}).$$

We show that $Y'' \setminus Y$ is small. Otherwise, let $D_1, \dots, D_s \subseteq Y''$ be the (nonempty) collection of prime divisors such that $D_i \setminus Y$ is dense in D_i. Choose non-zero functions $f, g \in \Gamma(Y', \mathcal{O})$ with

$$D_i \subseteq V\big(Y'', f\big), \qquad D_i \nsubseteq V\big(Y'', g\big), \qquad V(Y, f) \subseteq V(Y, g).$$

Then, for any $m \in \mathbb{Z}_{\geqslant 0}$, the function $g^m f^{-1}$ is not regular on Y'' and hence not on Y. On the other hand, for m big enough, we have $m\,\mathrm{div}(\pi^*(g)) > \mathrm{div}(\pi^* f)$ and thus $\pi^*(g^m f^{-1})$ belongs to $\Gamma(X, \mathcal{O})^G$. This contradicts (ii).

We check "(iii) ⇒ (ii)". Clearly, $\pi^* : \Gamma(Y, \mathcal{O}) \to \Gamma(X, \mathcal{O})^G$ is injective. To see surjectivity, let $f \in \Gamma(X, \mathcal{O})^G$ be given. Then we have $f = \pi^* g$ with a rational function $g \in \mathbb{K}(Y'')$. But condition (iii) ensures that g has no poles and thus, we have $g \in \Gamma(Y, \mathcal{O})^G$.

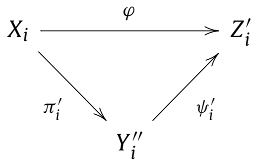
We show that (ii) and (iii) imply (i). Let $\varphi : X \to Z$ be a G-invariant morphism. Cover Z by open subspaces $Z_1, \dots, Z_r$ such that we have open embeddings $Z_i \subseteq Z_i'$ with affine varieties Z_i'. Then X is covered by the open subsets $X_i := \varphi^{-1}(Z_i)$, and we have

$$X \setminus X_i = D_i \cup B_i,$$

where $D_i \subset X$ is of pure codimension one, and $B_i \subset X$ is of codimension at least two. Choose G-invariant functions $f_i \in \Gamma(X, \mathcal{O})$ having precisely D_i as their set of zeroes. These f_i descend to Y, and, by Lemma 2.1, extend to Y''. Set $Y_i'' := Y''_{f_i}$. Then we have

$$\Gamma(X_i, \mathcal{O})^G = \Gamma(X, \mathcal{O})^G_{f_i} = \pi^* \Gamma(Y, \mathcal{O})_{f_i} = \left(\pi'\right)^* \Gamma\left(Y_i'', \mathcal{O}\right),$$

where, for the last equality, we again use Lemma 2.1. As a consequence, we obtain a commutative diagram



Consider $Y_i := \pi_i'(X_i) \subseteq Y_i''$. Then we have $Y_i = \pi(X_i)$. Moreover, because of $\psi_i'(Y_i) = \varphi(X_i) \subseteq Z_i$, we obtain morphisms $\psi_i : Y_i \to Z_i$, $y \mapsto \psi_i'(y)$ of constructible spaces. By construction, these morphisms glue together to the desired factorization $\psi : Y \to Z$.

In order to see that the categorical quotient $\pi : X \to Y$ is even strong, first note that for every principal open subset Y_f the restriction $\pi : X_{\pi^* f} \to Y_f$ is a categorical quotient, because it satisfies the second condition of the theorem. Then the desired property is obtained by gluing. □

Remark 2.2. Let G act on X as in Theorem 1.1. If there is a categorical quotient $\pi : X \to Y$ with a quasiaffine constructible space Y, then this quotient is obtained by the procedure of Theorem 1.1. Indeed, by the universal property of a categorical quotient, the pullback $\pi^* : \Gamma(Y, \mathcal{O}) \to \Gamma(X, \mathcal{O})^G$ is an isomorphism. Now choose an embedding $Y \subseteq Y'$ into an affine variety Y'. Then $A := \pi^* \Gamma(Y', \mathcal{O})$ is as wanted.

Note that, given a subalgebra A of the algebra of invariants as in Theorem 1.1, the equivalent conditions of 1.1 need not be fulfilled, see [15, Section 4].

3. Quotients in the category of varieties

Here, we prove Theorem 1.3. A first observation is existence of separating subalgebras; note that for affine G-varieties, an elementary proof is given in [6, Theorem 3.15].

