On 4-reflective complex analytic planar billiards

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Abstract

The famous conjecture of V.Ya.Ivrii [11] says that *in every billiard with infinitely-smooth boundary in a Euclidean space the set of periodic orbits has measure zero*. In the present paper we study its complex analytic version for quadrilateral orbits in two dimensions, with reflections from holomorphic curves. We present the complete classification of 4-reflective analytic counterexamples: billiards formed by four holomorphic curves in the projective plane that have open set of quadrilateral orbits. This extends the author's result [5] classifying 4-reflective planar algebraic counterexamples. We provide applications to real billiards: classification of 4-reflective real planar analytic pseudo-billiards; solution of the piecewise-analytic case of Tabachnikov's commuting planar billiard problem; solution of a particular case of Plakhov's Invisibility Conjecture. In particular, we retrieve the solution of Ivrii's Conjecture for quadrilateral orbits in planar billiards [7, 8] in piecewise-analytic case

Contents

1 Introduction

The famous V.Ya.Ivrii's conjecture [11] says that *in every billiard with infinitely-smooth boundary in a Euclidean space of any dimension the set of periodic orbits has measure zero*. As it was shown by V.Ya.Ivrii [11], his conjecture implies the famous H.Weyl's conjecture on the two-term asymptotics of the spectrum of Laplacian [21]. A brief historical survey of both conjectures with references is presented in [7, 8].

For the proof of Ivrii's conjecture it suffices to show that for every $k \in \mathbb{N}$ the set ot k-periodic orbits has measure zero. For $k = 3$ this was proved in [2, 17, 18, 20, 22]. For $k = 4$ in dimension two this was proved in [7, 8].

Remark 1.1 Ivrii's conjecture is open already for piecewise-analytic billiards, and we believe that this is its principal case. In the latter case Ivrii's conjecture is equivalent to the statement saying that for every $k \in \mathbb{N}$ the set of k-periodic orbits has empty interior.

An approach to Ivrii's conjecture via studying its complexification was suggested in [5, 6]. The *complexified planar Ivrii's conjecture* stated in loc.cit. and recalled below is the problem to classify all the so-called k-*reflective complex planar analytic billiards:* those collections of k complex analytic curves in $\mathbb{CP}^2$ for which the corresponding billiard has an open set of k-periodic orbits. Results on complexified Ivrii's conjecture have applications to another analogue of Ivrii's conjecture: Plakhov's Invisibility Conjecture, see [6]. In [5] the algebraic 4-reflective planar billiards were classified. In the present paper we give classification of *complex analytic* 4-reflective planar billiards. At the end of the paper we deduce classification of the so-called 4-reflective real analytic planar pseudo-billiards: 4-reflective billiards where the reflection law allows to change the side with respect to the reflecting tangent line. This generalizes the solution of Ivrii's Conjecture for quadrilateral orbits in planar billiards [7, 8] in the piecewise-analytic case. As applications of the new result, we give solutions to the piecewise-analytic cases of Tabachnikov's commuting billiard problem and Plakhov's Invisibility Conjecture for $k = 4$ in two dimensions. Basic definitions and statement of main result are given below.

1.1 Main result: classification of 4-reflective complex analytic planar billiards

To recall the complexified Ivrii's conjecture and state the main result, let us recall some basic definitions contained in [5, section 1]. We consider the complex plane $\mathbb{C}^2$ with the complexified Euclidean metric, which is the standard complex-bilinear quadratic form $dz_1^2 + dz_2^2$. This defines the notion of symmetry with respect to a complex line, reflections with respect to complex lines and more generally, reflections of complex lines with respect to complex analytic (algebraic) curves. The symmetry is defined by the same formula, as in the real case. More details concerning the complex reflection law are given in Subsection 2.1.

Definition 1.2 A complex projective line $l \subset \mathbb{CP}^2 \supset \mathbb{C}^2$ is *isotropic*, if either it coincides with the infinity line, or the complexified Euclidean quadratic form vanishes on l. Or equivalently, a line is isotropic, if it passes through some of two points with homogeneous coordinates $(1 : \pm i : 0)$: the so-called *isotropic points at infinity* (also known as *cyclic* (or *circular*) points).

Convention 1.3 *Everywhere below by an irreducible analytic curve in $\mathbb{CP}^n$ we mean a non-constant $\mathbb{CP}^n$- valued holomorphic function on a connected Riemann surface.*

Definition 1.4 [5, definition 1.3] A *complex analytic (algebraic) planar billiard* is a finite collection of complex analytic (algebraic) curves $a_1, \dots, a_k \subset \mathbb{CP}^2$ that are not isotropic lines; set $a_{k+1} = a_1$, $a_0 = a_k$. A *k-periodic billiard orbit* is a collection of points $A_j \in a_j$, $A_{k+1} = A_1$, $A_0 = A_k$, such that for every $j = 1, \dots, k$ one has $A_{j+1} \neq A_j$, the tangent line $T_{A_j} a_j$ is not isotropic and the complex lines $A_{j-1}A_j$ and $A_j A_{j+1}$ are symmetric with respect to the line $T_{A_j} a_j$ and are distinct from it. (Properly saying, we have to take points A_j together with prescribed branches of curves a_j at A_j: this specifies the line $T_{A_j} a_j$ in unique way, if A_j is a self-intersection point of the curve a_j.)

Definition 1.5 [5, definition 1.4] A complex analytic (algebraic) billiard $a_1, \dots, a_k$ is *k-reflective,* if it has an open set of k-periodic orbits. In more detail, this means that there exists an open set of pairs $(A_1, A_2) \in a_1 \times a_2$ extendable to k-periodic orbits $A_1 \dots A_k$. (Then the latter property automatically holds for every other pair of neighbor mirrors a_j, a_{j+1}.)

Problem (Complexified version of Ivrii's conjecture). *Classify all the k-reflective complex analytic (algebraic) billiards.*

Theorem 1.6 *A complex planar analytic billiard a, b, c, d is 4-reflective, if and only if it has one of the three following types:*

1) one of the mirrors, say a is a line, $c = a$, the curves b and d are symmetric with respect to the line a and distinct from it, see Section 5, Fig.8;

2) the mirrors are distinct lines through the same point $O \in \mathbb{CP}^2$, the pair of lines (a, b) is transformed to (d, c) by complex rotation around O, i.e., a complex isometry $\mathbb{C}^2 \to \mathbb{C}^2$ fixing O with unit Jacobian, see Section 5, Fig.9;

3) $a = c$, $b = d$, and they are distinct confocal conics, see Section 5, Fig.10–13.

4

Remark 1.7 Theorem 1.6 in the algebraic case is given by [5, theorem 1.11], which implies the 4-reflectivity of billiards of types 2) and 3). The proof of 4-reflectivity of billiards of type 1) repeats the proof in the algebraic case, see [5, example 1.7].

1.2 The plan of the proof of Theorem 1.6

Theorem 1.6 is obviously implied by the two following theorems.

Theorem 1.8 *Every 4-reflective complex planar analytic billiard with at least one algebraic mirror has one of the above types 1)–3).*

Theorem 1.9 *Let in a complex planar analytic 4-reflective billiard no mirror be a line. Then all the mirrors are algebraic curves.*

Theorems 1.8 and 1.9 are proved in Subsection 3.5 and Section 4 respectively.

Remark 1.10 Theorem 1.6 is local and can be formulated just for a *germ of 4-reflective analytic billiard:* a collection of irreducible germs of analytic curves (a, A), (b, B), (c, C), (d, D) in $\mathbb{CP}^2$ such that the marked quadrilateral $ABCD$ lies in an open set of quadrilateral orbits of the corresponding billiard.

For the proof of Theorem 1.6 we study the maximal analytic extensions of the mirrors. These are analytic curves parametrized by abstract connected Riemann surfaces, which we will denote by $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$. The latter are called the *maximal normalizations,* see the corresponding background material in Subsection 2.2. We represent the open set of quadrilateral orbits as a subset in $\hat{a} \times \hat{b} \times \hat{c} \times \hat{d}$ and will denote it by U_0. Its closure

$$U = \overline{U_0} \subset \hat{a} \times \hat{b} \times \hat{c} \times \hat{d}$$

in the usual topology is an analytic subset with only two-dimensional irreducible components. It will be called the *4-reflective set*, see [5, definition 2.13 and proposition 2.14]. The complement $U \setminus U_0$ consists of the so-called degenerate quadrilateral orbits: quadrilaterals $ABCD$ satisfying the reflection law that have either a pair of coinciding neighbor vertices, or a pair of coinciding adjacent edges, e.g., an edge tangent to a mirror through an adjacent vertex, or an isotropic tangency vertex.

One of the main ideas of the proof of Theorem 1.6 is similar to that from [7, 8, 5]: to study the degenerate orbit set $U \setminus U_0$. This idea itself

together with basic algebraic geometry allowed to treat the algebraic case in [5]. One of the key facts used in the proof was properness (and hence, epimorphicity) of the projection $U \to \hat{a} \times \hat{b}$ to the position of two neighbor vertices. In the algebraic case the properness is automatic (follows from Remmert's Proper Mapping Theorem [9, p.46 of Russian edition]), but in the general analytic case under consideration it isn't. We prove that the above projection is indeed proper in the analytic case. The most part of the proof of Theorem 1.6, and in particular, the proof of properness are based on studying restricted versions of Birkhoff distribution, which was introduced in [2]. All the Birkhoff distributions are briefly described below; more details are given in Subsection 2.7.

Definition 1.11 Let M be an n-dimensional (real or complex) analytic manifold. Let $\mathcal{D}$ be a d-dimensional analytic distribution on M, i.e., $\mathcal{D}(x) \subset T_x M$ is a d-dimensional subspace for every $x \in M$ and the map $x \mapsto \mathcal{D}(x)$ is analytic. Let $l \le d$. An l-dimensional surface $S \subset M$ is said to be an *integral surface* for the distribution $\mathcal{D}$, if $T_x S \subset \mathcal{D}(x)$ for every $x \in S$.

Consider the projectivization of the tangent bundle $T\mathbb{CP}^2$:

$$\mathcal{P} = \mathbb{P}(T\mathbb{CP}^2).$$

It is the space of pairs (A, L): $A \in \mathbb{CP}^2$, $L \subset T_A\mathbb{CP}^2$ is a one-dimensional subspace. The space $\mathcal{P}$ is three-dimensional and it carries the standard two-dimensional contact distribution $\mathcal{H}$: the plane $\mathcal{H}(A, L) \subset T_{(A,L)}\mathcal{P}$ is the preimage of the line $L \subset T_A\mathbb{CP}^2$ under the derivative of the bundle projection $\mathcal{P} \to \mathbb{CP}^2$. The product $\mathcal{P}^k$ carries the product distribution $\mathcal{H}^k$. Let $\mathcal{R}_{0,k} \subset \mathcal{P}^k$ denote the subset of points $((A_1, L_1), \dots, (A_k, L_k))$ such that for every j one has $A_{j\pm1} \ne A_j$, the lines $A_j A_{j-1}$, $A_j A_{j+1}$ are symmetric with respect to the line L_j, and the three latter lines are distinct and non-isotropic. The above product distribution induces the so-called *Birkhoff distribution* $\mathcal{D}^k$ on $\mathcal{R}_{0,k}$, see [2]. It is well-known [2] that for every analytic billiard $a_1, \dots, a_k$ *the natural lifting to $\mathcal{P}^k$ of any analytic family of its k-periodic orbits $A_1 \dots A_k$ with $L_j = T_{A_j} a_j$ lies in $\mathcal{R}_{0,k}$ and is tangent to Birkhoff distribution.* In particular, if the billiard is k-reflective, then *the lifting to $\mathcal{R}_{0,k}$ of an open set of its k-periodic orbits is an integral surface of Birkhoff distribution.*

We will study the following restricted versions $\mathcal{D}_a$ and $\mathcal{D}_{ab}$ of Birkhoff distribution that correspond respectively to 4-reflective billiards a, b, c, d with one given mirror a (or two given mirrors a and b). The products $\hat{a} \times \mathcal{P}^3$ and $\hat{a} \times \hat{b} \times \mathcal{P}^2$ admit natural inclusions to $\mathcal{P}^4$ induced by parametrizations

6

$\hat{a} \to a$, $\hat{b} \to b$. Let $M_a \subset \hat{a} \times \mathcal{P}^3$, $M_{ab} \subset \hat{a} \times \hat{b} \times \mathcal{P}^2$ denote the closures of the corresponding pullbacks of the set $\mathcal{R}_{0,4}$. The distributions $\mathcal{D}_a$, $\mathcal{D}_{ab}$ are the pullbacks of the Birkhoff distribution $\mathcal{D}^4$ on $\mathcal{R}_{0,4}$. They are singular analytic distributions on M_a and M_{ab} in the sense of Subsection 2.6. For every billiard as above the natural lifting to $\hat{a} \times \mathcal{P}^3$ ($\hat{a} \times \hat{b} \times \mathcal{P}^2$) of any open set of its quadrilateral orbits lies in M_a (M_{ab}) and is an integral surface of the corresponding distribution $\mathcal{D}_a$ (respectively, $\mathcal{D}_{ab}$).

The proof of Theorem 1.6 is split into the following steps.

Step 1. Case of two neighbor algebraic mirrors. In this case it is easy to show that all the mirrors are algebraic (Proposition 2.1 in Subsection 2.1). This together with [5, theorem 1.11] implies that the billiard under question is of one of the types 1)–3), see Remark 1.7.

From now on we consider that no two neighbor mirrors are algebraic.

Step 2. Preparatory description of the complement $U \setminus U_0$. In Subsection 2.4 we study degenerate quadrilaterals $ABCD \in U \setminus U_0$ with a pair of coinciding neighbor vertices, say $A = D$. Under mild additional assumptions, in particular, $B, C \neq A = D$, we show that the other mirrors b and c are special curves called *triangular spirals centered at A*. Namely, they are phase curves of algebraic line fields on $\mathbb{CP}^2$: the so-called triangular line fields centered at A introduced in the same subsection (Proposition 2.17; this result in the real case was proved in [8, p.320].) One of the key arguments used in the proof of Theorem 1.6 is Proposition 2.19, which says that every triangular spiral with at least two distinct centers is algebraic. In Subsections 2.3 and 2.5 we recall the results of [5, subsections 2.1, 2.2] on partial description of degenerate quadrilaterals in $U \setminus U_0$ with either an isotropic tangency vertex, or an edge tangent to a mirror through an adjacent vertex.

Step 3. Properness of the projection $U \to \hat{a} \times \hat{b}$ (Section 3, Corollary 3.4). To prove it, we study the Birkhoff distribution $\mathcal{D}_{ab}$ and prove its non-integrability in Subsection 3.1. Moreover, we show that *the closure in M_{ab} of the union of its integral surfaces (if any) is a two-dimensional analytic subset in M_{ab}* (Lemma 3.1 and Corollary 3.2.) The set U is naturally identified with either the above two-dimensional analytic subset in M_{ab}, or a smaller analytic subset. This together with Remmert's Proper Mapping Theorem implies properness of the projection $U \to \hat{a} \times \hat{b}$ (Corollary 3.4). The proof of Lemma 3.1 is done by contradiction. The contrary would imply the existence of at least three-dimensional invariant irreducible analytic subset $M \subset M_{ab}$ where the distribution $\mathcal{D}_{ab}$ is integrable. Then a complement $M^0 \subset M$ to a smaller analytic subset is saturated by open sets of quadrilateral orbits of 4-reflective billiards a, b, c, d with variable mirrors $c = c(x)$ and $d = d(x)$, $x \in M^0$. The tangent lines $L_D = T_{D(x)}d(x)$ form

an implicit multivalued function $\lambda : (A, B, D) \mapsto L_D = \lambda(A, B, D)$, which we call the *line correspondence*. We first show that there are two possible cases:

- the line correspondence is multivalued on an open set of triples $(A(x), B(x), D(x))$, $x \in M$;

- the line correspondence is meromorphic.

The first case will be treated in Subsection 3.3. We show that there exist $x, y \in M^0$ projected to the same vertices A, B, $D = D_0$ but with distinct tangent lines $T_{D_0} d(x) \neq T_{D_0} d(y)$, D_0 being not a cusp[1] of the curves $d(x)$ and $d(y)$. We then deduce that the billiard $d(y)$, $d(x)$, $c(x)$, $c(y)$ is 4-reflective (as in [5, proof of lemma 3.1]), and the mirror $c(x)$ is a triangular spiral with center D_0 (Proposition 2.17, Step 2). Then we slightly deform y with fixed vertices A and B to a point y' so that the corresponding mirror $d(y')$ intersects $d(x)$ at a point $D_1 \neq D_0$. We get analogously that the curve $c(x)$ is a triangular spiral with two distinct centers D_0 and D_1. This implies that $c(x)$ is algebraic (Proposition 2.19, Step 2). Similarly, we show that $c(y)$ is algebraic, fixing y and deforming x. Hence, the mirror $d(x)$ of the 4-reflective billiard $d(y)$, $d(x)$, $c(x)$, $c(y)$ is algebraic, as are $c(x)$ and $c(y)$ (Proposition 2.1, Step 1). Similarly, a and b are algebraic, as are $c(x)$ and $d(x)$. The contradiction thus obtained implies that the first case is impossible.

In the second case we show (in Subsection 3.4) that for an open set of points $x \in M^0$ the mirrors $c(x)$ and $d(x)$ are lines. Hence, the curves a and b are algebraic, by Step 1, – a contradiction. Finally, none of the above cases is possible. The contradiction thus obtained will prove Lemma 3.1.

Step 4. Case of one algebraic mirror, say a: proof of Theorem 1.8 (Subsection 3.5). Properness of the projection $U \to \hat{a} \times \hat{b}$ (Step 3) implies properness of the projection $U \to \hat{b}$ (algebraicity). Therefore, the preimage in U of every point $B \in \hat{b}$ is a compact holomorphic curve. This immediately implies that the mirror c is algebraic and there are two possibilities:

- either all the mirrors are algebraic, and we are done;

- or the projection of the above preimage to the position of the point D is constant for every B.

In the latter case we show that $a = c$ is a line and the mirrors $b, d \neq a$ are symmetric with respect to it: the billiard has type 1). This will prove Theorem 1.8.

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[1]Everywhere in the paper by *cusp* we mean the singularity of an arbitrary irreducible singular germ of analytic curve, not necessarily the one given by equation $x^2 = y^3 + \dots$ in appropriate coordinates.

From now on we consider that no mirror is algebraic. We show that this case is impossible. This will prove Theorem 1.9 and hence, Theorem 1.6.

Step 5. Case of intersected mirrors, say a and b intersect at a point A. The set U contains a non-empty at most one-dimensional compact analytic set of quadrilaterals $AACD$ (properness of projection, Step 3). We show in Subsection 3.6 that this is a discrete subset in U consisting of quadrilaterals with all the vertices coinciding with A. Indeed, otherwise, if the above subset were one-dimensional, this would immediately imply that some of the mirrors c or d is algebraic, – a contradiction. Under the additional assumption that A is regular and not an isotropic tangency point for both a and b we show that either $a = c$, or b is a line (Corollary 3.5 proved in Subsection 3.6).

In what follows we study the Birkhoff distribution $\mathcal{D}_a$ using the above results together with the involutivity theory of Pfaffian systems and a version of Cauchy–Kovalevskaya theorem. Recall that it is a three-dimensional singular analytic distribution on a 6-dimensional analytic set M_a. We fix a connected component of the open set of quadrilateral orbits of the billiard a, b, c, d. It is an integral surface, which we will denote S.

