LIMIT THEOREMS FOR NUMBER OF EDGES IN THE GENERALIZED RANDOM GRAPHS WITH RANDOM VERTEX WEIGHTS

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ABSTRACT. We get central limit type theorems for the total number of edges in the generalized random graphs with random vertex weights under different moment conditions on the distributions of the weights.

Complex networks attract increasing attention of researchers in various fields of science. In last years numerous network models have been proposed. Since the uncertainty and the lack of regularity in real-world networks, these models are usually random graphs. Random graphs were first defined by Paul Erdős and Alfréd Rényi in their 1959 paper "On Random Graphs", see [5], and independently by Gilbert in [7]. The suggested models are closely related: there are n isolated vertices and every possible edge occurs independently with probability $p : 0 < p < 1$. It is assumed that there are no self-loops. Later the models were generalized. A natural generalization of the Erdős and Rényi random graph is that the equal edge probabilities are replaced by probabilities depending on the vertex weights. Vertices with higher weights are more likely to have many neighbors than vertices with small weights. Vertices with extremely high weights could act as the hubs observed in many real-world networks.

The following generalized random graph model was first introduced by Britton et al., see [3]. Let $\{1, 2, ..., n\}$ be the set of vertices, and $W_i > 0$ be the weight of vertex $i, 1 \leq i \leq n$. The edge probability of the edge between any two vertices i and j is equal to

$$p_{ij} = \frac{W_i W_j}{L_n + W_i W_j},$$

where $L_n = \sum_{i=1}^{n} W_i$ denotes the total weight of all vertices, and the weights $W_i, i = 1, 2, \ldots, n$ can be taken to be deterministic or random. If we take all W_i-s as the same constant: $W_i \equiv n\lambda/(n - \lambda)$ for some $0 < \lambda < n$, it is easy to see that $p_{ij} = \lambda/n$ holds for all $1 \leq i < j \leq n$. That is, the Erdős–Rényi random graph with $p = \lambda/n$ is a special case

of the generalized random graph. There are many versions of the generalized random graphs, such as Poissonian random graph (introduced by Norros and Reittu in [11] and studied by Bhamidi et al.[1]), rank-1 inhomogeneous random graph (see [2]), random graph with given prescribed degrees (see [4]), etc. Under some common conditions (see [9]), all of the above mentioned random graph models are asymptotically equivalent, meaning that all events have asymptotically equal probabilities. The updated review on the results about these inhomogeneous random graphs see in Chapters 6 and 9 in [12].

In the present paper we assume that $W_i, i = 1, 2, \ldots, n$, are independent identically distributed random variables distributed as W. Let E_n be the total number of edges in a generalized random graph with vertex weights $W_1, W_2, \ldots, W_n$. In [8], under the conditions that W has a finite or infinite mean, several weak laws of large numbers for E_n are established, see also Ch.6, [12]. For instance, in [8] and Ch.6, [12], it is proved that E_n/n tends in probability to $\mathbb{E}W/2$, provided $\mathbb{E}W$ is finite.

Note that

$$E_n = \frac{1}{2} \sum_{i=1}^n D_i,$$

where $D_i, i = 1, 2, \ldots, n$ is a degree of vertex i, i.e. the number of edges coming out from vertex i. It is clear, the random variables $D_i, i = 1, 2, \ldots, n$ are dependent ones. The aim of the present paper is to refine the law of large numbers type results for E_n and to get central limit type theorems under different moment conditions for W. In Theorem 1 we assume that $\mathbb{E}W^2 < \infty$. It implies normal limit distribution for $\{E_n\}$ after proper normalization. In Theorem 2 we assume that the distribution of W belongs to the domain of attraction of a stable law F with characteristic exponent $\alpha : 1 < \alpha < 2$. Then we prove that the limit distribution for normalized E_n is F.