Proposition 3.1. *Let G be any affine algebraic group and X any G-variety. Then there exists a finitely generated separating subalgebra $A \subseteq \Gamma(X, \mathcal{O})^G$. Moreover, if X is normal and the G-action is factorial, then one may choose A to be normal and to have $\mathbb{K}(X)^G$ as its field of fractions.*

Lemma 3.2. *Let X be a normal variety with a factorial action of an affine algebraic group G. Then there is a finitely generated subalgebra $A \subseteq \Gamma(X, \mathcal{O})^G$ having $\mathbb{K}(X)^G$ as its field of fractions.*

Proof. Let $\mathbb{K}(X)^G = \mathbb{K}(g_1, \ldots, g_r)$ with $g_i \in \mathbb{K}(X)^G$. Then g_i is defined on an open invariant subset $U_i \subseteq X$. By factoriality of the action, the union of all one-codimensional components of $X \setminus U_i$ is the zero set of a function $f_i \in \Gamma(X, \mathcal{O})^G$. Normality of X implies $h_i := g_i f_i^{m_i} \in \Gamma(X, \mathcal{O})^G$ for some $m_i > 0$. Thus, the algebra $A := \mathbb{K}[f_i, h_i;\ 1 \leqslant i \leqslant r]$ is as wanted. □

Proof of Proposition 3.1. Assume that for every finitely generated subalgebra $B \subseteq \Gamma(X, \mathcal{O})^G$ there exist $x_1, x_2 \in X$ such that $F(x_1) = F(x_2)$ for all $F \in B$, but $f(x_1) \neq f(x_2)$ for some $f \in \Gamma(X, \mathcal{O})^G$. Then we may construct an infinite strictly increasing sequence of finitely generated subalgebras

$$B_1 \subset B_2 \subset B_2 \subset \cdots$$

in $\Gamma(X, \mathcal{O})^G$ such that for any $i \geqslant 1$ there exist $x_{1i}, x_{2i} \in X$ with $F(x_{1i}) = F(x_{2i})$ for all $F \in B_i$, but $f(x_{1i}) \neq f(x_{2i})$ for some $f \in B_{i+1}$. This sequence of subalgebras gives us the affine varieties $Y_i := \operatorname{Spec} B_i$ and the morphisms $\psi_i : X \to Y_i$ and $\varphi_i : Y_{i+1} \to Y_i$ defined by the inclusions $B_i \subset \Gamma(X, \mathcal{O})^G$ and $B_i \subset B_{i+1}$.

The images $V_i := \psi_i(X) \subseteq Y_i$ and the maps $\varphi_i : V_{i+1} \to V_i$ form a dominated inverse system of dc-subsets, see [5, Section 3]. By [5, Theorem O], there exists $m \geqslant 1$ such that the maps $\varphi_i : V_{i+1} \to V_i$ are bijective for any $i \geqslant m$. This implies that the fibers of the morphisms ψ_i and ψ_{i+1} coincide for any $i \geqslant m$, a contradiction.

The supplement is a simple consequence of Lemma 3.2 and finite generation of the integral closure. □

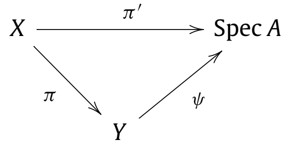
The basic property of a separating subalgebra $A \subseteq \Gamma(X, \mathcal{O})^G$ we will use is that it realizes the categorical closure of the equivalence relation given by the G-action on X in the following sense.

Proposition 3.3. *Let X be a normal variety with a factorial action of an affine algebraic group G. If $A \subseteq \Gamma(X, \mathcal{O})^G$ is a finitely generated separating subalgebra and $U \subseteq X$ a G-invariant open subset, then every G-invariant morphism $\varphi : U \to Z$ to a prevariety Z is constant along the fibers of the map $\pi' : X \to \operatorname{Spec} A$.*

Proof. Consider $x_1, x_2 \in U$ with $\varphi(x_1) \neq \varphi(x_2)$. Let $Z_1 \subseteq Z$ be an open affine neighborhood of $\varphi(x_1)$. Set $U_1 := \varphi^{-1}(Z_1)$, and write $U \setminus U_1 = D_1 \cup B_1$, where $D_1 \subset U$ is of pure codimension one, and $B_1 \subset U$ is of codimension at least two. Then the morphism $\varphi_1 : U_1 \to Z_1$ extends to a morphism $\varphi_1 : U \setminus D_1 \to Z_1$. Consequently, we must have $U_1 = U \setminus D_1$.