Step 6. We consider the minimal analytic subset $M \subset M_a$ containing S. The set M is irreducible and at least three-dimensional: otherwise it is two-dimensional, its fibers over points $A \in \hat{a}$ are compact holomorphic curves and the mirror b is then obviously algebraic, – a contradiction. We consider the restriction $\mathcal{D}_M$ to M of the distribution $\mathcal{D}_a$ and study it using Cartan–Kuranishi–Rashevsky involutivity theory of Pfaffian systems. The corresponding background material is recalled in Subsection 4.1. We treat two different cases: 1) the distribution $\mathcal{D}_M$ is either two-dimensional, or three-dimensional non-involutive (Subsection 4.2); 2) the distribution $\mathcal{D}_M$ is three-dimensional involutive (Subsection 4.3). The first case will be basically reduced to the two-dimensional case: we show that S is always tangent to a (single-valued or double-valued) singular integrable two-dimensional analytic distribution contained in $\mathcal{D}_M$, the integral plane distribution. We consider its integral surfaces through points $x \in M$ that correspond to 4-reflective billiards a, $b(x)$, $c(x)$, $d(x)$ with $b(x)$ intersecting a: their existence easily follows from definition and the transcendence of the curve a. Moreover, the latter integral surfaces saturate an open subset $V \subset M$. We show that either the mirror $b(x)$ is a line for all $x \in V$ (and hence, for all x regular for both M and $\mathcal{D}_M$), or the mirror $c(x)$ coincides with a for all x as above. This basically follows from Corollary 3.5, Step 5. The first subcase is impossible, since then the mirror b of the initial transcendental billiard would be a line, – a contradiction. In the second subcase the projection $\nu_C(M)$ of

the whole variety M to the position of the vertex C lies in a. For a generic $A \in \hat{a}$ we consider its preimage $W_A \subset M$ under the projection $\nu_a : M \to \hat{a}$, which is a projective algebraic variety. It follows that the projection $\nu_C(W_A)$ lies in a transcendental curve a, while it should be an algebraic subset in $\mathbb{CP}^2$ (Remmert's Proper Mapping and Chow's Theorems). Hence, $\nu_C(W_A)$ is discrete. On the other hand, it cannot be discrete, whenever b is neither a line, nor a conic, by [5, proposition 2.32]. The contradiction thus obtained shows that Case 1) is impossible. The three-dimensional involutive Case 2) is treated analogously.

2 Preliminaries

2.1 Case of two neighbor algebraic mirrors

Proposition 2.1 *Let in a 4-reflective billiard a, b, c, d the mirrors a and b be algebraic curves. Then all the mirrors are algebraic.*

Proof By symmetry, it suffices to prove algebraicity of the mirror c. Fix a quadrilateral orbit $A_0 B_0 C_0 D_0$. Consider the family of quadrilateral orbits $ABCD$ with fixed $D = D_0$. They are locally parametrized by the line $l = AD$, which lies in the space $\mathbb{CP}^1$ of lines through D. The point A depends algebraically on l, since a is algebraic. Similarly, the line AB, and hence, the point B depend algebraically on l, since a, b are algebraic and AB is symmetric to l with respect to the line $T_A a$. Analogously, the line BC, which is symmetric to AB with respect to the line $T_B b$, depends algebraically on l. The line DC also depends algebraically on l, being the reflected image of the line l with respect to the fixed line $T_D d$. Finally, the variable intersection point $C = BC \cap DC$ should also depend algebraically on l. Hence, c is algebraic. The proposition is proved. $\quad\square$

2.2 Maximal analytic extension

Recall that a germ $(a, A) \subset \mathbb{CP}^n$ of analytic curve is *irreducible*, if it is the image of a germ of analytic mapping $(\mathbb{C}, 0) \to \mathbb{CP}^n$.

Definition 2.2 [6, definition 5] Consider two holomorphic mappings of connected Riemann surfaces S_1, S_2 with base points $s_1 \in S_1$ and $s_2 \in S_2$ to $\mathbb{CP}^n$, $f_j : S_j \to \mathbb{CP}^n$, $j = 1, 2$, $f_1(s_1) = f_2(s_2)$. We say that $f_1 \leq f_2$, if there exists a holomorphic mapping $h : S_1 \to S_2$, $h(s_1) = s_2$, such that $f_1 = f_2 \circ h$. This defines a partial order on the set of classes of Riemann

surface mappings to $\mathbb{CP}^n$ up to conformal reparametrization respecting base points.

The following proposition is classical, see the proof, e.g., in [6].

Proposition 2.3 *[6, proposition 2]. Every irreducible germ of analytic curve in $\mathbb{CP}^n$ has maximal analytic extension. In more detail, let $(a, A) \subset \mathbb{CP}^n$ be an irreducible germ of analytic curve. There exists an abstract connected Riemann surface $\hat{a}$ with base point $\hat{A} \in \hat{a}$ (which we will call the* **maximal normalization** *of the germ a) and a holomorphic mapping $\pi_a : \hat{a} \to \mathbb{CP}^n$, $\pi_a(\hat{A}) = A$ with the following properties:*

- the image of germ at $\hat{A}$ of the mapping π_a is contained in a;

- π_a is the maximal mapping with the above property in the sense of Definition 2.2.

Moreover, the mapping π_a is unique up to composition with conformal isomorphism of Riemann surfaces respecting base points.

Corollary 2.4 *Let M be a complex manifold, and let $f : M \to \mathbb{CP}^n$ be a non-constant holomorphic mapping. Let $U \subset M$ be an irreducible analytic subset, and let the restriction $f|_U$ have rank one on an open subset. Let $x \in U$, and let $\pi_a : \hat{a} \to a$ be the maximal analytic curve containing the image of the germ of $f|_U$ at x. Let $\hat{U}$ be the normalization of the analytic set U (see [4, p.78]), $\pi_U : \hat{U} \to U$ be the natural projection (which is bijective outside the self-intersections of the set U). Then there exists a unique holomorphic lifting $F : \hat{U} \to \hat{a}$ such that $f \circ \pi_U = \pi_a \circ F$.*

Proof The corollary obviously holds for one-dimensional analytic subsets in open sets of the manifold M. For every point $x \in U$ and any point $y \in U$ close enough to x there exists an analytic curve in U through y and x. Applying the corollary to the restriction of the mapping f to each latter curve together with Hartogs' Theorem imply the corollary. $\square$

2.3 Complex reflection law

The material presented in this subsection is contained in [5, subsection 2.1].

We fix an Euclidean metric on $\mathbb{R}^2$ and consider its complexification: the complex-bilinear quadratic form $dz_1^2 + dz_2^2$ on the complex affine plane $\mathbb{C}^2 \subset \mathbb{CP}^2$. We denote the infinity line in $\mathbb{CP}^2$ by $\overline{\mathbb{C}}_\infty = \mathbb{CP}^2 \setminus \mathbb{C}^2$.

Definition 2.5 The *symmetry* $\mathbb{C}^2 \to \mathbb{C}^2$ with respect to a non-isotropic complex line $L \subset \mathbb{CP}^2$ is the unique non-trivial complex-isometric involution

fixing the points of the line L. It extends to a projective transformation of the ambient plane $\mathbb{CP}^2$. For every $x \in L$ it acts on the space $\mathbb{CP}^1$ of lines through x, and this action is called *symmetry at x*. If L is an isotropic line through a finite point x, then a pair of lines through x is called symmetric with respect to L, if it is a limit of symmetric pairs of lines with respect to non-isotropic lines converging to L.

Lemma 2.6 *[5, lemma 2.3] Let L be an isotropic line through a finite point x. A pair of lines (L_1, L_2) through x is symmetric with respect to L, if and only if some of them coincides with L.*

Convention 2.7 *For every irreducible analytic curve $a \subset \mathbb{CP}^2$ and a point $A \in \hat{a}$ the **local branch** a_A of the curve a at A is the germ of curve $\pi_a : (\hat{a}, A) \to \mathbb{CP}^2$, which is contained in a. By $T_A a$ we denote the tangent line to the local branch a_A at $\pi_a(A)$. Sometimes we idendity a point (subset) in a with its preimage in the normalization $\hat{a}$ and denote both subsets by the same symbol. In particular, given a subset in $\mathbb{CP}^2$, say a line l, we set $\hat{a} \cap l = \pi_a^{-1}(a \cap l) \subset \hat{a}$. If $a, b \subset \mathbb{CP}^2$ are two irreducible analytic curves, and $A \in \hat{a}$, $B \in \hat{b}$, $\pi_a(A) \neq \pi_b(B)$, then for simplicity we write $A \neq B$ and the line $\pi_a(A)\pi_b(B)$ will be referred to, as AB.*

Definition 2.8 A triple of points $BAD \in (\mathbb{CP}^2)^3$ satisfies the *complex reflection law* with respect to a given line L through A, if one of the following statements holds:

    - either $B, D \neq A$, the line L is non-isotropic and the lines AB, AD are symmetric with respect to L;

    - or $B, D \neq A$, the line L is isotropic and some of the lines AB, AD coincides with L;

    - or A coincides with some of the points B or D.

Definition 2.9 Let $a_1, \ldots, a_k \subset \mathbb{CP}^2$ be an analytic (algebraic) billiard, and let $\hat{a}_1, \ldots, \hat{a}_k$ be the maximal normalizations of its mirrors. Let $P_k \subset \hat{a}_1 \times \cdots \times \hat{a}_k$ denote the subset corresponding to k-periodic billiard orbits. The set P_k is contained in the subset $Q_k \subset \hat{a}_1 \times \cdots \times \hat{a}_k$ of (not necessarily periodic) k-orbits: the k-gons $A_1 \ldots A_k$ such that for every $2 \leq j \leq k-1$ one has $A_j \neq A_{j\pm 1}$, the line $T_{A_j} a_j$ is not isotropic and the lines $A_j A_{j-1}$, $A_j A_{j+1}$ are symmetric with respect to it and distinct from it. Let $U_0 = \mathrm{Int}(P_k)$ denote the interior of the subset $P_k \subset Q_k$. Set

$U = \overline{U_0} \subset \hat{a}_1 \times \ldots \hat{a}_k :$ the closure is taken in the usual product topology.

The set U will be called the *k-reflective set*.

Proposition 2.10 *[5, proposition 2.14]. The k-reflective set U is an analytic (algebraic) subset in $\hat{a}_1 \times \cdots \times \hat{a}_k$. The billiard is k-reflective, if and only if the k-reflective set U is non-empty; then each its irreducible component is two-dimensional. If the billiard is k-reflective, then for every point $A_1 \ldots A_k \in U$ each triple $A_{j-1}A_jA_{j+1}$ satisfies the complex reflection law from Definition 2.8 with respect to the line $T_{A_j}a_j$, and each projection $U \to \hat{a}_j \times \hat{a}_{j+1}$ is a submersion on an open dense subset in U.*

   Addendum. *For every k-reflective billiard the latter projections $U \to \hat{a}_j \times \hat{a}_{j+1}$ are local biholomorphisms on the set of those k-periodic orbits whose vertices are not cusps of the corresponding mirrors.*
   The addendum follows from definition.

2.4   Triangular algebraic line fields and spirals

Here we deal with a 4-reflective complex analytic billiard a, b, c, d whose k-reflective set U contains a quadrilateral $ABCD$ with coinciding vertices $A = D$. We show (Proposition 2.17) that under mild genericity assumptions (implying, e.g., that $ABCD$ is not a single-point quadrilateral) either the mirrors b and c are conics, or they are so-called triangular spirals centered at A: phase curves of algebraic line fields invariant under the rotations around A. We show (Proposition 2.19) that every triangular spiral with two distinct centers is algebraic.

   To define triangular spirals and state the above-mentioned results, we introduce yet another restricted Birkhoff distribution on the space of "framed triangles with fixed vertex". Let us fix a point $A \in \mathbb{C}^2$ (take it as the origin) and a non-trivial complex isometry $H \in SO(2, \mathbb{C}) \setminus Id$ fixing A. Recall that $\mathcal{P} = \mathbb{P}(T\mathbb{C}\mathbb{P}^2)$, and $\mathcal{H}$ is the standard contact plane field on $\mathcal{P}$, see Subsection 1.2. Namely, for every $x = (B, L) \in \mathcal{P}$, where $B \in \mathbb{C}\mathbb{P}^2$, $L \subset T_B\mathbb{C}\mathbb{P}^2$ is a one-dimensional subspace, the plane $\mathcal{H}(x) \subset T_x\mathcal{P}$ is the preimage of the line L under the differential of the bundle projection $\mathcal{P} \to \mathbb{C}\mathbb{P}^2$. Consider the product $\mathcal{P}^2$ equipped with the four-dimensional algebraic distribution $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$. Let $\mathcal{T}_{A,H} \subset \mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^2$ denote the subset of pairs (B, C) such that $B, C \neq A$, the lines AB, AC are distinct, non-isotropic and $AC = H(AB)$. Let $M^0_{A,H} \subset \mathcal{P}^2$ denote the subset of those pairs $((B, L_B), (C, L_C))$, for which $(B, C) \in \mathcal{T}_{A,H}$, the lines L_B, L_C are non-isotropic, the lines AB, BC are symmetric with respect to the line L_B; AC, BC are symmetric with respect to the line L_C; $AB \neq L_B$, $AC \neq L_C$. Set

$$M_{A,H} = \overline{M^0_{A,H}} \subset \mathcal{P}^2 : \text{ the closure in the usual topology.}$$

This is a three-dimensional projective algebraic variety, and $M^0_{A,H} \subset M_{A,H}$ is its Zariski open and dense subset.

Proposition 2.11 *The variety $M^0_{A,H}$ is smooth and transversal to the distribution $\mathcal{H}^2$.*

Proof The smoothness is obvious. The restriction $\nu : M^0_{A,H} \to \mathcal{T}_{A,H}$ of the bundle projection $\mathcal{P}^2 \to (\mathbb{CP}^2)^2$ is a local diffeomorphism, by construction. For every $x = ((B, L_B), (C, L_C)) \in M^0_{A,H}$ the subspace $\mathcal{H}^2(x) \subset T_x(\mathcal{P}^2)$ is the preimage of the direct sum $L_B \oplus L_C \subset T_{(B,C)}(\mathbb{CP}^2)^2$ under the differential of the bundle projection. Thus, it suffices to show that for every $(B, C) \in \mathcal{T}_{A,H}$ the space $T_{(B,C)}\mathcal{T}_{A,H}$ is transversal to $L_B \oplus L_C$. Here L_B, L_C are arbitrary lines such that AB and BC are symmetric with respect to the line L_B and AC, BC are symmetric with respect to the line L_C.

For every $(B, C) \in \mathcal{T}_{A,H}$ finitely punctured lines $AB \times C$ and $B \times AC$ are contained in $\mathcal{T}_{A,H}$, by definition. We identify AB and AC with the corresponding one-dimensional subspaces in $T_B\mathbb{CP}^2$ and $T_C\mathbb{CP}^2$ respectively. Thus, $AB \oplus AC \subset T_{(B,C)}\mathcal{T}_{A,H}$ and $AB \oplus AC$ is transversal to $L_B \oplus L_C$, since $AB \neq L_B$ and $AC \neq L_C$ by definition. This proves the proposition. □

Corollary 2.12 *For every $x \in M^0_{A,H}$ the intersection*

$$\mathcal{D}_{A,H}(x) = \mathcal{H}^2(x) \cap T_x M^0_{A,H}$$

is one-dimensional, and the subspaces $\mathcal{D}_{A,H}(x)$ form an analytic (even algebraic) line field on $M^0_{A,H}$.

Proof The transversal variety $M^0_{A,H}$ and distribution $\mathcal{H}^2$ in the ambient six-dimensional space $\mathcal{P}^2$ have dimensions 3 and 4 respectively. Hence, the intersections of their tangent spaces are one-dimensional. The algebraicity of the line field $\mathcal{D}_{A,H}$ is obvious. □

Proposition 2.13 *The line field $\mathcal{D}_{A,H}$ has an algebraic first integral: appropriate branch holomorphic on $M^0_{A,H}$ of the multivalued squared perimeter*

$$P^2 = P^2((B, L_B), (C, L_C)) = (|AB| + |BC| + |CA|)^2, \text{ where}$$

$$|AB| = \sqrt{|AB|^2}, \ |BC| = \sqrt{|BC|^2}, \ |CA| = \sqrt{|CA|^2} :$$

the complex distances defined up to sign. The space $M_{A,H}$, the line field $\mathcal{D}_{A,H}$ and the above squared perimeter are invariant with respect to the complex rotation group $SO(2, \mathbb{C})$ fixing A.

Proof The $SO(2, \mathbb{C})$-invariance follows from construction. Let us show that appropriate branch holomorphic on $M^0_{A,H}$ of the squared perimeter is a first integral of the distribution $\mathcal{D}_{A,H}$. Appropriate local branch of the perimeter is a first integral: this is a complexification of a classical real statement, see, e.g., [2, section 2], and its proof is analogous to the real case. Let us show how to choose the corresponding branch of the perimeter function. To do this, we use the following remark.

Remark 2.14 For every non-isotropic line $L \subset \mathbb{C}^2$ the complex length function $L \times L \to \mathbb{C}$: $(Q_1, Q_2) \mapsto |Q_1 Q_2|$ has two branches holomorphic on $L \times L$ that differ only by sign; the latter branches will be referred to, as (holomorphic) length functions. There exists a unique affine coordinate z on L up to sign and translation (a complex natural parameter) such that the holomorphic length functions are given by the formula $\pm(z(Q_2) - z(Q_1))$. In particular, each holomorphic length function is antisymmetric.

Take an arbitrary point $((B, L_B), (C, L_C)) \in M^0_{A,H}$. We identify the lines L_B and L_C with the corresponding projective lines in $\mathbb{CP}^2$. Fix a point $B' \in L_B \setminus B$, and let $B'_1 \in AB$ and $B'_2 \in BC$ denote its orthogonal projections to the lines AB and BC respectively, see Fig.1 in the case, when $A, B, C \in \mathbb{R}^2$. Let us fix a holomorphic length function on $AB \times AB$ in
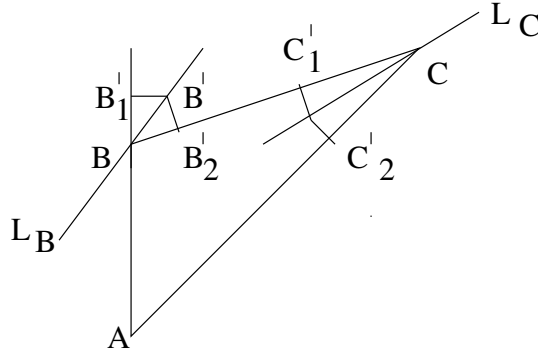
Figure 1: Length functions in the real case: here $P = |AB| + |BC| - |AC|$.

an arbitrary way. Now let us choose the holomorphic length function on $BC \times BC$ so that $|BB'_1| = |BB'_2|$. This is possible, since $|BB'_1|^2 = |BB'_2|^2$, by symmetry. This definition is independent on the choice of the point B'. Given the length function on $BC \times BC$ thus constructed, we choose the length function on $AC \times AC$ analogously to the above construction, with B'

15

replaced by $C' \in L_C$. The perimeter $|AB| + |BC| + |CA|$ thus constructed is well-defined up to sign, as is the initial length function on $AB \times AB$. Hence, its square is well-defined and holomorphic on the whole set $M_{A,H}^0$. Its level surfaces are tangent to the distribution $\mathcal{D}_{A,H}$ by construction, as in the real case, see [2, section 2]. The proposition is proved. $\quad\square$

Proposition 2.15 *Let $P^2 : M_{A,H}^0 \to \mathbb{C}$ be the squared perimeter function from the above proposition. Let $p \in \mathbb{C}$, $\mathcal{S}_p$ be an irreducible component of the level set $\{P^2 = p\}$ in $M_{A,H}^0$. The projections $\nu_G : \mathcal{S}_p \to \mathbb{CP}^2$ to the position of the vertex $G = B, C$ have discrete preimages, and thus, are submersions on Zariski open dense subsets. The restriction to $\mathcal{S}_p$ of the line field $\mathcal{D}_{A,H}$ is sent by each projection to an $SO(2,\mathbb{C})$-invariant algebraic line field on $\mathbb{CP}^2$ (depending on the choice of G) called* **triangular line field centered at A with parameters H, p.**

Proof The contrary to the discreteness of preimages of the projection, say ν_B would imply constance of the perimeter on a one-parameter family of triangles ABC with fixed vertices A and B, fixed line $AC = H(AB) = L$ and variable $C \in L$. This is obviously impossible. The algebraicity and invariance of the projected line field obviously follow from the algebraicity and invariance of the surface $\mathcal{S}_p$ and submersivity. $\quad\square$

Definition 2.16 A *triangular spiral centered at A* is a complex orbit of a triangular line field centered at A, see Fig.2a).