Theorem 1. *If $\mathbb{E}W^2 < \infty$, then*

$$\frac{2E_n - n\mathbb{E}W}{\sqrt{n\left(2\mathbb{E}W + Var(W)\right)}} \xrightarrow{d} N(0, 1).$$

Proof. Put for all integer $n \geq 1$

$$(1) \qquad b_n = \frac{1}{2} n\, \mathbb{E}W, \ c_n = \frac{1}{2} \sqrt{n\, \mathrm{Var}(W)}.$$

For any $t \in \mathbb{R}$, we have

$$\mathbb{E} \exp\left\{ it \frac{E_n - b_n}{c_n} \right\} = \mathbb{E} \exp\left\{ \frac{it}{c_n} \left(\sum_{1 \leq i < j \leq n} I_{ij} - b_n \right) \right\}$$

$$= \mathbb{E}\left(\mathbb{E}\left(\exp\left\{ \frac{it}{c_n} \left(\sum_{1 \leq i < j \leq n} I_{ij} - b_n \right) \right\} \Big| W_1, \cdots, W_n \right) \right)$$

$$= \mathbb{E}\Big(e^{-itb_n/c_n} \prod_{1 \le i < j \le n} \frac{L_n + e^{it/c_n}W_iW_j}{L_n + W_iW_j}\Big)$$

$$:= \mathbb{E}e^{Y_n},$$

where

$$Y_n = \sum_{1 \le i < j \le n} \log \frac{L_n + e^{it/c_n}W_iW_j}{L_n + W_iW_j} - \frac{itb_n}{c_n}$$

$$(2) \quad = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\log \frac{L_n + e^{it/c_n}W_iW_j}{L_n + W_iW_j} - \frac{itb_n}{c_n} - \sum_{i=1}^{n}\log \frac{L_n + e^{it/c_n}W_i^2}{L_n + W_i^2}$$

and $\log(\cdot)$ is the principal value of the complex logarithm function.

By using the Maclaurin series expansion of $\log(1 + x)$ for complex x with $|x| < 1$, we have that

$$\frac{|\log(1 + x)|}{|x|} \longrightarrow 1, \quad \frac{|\log(1 + x) - x|}{|x|^2} \longrightarrow \frac{1}{2} \text{ as } |x| \to 0.$$

Hence there exists some constant $c_0 > 0$ such that $|\log(1 + x)| \le 2|x|$ and $|\log(1 + x) - x| \le |x|^2$ hold for any $|x| \le c_0$.

Clearly, for any fixed t, there exists $n_0 = n_0(t) \in \mathbb{N}$ such that for all $n \ge n_0$ and any $1 \le i, j \le n$ one has

$$\Big|\frac{(e^{it/c_n} - 1)W_iW_j}{L_n + W_iW_j}\Big| \le |e^{it/c_n} - 1| \le |t|/c_n \le c_0.$$

Thus, since

$$(3) \qquad \frac{L_n}{n} \to \mathbb{E}W \ a.s. \text{ and } \frac{\sum_{i=1}^{n}W_i^2}{n} \to \mathbb{E}W^2 \ a.s.,$$

we have for any $n \ge n_0$

$$\Big|\sum_{i=1}^{n}\log \frac{L_n + e^{it/c_n}W_i^2}{L_n + W_i^2}\Big| \le \sum_{i=1}^{n}\Big|\log \Big(1 + \frac{(e^{it/c_n} - 1)W_i^2}{L_n + W_i^2}\Big)\Big|$$

$$\le 2|e^{it/c_n} - 1|\sum_{i=1}^{n}\frac{W_i^2}{L_n + W_i^2}$$

$$(4) \qquad \le 2\frac{|t|}{c_n}\frac{\sum_{i=1}^{n}W_i^2}{L_n} \to 0 \ a.s.$$

and

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\log \frac{L_n + e^{it/c_n}W_iW_j}{L_n + W_iW_j} - \frac{itb_n}{c_n}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\log \Big(1 + \frac{(e^{it/c_n} - 1)W_iW_j}{L_n + W_iW_j}\Big) - \frac{itb_n}{c_n}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(e^{it/c_n}-1)W_iW_j}{L_n+W_iW_j} - \frac{itb_n}{c_n} + O_1\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(e^{it/c_n}-1)^2W_i^2W_j^2}{(L_n+W_iW_j)^2}$$

$$= \frac{1}{2}\Big(e^{it/c_n}-1-\frac{it}{c_n}+\frac{t^2}{2c_n^2}\Big)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_iW_j}{L_n+W_iW_j}$$