Choose a function $f_1 \in \Gamma(X, \mathcal{O})^G$ having inside U precisely D_1 as its set of zeroes. If $x_2 \in D_1$ holds, then we obtain $f_1(x_2) = 0$ and $f_1(x_1) \neq 0$. If $x_2 \in U_1$ holds, then there is a function $f \in \Gamma(U_1, \mathcal{O})^G$ with $f(x_1) \neq f(x_2)$. Since $\Gamma(U_1, \mathcal{O})^G$ equals $\Gamma(U, \mathcal{O})^G_{f_1}$, we find a function $f' \in \Gamma(U, \mathcal{O})^G$ with $f'(x_1) \neq f'(x_2)$. □

Proof of Theorem 1.3. The implication "(iii) ⇒ (ii)" follows from Proposition 3.1. Moreover, "(ii) ⇒ (i)" and the supplement are clear by Theorem 1.1. To verify "(i) ⇒ (iii)", let $\pi : X \to Y$ be a categorical quotient. Given any normal separating subalgebra $A \subseteq \Gamma(X, \mathcal{O})^G$ as in (iii), the universal property yields a commutative diagram



By assumption, the morphism $\psi : Y \to \operatorname{Spec} A$ is birational. Moreover, using surjectivity of the categorical quotient $\pi : X \to Y$ and Proposition 3.3, we see that it is injective. Consequently, since $\operatorname{Spec} A$ is normal, Zariski's Main Theorem yields that $\psi : Y \to \operatorname{Spec} A$ is an open embedding. Using once more surjectivity of $\pi : X \to Y$, we conclude that $\pi'(X) = \psi(Y)$ is open in $\operatorname{Spec} A$. □

Every constructible subspace $X \subseteq X'$ of a quasiaffine variety has an open kernel, i.e., a unique maximal subset, which is open in the closure of X in X'. This kernel does not depend on the embedding $X \subseteq X'$. Thus, given an arbitrary constructible space X, we can define the *set of varietic points*

$X^{\mathrm{var}} \subseteq X$ as the union of the open kernels of its quasiaffine open subspaces. Note that $X^{\mathrm{var}} \subseteq X$ is the unique maximal open subspace of X, which is a prevariety. Based on this observation, we obtain a statement on categorical quotients similar to Rosenlicht's theorem on existence of an open subset with a geometric quotient.

Corollary 3.4. *Let X be a normal variety with a factorial action of an affine algebraic group G. Then there is a unique maximal invariant open subset $U \subseteq X$ that admits a categorical quotient $\pi : U \to V$ in the category of varieties.*

Proof. Let $A \subseteq \Gamma(X, \mathcal{O})$ be a finitely generated normal separating subalgebra. Then, by Proposition 3.3, this is a separating subalgebra for any invariant open subset of X. Now, set $Y' := \operatorname{Spec} A$, let $\pi' : X \to Y'$ be the canonical morphism and set $Y := \pi'(X)$. Then Theorem 1.3 tells us that $U := \pi'^{-1}(V)$ for $V := Y^{\mathrm{var}}$ has a categorical quotient in the category of varieties. If another G-invariant open set $W \subseteq X$ admits a categorical quotient in the category of varieties, then Theorem 1.3 yields that $\pi'(W)$ is open in Y' and hence $\pi'(W) \subseteq V$ holds. This implies $W \subseteq U$. □

Proof of Theorem 1.5. The supplement and the implication "(ii) ⇒ (i)" are direct consequences of Corollary 1.4.

Now assume that (i) holds. Then Theorem 1.3 provides a normal separating subalgebra $A \subseteq \Gamma(V, \mathcal{O})^G$ with quotient field $\mathbb{K}(X)^G$. Clearly, we may assume that A is generated by homogeneous polynomials, i.e. is a graded subalgebra of $\mathcal{O}(V)$. Then $\mathbb{K}^*$ acts on $Y = \operatorname{Spec} A$ and $\pi : V \to Y$ becomes equivariant. The image $\pi(V) \subseteq Y$ is invariant and, according to Theorem 1.3, open in Y. Since we have $A_0 = \mathbb{K}$, we can conclude $\pi(V) = Y$.