Proposition 2.17 *Let (a, A), (b, B), (c, C), (d, D) be germs of analytic curves in $\mathbb{CP}^2$ forming a 4-reflective analytic planar billiard: the quadrilateral $ABCD$ is contained in the 4-reflective set U, cf. Remark 1.10. Let the mirror germs a and d intersect: $A = D$. Let $B, C \neq A$, $AB \neq T_A a, T_B b$, $AC \neq T_D d, T_C c$, and let the lines AB, $T_A a$, $T_D d$, $T_B b$, $T_C c$ be not isotropic. If $AB \neq AC$, then the mirrors b and c are triangular spirals centered at A. Otherwise, if $AB = AC$, then the mirrors b and c are conics: complex circles centered at A, see Fig.2.*

Proof There exists an irreducible germ $\Gamma \subset U$ of analytic curve at $ABCD$ parametrized by local small complex parameter t and consisting of quadrilaterals $AB_t C_t D_t$ with fixed vertex A: $AB_0 C_0 D_0 = ABCD$. Let us fix it. One has $D_t \equiv D = A$. This follows from the fact that D_t is found as a point of intersection of the curve d with the line L_t symmetric to AB_t with respect to the tangent line $T_A a$. Indeed, the line L_0 is transverse to $T_D d$, as is AC,

16

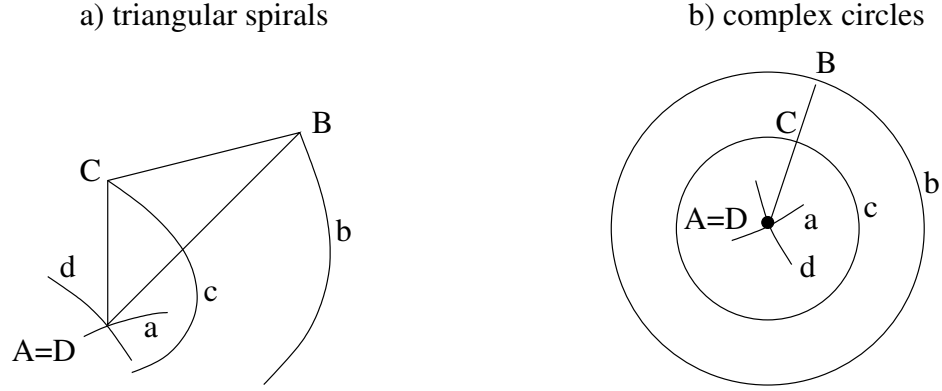a) triangular spirals   b) complex circles

Figure 2: Family of degenerate orbits with $A = D$: the mirrors b and c are either spirals, or conics

by assumption and since L_0 and AC are symmetric with respect to the line $T_D d$. Therefore, the intersection point $D_t \in L_t \cap d$ identically coincides with $D = D_0$. Let H denote the composition of symmetries with respect to the tangent lines, first $T_A a$, then $T_D d$. Thus, H is a complex isometry fixing A with unit Jacobian: a complex rotation around the point A. For every $AB_t C_t D \in \Gamma$ one has $H(AB_t) = AC_t$, by definition.

Case 1): $AB \neq AC$. Then $B \neq C$ and the germ Γ is embedded into $M^0_{A,H}$ via the mapping $t \mapsto ((B_t, T_{B_t} b), (C_t, T_{C_t} c))$, by construction and non-isotropicity condition. Its image is a phase curve of the line field $\mathcal{D}_{A,H}$, analogously to discussions in [2] and [8, p.320]. This together with Proposition 2.15 implies that the projection $\Gamma \to \mathbb{CP}^2$ to the position of each one of the vertices B and C sends Γ to a triangular spiral centered at A, see Fig.2a). Hence, b and c are triangular spirals.

Case 2): $AB = AC$. Then $H = \pm Id$, being a complex rotation around the point A that fixes three lines through A: two isotropic lines and the line AB. Therefore, $AB_t \equiv AC_t$. Note that at least one of vertices, either B_t, or C_t varies, since Γ is a curve. To treat the case under consideration, we use the following remark.

Remark 2.18 There exist no k-reflective analytic planar billiard such that some its two neighbor mirrors coincide with the same line. Indeed, in this case it would have no k-periodic orbits in the sense of Definition 1.4 (cf. [5, proof of corollary 2.19]).

Subcase 2a): $B_t \equiv C_t \not\equiv const$. This implies that $b = c$ and the line AB_t is tangent to b at variable point B_t, as in loc. cit. Therefore, $b = c = AB$, which is impossible by the above remark. Hence, this subcase is impossible.

Subcase 2b): $B_t \not\equiv C_t$. Without loss of generality we consider that $B \neq C$. Thus, for every t small enough the points A, B_t and C_t are distinct and lie on the same line. Note that $T_B b, T_C c \neq AB = AC$, by the condition of the proposition. Hence, $T_{B_t} b, T_{C_t} c \perp AB_t$ for all t. This implies that $B_t, C_t \not\equiv const$ and b, c are complex circles centered at A, see Fig.2b). This proves Proposition 2.17. □

Proposition 2.19 *Let a planar analytic curve be a triangular spiral with respect to two distinct centers. Then it is algebraic.*

Proof A triangular spiral is a phase curve of a triangular algebraic line field. The latter field is invariant under complex rotations: the isometries fixing the center of the spiral with unit Jacobian. Suppose the contrary: the spiral under consideration is not algebraic. Then the corresponding line field is uniquely defined: two algebraic line fields coinciding on a non-algebraic curve (which is Zariski dense) should coincide everywhere. Thus, the latter line field should be invariant under complex rotations around two distinct centers. The latter rotations generate the whole group of complex isometries of $\mathbb{C}^2$ with unit Jacobian. Thus, the line field is invariant under all the latter isometries, which is impossible. The contradiction thus obtained proves the proposition. □

2.5 Tangencies in k-reflective billiards

Here we recall the results of [5, subsection 2.4].

We deal with k-reflective analytic planar billiards $a_1, \ldots, a_k$ in $\mathbb{CP}^2$. Let $U \subset \hat{a}_1 \times \cdots \times \hat{a}_k$ be the k-reflective set. The results of loc.cit. presented below concern degenerate quadrilaterals in $U \backslash U_0$: limits $A_1 \ldots A_k$ of k-periodic orbits such that for a certain j with a_j being not a line the tangent line $T_{A_j} a_j$ and the adjacent edges $A_{j\pm1} A_j$ collide to the same non-isotropic limit. Then the limit vertex A_j will be called a *tangency vertex*. Proposition 2.21 shows that the latter cannot happen to be the only degeneracy of the limit k-gon. Its Corollary 2.24 presented at the end of the subsection concerns the case, when $k = 4$. It says that if the tangency vertex is distinct from its neighbor limit vertices, then its opposite vertex should be either also a tangency vertex or a cusp with a non-isotropic tangent line. Proposition 2.23 extends

18

Proposition 2.21 to the case, when some subsequent mirrors coincide and the corresponding subsequent vertices of a limiting orbit collide.

Definition 2.20 A point of a planar irreducible analytic curve is *marked*, if it is either a cusp, or an isotropic tangency point. Given a parametrized curve $\pi_a : \hat{a} \to a$, a point $A \in \hat{a}$ is marked, if it corresponds to a marked point of the local branch a_A, see Convention 2.7.

Proposition 2.21 *[5, proposition 2.16] Let $a_1, \ldots, a_k$ and U be as above. Then U contains no k-gon $A_1 \ldots A_k$ with the following properties:*
   - each pair of neighbor vertices correspond to distinct points, and no vertex is a marked point;
   - there exists a unique $s \in \{1, \ldots, k\}$ such that the line $A_s A_{s+1}$ is tangent to the curve a_s at A_s, and the latter curve is not a line, see Fig.3.

Remark 2.22 A real version of Proposition 2.21 is contained in [8] (lemma 56, p.315 for $k = 4$, and its generalization (lemma 67, p.322) for higher k).
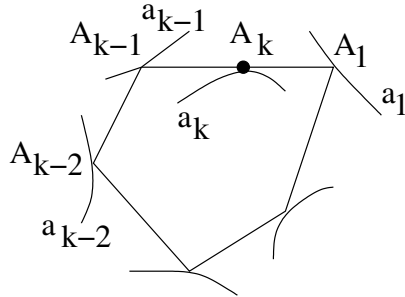
Figure 3: Impossible degeneracy of simple tangency: $s = k$.

Proposition 2.23 *[5, proposition 2.18] Let $a_1, \ldots, a_k$ and U be as at the beginning of the subsection. Then U contains no k-gon $A_1 \ldots A_k$ with the following properties:*
   1) each its vertex is not a marked point of the corresponding mirror;
   2) there exist $s, r \in \{1, \ldots, k\}$, $s < r$ such that $a = a_s = a_{s+1} = \cdots = a_r$, $A_s = A_{s+1} = \cdots = A_r$, and a is not a line;
   3) For every $j \notin \mathcal{R} = \{s, \ldots, r\}$ one has $A_j \neq A_{j\pm 1}$ and the line $A_{j-1} A_j$ is not tangent to a_j at A_j, see Fig.4.
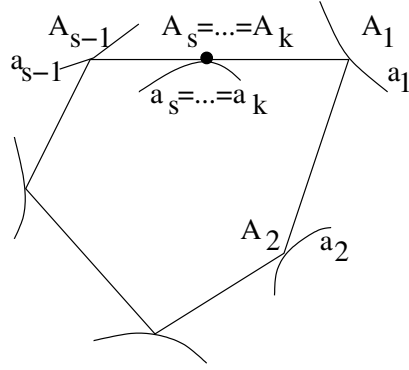
Figure 4: Coincidence of subsequent vertices and mirrors: $r = k$.

Corollary 2.24 *[5, corollary 2.20] Let a, b, c, d be a 4-reflective analytic billiard, and let b be not a line. Let $U \subset \hat{a} \times \hat{b} \times \hat{c} \times \hat{d}$ be the 4-reflective set. Let $ABCD \in U$ be such that $A \neq B$, $B \neq C$, the line $AB = BC$ is tangent to the curve b at B and is not isotropic. Then*

- either $AD = DC$ is tangent to the curve d at D, $\pi_a(A) = \pi_c(C)$, $a = c$ and the corresponding local branches coincide, i.e., $a_A = c_C$ (see Convention 2.7): "opposite tangency connection", see Fig.5a);

- or D is a cusp of the local branch d_D and the tangent line $T_D d$ is not isotropic: "tangency–cusp connection", see Fig.5b).

Figure 5: Opposite degeneracy to tangency vertex: tangency or cusp.

20

2.6 Singular analytic distributions
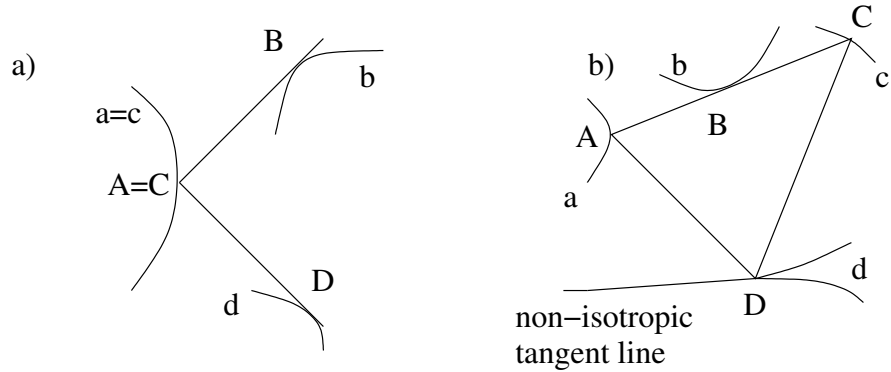
Here we recall the classical definitions and properties of singular analytic distributions.

Definition 2.25 Let W be a complex manifold, $n = dim W$, $\Sigma \subset W$ be a nowhere dense closed subset, $m \leq n$. Let $\mathcal{D}$ be an analytic field of codimension m vector subspaces $\mathcal{D}(y) \subset T_y W$, $y \in W \setminus \Sigma$. We say that $\mathcal{D}$ is a *singular analytic distribution* of codimension m (dimension $n - m$) with the singular set $\Sigma = Sing(\mathcal{D})$, if it extends analytically to no point of the set Σ and every $x \in W$ has a neighborhood U where there exists a finite collection Ω of holomorphic 1-forms such that $\mathcal{D}(y) = \{\Omega(y) = 0\}$ for every $y \in U \setminus \Sigma$.

Remark 2.26 Every k-dimensional singular analytic distribution on a complex manifold W is defined by a meromorphic[2] section of the Grassmanian k-subspace bundle $Gr_k(TW)$ of the tangent bundle TW, and vice versa: each meromorphic section defines a k-dimensional singular analytic distribution. Its singular set is an analytic subset in W of codimension at least two, being the indeterminacy locus of a meromorphic section of a bundle with compact fibers.

Example 2.27 Let M be a complex analytic manifold, $N \subset M$ be a connected complex submanifold, $\mathcal{D}$ be a (regular) analytic distribution on M. The intersection $\mathcal{D}|_N(x) = T_x N \cap \mathcal{D}(x)$ with $x \in N$ has constant and minimal dimension on an open and dense subset $N^0 \subset N$. The subspaces $\mathcal{D}|_N(x) \subset T_x N$ form a singular analytic distribution $\mathcal{D}|_N$ on N that is called the *restriction* to N of the distribution $\mathcal{D}$. Its singular set is contained in the complement $N \setminus N^0$: the set of those points x, where the above dimension is not minimal. The restriction to N of a singular analytic distribution $\mathcal{D}$ on M with $N \not\subset Sing(\mathcal{D})$ is defined analogously; it is also a singular analytic distribution on N whose singular set is contained in the union of the intersection $Sing(\mathcal{D}) \cap N$ and the set of those points $x \in N$, where the above dimension $dim(\mathcal{D}|_N(x))$ is not minimal.

---

[2]Recall that a mapping $V \to W$ of complex manifolds (or analytic sets in complex manifolds) is *meromorphic*, if it is well-defined and holomorphic on an open and dense subset in V, and the closure of its graph is an analytic subset in $V \times W$, see Convention 2.29. It is well-known that if W is compact and V is irreducible, then the set of indeterminacies of every meromorphic mapping $V \to W$ is contained in the union of the singular set of V and an analytic subset in V of codimension at least two. A mapping is *bimeromorphic*, if it is meromorphic together with its inverse.

Example 2.28 Let $\mathcal{D}$ be a singular distribution on a complex manifold W. Let M be another connected complex manifold, and let $\phi : M \to W$ be a non-constant holomorphic mapping. For every $x \in M$ set

$$\phi^* \mathcal{D}(x) = (d\phi(x))^{-1}(\mathcal{D}(\phi(x)) \cap d\phi(x)(T_x M)) \subset T_x M.$$

The subspaces $\phi^* \mathcal{D}(x)$ form a singular analytic distribution on M called the *pullback distribution*. In the case, when ϕ is an immersion on an open and dense subset, the dimension of the distribution $\phi^* \mathcal{D}$ equals the minimal dimension of the above intersection.

Convention 2.29 *Let W be a complex manifold, $M \subset W$ be an analytic subset. Everywhere below for simplicity we say that a **subset** $N \subset M$ **is analytic,** if it is an analytic subset of the ambient manifold W.*

Definition 2.30 Let W be a complex manifold, $M \subset W$ be an irreducible analytic subset, and let $\mathcal{D}$ be a singular analytic distribution on W, $M \not\subset Sing(\mathcal{D})$. There exists an open and dense subset of those points[3] $x \in M_{reg}$ regular for $\mathcal{D}$, for which the intersection $\mathcal{D}|_M(x) = \mathcal{D}(x) \cap T_x M$ has minimal dimension. Then we say that the subspaces $\mathcal{D}|_M(x)$ form a *singular analytic distribution $\mathcal{D}|_M$ on M*. It is regular on an open dense subset $M_{reg}^0 \subset M_{reg}$. Its *singular set* $M \setminus M_{reg}^0$ is the union of the set M_{sing} and the set of those points $x \in M_{reg}$ where the distribution $\mathcal{D}|_M$ does not extend analytically. The distribution $\mathcal{D}|_M$ is also called the *restriction* to M of the distribution $\mathcal{D}$. The restriction of a singular analytic distribution $\mathcal{D}|_M$ to an irreducible analytic subset $V \subset M$, $V \not\subset Sing(\mathcal{D}|_M)$ is a singular analytic distribution on V defined analogously: it coincides with $\mathcal{D}|_V$.

Example 2.31 The Birkhoff distribution $\mathcal{D}^k$ introduced at the end of Section 1 extends to a singular analytic distribution on the closure $\overline{\mathcal{R}_{0,k}} \subset \mathcal{P}^k$.

Definition 2.32 An *integral l-surface* of a singular analytic distribution $\mathcal{D}$ on an analytic variety[4] M is a holomorphic connected l-dimensional surface $S \subset M$ lying outside the singular set of $\mathcal{D}$ such that $T_x S \subset \mathcal{D}(x)$ for every $x \in S$. An m-dimensional singular analytic distribution is *integrable,* if there exists an integral m-surface through each its regular point.

---

[3]Everywhere below for an analytic set M by M_{reg} (M_{sing}) we denote the set of its smooth (respectively, singular) points

[4]Everywhere below by *analytic variety* we mean an analytic subset in a complex manifold

Remark 2.33 The singular set of a singular analytic distribution is always an analytic subset in the ambient variety (see Convention 2.29), as in the above remark. In general an integral surface is not an analytic set. Indeed, a generic linear vector field on $\mathbb{CP}^2$ has transcendental orbits. Hence, they are not analytic subsets in $\mathbb{CP}^2$, by Chow's Theorem [9, p.183 of Russian edition].

Proposition 2.34 *Let an m-dimensional singular analytic distribution $\mathcal{D}$ on an analytic subset N in a complex manifold have at least one m-dimensional integral surface. Given an arbitrary union S of m-dimensional integral surfaces, let $M \subset N$ denote the minimal analytic subset in N containing S. Then the restriction $\mathcal{D}|_M$ is an integrable singular m-dimensional distribution.*

Proof The set $\{x \in M^0_{reg} \mid \mathcal{D}(x) \subset T_x M\}$ coincides with all of M^0_{reg}, since it contains S and its closure is an analytic subset in N contained in M. Similarly, the set of those points in M^0_{reg} where the distribution $\mathcal{D}|_M$ satisfies the Frobenius integrability condition coincides with all of M^0_{reg}, since it contains S and its closure is an analytic subset in M. Thus, $\mathcal{D}|_M$ is an m-dimensional integrable distribution. The proposition is proved. □

2.7 Birkhoff distributions and periodic orbits

Here we recall the definition and basic properties of Birkhoff distribution and its restricted versions. Consider the space $\mathcal{P} = \mathbb{P}(T\mathbb{CP}^2)$, which consists of pairs (A, L), $A \in \mathbb{CP}^2$, L being a one-dimensional subspace in $T_A\mathbb{CP}^2$. Its natural projection to $\mathbb{CP}^2$ will be denoted by Π. The *standard contact structure* is the two-dimensional analytic distribution $\mathcal{H}$ on $\mathcal{P}$ given by the $d\Pi$-pullbacks of the lines L:

$$\mathcal{H}(A, L) = (d\Pi(A, L))^{-1}(L) \subset T_{(A,L)}\mathcal{P}.$$

The distribution $\mathcal{H}^k = \oplus_{j=1}^k \mathcal{H}$ is the $2k$-dimensional product distribution on $\mathcal{P}^k$. Recall that $\mathcal{R}_{0,k} \subset \mathcal{P}^k$ is the subset of k-tuples $((A_1, L_1), \ldots, (A_k, L_k))$ such that for every $j = 1, \ldots, k$ one has $A_j \neq A_{j\pm1}$, the lines A_jA_{j-1}, A_jA_{j+1} are symmetric with respect to the line L_j, and the three latter lines are distinct and non-isotropic. This is a $2k$-dimensional smooth quasiprojective variety. The *Birkhoff distribution* $\mathcal{D}^k$ is the restriction to $\mathcal{R}_{0,k}$ of the product distribution $\mathcal{H}^k$:

$$\mathcal{D}^k(x) = T_x\mathcal{R}_{0,k} \cap \mathcal{H}^k(x) \text{ for every } x \in \mathcal{R}_{0,k}. \tag{2.1}$$

This is a k-dimensional analytic distribution. It is the complexification of the real Birkhoff distribution introduced in [2]. For every two analytic curves $a, b \subset \mathbb{CP}^2$ with maximal normalizations $\pi_a : \hat{a} \to \mathbb{CP}^2$, $\pi_b : \hat{b} \to \mathbb{CP}^2$ we will denote

$$\mathcal{H}_a = \hat{a} \times \mathcal{P}^3; \ \mathcal{H}_{ab} = \hat{a} \times \hat{b} \times \mathcal{P}^2.$$

We consider the natural embeddings $\eta_a : \mathcal{H}_a \to \mathcal{P}^4$, $\eta_{ab} : \mathcal{H}_{ab} \to \mathcal{P}^4$:

$$\eta_a(A, (B, L_B), (C, L_C), (D, L_D)) = ((\pi_a(A), T_A a), (B, L_B), (C, L_C), (D, L_D)),$$

$$\eta_{ab}(A, B, (C, L_C), (D, L_D)) = ((\pi_a(A), T_A a), (\pi_b(B), T_B b), (C, L_C), (D, L_D)).$$

Remark 2.35 The critical points of the mappings η_a (η_{ab}) are contained in the sets $Cusp_a \subset \mathcal{H}_a$, $Cusp_{ab} \subset \mathcal{H}_{ab}$ of those points for which A (A or B) is a cusp of the corresponding curve (see Footnote 1 in Section 1). The mappings η_a and η_{ab} are immersions outside the sets $Cusp_a$ and $Cusp_{ab}$.

