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Description of domain structures in the Solar Corona by means multi-color graphs[1]

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Abstracts. Magnetic charging topology explains many energy processes (flares, prominences, etc.) in the solar corona by changing the domain structure associated with the appearance or disappearance of the separators. It is known that at most of the nulls of the magnetic field are prone. In this paper it is proved that a topology of the domains of a field with the prone nulls is completely described by a multi-color graph. In addition, we give an efficient algorithm for distinguishing of these graphs.

Keywords: magnetic fields, model of corona, photosphere magnetic reconnection, dynamics prominences, multi-color graph, polynomial-time algorithm.

1. Introduction and the formulation of the results

Understanding the energy processes in the corona of the sun is very important to explain many of the laws of nature. This paper considered a possible model to explain such effects in the photosphere as the flares and the prominences. Their origin is connected with the restructuring of regions (domains), on which the fans and the spines of the null points of the magnetic field divide the corona of the sun — reconnection. Therefore, the main questions for this approach are the qualitative partition of the solar corona into domains, as well as the existence of the separators (the lines of intersection of fans) — marks of upcoming or already occurred reconnection. There are different approaches to the study of the topology of domains, such as the construction of graphs that reflect the structure and the relative position of the domains [3] or footprints — traces of spines and fans on the photosphere [8]. We have proposed a new approach consisting in distinguishing of traces of fans on some circle on the photosphere. We describe these trace on a language of multi-color graph whose isomorphic class is a complete invariant for the topology of domains and gives information on the number of the separators. In more detail.

By the topological approach the magnetic field in the corona is believed to arise from a large number of dipoles in the solar interior. The dipoles are interpreted as locations where flux tubes originating in the solar interior break through the surface and spread out into the atmosphere (see figure 1). We use the assumptions of Magnetic Charge Topology [7], where photospheric flux patches are modeled as point sources (charges) on the photosphere. Although this suggestion violate the solenoidal condition, but each source is considered to represent a flux tube passing through the solar surface and spreading out into the overlying corona, then this simplification is allowable. Following [2] for a model of the magnetic field

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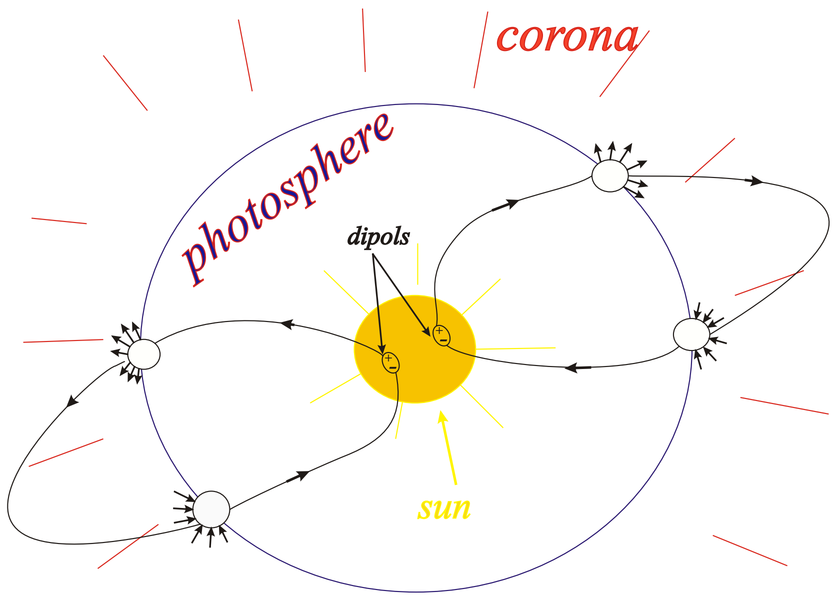

Рис. 1. The dipoles in the solar interior

B with point sources the two-dimensional sphere $P = \{(x, y, z, w) \in \mathbb{S}^3 \mid w = 0\}$ in three-dimensional sphere $\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$ is used as the photosphere and the region $\{(x, y, z, w) \in \mathbb{S}^3 \mid w > 0\}$ as solar corona. Moreover we suppose that **B** is symmetrically extended to the region $\{(x, y, z, w) \in \mathbb{S}^3 \mid w < 0\}$ being termed the mirror corona and, hence, it is defined on $M = \mathbb{S}^3 \setminus \bigcup\limits_{i=1}^{k} q_i$ where $q_1, \ldots, q_k$ are the points on the photosphere where the charges are situated.