By [8, Section 3], there are $f_1, \ldots, f_r \in A$ with $A_{f_i} = \Gamma(V, \mathcal{O})^G_{f_i}$ such that the zero set $V_V(f_1, \ldots, f_r)$ is of codimension at least two in V and the pullback homomorphism

$$\pi^* : \mathcal{O}\big(Y \setminus V_Y(f_1, \ldots, f_r)\big) \to \Gamma(V, \mathcal{O})^G$$

is an isomorphism. As seen before, $\pi : V \to Y$ is surjective. Consequently, the zero set $V_Y(f_1, \ldots, f_r)$ is of codimension at least two in Y. Since Y is normal, this implies $\Gamma(V, \mathcal{O})^G = A$ which finally gives (ii). □

4. Examples

Our first example is an action of the additive group $\mathbb{K}$ on a four-dimensional vector space having a finitely generated algebra of invariants but no categorical quotient in the category of varieties.

Example 4.1. See [11, Section 4.3] and [10, Example 6.4.10]. We regard $X := \mathbb{K}^4$ as the space of (2×2)-matrices and consider the linear action of the additive group $G = \mathbb{K}$ given by

$$\lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + \lambda c & b + \lambda d \\ c & d \end{pmatrix}.$$

This action fulfills the assumptions of Theorem 1.1. The algebra of invariants is generated by c, d and $ad - bc$. The corresponding morphism $\pi' : \mathbb{K}^4 \to \mathbb{K}^3$ is not surjective, its image is not open, it is the constructible set

$$Y = V \cup \big\{(0, 0, 0)\big\}, \qquad V := \mathbb{K}^* \times \mathbb{K} \times \mathbb{K} \cup \mathbb{K} \times \mathbb{K}^* \times \mathbb{K}.$$

According to Theorem 1.5, there is no categorical quotient in the category of varieties. However, by Corollary 1.2 the map $\pi : X \to Y$, $x \mapsto \pi'(x)$ is a strong categorical quotient in the category of constructible spaces. Moreover the set $U \subseteq X$ of Corollary 3.4 is $\pi^{-1}(V)$.

By a result of Sumihiro, every free torus action on a variety admits a geometric quotient with a possibly non-separated orbit space. The following example shows that this is not true for actions of the additive group $\mathbb{K}$, even if they admit a categorical quotient in the category of constructible spaces.

Example 4.2. See [14, Section 5]. Consider the (non-linear) action of the additive group $G = \mathbb{C}$ on $X = \mathbb{C}^4$ defined by

$$\lambda \cdot (x_1, x_2, x_3, x_4) := \left(x_1, x_2 + \lambda x_1, x_3 + \lambda x_2 + \frac{1}{2}\lambda^2 x_1, x_4 + \lambda\left(x_2^2 - 2x_1x_3 - 1\right)\right).$$

Then this action is free, and, according to [14, Lemma 10] the algebra of invariants is generated by

$$f_1 := x_1, \qquad f_3 := x_1x_4 - x_2\left(x_2^2 - 2x_1x_3 - 1\right),$$
$$f_2 := x_2^2 - 2x_1x_3, \qquad f_4 := \frac{1}{f_1}\left(f_3^2 - f_2(1 - f_2)^2\right).$$

The variety $Y' = \operatorname{Spec} \Gamma(\mathbb{C}^4, \mathcal{O})^G = V(\mathbb{C}^4; f_1f_4 - f_3^2 + f_2(1 - f_2)^2)$ is smooth, and the image of the canonical morphism $\pi' : \mathbb{C}^4 \to Y'$ is

$$Y = Y' \setminus \{f_1 = 0,\ f_2 = 1,\ f_3 = 0,\ f_4 \neq 0\}.$$

Thus, Theorem 1.1 says that $\pi : X \to Y$, $x \mapsto \pi'(x)$ is a categorical quotient in the category of constructible spaces. Since π does not separate the orbits of the points $(0, 1, 0, 0)$ and $(0, -1, 0, 0)$, a geometric quotient cannot exist, even if we allow a non-separated orbit space.

So far, we saw examples of unipotent group actions having no categorical quotients in the category of varieties. Here comes a semisimple group action on a smooth quasiaffine variety.