Consider the subsets

$$M_a^0 = \eta_a^{-1}(\mathcal{R}_{0,4}) \setminus Cusp_a \subset \mathcal{H}_a, \ M_{ab}^0 = \eta_{ab}^{-1}(\mathcal{R}_{0,4}) \setminus Cusp_{ab} \subset \mathcal{H}_{ab}, \quad (2.2)$$

$$M_a = \overline{M_a^0} \subset \mathcal{H}_a; \ M_{ab} = \overline{M_{ab}^0} \subset \mathcal{H}_{ab} :$$

the closures are taken in the usual topology. The subsets $M_a \subset \mathcal{H}_a$ and $M_{ab} \subset \mathcal{H}_{ab}$ are obviously analytic. The *restricted (pullback) Birkhoff distributions* $\mathcal{D}_a$ on M_a^0 and $\mathcal{D}_{ab}$ on M_{ab}^0 respectively are the pullbacks of the Birkhoff distribution $\mathcal{D}^4$:

$$\mathcal{D}_a(x) = (d\eta_a(x))^{-1}(\mathcal{D}^4(\eta_a(x)) \cap d\eta_a(x)(T_x M_a^0)) \subset T_x M_a^0, \ x \in M_a^0; \quad (2.3)$$

$$\mathcal{D}_{ab}(x) = (d\eta_{ab}(x))^{-1}(\mathcal{D}^4(\eta_{ab}(x)) \cap d\eta_{ab}(x)(T_x M_{ab}^0)) \subset T_x M_{ab}^0, \ x \in M_{ab}^0. \tag{2.4}$$

They extend to singular analytic distributions on M_a and M_{ab} respectively in the sense of Subsection 2.6. For example, $\mathcal{D}_a$ is the restriction to M_a of the distribution $T\hat{a} \oplus \mathcal{H}^3$ on $\mathcal{H}_a = \hat{a} \times \mathcal{P}^3$. One has

$$dim M_a = 6, \ dim\mathcal{D}_a = 3; \ \ dim M_{ab} = 4, \ dim\mathcal{D}_{ab} = 2.$$

Definition 2.36 (complexification of [8, definition 14]) Let $k \in \mathbb{N}$. A k-gon $A_1 \dots A_k \in (\mathbb{CP}^2)^k$ is said to be *non-degenerate,* if for every $j = 1, \dots, k$ (we set $A_{k+1} = A_1$, $A_0 = A_k$) one has $A_{j+1} \neq A_j$, $A_{j-1}A_j \neq A_j A_{j+1}$ and the line $A_j A_{j+1}$ is not isotropic. We will call the complex lines $A_j A_{j\pm1}$ the *edges adjacent to* A_j.

Remark 2.37 The above sets $\mathcal{R}_{0,k}$, M_a^0, M_{ab}^0 are projected to the sets of non-degenerate k-gons (quadrilaterals). A periodic billiard orbit in the sense of Definition 1.4 is non-degenerate, provided that its edges are non-isotropic and every two adjacent edges are distinct. The k-reflective set U of a k-reflective billiard contains an open and dense subset $U_1 \subset U$ of those non-degenerate orbits whose vertices are not marked points of the corresponding mirrors.

Definition 2.38 (cf. [8, definition 16]) An integral surface of one of the above (restricted) Birkhoff distributions is *non-trivial*, if its projection to the position of each vertex is non-constant. (Then its projection image is a holomorphic curve, since the distribution subspaces are projected to lines.)

For every analytic billiard a, b, c, d there exist natural analytic embeddings $\Psi_a : \hat{a} \times \hat{b} \times \hat{c} \times \hat{d} \to \mathcal{H}_a$, $\Psi_{ab} : \hat{a} \times \hat{b} \times \hat{c} \times \hat{d} \to \mathcal{H}_{ab}$:

$$\Psi_a(ABCD) = (A, (B, T_B b), (C, T_C c), (D, T_D d));$$
$$\Psi_{ab}(ABCD) = (A, B, (C, T_C c), (D, T_D d)). \tag{2.5}$$

Proposition 2.39 *Let a, b, c, d be a 4-reflective billiard. The mappings Ψ_a, Ψ_{ab} send the subset $U_1 \subset U$ (see Remark 2.37) to M_a^0, M_{ab}^0, and the images of its connected components are non-trivial integral surfaces of the restricted Birkhoff distributions $\mathcal{D}_a$ and $\mathcal{D}_{ab}$ respectively. Vice versa, each non-trivial integral surface of any of the latter distributions is the image of an open set of quadrilateral orbits of a 4-reflective billiard a, b, c, d with given mirror a (respectively, given mirrors a and b).*

The proposition is the direct complexification of an analogous result from [2] and [8, section 2, lemmas 17, 18].

Everywhere below for every $x \in M_{ab}^0 \subset \mathcal{H}_{ab} = \hat{a} \times \hat{b} \times \mathcal{P}^2$ we denote

$$l_a = l_a(x) = A(x)D(x), \ l_b = l_b(x) = B(x)C(x).$$

Remark 2.40 The lines l_a and l_b depend only on $(A, B) = (A(x), B(x))$: these are the lines symmetric to AB with respect to the lines $T_A a$ and $T_B b$ respectively. Sometimes we will write $l_a = l_a(A, B)$, $l_b = l_b(A, B)$.

For every $x \in M_{ab}^0$ ($x \in M_a^0$) the corresponding lines L_G, $G = (B,)C, D$, will be denoted by $L_G(x)$. The projections to the positions of vertices will be denoted by

$$\nu_a : \mathcal{H}_a \to \hat{a}, \ \nu_{ab} : \mathcal{H}_{ab} \to \hat{a} \times \hat{b},$$

$\nu_G : \mathcal{H}_a, \mathcal{H}_{ab} \to \mathbb{CP}^2$, $x \mapsto G(x)$ for $G = B, C, D$ (respectively, $G = C, D$).

Remark 2.41 The above projections ν_a and ν_{ab} are proper and epimorphic. The corresponding preimages of points are compact and naturally identified with projective algebraic varieties.

Proposition 2.42 *Each integral surface of the Birkhoff distribution $\mathcal{D}_{ab}$ is non-trivial.*

Proof Suppose the contrary: there exists a trivial integral surface $S \subset M_{ab}^0$. Then its projection to the position of some vertex is constant.

Case 1): the projection $\nu_{ab} : S \to \hat{a} \times \hat{b}$ has rank 1 at a generic point. Then S is fibered by holomorphic curves on which $A, B \equiv const$, hence $l_a, l_b \equiv const$, see Remark 2.40. Some of the vertices D or C, say D should be non-constant along some one-dimensional fiber Γ of the latter fibration. Thus, while y moves along the curve Γ, the line $l_a(y) = A(y)D(y)$ remains constant and distinct from $C(y)D(y)$, while the point $D(y)$ moves along the constant line l_a. Hence, $L_D(y) = l_a = A(y)D(y)$, by definition and since Γ is tangent to the distribution $\mathcal{D}_a$. On the other hand, $L_D(y) \neq A(y)D(y)$, by definition and since $y \in M_{ab}^0 \subset \eta_{ab}^{-1}(\mathcal{R}_{0,4})$. The contradiction thus obtained shows that Case 1) is impossible.

Case 2): the projection ν_{ab} is constant on S. This case is treated analogously to Case 1) and is also impossible.

Case 3): the projection $\nu_{ab}|_S$ has rank 2 at a generic point, while some of the vertices D or C, say D is constant along the surface S. This means that the lines $A(y)B(y)$ with $y \in S$ form a two-dimensional family, while their reflection images $l_a(y)$ from $T_A a$ pass through the same point D and hence, form a one-dimensional family. This is obviously impossible. Proposition 2.42 is proved. $\square$

3 Non-integrability of the Birkhoff distribution $\mathcal{D}_{ab}$ and corollaries

3.1 Main lemma, corollaries and plan of the proof

In the present section we prove the following lemma on the non-integrability of the two-dimensional Birkhoff distribution $\mathcal{D}_{ab}$ and corollaries.

Lemma 3.1 *For every pair of analytic curves $a, b \subset \mathbb{CP}^2$ distinct from isotropic lines that are not both lines the corresponding Birkhoff distribution $\mathcal{D}_{ab}$ is non-integrable. Moreover, there is no three-dimensional irreducible*

analytic subset $M \subset M_{ab}$ (see Convention 2.29) tangent to $\mathcal{D}_{ab}$ such that $M \cap M_{ab}^0 \neq \emptyset$ and the restriction $\mathcal{D}_{ab}|_M$ is two-dimensional and integrable.

Corollary 3.2 *In the conditions of Lemma 3.1 the union of all the integral surfaces of the Birkhoff distribution $\mathcal{D}_{ab}$ in M_{ab}^0 is contained in a two-dimensional analytic subset in M_{ab}.*

Proof The minimal analytic subset $M \subset M_{ab}$ containing all the integral surfaces should be tangent to $\mathcal{D}_{ab}$, and the distribution $\mathcal{D}_{ab}$ should be integrable there (Proposition 2.34). Hence, $dim M = 2$, by Lemma 3.1. This proves the corollary. □

Remark 3.3 In the case, when a and b are lines, the statements of Lemma 3.1 and the corollary are false. In this case there exists a one-parametric family of 4-reflective billiards a, b, c, d of type 2) from Theorem 1.6. The corresponding open sets of quadrilateral orbits form a one-parametric family of integral surfaces of the distribution $\mathcal{D}_{ab}$. They saturate an open and dense subset in a three-dimensional analytic subset in M_{ab}.

Let $\Psi = \Psi_{ab} : \hat{a} \times \hat{b} \times \hat{c} \times \hat{d} \to \mathcal{H}_{ab}$ be the embedding from (2.5).

Corollary 3.4 *Let a, b, c, d be a 4-reflective planar analytic billiard, and let U be the 4-reflective set. Then the image $\Psi(U) \subset \mathcal{H}_{ab}$ is a two-dimensional analytic subset lying in M_{ab}. The natural projection $U \to \hat{a} \times \hat{b}$ is a proper epimorphic mapping.*

Proof It suffices to prove the first statement of the corollary. Then its second statement, which is equivalent to the properness and the epimorphicity of the analytic set projection $\nu_{ab} : \Psi(U) \to \hat{a} \times \hat{b}$, follows from the properness of the projection $\mathcal{H}_{ab} \to \hat{a} \times \hat{b}$, Proposition 2.10 and Remmert's Proper Mapping Theorem [9, p.46 of Russian edition]. The image $\Psi(U)$ lies in M_{ab}, which follows from definition. Recall that $U_1 \subset U$ denote the open and dense subset of non-degenerate orbits whose vertices are not marked points. Let $\mathcal{S} \subset \mathcal{H}_{ab}$ denote the minimal analytic subset containing $\Psi(U_1)$, which obviously contains $\Psi(U)$. Each its irreducible component is two-dimensional, as is U_1, by Corollary 3.2 and since $\Psi(U_1)$ is a union of integral surfaces of the distribution $\mathcal{D}_{ab}$, see Proposition 2.39. The projections $\nu_C, \nu_D : \mathcal{S} \to \mathbb{CP}^2$ to the positions of the vertices C and D have rank one, and $\nu_C(\mathcal{S}) \subset c$, $\nu_D(\mathcal{S}) \subset d$: this holds on $\Psi(U_1)$, and hence, on each irreducible component of the set $\mathcal{S}$. Let $\hat{\mathcal{S}}$ denote the normalization of the analytic set $\mathcal{S}$, and $\pi_{\mathcal{S}} : \hat{\mathcal{S}} \to \mathcal{S}$ denote the natural projection. The above projections lift to

holomorphic mappings $\nu_{\hat{g}} : \hat{\mathcal{S}} \to \hat{g}$, $g = c, d$: $\nu_G \circ \pi_{\mathcal{S}} = \pi_g \circ \nu_{\hat{g}}$ on $\hat{\mathcal{S}}$ (Corollary 2.4). This yields an "inverse" mapping $\Psi^{-1} = \nu_{ab} \times \nu_{\hat{c}} \times \nu_{\hat{d}} : \hat{\mathcal{S}} \to \hat{a} \times \hat{b} \times \hat{c} \times \hat{d}$. Its image is contained in U, by analyticity and since the image U_1 of the set $\Psi(U_1)$ (lifted to $\hat{\mathcal{S}}$) is contained in U. This together with the inclusion $\Psi(U) \subset \mathcal{S}$ implies that $\Psi(U) = \mathcal{S}$ and proves the corollary. □

Corollary 3.5 *Let a, b, c, d be a 4-reflective planar analytic billiard, and none of the mirrors a, b be a line. Let a and b intersect at a point A represented by some non-marked points in $\hat{a}$ and $\hat{b}$. Then $a = c$ and $a \neq b$.*

At the end of the section we prove Theorem 1.8 and Corollary 3.5. Both of them will be used further on in the proof of Theorem 1.9.

Plan of the proof of Lemma 3.1. Recall that a and b are not both lines. In the case, when both a and b are algebraic curves, there exist at most unique analytic curves c and d such that the billiard a, b, c, d is 4-reflective, and if they exist, they are algebraic (Proposition 2.1 and Remark 1.7). Thus, the only integral surfaces of the distribution $\mathcal{D}_{ab}$ are given by the open set of its quadrilateral orbits, by Propositions 2.39 and 2.42. Moreover, the latter orbit set is a Zariski open dense subset in a projective algebraic surface. This immediately implies the statement of Lemma 3.1. Everywhere below we consider that some of the curves a or b is transcendental and prove the lemma by contradiction. Suppose the contrary to Lemma 3.1: there exists a three- or four-dimensional irreducible analytic subset $M \subset \mathcal{H}_{ab}$ contained in M_{ab} such that $M \cap M_{ab}^0 \neq \emptyset$ and the restriction $\mathcal{D}_M$ to $M^0 = M \cap M_{ab}^0$ of the distribution $\mathcal{D}_{ab}$ is two-dimensional and integrable. (In the second case $M = M_{ab}$.) The complement

$$\Sigma^0 = M \setminus M^0 = M \setminus M_{ab}^0 \subset M \qquad (3.1)$$

is an analytic subset of positive codimension in M, and M^0 is dense in M. Thus, every $x \in M^0$ is contained in an integral surface, and the latter is formed by quadrilateral orbits of a 4-reflective billiard a, b, $c(x)$, $d(x)$ (Propositions 2.39 and 2.42). We show that there exists an $x \in M^0$ such that the corresponding mirrors $c(x)$, $d(x)$ are algebraic. This together with Proposition 2.1 implies that a and b are algebraic. The contradiction thus obtained will prove Lemma 3.1.

For the proof of Lemma 3.1 we study the projections of the set M to the positions of three vertices (A, B, D) and to the positions of the same vertices and the line L_D: set

$$\nu_{ab,D} : x \mapsto (A(x), B(x), D(x)); \ \nu_{ab,D}^L : x \mapsto (A(x), B(x), D(x), L_D(x)),$$

$$M_D = \nu_{ab,D}(M) \subset \hat{a} \times \hat{b} \times \mathbb{CP}^2, \ M_{D,L_D} = \nu^L_{ab,D}(M) \subset \hat{a} \times \hat{b} \times \mathbb{CP}^2 \times \mathbb{CP}^{2*},$$

$$p_{L,D} : M_{D,L_D} \to M_D : \ (A, B, D, L_D) \mapsto (A, B, D).$$

Analogous projections and spaces are defined with D replaced by C.

Remark 3.6 The images $M_D = \nu_{ab,D}(M)$ and $M_{D,L_D} = \nu^L_{ab,D}(M)$ are analytic subsets in $\hat{a} \times \hat{b} \times \mathbb{CP}^2$ and $\hat{a} \times \hat{b} \times \mathbb{CP}^2 \times \mathbb{CP}^{2*}$ respectively, by Remark 2.41 and Remmert's Proper Mapping Theorem. Analogous statements hold for similarly defined sets M_C and M_{C,L_C}. All the latter analytic subsets are irreducible, as is M.

For every $G = D, C$ the correspondence $\lambda : M_G \to \mathbb{CP}^{2*}$: $(A, B, G) \mapsto L_G$ induced by the inverse $p^{-1}_{L,G}$ will be called the *line correspondence*. In Subsections 3.3 and 3.4 respectively we treat separately the following cases:
    - some of the above line correspondences is multivalued on an open set;
    - both line correspondences are meromorphic (see Footnote 2 in Subsection 2.6).
    Basic properties of the above projections and line correspondences used in the proof of Lemma 3.1 are presented in Subsection 3.2.

3.2   Line correspondence

Recall that by $\mathcal{D}_M$ we denote the restriction to M of the distribution $\mathcal{D}_{ab}$. In what follows, we denote $\Sigma^1 \subset M^0$ the subset of points $x \in M^0$ such that
    - either x is a singular point of the variety M,
    - or it is a singular point of the distribution $\mathcal{D}_M$,
    - or the restriction to $\mathcal{D}_M(x)$ of the differential $d\nu_{ab}(x)$ has rank less than two,
    - or x is a critical point of some of the projections $\nu^L_{ab,D}$, $\nu_{ab,D}$: a point where the rank of differential is not maximal,
    - or its image under some of the latter is a singularity of the image,
    - or the differential of the projection $\nu_D : M^0_{ab} \to \mathbb{CP}^2$: $y \mapsto D(y)$ vanishes on the distribution plane $\mathcal{D}_M(x)$,
    - or one of the three latter statements holds with D replaced by C.
    Let Σ^0 be the same, as in (3.1). Set

$$\Sigma = \Sigma^0 \cup \Sigma^1 \subset M. \tag{3.2}$$

This is an analytic subset in $\mathcal{H}_{ab}$ that has positive codimension in M. Its complement in M is contained in M^0 and dense in M.