$$+\frac{1}{2}\Big(\frac{it}{c_n}-\frac{t^2}{2c_n^2}\Big)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_iW_j}{L_n+W_iW_j} - \frac{itb_n}{c_n}$$

$$+O_1\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(e^{it/c_n}-1)^2W_i^2W_j^2}{(L_n+W_iW_j)^2}$$

$$(5)= I_1 + I_2 + I_3,$$

where $|O_1| \leq 1/2$. By (3) and the inequality $|e^{ix}-1-ix+x^2/2| \leq |x|^3/6$ for any $x \in \mathbb{R}$, we have

$$(6) \qquad |I_1| \leq \frac{|t|^3}{12c_n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_iW_j}{L_n} = \frac{|t|^3 L_n}{12c_n^3}\longrightarrow 0 \ a.s.$$

Similarly, by (3) and the inequality $|e^x - 1| \leq |x|$, we get

$$(7)\,|I_3| \leq \frac{t^2}{2c_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_i^2W_j^2}{L_n^2} = \frac{t^2}{c_n^2}\Big(\frac{1}{n}\sum_{i=1}^{n}W_i^2\Big)^2\Big(\frac{n}{L_n}\Big)^2\longrightarrow 0 \ a.s.$$

Recalling the definition (1) for b_n and c_n, we have

$$I_2 = \frac{1}{2}\Big(\frac{it}{c_n}-\frac{t^2}{2c_n^2}\Big)\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_iW_j}{L_n} - \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_i^2W_j^2}{L_n(L_n+W_iW_j)}\Big) - \frac{itb_n}{c_n}$$

$$= it\frac{L_n-n\mathbb{E}W}{\sqrt{n\mathrm{Var}(W)}} - \frac{t^2 L_n}{n\mathrm{Var}(W)} - \frac{1}{2}\Big(\frac{it}{c_n}-\frac{t^2}{2c_n^2}\Big)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_i^2W_j^2}{L_n(L_n+W_iW_j)}.$$

Moreover, by (3) we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_i^2W_j^2}{L_n(L_n+W_iW_j)} \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_i^2W_j^2}{L_n^2}$$

$$= \frac{\Big(\sum_{i=1}^{n}W_i^2\Big)^2}{L_n^2} \to \Big(\frac{\mathbb{E}W^2}{\mathbb{E}W}\Big)^2 \ a.s.$$

The central limit theorem yields

$$(8) \qquad\qquad I_2 \xrightarrow{d} it\mathbf{N} - t^2\mathbb{E}W/\mathrm{Var}(W),$$

where $\mathbf{N}$ is a standard normal random variable. Now, it follows from (2)–(8) that

$$Y_n \xrightarrow{d} it\mathbf{N} - t^2\mathbb{E}W/\mathrm{Var}(W).$$

Hence, by noting that $|e^{Y_n}| \leq 1$ and applying the Lebesgue dominated convergence theorem, we get that, for any $t \in \mathbb{R}$,

$$
\begin{aligned}
\mathbb{E} \exp\left\{ it\frac{E_n - b_n}{c_n} \right\} &= \mathbb{E}e^{Y_n} \to \mathbb{E}\exp\{it\mathbf{N} - t^2\mathbb{E}W/\mathrm{Var}(W)\} \\
&= \exp\{-(1/2)t^2(1 + 2\mathbb{E}W/\mathrm{Var}(W))\}.
\end{aligned}
$$

Thus, Theorem 1 is proved. □

In the following theorem we get convergence of the sequence $\{E_n\}$ under weaker moment conditions on W_i's.

Theorem 2. *Let $W, W_1, W_2, \cdots$ be a sequence of i.i.d. nonnegative random variables and*

$$
(9) \qquad \frac{W_1 + \cdots + W_n - n\mathbb{E}W}{a_n} \xrightarrow{d} F,
$$

where F is a stable distribution with characteristic exponent $\alpha : 1 < \alpha < 2$, then

$$
\frac{2E_n - n\mathbb{E}W}{a_n} \xrightarrow{d} F.
$$

Before we start to prove the theorem, let us state some properties of the distribution of W.