Example 4.3. Let V be the space of (2×3)-matrices with the SL_2-action by multiplication from the left. The algebra of invariants is generated by (2×2)-minors Δ_{12}, Δ_{23}, Δ_{13}, and the canonical morphism

$$\pi' : V \to \mathbb{K}^3, \qquad M \to \left(\Delta_{12}(M), \Delta_{23}(M), \Delta_{13}(M)\right)$$

is surjective. Consider the open invariant subset $X \subset V$ consisting of matrices with non-zero first column. It has the same algebra of invariants as V. However, by Corollary 1.4, it has no categorical quotient in the category of varieties, because the image $Y = \pi'(X) \subset \mathbb{K}^3$ is not open: it is given by

$$\left(\mathbb{K}^3 \setminus V\left(\mathbb{K}^3; \Delta_{12}, \Delta_{13}\right)\right) \cup \left\{(0, 0, 0)\right\}.$$

We now provide a class of examples, showing that the conditions of Theorem 1.3 may be fulfilled even without finite generation of the ring of invariants.

Example 4.4. Let F be a connected simply connected semisimple algebraic group and $G \subseteq F$ a closed subgroup with $\mathbb{X}(G) = 0$, and let G act on F by multiplication from the right. Then, in general, $\Gamma(F, \mathcal{O})^G$ is not finitely generated. Choose any finitely generated normal subalgebra $A \subseteq \Gamma(F, \mathcal{O})^G$ having $\mathbb{K}(F)^G$ as its field of fractions and being invariant with respect to the F-action by multiplication from the left. Then the morphism $\pi' : F \to Y' := \operatorname{Spec} A$ is F-equivariant and its image coincides with an open F-orbit.

The next example shows that without the assumption of a "factorial action", even a surjective morphism $\pi' : X \to \operatorname{Spec} \Gamma(X, \mathcal{O})^G$ need not be a categorical quotient.

Example 4.5. Consider the action of the additive group $G = \mathbb{K}$ on the smooth quasiaffine variety

$$X = V\left(\mathbb{K}^4; x_1x_4 - x_2x_3\right) \setminus \left\{(0,0,0,0)\right\}$$

given by

$$\lambda \cdot (x_1, x_2, x_3, x_4) := (x_1, x_2, x_3 + \lambda x_1, x_4 + \lambda x_2).$$

The algebra of invariants is generated by x_1 and x_2, and the canonical morphism $\pi' : X \to \operatorname{Spec} \Gamma(X, \mathcal{O})^G$ is surjective. However, the following G-invariant morphism does not factor through π':

$$X \to \mathbb{P}_1, \qquad x \mapsto [x_1, x_2] = [x_3, x_4].$$

Finally, we give an example without quotient in the category of varieties, where we don't know, if it has a quotient in the category of constructible spaces:

Example 4.6. Fix a number $m \in \mathbb{Z}_{\geqslant 2}$ and consider the action of the additive group $G = \mathbb{C}$ on $X = \mathbb{C}^7$ given by

$$\lambda \cdot (x, y, z, s, t, u, v) := \left(x, y, z, s + \lambda x^{m+1}, t + \lambda y^{m+1}, u + \lambda z^{m+1}, v + \lambda x^m y^m z^m\right).$$

As observed in [1], the algebra of invariants is Roberts' algebra [13]; in particular, it is not finitely generated. By [13, Lemma 2], any non-constant term of a G-invariant polynomial contains at least one of the variables x, y and z. Let

$$f_1 = x,\ f_2 = y,\ f_3 = z,\ f_4, \ldots, f_n \in \Gamma(X, \mathcal{O})^G$$

generate a normal separating subalgebra and suppose that none of the f_i has a constant term. Consider the morphism $\pi' : \mathbb{C}^7 \to \mathbb{C}^n$ given by

$$(x, y, z, s, t, u, v) \mapsto \left(x, y, z, f_4(x, y, z, s, t, u, v), \ldots, f_n(x, y, z, s, t, u, v)\right).$$

We claim that the image $Y = \pi'(\mathbb{C}^7)$ is not open in its closure. Otherwise, it were a 6-dimensional variety. But if we restrict the projection

$$r : \mathbb{C}^n \to \mathbb{C}^3, \qquad (x, y, z, \ldots) \mapsto (x, y, z)$$

to Y, then the preimage $r^{-1}(0,0,0)$ intersected with Y is just one point; a contradiction to semicontinuity of the fiber dimension. Thus, by Theorem 1.3, there is no categorical quotient in the category of varieties.