Remark 3.7 For every point $x \in M \setminus \Sigma$ the corresponding germs $(g, G(x))$ of mirrors $g = a, b, c(x), d(x)$, $G = A, B, C, D$, are regular, and the points $G(x)$ are not marked. This follows from the definition of the set $\Sigma^0 \subset \Sigma$ (for the mirrors a and b) and from the two last conditions in the definition of the set $\Sigma^1 \subset \Sigma$ (for the mirrors $c(x)$ and $d(x)$). The projection $\nu_{ab} : S \to \hat{a} \times \hat{b}$ of each integral surface S of the distribution $\mathcal{D}_M$ in $M \setminus \Sigma$ is a local diffeomorphism, by the Addendum to Proposition 2.10.

Proposition 3.8 *The projections $\nu^L_{ab,D}, \nu^L_{ab,C} : M \to M_{D,L_D}, M_{C,L_C}$ are biholomorphisms on the complement $M \setminus \Sigma$.*

Proof Let us prove the statement of the proposition for the projection $\nu^L_{ab,D}$: the case of $\nu^L_{ab,C}$ is symmetric. Recall that its restriction to $M \setminus \Sigma$ has no critical points, by the definition of the set Σ. Fix a $y = (A, B, D, L_D) \in \nu^L_{ab,D}(M \setminus \Sigma)$. Let us show that a point $x \in (\nu^L_{ab,D})^{-1}(y) \cap (M \setminus \Sigma)$ is uniquely determined by y. The lines BC and DC are uniquely determined by y: BC is symmetric to the line AB with respect to the line $T_B b$; DC is symmetric to the line AD with respect to the line L_D. Therefore, so is their intersection point $C = BC \cap DC$. (Note that $BC \neq DC$, since the quadrilateral corresponding to a point $x \notin \Sigma$ should be non-degenerate.) Consider the germ at $ABCD$ of one-parametric family of quadrilaterals $A'B'C'D$ with $A' \in a$, $B' \in b$ and fixed D that satisfy the latter symmetry conditions, and in addition, we require that the line $A'B'$ be symmetric to $A'D$ with respect to the line $T_{A'} a$. The vertex C' varies along a holomorphic curve γ that depends only on y. One should obviously have $\gamma = c(x)$ (as in [8, proof of lemma 41]), and $T_C \gamma = L_C(x)$. Hence, x is uniquely defined by y. The proposition is proved. $\square$

Proposition 3.9 *The projections $\nu_{ab,D} : M \to M_D$, $\nu_{ab,C} : M \to M_C$ have rank three on an open and dense subset; hence $dim M_D = dim M_C = 3$.*

Proof The latter projections have rank at least two, since the projection ν_{ab} is epimorphic and M is irreducible. Suppose the contrary: one of the latter projections, say $\nu_{ab,D}$ has rank two on an open subset $V \subset M \setminus \Sigma$. We can and will choose V so that the integral surfaces of the distribution $\mathcal{D}_M$ in V are diffeomorphically projected by ν_{ab} onto $Y = \nu_{ab}(V) \subset \hat{a} \times \hat{b}$ (see the above remark) and the two-dimensional projection image $\nu_{ab,D}(V) \subset M_D$ is a graph of an implicit function $D = \Delta(A, B) \in l_a = l_a(A, B) \subset \mathbb{CP}^2$ (see Remark 2.40), Δ being holomorphic on Y. For every $(A, B) \in Y$ and every $x \in \nu_{ab}^{-1}(A, B) \cap V$ the germ at $D = \Delta(A, B)$ of the mirror $d(x)$ is uniquely

defined by (A, B): it coincides with the germ of holomorphic curve $\Delta(A \times \hat{b})$. Therefore, the line $L_D(x) = T_D d(x)$ also depends only on (A, B). Finally, the projection $\nu_{ab,D}^L(x)$ is completely defined by (A, B). This together with Proposition 3.8 implies that $dim M = dim M_{D,L_D} = 2$. The contradiction thus obtained proves Proposition 3.9.   □

Corollary 3.10 *For every $G = C, D$ one of the following statements holds (depending on the choice of G):*

1) $dim M = dim M_{G,L_G} = 4$; then the projection $p_{L,G} : M_{G,L_G} \to M_G$ is a fibration by algebraic curves over an open and dense subset in M_G.

2) $dim M = dim M_{G,L_G} = 3 = dim M_G$, $dim \Sigma \leq 2$, set

$$\sigma_G = \nu_{ab,G}(\Sigma), \ \ M_G^0 = M_G \setminus \sigma_G; \tag{3.3}$$

then $p_{L,G} : p_{L,G}^{-1}(M_G^0) \to M_G^0$ is a non-ramified finite covering.

a) Either the degree of the latter covering is greater than one;

b) Or it equals one: then the projection $p_{L,G} : M_{G,L_G} \to M_G$ is bimeromorphic; its inverse is holomorphic on M_G^0.

The corollary follows immediately from the above propositions.

3.3 Case of multivalued line correspondence

Here we consider that the line correspondence $\lambda : (A, B, D) \mapsto L_D$ is multivalued (Cases 1) or 2a) in Corollary 3.10). We show that both curves a and b are algebraic, and hence, this case is impossible. (The case of vertex C is symmetric.) The proof is based on the following key observation.

Proposition 3.11 *Let Σ be the same, as in (3.2). For every two points $x, y \in M \setminus \Sigma$ projected to the same $(A, B) \in \hat{a} \times \hat{b}$ such that either $(C(x), L_C(x)) \neq (C(y), L_C(y))$, or $(D(x), L_D(x)) \neq (D(y), L_D(y))$, the billiard $c(x), d(x), d(y), c(y)$ is 4-reflective.*

Proof The proof of the proposition repeats the final argument from [5, proof of lemma 3.1].

Case (i): $C(x) \neq C(y)$ and $D(x) \neq D(y)$. Then the quadrilateral $C(x)D(x)D(y)C(y)$ is an orbit of the billiard $c(x), d(x), d(y), c(y)$, by definition and reflection law, see Fig.6. Let us deform x and y along their integral surfaces of the distribution $\mathcal{D}_M$ so that $\nu_{ab}(x) = \nu_{ab}(y)$: this does not change the billiards. The quadrilateral orbits $ABC(z)D(z)$ of the billiards a, b, $c(z)$, $d(z)$, $z = x, y$ depend locally holomorphically on two parameters

$(A, B) = \nu_{ab}(z) \in \hat{a} \times \hat{b}$ (the Addendum to Proposition 2.10). Therefore, the above quadrilateral orbits $C(x)D(x)D(y)C(y)$ form a two-dimensional family. Hence, the billiard $c(x)$, $d(x)$, $d(y)$, $c(y)$ is 4-reflective.
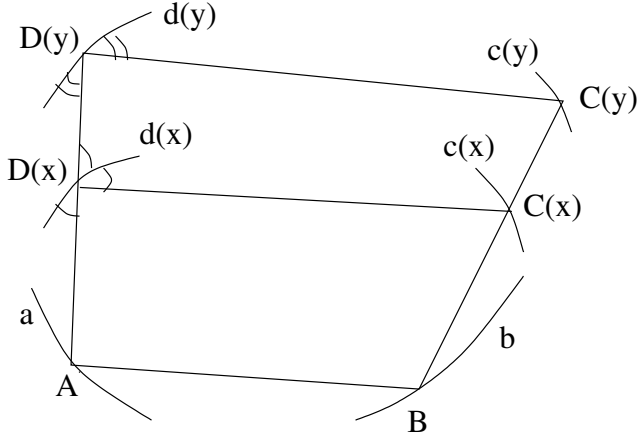

Figure 6: The 4-reflective billiard $c(x)$, $d(x)$, $d(y)$, $c(y)$: open set of quadrilateral orbits $C(x)D(x)D(y)C(y)$.

Case (ii): $D(x) = D(y) = D_0$ but $L_D(x) = T_{D(x)}d(x) \neq L_D(y) = T_{D(y)}d(y)$ (the same case with D replaced by C is symmetric). Let us show that one can achieve the inequalities of Case (i) by deforming x and y as above. The mirrors $d(x)$ and $d(y)$ intersect transversely at D_0. Therefore, deforming (A, B), one can achieve that the line $l_a = l_a(A, B)$ intersects $d(x)$ and $d(y)$ at two distinct points close to D_0. This lifts to deformation of x and y along their integral surfaces (the last statement of Remark 3.7), and we get new x, y with $\nu_{ab}(x) = \nu_{ab}(y)$ and $D(x) \neq D(y)$. One has $(C(x), L_C(x)) \neq (C(y), L_C(y))$, since otherwise, $\nu_{ab,C}^L(x) = \nu_{ab,C}^L(y)$ but $x \neq y$, – a contradiction to Proposition 3.8. Hence, one can achieve that $C(x) \neq C(y)$ via small deformation, by the above argument. This reduces us to Case (i) and proves the proposition. $\qquad\square$

In what follows, we fix arbitrary

$$x, y \in M \setminus \Sigma \text{ with } \nu_{ab,D}(x) = \nu_{ab,D}(y), \ L_D(x) \neq L_D(y).$$

They exist by multivaluedness of the line correspondence. Set

$$D_0 = D(x) = D(y), \ (A, B) = \nu_{ab}(x) = \nu_{ab}(y).$$

Claim 1. *The curves $c(x)$, $c(y)$ are either both triangular spirals centered at D_0, or both conics: complex circles centered at D_0.*

Proof The germ at $C(x)D_0D_0C(y)$ of billiard $c(x)$, $d(x)$, $d(y)$, $c(y)$ is 4-reflective (Proposition 3.11). It satisfies the non-isotropicity and line non-coincidence conditions of Proposition 2.17, since x, y correspond to non-degenerate quadrilaterals. For example, the tangent line to a mirror through each vertex is non-isotropic and distinct from the adjacent edges, by non-degeneracy. This together with Proposition 2.17 implies the claim.   □

Corollary 3.12 *The mirrors $c(z)$ and $d(z)$ are algebraic for $z = x, y$.*

Proof It suffices to prove that the mirrors $c(z)$, $z = x, y$, are both algebraic: then so are $d(z)$, by 4-reflectivity of the billiard $c(x)$, $d(x)$, $d(y)$, $c(y)$ and Proposition 2.1. Suppose the contrary: say $c(x)$ is not algebraic. Take an arbitrary $y' \in M \setminus \Sigma$ close to y with $\nu_{ab}(y') = (A, B) = \nu_{ab}(y)$ and $D(y') \neq D_0$. It exists, since the projection $\nu_{ab,D}$ is a local submersion at y. The mirror $d(y')$ intersects $d(x)$ at a point $D_1 \neq D_0$ close to D_0, since both mirrors $d(x)$, $d(y)$ are regular at D_0, see Remark 3.7, and the lines AD_0, $L_D(x) = T_{D_0}d(x)$, $L_D(y) = T_{D_0}d(y)$ are pairwise transverse. The billiard $c(x)$, $d(x)$, $d(y')$, $c(y')$ is 4-reflective, by Proposition 3.11, and its 4-reflective set contains a degenerate quadrilateral $C_1D_1D_1C_2$ close to $C(x)D_0D_0C(y)$. This together with Proposition 2.17 implies that $c(x)$ is a triangular spiral centered at D_1, as in Claim 1. Thus, $c(x)$ is a triangular spiral with two distinct centers D_0 and D_1. Hence, it is algebraic, by Proposition 2.19. The contradiction thus obtained proves the corollary.   □

Thus, the 4-reflective billiard a, b, $c(x)$, $d(x)$ has two neighbor algebraic mirrors $c(x)$ and $d(x)$. Hence, a and b are also algebraic, by Proposition 2.1. The contradiction thus obtained shows that for every $G = C, D$ the line correspondence $\lambda : (A, B, G) \mapsto L_G$ cannot be multivalued.

3.4 Meromorphic line correspondence

Here we consider that Case 2b) of Corollary 3.10 holds for both $G = D, C$: $dimM = dimM_{G,L_G} = dimM_G = 3$, and the inverse $p_{L,G}^{-1} : M_G \to M_{G,L_G}$ is holomorphic on $M_G^0 = M_G \setminus \sigma_G$, see (3.3). Thus, the line correspondence

$$\lambda : M_G \to \mathbb{CP}^{2*} : (A, B, G) \mapsto L_G,$$

which is the composition of the latter inverse with the projection to the position of the line L_G, is also holomorphic on M_G^0. We show that there exists an open and dense set of points $x \in M \setminus \Sigma$ such that both $c(x)$ and $d(x)$ are lines. This will bring us to contradiction as above and prove Lemma 3.1. It suffices to prove the above statement for the mirrors $d(x)$ only. We identify the line $L_D = \lambda(A, B, D)$ with the corresponding tangent line in $T_D\mathbb{CP}^2$. We first show (the next proposition) that the line L_D locally depends only on $D \in \mathbb{CP}^2$. We then deduce (the next corollary) that for every (A, B) from an open and dense subset $\mathcal{R} \subset \hat{a} \times \hat{b}$ the lines $L_D = \lambda(A', B', D) \subset T_D\mathbb{CP}^2$ with (A', B') close to (A, B) induce a holomorphic line field on a neighborhood of the projective line $l_a = l_a(A, B)$. Afterwards we show that the latter line field is transverse to l_a (Proposition 3.16). This implies (Lemma 3.17) that its phase curves, which coincide with the mirrors $d(x)$, form a pencil of lines through the same point.

Let $Is \subset \hat{a} \times \hat{b}$ denote the set of those points (A, B), for which the line $T_A a$ is isotropic and contains B. This is a discrete set, by definition. Let $\mu_{ab} : M_D \to \hat{a} \times \hat{b}$ denote the natural projection. The set M_D is a regular fibration by lines $A \times B \times l_a(A, B) = \mu_{ab}^{-1}(A, B)$ over the complement $\hat{a} \times \hat{b} \setminus Is$, by definition and since $dim M_D = 3$. Thus, its singular set is contained in $\mu_{ab}^{-1}(Is)$. The mapping $\lambda : M_D \to \mathbb{CP}^{2*}$ is meromorphic, by assumption (see Corollary 3.10, case 3b)). Hence, the set Ind of its indeterminacies (if non-empty) is contained in the union of the set $\mu_{ab}^{-1}(Is)$ and an analytic set of codimension at most two (see Footnote 2 in Subsection 2.6). Let $\Sigma_{ab} \subset \hat{a} \times \hat{b}$ denote the union of the sets $\mu_{ab}(Ind)$, Is and the set of those points (A, B) for which either $W_{AB} = \nu_{ab}^{-1}(A, B) \cap M \subset \Sigma$, or the line AB coincides with $l_a(A, B)$ (e.g., if $AB = T_A a = l_a(A, B)$), or some of the points A, B is a marked point of the corresponding curve a (respectively, b). Set

$$\mathcal{R} = \hat{a} \times \hat{b} \setminus \overline{\Sigma_{ab}}, \ \ M'_D = \mu_{ab}^{-1}(\mathcal{R}) \subset M_D. \tag{3.4}$$

Remark 3.13 The set $\overline{\Sigma_{ab}}$ is analytic of positive codimension, by Remmert's Proper Mapping Theorem and Chevalley–Remmert Theorem [10, p.189, statement (10)]. Hence its complement $\mathcal{R}$ is open, connected and dense. The set M'_D is a regular three-dimensional complex analytic manifold dense in M_D, and λ is analytic on M'_D, by construction. The projection

$$\mu_D : M'_D \to \mathbb{CP}^2, \ (A, B, D) \mapsto D$$

is a submersion.

Let $\mathcal{F}$ denote the foliation of the manifold M'_D by connected components of level curves $\mu_D = const$. Its leaf through a point y will be denoted $F(y)$.

Proposition 3.14 *The function λ is constant on leaves of the foliation $\mathcal{F}$.*

Proof It suffices to prove the constance of the function λ on the leaves $F(y)$ through $y = \nu_{ab,D}(x)$ with $x \in M \setminus \Sigma$, since the latter y are dense in M'_D. Fix an $x \in M \setminus \Sigma$ with $y = \nu_{ab,D}(x) \in M'_D$. The integral surface of the distribution $\mathcal{D}_M$ through x contains an analytic curve $\Gamma(x)$ through x where $D \equiv D(x)$. It corresponds to a one-parametric family of quadrilateral orbits of the billiard a, b, $c(x)$, $d(x)$ with fixed vertex $D = D(x)$. The projection $\nu_{ab,D}$ is non-constant on $\Gamma(x)$ and sends its germ at x to the leaf $F(y)$. The line $L_D = T_{D(x)}d(x)$ being obviously constant along the curve $\Gamma(x)$, the function λ is constant on the leaf $F(y)$. This together with the above discussion proves the proposition. $\qquad\square$

Corollary 3.15 *For every $(A, B) \in \mathcal{R}$, set $l_a = l_a(A, B)$, and every small neighborhood V of the line $A \times B \times l_a \subset M'_D$ the lines $\lambda(A', B', D) \subset T_D\mathbb{CP}^2$ with $(A', B', D) \in V$ form an analytic line field $\Lambda = \Lambda(D)$ on a neighborhood of the projective line l_a in $\mathbb{CP}^2$.*

Proof If V is small enough, then any its two points with the same μ_D-image lie in the same leaf of the foliation $\mathcal{F}$, by submersivity of the projection μ_D. This together with Proposition 3.14 implies the corollary. $\qquad\square$

Proposition 3.16 *The above line field Λ is transverse to the line l_a.*

Proof We prove the proposition by contradiction. Suppose the contrary: the line field Λ is tangent to l_a at some point. This means that there exist $x \in \nu_{ab}^{-1}(A, B) \setminus \Sigma$ with $L_D(x) = T_{D(x)}d(x)$ being arbitrarily close to l_a in the Fubini-Studi metric of the dual projective plane. Fix an x as above such that in addition $D(x) \notin a$ and the mirror $d(x)$ be not a line. The latter condition is possible to achieve, since otherwise, if all the corresponding mirrors $d(x)$ were lines, then l_a would be a phase curve of the line field Λ (tangency assumption). This is impossible: $x \in \nu_{ab}^{-1}(A, B) \setminus \Sigma$ correspond to non-degenerate quadrilateral billiard orbits, and the corresponding lines $L_D(x)$ are obviously transverse to l_a. We show that the 4-reflective set of the corresponding billiard a, b, $c(x)$, $d(x)$ contains a degenerate quadrilateral with tangency that is forbidden by one of Propositions 2.21 or 2.23. The contradiction thus obtained will prove the proposition.

Set $y = \nu_{ab,D}(x) \in M'_D$, $D_0 = D(x)$. The leaf $F(y)$ of the foliation $\mathcal{F}$ contains a point $z = (A_1, B_1, D_0)$ such that $L_D(z) = \Lambda(D_0) = A_1D_0$; with A_1, B_1 close to A, B, as $L_D(x)$ is close to $l_a = AD_0$. The leaf $F(y)$

35

lifts to an analytic family of quadrilaterals $A'B'C'D_0$ in the 4-reflective set U of the billiard a, b, $c(x)$, $d(x)$. Indeed, the three vertices A', B', D_0 are given by the point of the leaf. The fourth vertex C' is found as the intersection of the lines symmetric to $A'B'$, $A'D_0$ with respect to $T_{B'}b$ and $L_D(x) = T_{D_0}d(x)$ respectively; it depends analytically on the parameter of the curve $F(y)$. The point z corresponds to a degenerate quadrilateral $A_1B_1C_1D_0$ with $A_1D_0 = T_{D_0}d(x)$. There exists an irreducible germ at $A_1B_1C_1D_0$ of analytic curve $\gamma \subset U$ consisting of quadrilaterals $A'B'C'D'$ with $A'D' = D'C' = T_{D'}d(x)$. Let us now check that a generic quadrilateral in the curve γ satisfies the conditions of some of Propositions 2.21 or 2.23.

Recall that the quadrilateral $ABCD_0$ corresponding to $x \in M \setminus \Sigma$ is non-degenerate and thus, has no vertex collisions. Moreover, its vertices are not marked points, by assumption. Hence, $A_1 \neq B_1$ and A_1, B_1 are not marked, being close to (A, B), $D_0 \neq A_1$ since $D_0 \notin a$. Note that $C' \not\equiv const$ along the curve γ, since C' is a point of intersection of the curve $c(x)$ with a variable tangent line $T_{D'}d(x)$.