If (9) holds true, then a_n (see e.g. [6], ch.XVII, §5) is a regularly varying function with exponent $1/\alpha$ satisfying

$$
(10) \qquad n\mathbb{E}W^2 I(W \leq a_n) \sim a_n^2,
$$

and there exists some constant $c > 0$ and $h(x)$, a slowly varying function at ∞, such that

$$
(11) \qquad P(W > x) \sim cx^{-\alpha}h(x).
$$

We shall use the following lemma.

Lemma 1. *If (11) holds with $\alpha : 1 < \alpha < 2$, then we have*

$$
\mathbb{E}W^2 I(W \leq x) \sim \frac{c\alpha}{2 - \alpha}x^{2-\alpha}h(x),
$$

$$
\mathbb{E}WI(W \geq x) \sim c\frac{2 - \alpha}{\alpha - 1}x^{1-\alpha}h(x).
$$

The proof of the lemma see e.g. [6], ch.XVII, §5.
Now we are ready to prove Theorem 2.

Proof. Let $b_n = (1/2)\, n\, \mathbb{E}W$ and $c_n = (1/2)\, a_n$ with a_n from (10). As in the proof of Theorem 1, for any $t \in \mathbb{R}$, we also write

$$
\mathbb{E}\exp\left\{ it\frac{E_n - b_n}{c_n} \right\} = \mathbb{E}e^{Y_n}
$$

with new definition for c_n and

$$Y_n = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\log\frac{L_n + e^{it/c_n}W_iW_j}{L_n + W_iW_j} - \frac{itb_n}{c_n} - \sum_{i=1}^{n}\log\frac{L_n + e^{it/c_n}W_i^2}{L_n + W_i^2}.$$

For the last sum for any $n \geq n_0$, where $n_0 = n_0(t)$ is defined in the proof of Theorem 1, we have (cp. (4))

$$\left|\sum_{i=1}^{n}\log\frac{L_n + e^{it/c_n}W_i^2}{L_n + W_i^2}\right| \leq 2\frac{|t|}{c_n}\frac{\sum_{i=1}^{n}W_i^2}{L_n}.$$

Similarly to (5), we get

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\log\frac{L_n + e^{it/c_n}W_iW_j}{L_n + W_iW_j} - \frac{itb_n}{c_n}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(e^{it/c_n}-1)W_iW_j}{L_n + W_iW_j} - \frac{itb_n}{c_n} + O_1\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(e^{it/c_n}-1)^2W_i^2W_j^2}{(L_n + W_iW_j)^2}$$

$$= \frac{1}{2}\left(e^{it/c_n}-1-\frac{it}{c_n}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_iW_j}{L_n + W_iW_j}$$

$$+ \frac{1}{2}\frac{it(L_n - 2b_n)}{c_n} - \frac{1}{2}\frac{it}{c_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_i^2W_j^2}{L_n(L_n + W_iW_j)}$$

$$+ O_1\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(e^{it/c_n}-1)^2W_i^2W_j^2}{(L_n + W_iW_j)^2}$$

with $|O_1| \leq 1/2$. Due to Theorem's condition we have $(L_n - 2b_n)/(2c_n) \xrightarrow{d} F$. Since

$$|e^{ix}-1| \leq |x|, \ |e^{ix}-1-ix| \leq |x|^2/2 \text{ for all } x \in \mathbb{R},$$

in order to prove Theorem 2, we only need to show that

$$(12) \qquad \frac{1}{a_n}\frac{\sum_{i=1}^{n}W_i^2}{L_n} \xrightarrow{p} 0,$$

$$(13) \qquad \frac{1}{a_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_iW_j}{L_n + W_iW_j} \xrightarrow{p} 0,$$

$$(14) \qquad \frac{1}{a_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_i^2W_j^2}{L_n(L_n + W_iW_j)} \xrightarrow{p} 0,$$

$$(15) \qquad \frac{1}{a_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_i^2W_j^2}{(L_n + W_iW_j)^2} \xrightarrow{p} 0.$$