5. A combinatorial GIT-type construction

Let G be an affine algebraic group with trivial character group $\mathbb{X}(G)$. We consider an action of G on a $\mathbb{Q}$-factorial variety X with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group $\mathrm{Cl}(X)$. For simplicity, let us assume that $\mathrm{Cl}(X)$ is free, though m.m. everything works as well if torsion appears. Our aim is to present a construction of open G-invariant subsets $U \subseteq X$ that admit a strong categorical quotient $U \to Y$. Passing, if necessary, to the action of the simply connected covering group, we may assume that G itself is simply connected.

The idea is to lift the G-action to the characteristic space over X and then reduce the problem to the case of a torus action on an affine variety by means of the results obtained so far. More precisely, the procedure is the following; we refer to [3] for details. Choose any subgroup $K \subseteq \mathrm{WDiv}(X)$ of the group of Weil divisors projecting isomorphically onto the divisor class group $\mathrm{Cl}(X)$ and define a sheaf of K-graded $\mathcal{O}_X$-algebras by

$$\mathcal{R} := \bigoplus_{D \in K} \mathcal{O}_X(D).$$

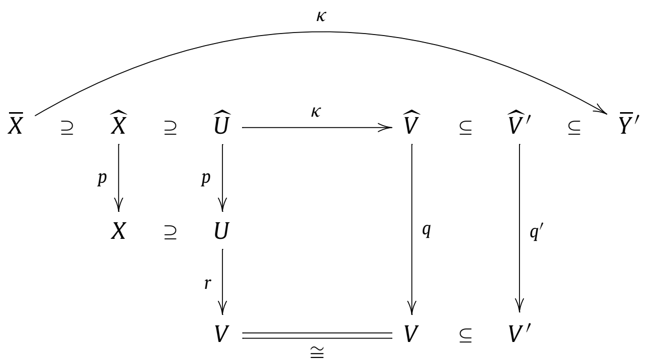
Then the K-grading of $\mathcal{R}$ defines an action of the torus $H := \mathrm{Spec}\, \mathbb{K}[K]$ on the relative spectrum $\widehat{X} := \mathrm{Spec}_X \mathcal{R}$ and the canonical morphism $p: \widehat{X} \to X$ is a geometric quotient for this action. We call $p: \widehat{X} \to X$ the characteristic space over X; for smooth X, we obtain the well-known universal torsor. Using G-linearization of the homogeneous components of $\mathcal{R}$, we may lift the G-action to $\widehat{X}$ such that it commutes with the H-action and $p: \widehat{X} \to X$ becomes G-equivariant, see [4, Section 4].

The variety $\widehat{X}$ is quasiaffine and the Cox ring $\mathcal{R}(X) = \Gamma(\widehat{X}, \mathcal{O})$ is factorial. In particular, the G-action on $\widehat{X}$ satisfies the assumptions of Theorems 1.1 and 1.3. Suppose that the Cox ring $\mathcal{R}(X)$ and the algebra of invariants $\mathcal{R}(X)^G$ are finitely generated. This gives us factorial affine varieties

$$\overline{X} := \mathrm{Spec}\, \mathcal{R}(X), \qquad \overline{Y}' := \mathrm{Spec}\, \mathcal{R}(X)^G,$$

see [11, Theorem 3.17]. The variety $\widehat{X}$ is a $(G \times H)$-invariant open subset of $\overline{X}$ and, by Corollary 1.2, there is a strong categorical quotient $\kappa: \overline{X} \to \overline{Y}$ with a constructible subset $\overline{Y} \subseteq \overline{Y}'$ such that $\overline{Y}' \setminus \overline{Y}$ is of codimension at least two. Moreover, since $\mathcal{R}(X)^G$ is K-graded, the H-action on $\overline{X}$ descends to an H-action on $\overline{Y}'$ leaving $\overline{Y}$ invariant.

Construction 5.1. Let $\widehat{V}' \subseteq \overline{Y}'$ be an H-invariant open subset with $\kappa^{-1}(\widehat{V}') \subseteq \widehat{X}$ admitting a good quotient $q': \widehat{V}' \to V'$ for the action of H. Set $\widehat{V} := \overline{Y} \cap \widehat{V}'$ and suppose we have $(*)$: for each $v \in V := q(\widehat{V})$, the closed H-orbit of $q'^{-1}(v)$ lies in $\widehat{V}$. Then $U := p(\widehat{U})$, where $\widehat{U} := \kappa^{-1}(\widehat{V})$, is open in X, admits a strong categorical quotient $r: U \to V$ for the action of G in the category of constructible spaces and U is covered by r-saturated affine open subsets. For convenience, we summarize the data in a commutative diagram