Case 1): $C' \not\equiv B', D'$ along the curve γ. Then a generic quadrilateral in γ has no coinciding neighbor vertices and no marked vertices, by the above statements. Therefore, the latter quadrilateral is forbidden by Proposition 2.21, – a contradiction.

Case 2): $C' \equiv D' \not\equiv B'$ along the curve γ; $c(x) = d(x)$. Then the quadrilateral $A_1B_1C_1D_0 \in \Gamma$ is forbidden by Proposition 2.23, – a contradiction, The case $C' \equiv B' \equiv D'$ is analogous.

Case 3): $C' \equiv B' \not\equiv D'$ along γ; $c(x) = b(x)$. Then the variable tangent line $T_{D'}d(x)$ would be tangent to the curves $c(x)$ and $d(x)$ at two distinct points C' and D', which is impossible. Proposition 3.16 is proved. $\qquad\square$

Lemma 3.17 *Let a holomorphic line field Λ on a neighborhood of a projective line $L \subset \mathbb{CP}^2$ be transverse to L. Then Λ is tangent to a pencil of lines through the same point.*

As it is shown below, the lemma is implied by the two following propositions.

Proposition 3.18 *Let $L \subset \mathbb{CP}^2$ be a projective line, and let $\Lambda : L \to \mathbb{CP}^{2*}$, $D \mapsto \Lambda(D)$ be a holomorphic family of lines through points $D \in L$ transverse to L. Then the lines $\Lambda(D)$ pass through the same point.*

Proof The lines $\Lambda(D)$ form a rational curve $\lambda \subset \mathbb{CP}^{2*}$. Its dual curve $\lambda^* \subset \mathbb{CP}^2$ is disjoint from the line L. Indeed, both curves are analytically

parametrized by $D \in L$, let us write the parametrization $\lambda^* = \lambda^*(D)$. Suppose the contrary, λ^* intersects L at some point $\lambda^*(D)$, see Fig.7. By defi-
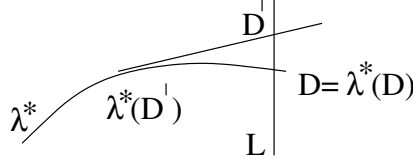

Figure 7: The curve λ^* intersecting the line L transversely: $dist(D', D) = o(dist(\lambda^*(D'), \lambda^*(D)))$

nition, $T_{\lambda^*(D)}\lambda^* = \Lambda(D)$, $\Lambda(D) \cap L = D$ (transversality), hence $D = \lambda^*(D)$. Consider the parametrized germ of the curve λ^* at $\lambda^*(D)$, which is transverse to L. For every parameter value $D' \in L$ close to D the tangent line $T_{\lambda^*(D')}\lambda^* = \Lambda(D')$ intersects L at D'. On the other hand, the distance of the intersection point D' to D should be $o(dist(\lambda^*(D'), \lambda^*(D))) = o(dist(D', D))$, as $D' \to D$, by transversality and tangency. This is obviously impossible. Thus, λ^* is an irreducible algebraic set disjoint from the line L. Hence, it is a point and Λ is a pencil of lines through it. This proves the proposition. $\square$

Proposition 3.19 *Let Λ be a holomorphic line field on a domain $V \subset \mathbb{CP}^2$ that satisfies the following*

   collinearity property: *for every three points $D_1, D_2, D_3 \in V$ lying on the same line the corresponding projective lines $\Lambda(D_j)$, $j = 1, 2, 3$ intersect at the same point.*

   Then the field Λ is tangent to a pencil of lines through the same point.

Proof It suffices to show that the phase curves of the field Λ are lines: then the statement of the proposition follows immediately. Suppose the contrary: there exists a non-linear phase curve l. Fix a point $D \in l$. The projective tangent line $L = T_D l = \Lambda(D)$ is obviously transverse to the field Λ in a punctured neighborhood of the point D in L. Therefore, for every two points $D_1, D_2 \in L$ in the latter neighborhood the intersection point $\Lambda(D_1) \cap \Lambda(D_2)$ lies outside the line L, while the three lines $L = \Lambda(D)$, $\Lambda(D_1)$, $\Lambda(D_2)$ should intersect at the same point by the collinearity property. The contradiction thus obtained proves the proposition. $\square$

Proof of Lemma 3.17. The line field $\Lambda(D)$ satisfies the above collinearity property for triples of points on each line close enough to L, by Proposition

3.18. Hence, the collinearity property holds on a neighborhood of the line L. This together with Proposition 3.19 proves the lemma. □

The phase curves of the line field Λ from Corollary 3.15 form a pencil of lines through the same point, by Proposition 3.16 and Lemma 3.17. This implies that for every x from open and dense subset $\nu_{ab}^{-1}(\mathcal{R}) \setminus \Sigma \subset M$ the mirror $d(x)$ is a line. This together with the similar statement for the mirrors $c(x)$ (proved analogously) and Proposition 2.1 implies that a and b are algebraic curves. The contradiction thus obtained proves Lemma 3.1.

3.5 Case of one algebraic mirror. Proof of Theorem 1.8

In the present subsection we consider that a, b, c, d is a 4-reflective analytic planar billiard, and the mirror a is algebraic. Without loss of generality we consider that the curve b is transcendental: in the opposite case Theorem 1.8 follows immediately from Proposition 2.1 and [5, theorem 1.11]. As it is shown below, Theorem 1.8 is implied by the following proposition.

Proposition 3.20 *In the above conditions the mirror c is also algebraic.*

Proof The projection $\nu_b : U \to \hat{b}$ of the 4-reflective set U is proper and epimorphic, by Corollary 3.4 and since a is algebraic. This implies that for every non-marked $B \in \hat{b}$ the preimage $\nu_b^{-1}(B) \subset U$ is a compact analytic curve with non-constant holomorphic projection to $\hat{c}$. Hence, $\hat{c}$ is compact and c is algebraic. The proposition is proved. □

Now let us prove Theorem 1.8. If the mirror d is algebraic, then all the mirrors are algebraic (Proposition 2.1), and we are done. Let now d be non-algebraic. Fix a non-marked point $B \in \hat{b}$ and consider an irreducible component Γ_B of the above compact analytic curve $\nu_b^{-1}(B) \subset U$. The image of its projection to the position of the vertex D in $\mathbb{CP}^2$ is either an algebraic curve, or a single point. The former case is impossible, by the non-algebraicity of the mirror d. Hence, for an open and dense set of points $B \in \hat{b}$ the projection of the curve Γ_B to the position of the vertex D is constant and is determined by B. Thus, there exists a mapping $\hat{b} \to \hat{d}$, $B \mapsto D_B$, defined on an open set $V \subset \hat{b}$ such that for every fixed $B \in V$ and variable $A \in \hat{a}$ the lines AB and AD_B are symmetric with respect to the tangent line $T_A a$. This implies that either a is a line and B, D_B are symmetric with respect to a for every $B \in \hat{b}$, or a is a conic with one-dimensional family of foci pairs (B, D_B), see [5, proposition 2.32]. The latter case being obviously impossible, the curves b, d are symmetric with respect to the line a. Applying the above argument to the algebraic mirror c instead of a, we

get that B and D_B are symmetric with respect to the line c. Thus, the above pairs (B, D_B) are symmetric with respect to both lines a and c, by construction. Therefore, $a = c \neq d$, and the billiard is of type 1) from Theorem 1.6. Theorem 1.8 is proved.

3.6 Intersected neighbor mirrors. Proof of Corollary 3.5

In the conditions of the corollary no mirror is a line, by Theorem 1.8 and since a, b are not lines. Without loss of generality we consider that all the mirrors are transcendental, since otherwise, all of them are algebraic and $a = c$, by Theorem 1.8. Let U be the 4-reflective set. Its projection to $\hat{a} \times \hat{b}$ is proper and epimorphic, by Corollary 3.4.

Claim 1. $a \neq b$.
Proof Suppose the contrary: $a = b$. Then U contains a one-parametric analytic family $\mathcal{T}$ of quadrilaterals $AACD$ with variable A, C, D, by the above epimorphicity statement. A generic quadrilateral in $\mathcal{T}$ is forbidden by Proposition 2.23. The contradiction thus obtained proves the claim. $\square$

The projection preimage of the pair (A, A) in U is a non-empty compact analytic subset $\Gamma \subset U$ of dimension at most one.

Case 1): $dim\Gamma = 1$. Then at least one of the curves $\hat{c}$, $\hat{d}$, say $\hat{c}$ is a compact Riemann surface, – a contradiction to our non-algebraicity assumption. Thus, this case is impossible.

Case 2): $dim\Gamma = 0$: Γ is a finite set.

Claim 2. *Every quadrilateral $AACD \in \Gamma$ is a single-point quadrilateral: the mirrors c and d pass through the same point A; $\pi_c(C) = \pi_d(D) = A$.*
Proof Suppose the contrary: say, $\pi_c(C) \neq A$. The projection $U \to \hat{a} \times \hat{b}$ is open on a neighborhood of the point $AACD$, since it contracts no curve to (A, A) by assumption. Therefore, each converging sequence $(A^k, B^k) \to (A, A)$ lifts to a converging sequence $A^k B^k C^k D^k \to AACD$ in U. Let us take two sequences $(A_j^k, B_j^k) \to (A, A)$, $j = 1, 2$, with lines $A_j^k B_j^k$ converging to different limits for $j = 1, 2$; this is possible, since $a \neq b$. We get two sequences of quadrilaterals $A_j^k B_j^k C_j^k D_j^k$ converging to the same quadrilateral $AACD$. On the other hand, the lines symmetric to $A_j^k B_j^k$ with respect to the tangent lines $T_{B_j^k} b$ should converge to two distinct limits H_j, $j = 1, 2$, which follows from assumption and since A is not a marked point of the curve b. The distinct limit lines H_1 and H_2 pass through the same two distinct points A and $\pi_c(C)$, by construction. The contradiction thus obtained proves the claim. $\square$

Thus, Γ is a finite set of points corresponding to the single-point quadrilateral $AAAA$. Fix one of them and denote it $AAAA$: the corresponding vertices $C \in \pi_c^{-1}(A)$ and $D \in \pi_d^{-1}(A)$ will be denoted by A. The projection $U \to \hat{c} \times \hat{d}$ is open on a neighborhood of the point $AAAA$, as in the above discussion. Let $\gamma \subset \hat{c} \times \hat{d}$ be an irreducible germ at (A, A) of analytic curve consisting of pairs (C', D') with variable C' and D' for which $C' \in T_{D'}d$. The germ γ lifts to an irreducible germ $\widetilde{\gamma}$ of analytic curve through $AAAA$ in U. The curve $\widetilde{\gamma}$ consists of quadrilaterals $A'B'C'D' \in U$ for which $B', D' \not\equiv C'$ (Proposition 2.23). Therefore, $A' \equiv C'$, by Corollary 2.24 and since A is not a marked point of the mirror b. Hence, $a = c$. Corollary 3.5 is proved.

4   Algebraicity: proof of Theorem 1.9

Theorem 1.6, and thus, Theorem 1.9 are already proved in the case, when at least one mirror is algebraic (Theorem 1.8). Here we prove Theorem 1.9 in the general case by contradiction. Suppose the contrary: there exists a 4-reflective billiard a, b, c, d with no algebraic mirrors. We study Birkhoff distribution $\mathcal{D}_a$ on the space M_a and consider its integral surface S formed by an open set of quadrilateral orbits of the billiard. Recall that $M_a \subset \mathcal{H}_a$ is a six-dimensional analytic subset, and $\mathcal{D}_a$ is a singular three-dimensional distribution on M_a, see Subsection 2.7. Set

$$M = \text{ the minimal analytic subset in } M_a \text{ containing } S.$$

This is an irreducible analytic subset in $\mathcal{H}_a$, by definition, see Convention 2.29. The intersections

$$\mathcal{D}_M(x) = \mathcal{D}_a(x) \cap T_x M, \ x \text{ being a smooth point of the variety } M,$$

induce a singular analytic distribution $\mathcal{D}_M$ on M, for which S is an integral surface. This is either two- or three-dimensional distribution, since $dim S = 2$ and $dim \mathcal{D}_a = 3$. We study the corresponding Pfaffian system: the problem to find two-dimensional surfaces of the distribution $\mathcal{D}_M$. The cases, when either $dim \mathcal{D}_M = 2$, or $dim \mathcal{D}_M = 3$ and the Pfaffian system is non-involutive in the sense of Cartan–Kuranishi–Rashevsky theory are treated in Subsection 4.2. The case, when $dim \mathcal{D}_M = 3$ and the Pfaffian system is involutive is treated in Subsection 4.3. The methods of proof of Theorem 1.9 in both cases are similar. We show that an open set of points $x \in M$ lie in integral surfaces corresponding to 4-reflective billiards a, $b(x)$, $c(x)$, $d(x)$ with regularly intersected mirrors a and $b(x)$. Then either $b(x)$ is

a line, or $c(x) = a$, by Corollary 3.5. We then show that either $\nu_C(M) \subset a$, or the mirror b of the initial billiard is a line. We get a contradiction in both cases.

The background material on involutive Pfaffian systems is recalled in the next subsection.

Recall that $\nu_a : \mathcal{H}_a \to \hat{a}$ is the natural projection. It is proper, epimorphic and for every $A \in \hat{a}$ the projection preimage

$$W_A = \nu_a^{-1}(A) \cap M$$

is a projective algebraic set. One of the key statements used in the proof of Theorem 1.9 in both cases is the following proposition.

Proposition 4.1 *There exists a complement $\hat{a}_0 \subset \hat{a}$ to a discrete subset in $\hat{a}$ such that for every $A \in \hat{a}_0$ the projection $\nu_B : W_A \to \mathbb{CP}^2$ is epimorphic and has rank two on a non-empty Zariski open and dense subset in W_A.*

Proof For every $A \in \hat{a}$ the image of the projection $\nu_B : W_A \to \mathbb{CP}^2$ is either the whole projective plane, or an algebraic subset of dimension at most one (Remmert's Proper Mapping and Chow's Theorems). Either $\nu_B(W_A) = \mathbb{CP}^2$ for all but a discrete set of points A (and then the statement of the proposition obviously holds), or it is at most one-dimensional algebraic set for an open and dense set Q of points $A \in a$, by analyticity. The latter case cannot happen, since otherwise for every non-marked $A \in \nu_a(S) \cap Q$ the set $\nu_B(W_A \cap S)$ would be an analytic curve lying simultaneously in the transcendental curve b and in at most one-dimensional algebraic set $\nu_B(W_A)$, – a contradiction. This proves the proposition. □

Corollary 4.2 *For every open dense subset $N \subset M$ whose complement $M \setminus N \subset M$ is an analytic subset there exists an open dense subset $\hat{a}_N \subset \hat{a}$ such that for every $A \in \hat{a}_N$ the intersection $\nu_B(W_A \cap N) \cap a$ contains the π_a- image of an open dense subset in $\hat{a}$: a complement to a discrete subset.*

Proof Let $\hat{a}_0$ be the same, as in Proposition 4.1. There exists an open dense subset $\hat{a}_N \subset \hat{a}_0$ such that for every $A \in \hat{a}_N$ the subset $W_A^N = W_A \cap N \subset W_A$ is dense, since N is open and dense. Then for the same A the complement $W_A \setminus W_A^N \subset W_A$ is an algebraic subset, since it is analytic (as is $M \setminus N$) and by Chow's Theorem. Therefore, the complement $\mathbb{CP}^2 \setminus \nu_B(W_A^N)$ is contained in an algebraic subset in $\mathbb{CP}^2$ of positive codimension (Proposition 4.1 and Chevalley–Remmert and Chow's Theorems). Its π_a-preimage is at most discrete, since a is non-algebraic. This implies the statement of the corollary. □

4.1 Background material: Phaffian systems and involutivity

Everywhere below in the present subsection whenever the contrary is not specified, $\mathcal{F}$ is a k-dimensional analytic distribution on an analytic manifold M, $\mathcal{F}(x) \subset T_x M$ are the corresponding subspaces.

Definition 4.3 [16, p.290] Let $k, l \in \mathbb{N}$, $k \geq l$, and let $\mathcal{F}$ be as above. A *Pfaffian system* $\mathcal{F}_{k,l}$ is the problem to find l-dimensional analytic integral surfaces of the distribution $\mathcal{F}$.

Definition 4.4 [16, p.298] An m-dimensional *integral element* of the distribution $\mathcal{F}$ is an m-dimensional vector subspace $E_m(x) \subset \mathcal{F}(x)$ such that for every 1-form ω on the ambient manifold vanishing on the subspaces of the distribution $\mathcal{F}$ its differential $d\omega$ vanishes on $E_m(x)$.

Definition 4.5 [16, p.300] A Pfaffian system $\mathcal{F}_{k,l}$ is *in involution* (or briefly, involutive), if for every $x \in M$, $p < l$ each p-dimensional integral element in $T_x M$ is contained in some $(p+1)$-dimensional integral element.

Example 4.6 A tangent subspace to an integral surface is an integral subspace. Every Pfaffian system defined by a Frobenius integrable distribution is involutive.

Remark 4.7 A Pfaffian system on a connected manifold is involutive, if and only if it is involutive on an open subset. Let a Pfaffian system $\mathcal{F}_{k,l}$ be involutive. Then for every $p < l$ and every p-dimensional integral element $E_p(x)$ the set of $(p+1)$-dimensional integral elements containing $E_p(x)$ is a projective space. An integral element $E_p(x)$ is said to be *nonsingular*, if the dimension of the latter projective space is minimal, and singular otherwise [16, p.306]. The p-dimensional integral elements form an analytic subset $\mathcal{I}_p \subset Gr_p(\mathcal{F})$ in the p-Grassmanian bundle of the subbundle $\mathcal{F} \subset TM$. The nonsingular integral elements form a non-empty open subset in $\mathcal{I}_p$. The singular integral elements form an analytic set. The two latter statements follow from definition, compactness of Grassmanian fibers and Remmert's Proper Mapping Theorem. See also loc. cit. and [16, formula (58.13)], which characterizes $p + 1$-dimensional integral elements containing a given p-dimensional one.

Theorem 4.8 *(a version of Cauchy–Kovalevskaya Theorem; implicitly contained in [16, section 60]). Let an analytic Pfaffian system $\mathcal{F}_{k,l}$ on a manifold M be involutive. Let $p \leq l$, $\Gamma \subset M$ be a $(p-1)$-dimensional ana-*

lytic integral surface such that all its tangent spaces be nonsingular $(p-1)$-dimensional integral elements. Then for every $x \in \Gamma$ there exists a germ of p-dimensional integral surface through x that contains the germ of Γ at x.

Remark 4.9 The above definitions and theorem obviously extend to the case of *singular analytic Pfaffian systems* corresponding to singular analytic distributions on (singular) analytic varieties, i.e., analytic subsets in complex manifolds: we assume that the corresponding x lies in the open set of points regular for both the variety and the distribution. In this case we use the following stronger analyticity properties of the above-defined sets of integral elements. To formulate them, let us recall the following definition.

Definition 4.10 [10, p.188]. A subset N of a complex manifold V is called *analytically constructible,* if each point of the manifold V has a neighborhood U such that the intersection $N \cap U$ is a finite union of subsets defined by finite systems of equations $f_j = 0$ and inequalities $g_i \neq 0$; f_j and g_i are holomorphic functions on U.

Recall that for a singular analytic distribution $\mathcal{D}_M$ on an analytic subset M in a complex manifold V by $M_{reg}^0 \subset M$ we denote the open and dense subset of points regular both for M and $\mathcal{D}_M$; the complement $M \setminus M_{reg}^0 \subset V$ is an analytic subset.