For any $\gamma : \alpha > \gamma > 0$, we have $\mathbb{E}(W^2)^{(\alpha-\gamma)/2} = \mathbb{E}W^{\alpha-\gamma} < \infty$. Then by Marcinkiewicz–Zygmund's strong law of large numbers (see e.g. Theorem 4.23 in [10]) we have

$$n^{-2/(\alpha-\gamma)} \sum_{i=1}^{n} W_i^2 \to 0 \ a.s.$$

Since a_n is a regularly varying function with exponent $1/\alpha$, then we have $1/a_n = o(n^{-1/\alpha+\gamma})$. Now choose $\gamma > 0$ such that

$$2/(\alpha - \gamma) - 1 - 1/\alpha + \gamma < 0 \ \text{ and } -2/\alpha + 1 + 2\gamma < 0.$$

Then we have

$$\frac{1}{a_n} \frac{\sum_{i=1}^{n} W_i^2}{L_n} = \frac{n^{2/(\alpha-\gamma)-1}}{a_n} \frac{\sum_{i=1}^{n} W_i^2/n^{2/(\alpha-\gamma)}}{L_n/n} = o(n^{2/(\alpha-\gamma)-1-1/\alpha+\gamma}) \longrightarrow 0 \ a.s.$$

and

$$\frac{1}{a_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W_i W_j}{L_n + W_i W_j} \leq \frac{1}{a_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W_i W_j}{L_n} = \frac{n}{a_n^2} \frac{L_n}{n} = o(n^{-2/\alpha+1+2\gamma}) \longrightarrow 0 \ a.s.$$

Thus we get (12) and (13).

To prove (14), we write

$$\frac{1}{a_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W_i^2 W_j^2}{L_n(L_n + W_i W_j)}$$

$$= \frac{1}{a_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W_i^2 W_j^2 I(W_i W_j \leq n)}{L_n(L_n + W_i W_j)} + \frac{1}{a_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W_i^2 W_j^2 I(W_i W_j > n)}{L_n(L_n + W_i W_j)}$$

$$\leq \frac{1}{a_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W_i^2 W_j^2 I(W_i W_j \leq n)}{L_n^2} + \frac{1}{a_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W_i W_j I(W_i W_j > n)}{L_n}$$

$$\leq \frac{n^2}{a_n L_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W_i^2 W_j^2 I(W_i W_j \leq n)}{n^2} + \frac{n}{a_n L_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W_i W_j I(W_i W_j > n)}{n}.$$

Further, by (3) and by using the fact that $\mathbb{E}|X_n| \to 0$ implies $X_n \xrightarrow{p} 0$, in order to prove (14), it is sufficient to show that

$$(16) \qquad \frac{1}{a_n} \mathbb{E}W_1^2 W_2^2 I(W_1 W_2 \leq n) \longrightarrow 0,$$

$$(17) \qquad \frac{n}{a_n} \mathbb{E}W_1 W_2 I(W_1 W_2 > n) \longrightarrow 0.$$

For any $\alpha \in (1, 2)$, we can choose $\delta > 0$ satisfying $2 - \alpha - 1/\alpha + 2\delta < 0$. By Lemma 1, there exists some constant $c_1 = c_1(\alpha, \delta) > 0$ such that

$$\mathbb{E}W^2 I(W \leq x) \leq c_1 x^{2-\alpha+\delta}, \ \mathbb{E}W I(W \geq x) \leq c_1 x^{1-\alpha+\delta}$$

hold for all $x > 1$. Hence

$$\frac{1}{a_n} \mathbb{E}W_1^2 W_2^2 I(W_1 W_2 \leq n)$$

$$= \frac{1}{a_n}\mathbb{E}\Big(W_2^2 I(W_2 \leq n)\mathbb{E}(W_1^2 I(W_1 \leq n/W_2)|W_2)\Big)$$

$$+ \frac{1}{a_n}\mathbb{E}\Big(W_2^2 I(W_2 > n)\mathbb{E}(W_1^2 I(W_1 \leq n/W_2)|W_2)\Big)$$

$$\leq \frac{c_1}{a_n}\mathbb{E}\Big(W_2^2 (n/W_2)^{2-\alpha+\delta}\Big) + \frac{1}{a_n}\mathbb{E}\Big(W_2^2 I(W_2 > n)(n/W_2)^2\Big)$$

$$= \frac{c_1 n^{2-\alpha+\delta}}{a_n}\mathbb{E}W^{\alpha-\delta} + \frac{n^2}{a_n}P(W > n).$$