Lemma 5.2. *Let a reductive group H act on a normal variety $\widehat{V}'$ with good quotient $q' : \widehat{V}' \to V'$ and let $\widehat{V} \subseteq \widehat{V}'$ be an H-invariant constructible subset. If $\widehat{V}' \setminus \widehat{V}$ is of codimension at least two in $\widehat{V}'$ and for every $v \in V := q'(\widehat{V})$ the closed H-orbit of $q'^{-1}(v)$ lies in $\widehat{V}$, then $q : \widehat{V} \to V$, $x \mapsto q'(x)$ is a strong categorical quotient for the action of H on $\widehat{V}$ in the category of constructible spaces.*

Proof. Let $\varphi : \widehat{V} \to Z$ be any H-invariant morphism to a constructible space. By assumption, we have $\varphi = \psi \circ q$ with a set-theoretical map $\psi : V \to Z$. In order to see that this map is a morphism note that firstly V carries the quotient topology with respect to $q : \widehat{V} \to V$, because V' carries the quotient topology with respect to $q' : \widehat{V}' \to V'$, and secondly, that due to the fact that $\widehat{V}' \setminus \widehat{V}$ is of codimension at least two, the canonical morphism $\mathcal{O}_V \to q_* \mathcal{O}_{\widehat{V}}^H$ is an isomorphism. Clearly, the arguments work as well locally with respect to V, and thus we have even a strong categorical quotient. □

Proof of Construction 5.1. Since $q : \widehat{V}' \to V'$ is a good quotient, the set $\widehat{V}'$ is covered by q-saturated affine open subsets, and these are of the form $\overline{Y}'_{g_i}$. Thus, $\widehat{U} := \kappa^{-1}(\widehat{V}')$ is covered by the $q \circ \kappa$-saturated open subsets $\overline{X}_{f_i}$, where $f_i := \kappa^*(g_i)$. Since $\widehat{U}$ is $(G \times H)$-invariant, its image $U = p(\widehat{U})$ is open and G-invariant. Moreover, U is covered by the G-invariant affine open subsets $U_i := p(\overline{X}_{f_i})$. Since $p : \widehat{U} \to U$ is a categorical quotient, we have an induced morphism $r : U \to V$. By Lemma 5.2 and Theorem 1.1, this is a strong categorical quotient. Moreover, by construction, the sets U_i give the desired r-saturated affine covering. □

Now, in addition to the assumptions made so far, let X be projective. For every Weil divisor $D \in K$, we may define the associated set of semistable points $X^{ss}(D)$ as the union of all the affine sets X_f, where $n > 0$ and $f \in \mathcal{R}(X)^G_{nD}$. Then, for any ample divisor $D \in K$, we have

$$X^{ss}(D) = p\bigl(\kappa'^{-1}\bigl(\overline{Y}'^{ss}(D)\bigr)\bigr), \quad \overline{Y}'^{ss}(D) := \bigcup_{\substack{f \in \mathcal{R}(X)^G_{nD} \\ n>0}} \overline{Y}'_f.$$

Note that due to our finiteness assumptions on the Cox ring $\mathcal{R}(X)$ and the ring $\mathcal{R}(X)^G$ of invariants, the set $X^{ss}(D)$ coincides with the set of finitely generated semistable points introduced in [7, Definition 4.2.6]. Applying Construction 5.1 shows existence of a categorical quotient.

Corollary 5.3. *Let $D \in K$ and suppose that $\widehat{V}' = \overline{Y}'^{ss}(D)$ satisfies Condition $(*)$ of Construction 5.1, e.g., all points of $\widehat{V}'$ are stable. Then there is a strong categorical quotient $X^{ss}(D) \to V$ for the G-action, where $V = q(\overline{Y} \cap \widehat{V}')$ and $q : \widehat{V}' \to \widehat{V}' /\!\!/ H$ is the good quotient.*

Now one may apply the combinatorial description of GIT-equivalence for torus actions on factorial affine varieties, see [4, Section 3], to the action of H on $\overline{Y}'$, and thus compute the variation of the Doran–Kirwan GIT-quotients. We demonstrate this by means of the following example.