Proposition 4.11 *For every involutive singular analytic Pfaffian system on an analytic subset M in a complex manifold V (let $\mathcal{D}_M$ denote the corresponding distribution) the set of its one-dimensional nonsingular integral elements (see Remark 4.7) is open and dense in the projectivized bundle $\mathbb{P}(\mathcal{D}_M|_{M_{reg}^0})$. It is an analytically constructible subset in $\mathbb{P}(TV)$.*

Proof Each one-dimensional subspace in the bundle $\mathcal{D}_M|_{M_{reg}^0}$ is an integral element. The set of nonsingular integral elements is open and dense in its projectivization. Both statements follow from definition. Now for the proof of the proposition it suffices to show that the set of singular integral elements is analytically constructible. The singular distribution $\mathcal{D}_M$ is the restriction to M of a singular analytic distribution $\mathcal{D}$ on the ambient manifold V; the latter distribution is locally defined by zeros of finite collections of holomorphic 1-forms $\omega_1, \ldots, \omega_s$. A one-dimensional singular integral element is a one-dimensional subspace in $\mathcal{D}_M(x)$, $x \in M_{reg}^0$, for which the projective space of ambient two-dimensional integral elements is not of minimal dimension. For every $x \in M_{reg}^0$ and every one-dimensional

subspace $E_1(x) \subset \mathcal{D}_M(x)$ the latter condition is locally defined by vanishing of appropriate collection of minors of appropriate holomorphic matrix function: the latter matrix function depends meromorphically on $E_1(x)$ and the coefficients of the forms ω_j and their differentials, analogously to [16, formula (58.13) and p.306]. The above vanishing of minors may be written as a system of meromorphic equations in a neighborhood of each point $E_1 \in \overline{\mathbb{P}(\mathcal{D}_M|_{M^0_{reg}})}$ in $\mathbb{P}(TV)$. This proves analytic constructibility. Another, more explicit argument deals with the set of all the two-dimensional integral elements $E_2 \subset D_M|_{M^0_{reg}}$. It is analytically constructible, being locally defined by the system of meromorphic equations $d\omega_j|_{E_2} = 0$, $E_2 \in Gr_2(TV)$, and the inclusion $E_2 \in Gr_2(D_M)|_{M^0_{reg}}$ into an analytically constructible set. The set of pairs (L, E_2), $L \in \mathbb{P}(D_M|_{M^0_{reg}})$, $E_2 \supset L$ being a two-dimensional integral subspace in $D_M|_{M^0_{reg}}$, is an analytically constructible subset $\mathcal{W} \subset \overline{\mathbb{P}(D_M|_{M^0_{reg}})} \times Gr_2(TV)$. The singular elements are those points in $\mathbb{P}(D_M|_{M^0_{reg}})$, whose product projection preimages in $\mathcal{W}$ do not have the minimal possible dimension. Hence, they form an analytically constructible set (properness of the product projection, Remmert's Proper Mapping and Chevalley-Remmert Theorems). The proposition is proved. $\quad\square$

Proposition 4.12 *Every three-dimensional singular analytic distribution $\mathcal{F}$ on an irreducible analytic variety M satisfies one of the two following incompatible statements:*

- either the corresponding Pfaffian system $\mathcal{F}_{3,2}$ is involutive;

- or there exists an analytic subset $\Sigma \subset M$ such that for every $x \in M \setminus \Sigma$ the corresponding subspace $\mathcal{F}(x) \subset T_x M$ contains one and the same number (one or two) of integral planes, i.e., two-dimensional integral elements.

Proof For every point $x \in M^0_{reg}$ set

$$I(x) = \{v \in \mathcal{F}(x) \mid v \text{ is contained in an integral plane of } \mathcal{F}\}.$$

Claim. *For every $x \in M^0_{reg}$ either $I(x) = \mathcal{F}(x)$, or $I(x)$ is a union of at most two integral planes of the distribution $\mathcal{F}$.*

Proof The distribution $\mathcal{F}$ is locally defined as the field of kernels of a finite system of holomorphic 1-forms $\omega_1, \ldots, \omega_k$. By definition, a vector $v \in \mathcal{F}(x)$ is contained in $I(x)$, if and only if the 1-forms $\nu_j(x) = i_v d\omega_j(x) \in T_x^* M$, $j = 1, \ldots, k$, vanish simultaneously on some two-dimensional plane $E_x(v) \subset \mathcal{F}(x)$ containing v. Since the plane $E_x(v)$ is a hyperplane in $\mathcal{F}(x)$, the latter is equivalent to the statement that the restrictions to $\mathcal{F}(x)$ of the forms ν_j, $j = 1, \ldots, k$, are proportional. This holds if and only if the matrix formed

by the coordinate components of the latter restrictions has zero two-minors (cf. [16, p.300]). Vanishing of two-minors yields a system of homogeneous quadratic equations on the components of the vector v. The set $I(x)$ is a union of integral planes (by definition), and it is defined by a system of quadratic equations in the ambient three-dimensional space $\mathcal{F}(x)$. This is possible, if and only if either $I(x) = \mathcal{F}(x)$, or $I(x)$ is a union of at most two integral planes. The claim is proved. $\qquad\square$

If $I(x) = \mathcal{F}(x)$ for every $x \in M_{reg}^0$, then the Pfaffian system $\mathcal{F}_{3,2}$ is involutive, by definition. Otherwise, the set of those points $x \in M_{reg}^0$, for which $I(x) = \mathcal{F}(x)$, is an analytically constructible subset in M; its closure $Inv \subset M$ is an analytic subset in M of positive codimension, and so is

$$\Sigma = Inv \cup (M \setminus M_{reg}^0).$$

This follows by the above discussion with forms ω_j, as in the proof of Proposition 4.11. For every $y \in M \setminus \Sigma$ the set $I(y)$ is a union of at most two integral planes, by the above claim. Passing to a complement of a bigger analytic subset one can achieve that for every $y \in M \setminus \Sigma$ the set $I(y)$ is a union of one and the same number of integral planes: one or two. This proves Proposition 4.12. $\qquad\square$

Corollary 4.13 *Let $\mathcal{D}_M$ be a three-dimensional singular analytic distribution on an irreducible analytic subset M of a complex manifold V. Let the corresponding Pfaffian system $(\mathcal{D}_M)_{3,2}$ be non-involutive. Then there exist an analytic subset $\widetilde{M} \subset Gr_2(TV)$ and a singular two-dimensional analytic distribution $\widehat{\mathcal{D}}_M$ on $\widetilde{M}$ (i.e., on each its irreducible component) satisfying the following statements:*
1) the bundle projection $\pi : Gr_2(TV) \to V$ maps $\widetilde{M}$ onto M with degree at most two (the number of preimages of a generic point);
2) each integral surface of the three-dimensional distribution $\mathcal{D}_M$ lifts to an integral surface of the two-dimensional distribution $\widehat{\mathcal{D}}_M$.

Proof There exists an analytic subset $\Sigma \subset M$ such that for every $x \in M \setminus \Sigma$ the corresponding subspace $\mathcal{D}_M(x) \subset T_xM$ contains one and the same number (one or two) of integral planes, by Proposition 4.12. Let $\widetilde{M} \subset Gr_2(TV)$ denote the closure of the set $\widetilde{M}^0$ of all the integral planes in $\mathcal{D}_M|_{M \setminus \Sigma}$. The set $\widetilde{M}$ is analytic, since $\widetilde{M}^0$ is analytically constructible, see the proof of Proposition 4.11. Statement 1) follows from construction. Consider the canonical distribution $\mathcal{C}$ on $Gr_2(TV)$: for every $y \in Gr_2(TV)$ the subspace $\mathcal{C}(y) \subset T_yGr_2(TV)$ is the $d\pi(y)$-preimage of the plane $y \subset$

$T_{\pi(y)}V$. Let $\widetilde{\mathcal{D}}_M$ denote the restriction $\mathcal{C}|_{\widetilde{M}}$. By construction, this is a two-dimensional singular analytic distribution, and its subspaces are projected to all the integral planes of the distribution $\mathcal{D}_M$. The tangent planes to each integral surface of the distribution $\mathcal{D}_M$ are integral planes. Hence, its appropriate lifting to $\widetilde{M}$ is an integral surface of the distribution $\widetilde{\mathcal{D}}_M$. The corollary is proved. □

4.2 Two-dimensional and non-involutive cases

Here we treat the cases, when either $dim\mathcal{D}_M = 2$, or $\mathcal{D}_M$ is three-dimensional and the Pfaffian system $(\mathcal{D}_M)_{3,2}$ is non-involutive. The latter case is reduced to the two-dimensional one. Namely, consider the corresponding analytic set $\widetilde{M}$ and two-dimensional distribution $\widetilde{\mathcal{D}}_M$ from Corollary 4.13. The integral surface $S \subset M$ of the distribution $\mathcal{D}_M$ formed by open set of quadrilateral orbits of the initial billiard a, b, c, d lifts to an integral surface $\widetilde{S} \subset \widetilde{M}$ of the distribution $\widetilde{\mathcal{D}}_M$, by the corollary. The irreducible component containing $\widetilde{S}$ of the set $\widetilde{M}$ is the minimal analytic set containing $\widetilde{S}$, by the similar assumption for S and M. Then we denote the latter component by M, the distribution $\widetilde{\mathcal{D}}_M$ by $\mathcal{D}_M$ and the lifted surface $\widetilde{S}$ by S. Thus, in both cases
   - M is an irreducible analytic set in a complex manifold;
   - $\mathcal{D}_M$ is a two-dimensional singular analytic distribution on M;
   - S is its two-dimensional integral surface;
   - M is the minimal analytic set containing S;
   - the distribution $\mathcal{D}_M$ is integrable.

The four former statements follow from construction. The fifth one follows from Proposition 2.34.

We keep the previous notations ν_a, ν_G, $G = B, C, D$ for the projections to the positions of vertices. Set

$$M' = \{x \in M^0_{reg} \mid A(x)B(x)C(x)D(x) \text{ is non-degenerate and}$$

$$d\nu_a(x), d\nu_G(x) \not\equiv 0 \text{ on } \mathcal{D}_M(x) \text{ for } G = B, C, D\}. \tag{4.1}$$

This is a non-empty set: it contains an open subset in the surface S. It is an open and dense subset in M, and its complement $\Sigma = M \setminus M'$ is analytic.

Remark 4.14 The integral surface of the distribution $\mathcal{D}_M$ through each point $x \in M'$ is non-trivial. The germ of its image under each one of the projections ν_G, $G = B, C, D$, is a germ of analytic curve at its non-marked point. The regularity of germ follows by definition from the inequalities in (4.1). The non-isotropicity of tangent line to the germ follows from the

definition of the distribution $\mathcal{D}_M$ and non-degeneracy of the quadrilateral corresponding to x. Thus, the above integral surface corresponds to an open set of quadrilateral orbits of a 4-reflective billiard a, $b(x)$, $c(x)$, $d(x)$, and the above germs are germs of mirrors at non-marked points.

We show that there exists an open subset $V \subset M'$ of those x for which the mirrors a and $b(x)$ intersect at some their common regular point (easily follows from Corollaries 4.2 and 3.5). Afterwards we deduce that either $c(x) = a$ for all x, or $b(x)$ is a line for all x. We show that this contradicts either Proposition 4.1, or non-algebraicity of the curve b.

Recall that we denote $W_A = \nu_a^{-1}(A) \cap M$. Set $W_A^0 = W_A \cap M'$.

There exists an open dense subset $\hat{a}' \subset \hat{a}$ such that for every $A \in \hat{a}'$ the intersection $\nu_B(W_A^0) \cap a$ contains a regularly embedded disk $\alpha \subset a$ without marked points (Corollary 4.2). Fix α and an $x_0 \in M' \cap \nu_B^{-1}(\alpha)$. The point $\nu_B(x_0) \in b(x_0) \cap \alpha$ is a non-marked point of the corresponding local branches of the curves $b(x_0)$ and α (Remark 4.14). The latter local branches are distinct, by Corollary 3.5. Therefore, there exists a neighborhood $V = V(x_0) \subset M'$ such that for every $x \in V$ a regular branch of the curve $b(x)$ intersects α at a non-marked point for both curves (Remark 4.14 and analyticity of the foliation by integral surfaces). Then either $c(x) = a$ for all $x \in V$, or the curve $b(x)$ is a line for all $x \in V$, by Corollary 3.5 and analyticity. Hence, either $\nu_C(M) \subset a$, or the mirror b of the initial billiard is a line. The second case is obviously impossible, since b is non-algebraic by assumption. To treat the first case, we use the following proposition.

Proposition 4.15 *For every analytic billiard a, b, c, d with a non-algebraic mirror b in every one-parametric family of quadrilateral orbits $ABCD$ with fixed non-isotropic vertex $A \neq I_1, I_2$ the vertex C is non-constant.*

Proof If $C \equiv const$, then b would be either a line, or a conic, by [5, proposition 2.32], – a contradiction. □

Thus, we assume that $\nu_C(M) \subset a$. Fix an $A \in \hat{a}$ such that $\pi_a(A)$ is not an isotropic point at infinity and there exists a one-parametric family of quadrilateral orbits $ABCD$ of the initial billiard a, b, c, d with the given vertex A. The subset $\nu_C(W_A) \subset \mathbb{CP}^2$ is non-discrete, by Proposition 4.15. On the other hand, it is an algebraic subset in $\mathbb{CP}^2$ (Remmert's Proper Mapping and Chow's Theorems). It lies in a transcendental curve a. Hence, it is discrete. The contradiction thus obtained proves Theorem 1.9.

4.3 Case of involutive three-dimensional distribution

Here we treat the case, when the distribution $\mathcal{D}_M$ is three-dimensional and involutive.

Corollary 4.2 easily implies the next proposition and corollary, which state that there exist a connected open subset $V \subset M_{reg}^0$, a regular analytic hypersurface $V_a \subset V$, $\nu_B(V_a) \subset a$, and an analytic field $\mathcal{L}$ of one-dimensional nonsingular integral elements in $\mathcal{D}_M$ on V whose all complex orbits intersect V_a transversely (plus a mild genericity condition (4.2)). The germ through every $x \in V$ of complex orbit is included into an integral surface of the distribution $\mathcal{D}_M$, by Theorem 4.8. Condition (4.2) implies that the integral surface is non-trivial, and hence, corresponds to an open set of quadrilateral orbits of a 4-reflective billiard a, $b(x)$, $c(x)$, $d(x)$. If $x \in V_a$, then the mirrors a and $b(x)$ intersect at $\nu_B(x)$, and we deduce that either $c(x) = a$, or $b(x)$ is a line (Corollary 3.5). This easily implies that either $\nu_C(M) \subset a$, or the image under the projection ν_B of every analytic curve tangent to $\mathcal{D}_M$ is a line. We show that none of the latter cases is possible. The contradiction thus obtained will prove Theorem 1.9.

Let $M' \subset M_{reg}^0$ be the subset from (4.1) defined for our three-dimensional distribution $\mathcal{D}_M$. It is open, dense and the complement $M \setminus M'$ is analytic, as at the same place.

Proposition 4.16 *There exist an $x \in M'$ and a one-dimensional nonsingular integral element $\mathcal{L}_x \subset \mathcal{D}_M(x)$ such that $\nu_B(x) \in a$ and*

$$(d\nu_a(x))(\mathcal{L}_x) \neq 0, \ (d\nu_G(x))(\mathcal{L}_x) \neq 0 \text{ for every } G = B, C, D; \qquad (4.2)$$

$$\text{the line } (d\nu_B(x))(\mathcal{L}_x) \text{ is transverse to } T_{\nu_B(x)}a. \qquad (4.3)$$

Proof The set $\widetilde{Q}$ of nonsingular integral elements $L \subset \mathcal{D}_M$ satisfying (4.2) is open and dense in the projectivization of the subbundle $\mathcal{D}_M \subset TM_{reg}^0$, by definition and density of the set M'. It is an analytically constructible subset in $\mathbb{P}(T\mathcal{H}_a)$, see Proposition 4.11 and its proof. Let Q denote the projection of the set $\widetilde{Q}$ to M. (One has $Q \subset M'$, by definition.) The subset $Q \subset M$ is open, dense and analytically constructible, as is $\widetilde{Q}$, by properness of the projection $\mathbb{P}(T\mathcal{H}_a)|_M \to M$ and Chevalley–Remmert Theorem. Therefore, the intersection $\nu_B(Q) \cap a$ contains a regularly embedded disk $\alpha \subset a$ without isotropic tangent lines (Corollary 4.2). Fix an $x \in Q$ with $\nu_B(x) \in \alpha$. By definition, there exists a one-dimensional non-singular integral element $\mathcal{L}_x \subset \mathcal{D}_M(x)$ satisfying (4.2). Let us show that one can achieve transversality condition (4.3) as well. Suppose the contrary. Then for every

$x \in Q \cap \nu_B^{-1}(a)$ each one-dimensional nonsingular integral element in $\mathcal{D}_M(x)$ is tangent to the analytic hypersurface $\sigma(a) = \nu_B^{-1}(a)$. This implies that the distribution $\mathcal{D}_M$ is tangent to $\sigma(a)$: the tangent spaces to $\sigma(a)$ contain the distribution subspaces. Therefore, every irreducible germ γ of integral curve of the distribution $\mathcal{D}_M$ at a point in $\sigma(a)$ is contained in $\sigma(a)$. Let us choose the latter germ γ to be tangent to a nonsingular one-dimensional integral element satisfying inequalities (4.2). Then γ is contained in a germ of integral surface (Theorem 4.8), and the latter is non-trivial by the same inequalities. Note that $\nu_B(\gamma) \subset a$. Therefore, the latter surface is formed by an open set of quadrilateral orbits of a 4-reflective billiard a, a, c', d' with two coinciding nonlinear neighbor mirrors, – a contradiction to Corollary 3.5. This proves the proposition. □

Corollary 4.17 *There exist an open subset $V \subset M'$, a regularly embedded disk $\alpha \subset a$ without isotropic tangent lines, a non-empty analytic hypersurface $V_a \subset V$ with $\nu_B(V_a) \subset \alpha$ and an analytic line field $\mathcal{L}$ on V transverse to V_a such that for every $x \in V$ the line $\mathcal{L}_x$ is a nonsingular integral element satisfying inequalities (4.2) and each complex orbit of the line field $\mathcal{L}$ in V intersects V_a.*

The corollary follows immediately from the proposition and openness of the set of nonsingular integral elements satisfying (4.2).

Proposition 4.18 *Let V, V_a and $\mathcal{L}$ be as in the above corollary. Then there are two possible cases:*

Case 1): $\nu_C(V) \subset a$;

Case 2): the projection ν_B sends each complex orbit of the field $\mathcal{L}$ to a line.

Proof For every $x \in V$ the germ of the orbit of the field $\mathcal{L}$ through x lies in a germ of integral surface of the distribution $\mathcal{D}_M$ (Theorem 4.8). The latter surface is non-trivial by the inequalities from (4.2), and hence, is given by an open set of quadrilateral orbits of a 4-reflective billiard a, $b(x)$, $c(x)$, $d(x)$. If $x \in V_a$, then the mirrors a and $b(x)$ intersect at the point $B(x) = \nu_B(x)$, and the latter is not marked for both corresponding local branches of curves a and $b(x)$. This follows from construction and the inequalities from (4.2). Hence, for every $x \in V_a$ either $c(x) = a$, or $b(x)$ is a line (Corollary 3.5). Recall that each orbit of the field $\mathcal{L}$ intersects V_a, by assumption. Hence, either the projection ν_C sends it to a, or the projection ν_B sends it to a line. One of the two latter statements holds for all the orbits together, by analyticity. This proves the proposition. □

Now for the proof of Theorem 1.9 it suffices to show that none of the cases from the above proposition is possible.

Case 1): $\nu_C(V) \subset a$. Then $\nu_C(M) \subset a$, and we get a contradiction, as at the end of the previous subsection. Hence, Case 1) is impossible.

Case 2): $\nu_C(M) \not\subset a$. Let V and $\mathcal{L}$ be the same, as in the above corollary. Let us deform $\mathcal{L}$. The set of line fields $\mathcal{L}$ satisfying the conditions of the corollary is open in the space of line fields contained in the distribution $\mathcal{D}_M$. This together with the corollary implies that for every line field $\mathcal{L}$ contained in $\mathcal{D}_M$ the projection ν_B sends each its complex orbit to a line. This is equivalent to say that each analytic curve in M_{reg}^0 tangent to $\mathcal{D}_M$ is sent to a line by ν_B. In particular, this holds for every one-parametric family of quadrilateral orbits lifted to M of the initial billiard a, b, c, d with variable $B \in b$. This implies that the curve b is a line. The contradiction thus obtained proves Theorem 1.9. The proof of Theorem 1.6 is complete.