Since by (11) we have $P(W > n) \sim c n^{-\alpha}h(n) = o(n^{-\alpha+\delta})$ and $1/a_n = o(n^{-1/\alpha+\delta})$, we get

$$\frac{1}{a_n}\mathbb{E}W_1^2 W_2^2 I(W_1 W_2 \leq n) = o(n^{2-\alpha-1/\alpha+2\delta}) \to 0 \text{ as } x \to \infty.$$

Thus, we get (16).

Similarly, we have

$$\frac{n}{a_n}\mathbb{E}W_1 W_2 I(W_1 W_2 > n)$$

$$= \frac{n}{a_n}\mathbb{E}\Big(W_2 I(W_2 \leq n)\mathbb{E}(W_1 I(W_1 > n/W_2)|W_2)\Big)$$

$$+ \frac{n}{a_n}\mathbb{E}\Big(W_2 I(W_2 > n)\mathbb{E}(W_1 I(W_1 > n/W_2)|W_2)\Big)$$

$$\leq \frac{c_1 n}{a_n}\mathbb{E}\Big(W_2 (n/W_2)^{1-\alpha+\delta}\Big) + \frac{n}{a_n}\mathbb{E}\Big(W_2 I(W_2 > n)\mathbb{E}W_1\Big)$$

$$= \frac{c_1 n^{2-\alpha+\delta}}{a_n}\mathbb{E}W^{\alpha-\delta} + \frac{n}{a_n}\mathbb{E}W\mathbb{E}(WI(W > n))$$

$$\leq \frac{c_1 n^{2-\alpha+\delta}}{a_n}\mathbb{E}W^{\alpha-\delta} + \frac{c_1 n^{2-\alpha+\delta}}{a_n}\mathbb{E}W = o(n^{2-\alpha-1/\alpha+2\delta}) \to 0.$$

Hence (17), and then (14), are proved.

And (15) follows from (14). The proof of Theorem 2 is complete. $\square$

REFERENCES

[1] S. Bhamidi, R. van der Hofstad, J.S.H. van Leeuwaarden, "Novel scaling limits for critical inhomogeneous random graphs", *Ann. Probab.*, **40**, 2299–2361 (2012).

[2] B. Bollobás, S. Janson, O. Riordan, "The phase transition in inhomogeneous random graphs", *Random Struct. Algorithms*, **31**, 3–122 (2007).

[3] T. Britton, M. Deijfen, A. Martin-Löf, "Generating simple random graphs with prescribed degree distribution", *J. Stat. Phys.*, **124**, 1377–1397 (2006).

[4] F. Chung, L. Lu, "The volume of the giant component of a random graph with given expected degrees", *SIAM J. Discrete Math.*, **20**, 395–411 (2006) (electronic).

[5] P. Erdős, A. Rényi, "On Random Graphs", *Publ. Math. Debrecen*, **6**, 290–297 (1959).

[6] W. Feller, *An Introduction to Probability Theory and Its Applications*, Wiley, New-York, vol.2 (1971).

[7] E.N. Gilbert, "Random graphs", *Annals of Mathematical Statistics*, **30**, 1141–1144, (1959).

[8] Z.S. Hu, W. Bi, Q.Q. Feng, "Limit laws in the generalized random graphs with random vertex weights", *Statistics and Probability Letters*, **89**, 65–76 (2014).

[9] S. Janson, "Asymptotic equivalence and contiguity of some random graphs", *Random Struct. Algorithms*, **36**, 26–45, (2010).

[10] O. Kallenberg, *Foundations of Modern Probability*, Springer-Verlag, New York (2002).

[11] I. Norros, H.Reittu " On a conditionally Poisson graph process", *Adv. in Appl. Probab.*, **38**, 59–75, (2006).

[12] R. van der Hofstad, *Random graphs and complex networks*, Unpublished Manuscript. Available at: http://www.win.tue.nl/ rhofstad/NotesRGCN.pdf (2016).

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