Example 5.4. Compare also [7, Example 4.1.10]. Consider the action of the additive group $G = \mathbb{K}$ on $X = \mathbb{P}_1 \times \mathbb{P}_1$ given by

$$\lambda \cdot \bigl([a, c], [b, d]\bigr) := \bigl([a + \lambda c, c], [b + \lambda d, d]\bigr).$$

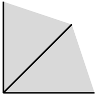
We have an obvious lifting of the action to the characteristic space $\widehat{X}$: The extension to $\overline{X} = \mathbb{K}^2 \times \mathbb{K}^2$ was discussed in Example 4.1, we have $\overline{Y}' = \mathbb{K}^3$ and the quotient map is

$$\pi' : \overline{X} \to \overline{Y}', \qquad \bigl((a, c), (b, d)\bigr) \mapsto (c, d, ad - bc).$$

The image is $\overline{Y} = \mathbb{K}^* \times \mathbb{K} \times \mathbb{K} \cup \mathbb{K} \times \mathbb{K}^* \times \mathbb{K} \cup \{(0,0,0)\}$. Now, the torus H is $\mathbb{K}^* \times \mathbb{K}^*$ and it acts on $\overline{X}$ via

$$(t_1, t_2) \cdot \big((a, c), (b, d)\big) = \big((t_1 a, t_1 c), (t_2 b, t_2 d)\big).$$

The induced H-action on $\overline{Y}'$ is given by $t \cdot (u, v, w) = (t_1 u, t_2 v, t_1 t_2 w)$. Its GIT-fan in $\mathbb{X}(H) = \mathbb{Z}^2$ looks like



The two full-dimensional chambers correspond via Construction 5.1 to the two sets $U_1 := \mathbb{P}_1 \times \mathbb{K}$ and $U_2 := \mathbb{K} \times \mathbb{P}_1$ of semistable points. Both of them have a strong categorical quotient $U_i \to \mathbb{P}_1$ in the category of varieties.

References

[1] A. A'Campo-Neuen, Note on a counterexample to Hilbert's fourteenth problem given by P. Roberts, Indag. Math. (N.S.) 5 (3) (1994) 253–257.

[2] A. A'Campo-Neuen, J. Hausen, Examples and counterexamples for existence of categorical quotients, J. Algebra 231 (1) (2000) 67–85.

[3] I. Arzhantsev, U. Derenthal, J. Hausen, A. Laface, Cox rings, arXiv:1003.4229, see also the authors' webpages.

[4] I.V. Arzhantsev, J. Hausen, Geometric invariant theory via Cox rings, J. Pure Appl. Algebra 213 (1) (2009) 154–172.

[5] A. Białynicki-Birula, Categorical quotients, J. Algebra 239 (1) (2001) 35–55.

[6] H. Derksen, G. Kemper, Computational Invariant Theory. Invariant Theory and Algebraic Transformation Groups, I, Encyclopaedia Math. Sci., vol. 130, Springer-Verlag, Berlin, 2002.

[7] B. Doran, F. Kirwan, Towards non-reductive geometric invariant theory, Pure Appl. Math. Q. 3 (1, part 3) (2007) 61–105.

[8] E. Dufresne, Finite separating sets and quasi-affine quotients, arXiv:1102.2132v1, 2011.

[9] A. Fauntleroy, Categorical quotients of certain algebraic group actions, Illinois J. Math. 27 (1) (1983) 115–124.

[10] W. Ferrer Santos, A. Rittatore, Actions and Invariants of Algebraic Groups, Pure Appl. Math. (Boca Raton), vol. 269, Chapman and Hall/CRC, Boca Raton, FL, 2005.

[11] V.L. Popov, E.B. Vinberg, Invariant theory, in: Algebraic Geometry IV, VINITI, Moscow, 1989, pp. 137–314 (in Russian); English transl. in: Algebraic Geometry IV, Encyclopaedia Math. Sci., vol. 55, Springer-Verlag, Berlin, 1994, pp. 123–284.

[12] L. Renner, A. Rittatore, Observable actions of algebraic groups, Transform. Groups 14 (4) (2009) 985–999.

[13] P. Roberts, An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's fourteenth problem, J. Algebra 132 (2) (1990) 461–473.

[14] J. Winkelmann, On free holomorphic **C**-actions on C^n and homogeneous Stein manifolds, Math. Ann. 286 (1–3) (1990) 593–612.

[15] J. Winkelmann, Invariant rings and quasiaffine quotients, Math. Z. 244 (1) (2003) 163–174.