5 Applications to real pseudo-billiards

In Subsection 5.1 we introduce and classify the 4-reflective real planar analytic pseudo-billiards. In Subsections 5.2, 5.3 we present applications respectively to Tabachnikov's commuting billiard problem and Plakhov's invisibility conjecture.

5.1 Classification of real planar analytic 4-reflective pseudo-billiards

Here by *real analytic curve* we mean a curve in $\mathbb{RP}^2$ analytically parametrized by either $\mathbb{R}$, or S^1 and distinct from the infinity line. For each real analytic curve $a \subset \mathbb{RP}^2$ (which may have singularities: cusps or self-intersections) we consider its maximal real analytic extension $\pi_a : \hat{a} \to a$, where $\hat{a}$ is either $\mathbb{R}$, or S^1, see [8, lemma 37, p.302]. The parametrizing curve $\hat{a}$ will be called here the *real normalization*. The affine plane $\mathbb{R}^2 \subset \mathbb{RP}^2$ is equipped with Euclidean metric.

Definition 5.1 [5, remark 1.6] Let a line $L \subset \mathbb{R}^2$ and a triple of points $A, B, C \in \mathbb{R}^2$ be such that $B \in L$, $A, C \neq B$ and the lines AB, BC are symmetric with respect to the line L. We say that the triple A, B, C and the line L satisfy the *usual reflection law*, if the points A and C lie on the same side from the line L. Otherwise, if they are on different sides from the line L, we say that the *skew reflection law* is satisfied.

Example 5.2 In every planar billiard orbit each triple of consequent vertices satisfies the usual reflection law with respect to the tangent line to the boundary of the billiard at the middle vertex.

Definition 5.3 (cf. [5, definition 6.1]) A *planar pseudo-billiard* is a collection of k real curves $a_1, \ldots, a_k \subset \mathbb{RP}^2$, none of them being the infinity line. Its *k-periodic orbit* is a k-gon $A_1 \ldots A_k$, $A_j \in a_j \cap \mathbb{R}^2$, such that for every $j = 1, \ldots, k$ one has $A_j \neq A_{j\pm1}$, $A_j A_{j\pm1} \neq T_{A_j} a_j$ and the lines $A_j A_{j-1}$, $A_j A_{j+1}$ are symmetric with respect to the tangent line $T_{A_j} a_j$. The latter means that for every j the triple A_{j-1}, A_j, A_{j+1} and the line $T_{A_j} a_j$ satisfy either the usual, or the skew reflection law. For brevity, we then say that the usual (skew) reflection law is satisfied at the vertex A_j. Here we set $a_{k+1} = a_1$, $A_{k+1} = A_1$, $a_0 = a_k$, $A_0 = A_k$. A real pseudo-billiard is called *(piecewise-)analytic/smooth*, if so are its curves. It is called *k-reflective*, if the set of its k-periodic orbits has positive two-dimensional Lebesgue measure.

Remark 5.4 In the piecewise-analytic case k-reflectivity is equivalent to the non-emptiness of the interior of the set of k-periodic orbits, i.e., existence of a two-parameter family of k-periodic orbits. The orbits from the latter interior will be called *k-reflective*, cf. loc.cit. The complexification of each k-reflective planar analytic pseudo-billiard is a k-reflective complex billiard.

Theorem 5.5 *A collection of 4 real planar analytic curves a, b, c, d is a 4-reflective pseudo-billiard, if and only if it has one of the following types:*

1) $a = c$ is a line, the curves $b, d \neq a$ are symmetric with respect to it;

2) a, b, c, d are distinct lines through the same point $O \in \mathbb{RP}^2$, the line pairs (a, b), (d, c) are transformed one into the other by rotation around O (translation, if O is an infinite point), see Fig.9;

3) $a = c$, $b = d$ and they are distinct confocal conics: either ellipses, or hyperbolas, or ellipse and hyperbola, or parabolas.

In every 4-reflective orbit the reflection law at each pair of opposite vertices is the same; it is skew for at least one opposite vertex pair.

Addendum 1. *In every pseudo-billiard of type 1) from Theorem 5.5 each quadrilateral orbit $ABCD$ has the same type, as at Fig.8. It is symmetric with respect to the line a, and the reflection law at A, C is skew. The reflection law at B, D is either usual at both, or skew at both.*

Addendum 2 [5, addendum 3 to theorem 6.3]. *In every pseudo-billiard of type 3) the 4-reflective orbits have the same types, as at Fig.8–13.*

Remark 5.6 The main result of paper [8] (theorem 2) concerns usual real planar billiards with piecewise-smooth boundary; the reflection law is usual. It says that the quadrilateral orbit set has measure zero. In the particular case of billiard with piecewise-analytic boundary this result follows from the last statement of Theorem 5.5.
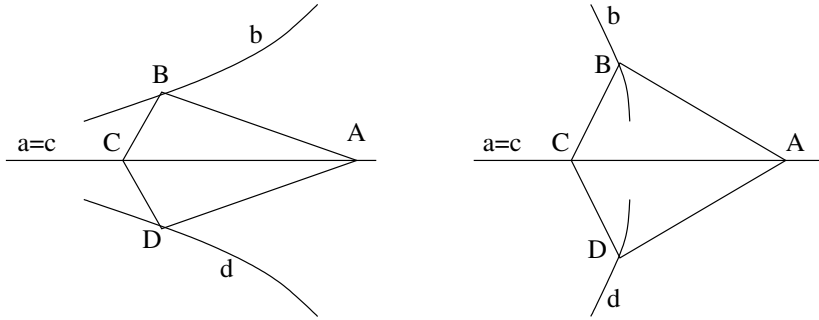

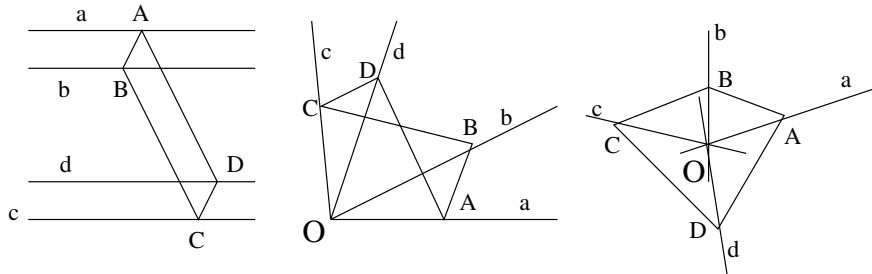Figure 8: 4-reflective pseudo-billiards symmetric with respect to a line mirror

Figure 9: 4-reflective pseudo-billiards on two positively-isometric line pairs

Proof of Theorem 5.5 and Addendum 1. The pseudo-billiard a, b, c, d under question being 4-reflective, its complexification is obviously 4-reflective, by Remark 5.4. This together with Theorem 1.6 implies that it is one of the above types 1)–3) (up to cyclic renaming of the mirrors). The 4-reflectivity of each pseudo-billiard of types 2), 3) and the classification of open sets of their quadrilateral orbits and reflection law configurations was proved in [5, section 6]. The proof of 4-reflectivity of pseudo-billiards of type 1) and Addendum 1 (symmetry of quadrilateral orbits) repeats the proof of analogous addendum 1 (algebraic case) from loc.cit. □
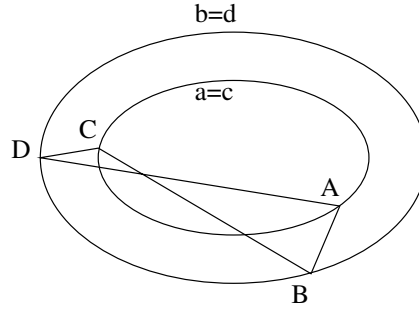
Figure 10: A 4-reflective pseudo-billiard on confocal ellipses

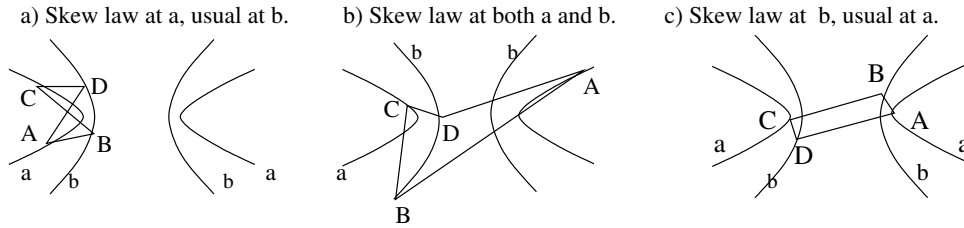a) Skew law at a, usual at b. b) Skew law at both a and b. c) Skew law at b, usual at a.

Figure 11: Open sets of orbits on confocal hyperbolas: three types

5.2 Application 1: Tabachnikov's commuting billiard problem

The following theorem solves the piecewise-analytic case of S.Tabachnikov's problem on commuting convex planar billiards. It deals with two billiards in nested convex compact domains $\Omega_1 \Subset \Omega_2 \Subset \mathbb{R}^2$, set $a = \partial\Omega_1$, $b = \partial\Omega_2$. We consider that both a and b are piecewise-smooth. For every Ω_j consider the corresponding billiard transformation acting on the space of oriented lines in the plane. It acts as identity on the lines disjoint from Ω_j. Each oriented line l intersecting Ω_j is sent to its image under the reflection from the boundary $\partial\Omega_j$ at its last intersection point x with $\partial\Omega_j$ in the sense of orientation: the orienting arrow of the line l at x is directed outside Ω_j. The reflected line is oriented by a tangent vector at x directed inside Ω_j. This is a continuous dynamical system, if the boundary $\partial\Omega_j$ is smooth and piecewise-continuous (measurable) otherwise.

Theorem 5.7 *Let two nested planar convex piecewice-analytic Jordan curves be such that the corresponding billiard transformations commute. Then they*
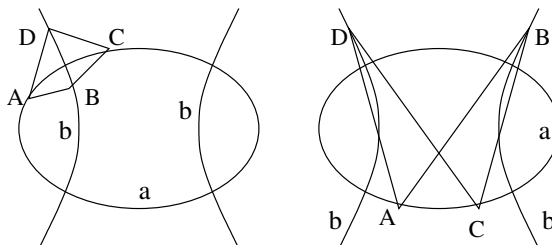
Figure 12: Open sets of orbits on confocal ellipse and hyperbola: two types
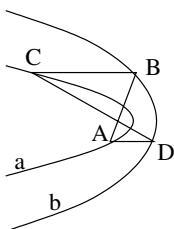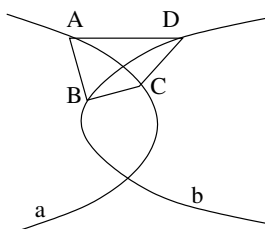

Figure 13: Open set of orbits on confocal parabolas: one type

are confocal ellipses.

In the proof of Theorem 5.7 we use the following commutativity criterion.

Proposition 5.8 *Let $a, b \subset \mathbb{R}^2$ be nested convex Jordan curves, as at the beginning of the subsection. The corresponding billiard transformations commute, if and only if each pair $(A, B) \in a \times b$ extends to a quadrilateral orbit $ABCD$ of the pseudo-billiard a, b, a, b as at Fig.10: the reflection law is usual at b and skew at a; only one of the segments AB, BC intersects the domain bounded by the curve a, if both ambient lines intersect it.*

The proposition follows from definition.

Proof of Theorem 5.7. The interior of the set of quadrilateral orbits $ABCD$ from the proposition contains an open and dense subset of quadrilateral orbits whose vertices lie in analytic pieces of the curves a and b. Thus, each pair of their analytic pieces $a' \subset a$, $b' \subset b$ extends to a 4-reflective pseudo-billiard a', b', c', d'; $c' \subset a$, $d' \subset b$. Hence, the latter pseudo-billiards have some of types 1)–3) from Theorem 5.5.

Case 1): the curve a has only one analytic arc (at most one singular point). Note that no analytic piece $b' \subset b$ is a line. Indeed, otherwise a quadrilateral orbit $ABCD$ with $B \in b'$ would be symmetric with respect to b' (Addendum 1), which is impossible by convexity and the obvious inclusion $ABCD \subset \overline{\Omega_2}$. Hence, a is an ellipse and each analytic piece $b' \subset b$ is a conic confocal to a. Note that the tangent lines to b' are disjoint from a and hence, from the focal segment, by convexity. This implies that b' cannot be a hyperbola, and thus, b' is an ellipse. Finally, each analytic piece $b' \subset b$ is an ellipse confocal to a, and hence, so is b.

Case 2): the curve a has at least two singular points. Fix one of them, A. Then the interior of the set of quadrilateral orbits of the pseudo-billiard a, b, a, b accumulates to a finite union of one-parametric analytic families of quadrilaterals $AB_{j,t}C_{j,t}D_{j,t}$ with the given vertex A and regular $B_{j,t}, D_{j,t} \in b$ (analyticity points) arranged as in Proposition 5.8; $t \in [0,1]$, $j = 1, \ldots, N$, $N \in \mathbb{N}$. In each one of the latter families the vertex $C_j = C_{j,t} \in a$ is singular and constant, as is A. Indeed, suppose the contrary: for some t the germs of the curve b at $B_{j,t}$, $D_{j,t}$ and the germ of the curve a at $C_{j,t}$ are regular, while that of the curve a at A is singular. The latter germs represent germs of mappings acting on the space of lines by reflections; their product should be identity, by 4-reflectivity. This is impossible, since all of them are regular except for the reflection from the singular germ (a, A) (cf. [5, proof of proposition 2.16]), – a contradiction. Therefore, the variable lines $AB_{j,t}$ and $B_{j,t}C_j$ through fixed finite points A and C_j are symmetric with respect to the line $T_{B_{j,t}}b$. This together with the discussion from the previous case implies that the analytic arc of the curve b containing the vertices $B_{j,t}$ is an ellipse arc with foci A and C_j. This together with Proposition 5.8 implies that each analytic arc of the curve b is an ellipse with focus A. Taking another singular point $A' \in a$ and applying the above arguments, we get that each analytic arc of the curve b is an ellipse with focus at A'. This implies that the curve a contains at most two distinct singular points and each analytic arc of the curve b is an ellipse with foci at them. This together with continuity implies that b is an ellipse. The curve a has two distinct analytic pieces and each of them is bounded by its singular points: the foci A and A' of the ellipse b. At least one of the latter pieces is non-linear and hence, it is an arc of conic confocal to b (Theorem 5.5). It passes through its own foci A and A', which is obviously impossible. Therefore, the case under consideration is impossible. This proves Theorem 5.7. $\qquad \square$

5.3 Application 2: planar Plakhov's Invisibility Conjecture with four reflections

This subsection is devoted to Plakhov's Invisibility Conjecture: the analogue of Ivrii's conjecture in the invisibility theory [13, conjecture 8.2]. We recall it below and show that it follows from a conjecture saying that no finite collection of germs of smooth curves can form a k-reflective billiard for appropriate "invisibility" reflection law. In the case, when the curves are analytic, the invisibility reflection law is a real form of complex reflection law. As it was shown in [6, subsection 5.2, proposition 8], both Plakhov's and Ivrii's conjectures have the same complexification. We prove the piecewise-analytic case of planar Plakhov's Invisibility Conjecture for four reflections as an immediate corollary of Theorem 5.5.

Definition 5.9 [13, chapter 8] Consider an arbitrary perfectly reflecting (may be disconnected) closed bounded body B in a Euclidean space. For every oriented line (ray) R take its first intersection point A_1 with the boundary ∂B and reflect R from the tangent hyperplane $T_{A_1}\partial B$. The reflected ray goes from the point A_1 and defines a new oriented line, as in billiards (see the previous subsection). Then we repeat this procedure. Let us assume that after a finite number k of reflections the output oriented line coincides with the input line R and will not hit the body any more. Then we say that the body B *is invisible for the ray R,* see Fig.14. We call R a *ray of invisibility* with k reflections.

Invisibility Conjecture (A.Plakhov, [13, conjecture 8.2, p.274].) *There is no body with piecewise C^∞ boundary for which the set of rays of invisibility has positive measure.*

Remark 5.10 As is shown by A.Plakhov in his book [13, section 8], there exist no body invisible for all rays. The same book contains a very nice survey on invisibility, including examples of bodies invisible in a finite number of (one-dimensional families of) rays. See also papers [1, 14, 15] for more results. The Invisibility Conjecture is equivalent to the statement saying that there are no k-reflective bodies for every k, see the next definition. It is open even in dimension 2.

Definition 5.11 [6, subsection 5.2, definition 12] A body B with piecewise-smooth boundary is called *k-reflective,* if the set of invisibility rays with k reflections has positive measure.
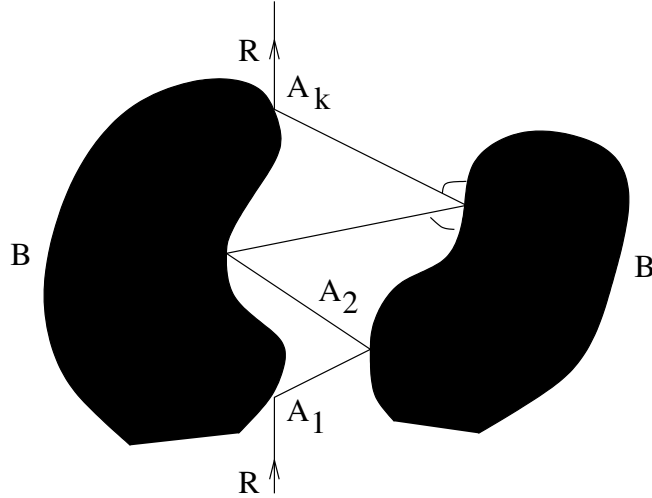
Figure 14: A body invisible for one ray.

Definition 5.12 (cf. [6, subsection 5.2, definition 13]) Let $a_1, \ldots, a_k$ be a real planar smooth pseudo-billiard. A k-gon $A_1 \ldots A_k$ with $A_j \in a_j$, $A_{k+1} = A_1$, $A_0 = A_k$ is said to be a *k-invisible orbit*, if it is a k-periodic orbit of the pseudo-billiard with usual reflection law at a_j for $j \neq 1, k$ and skew reflection law at a_1 and a_k, see Fig.15. We say that the pseudo-billiard $a_1, \ldots, a_k$ is *k-invisible*, if the set of its k-invisible orbits has positive measure. (in particular, the pseudo-billiard should be k-reflective).

Proposition 5.13 *Let $k \in \mathbb{N}$ and $B \subset \mathbb{R}^2$ be a body with piecewise-smooth (piecewise-analytic) boundary such that no collection of k germs of its boundary forms a k-invisible smooth (analytic) pseudo-billiard. Then the body B is not k-reflective.*

Proposition 5.13 is implicitly contained in [13, section 8].

Theorem 5.14 *There are no 4-reflective bodies in $\mathbb{R}^2$ with piecewise-analytic boundary.*

Proof The existence of a 4-reflective body as above implies the existence of a 4-invisible planar analytic pseudo-billiard (Proposition 5.13). This is a 4-reflective planar analytic pseudo-billiard having an open set of quadrilateral orbits with skew reflection law at some pair of neighbor vertices and usual
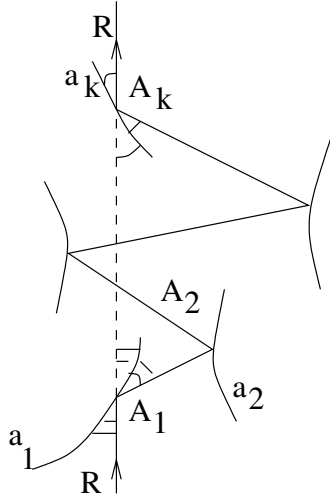
Figure 15: A k-invisible k-gon: skew reflection law at A_1 and A_k.

reflection law at the other vertices. Thus, in these quadrilateral orbits the reflection laws at each pair of opposite vertices are different, – a contradiction to the last statement of Theorem 5.5. This proves Theorem 5.14. □

6 Acknowledgements

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