Magnetic *nulls* are the points where the magnitude of magnetic field vector vanishes. Due to the solenoidal condition $\nabla \cdot \mathbf{B} = 0$ three eigenvalues λ_1, λ_2, λ_3 of the critical point satisfy the equality $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Since **B** is potential then all eigenvalues are real number. Generically each eigenvalue is different from 0, thus each null of **B** is a saddle point. Two quite distinct families of field lines tends to a null point: the *spine* is a line and the *fan* is a surface. For a null p denote by S_p the spine and by F_p the fan of p. The spines of different nulls have no intersections in general position. A null is called *positive (negative)* if $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 > 0$ ($\lambda_1 \cdot \lambda_2 \cdot \lambda_3 < 0$). The topological structure of a magnetic field **B** is largely defined by null points, spines, fans, and separators, the union of which forms the so-called *skeleton* of the magnetic field. There are several types of nulls. A null which belongs to the photosphere is called *photospheric*. A photospheric null point whose spine lies in the photosphere is called *prone*, whereas a photospheric null with a spine directed vertically is called *upright. The coronal null* is a null above the photosphere. It follows from [1] that the most nulls are prone.

When two fans have intersection they form a *separator*, which joins two oppositely signed null points. Fans divide the corona into different regions which called *domains*. Appearance and disappearance of separators change the topology of domains splitting. Such situation is called *separator reconnection*, which is one of the major reconnection mechanisms [15]. Much papers [3], [10], [11], [12] were devoted to classification of the magnetic field configurations that arise from such point-source models. It is naturally to introduce the following definition which goes back to the classic paper [14], see also [16].

Definition 1. One says that two coronal magnetic fields $\mathbf{B}, \mathbf{B}'$ are *topologically equivalent* if there is a homeomorphism $H : M \to M$ sending magnetic lines of $\mathbf{B}$ to magnetic lines of $\mathbf{B}'$ with preserving orientation on the lines.

Denote by $\mathcal{B}$ the set of the magnetic fields $\mathbf{B}$ with the following properties:

1) each null of $\mathbf{B}$ is prone;

2) if two fans of $\mathbf{B}$ are intersected then they are either coincide, either have contact along one curve on the photosphere or have transversal intersection along two symmetric with respect to the photosphere curves;

3) the closures of the spines of different nulls have no intersection.

Now let $\mathbf{B} \in \mathcal{B}$.

Theorem 1. *For each magnetic field $\mathbf{B} \in \mathcal{B}$ there is a circle $C \subset P$ which is transversal to the flow generated by $\mathbf{B}$ on P and such that each fan intersects C at exactly two points.*

We will called such circle C by *photosphere section*. Denote by N the set of nulls of $\mathbf{B}$. Set $W = P \setminus \bigcup\limits_{p\in N} S_p$, $\mathcal{F} = \bigcup\limits_{p\in N} F_p$ and $X = C \cap \mathcal{F}$. Denote by N^u (N^s) the set of positive (negative) nulls of $\mathbf{B}$. Set $\mathcal{F}^u = \bigcup\limits_{p\in N_u} F_p$ ($\mathcal{F}^s = \bigcup\limits_{p\in N_s} F_p$), $X^u = C \cap \mathcal{F}^u$ ($X^s = C \cap \mathcal{F}^s$) and $X^t = X^u \cap X^s$.

In order to introduce a combinatorial topological invariant of the magnetic field $\mathbf{B} \in \mathcal{B}$ we recall the following definitions.

A *finite graph* Γ is an ordered pair (V, E), such that the following conditions hold: V is a non-empty finite set of *vertices*; E is a set of pairs of vertices called *edges*.

If a graph contains an edge $e = (a, b)$, then each of the vertices a, b is said to be *incident to the edge* e and the vertices a and b are said to be connected by the edge e.

A path in a graph is a finite sequence of its vertices and edges of the form: $b_0, (b_0, b_1), b_1, \cdots, b_{i-1}, (b_{i-1}, b_i), b_i, \cdots b_{k-1}, (b_{k-1}, b_k), b_k$, $k \geq 1$. The number k is called *the length of the path*, it is equal to the number of edges involved in the path.

A cycle of length k, $k \in \mathbb{N}$ in a graph is a finite subset of vertices and edges of the form $\{b_0, (b_0, b_1), b_1, \cdots, b_{i-1}, (b_{i-1}, b_i), b_i, \cdots b_{k-1}, (b_{k-1}, b_0)\}$. *A simple cycle* is a cycle all of whose vertices and edges are pairwise distinct.

A graph Γ is called *multi-color* graph if the set of vertices or edges of Γ is the union of finite number subsets each of which consists of the vertices or edges of the same color.

Two multi-color graphs Γ and Γ' are said to be *isomorphic* if there exists a one-to-one correspondence ξ between the sets of their vertices which preserve the relations of incidence and the color.

For our invariant we will use three colors, we denote these colors by the letters s, t, u and, for brevity, refer to these vertices or edges as s-, t-, u-vertices or s-, t-, u-edges. We construct a multi-color graph Γ_B, corresponding to a magnetic field $\mathbf{B} \in \mathcal{B}$ as follows (see figure 2 where s, t, u are green, blue, red, accordingly):

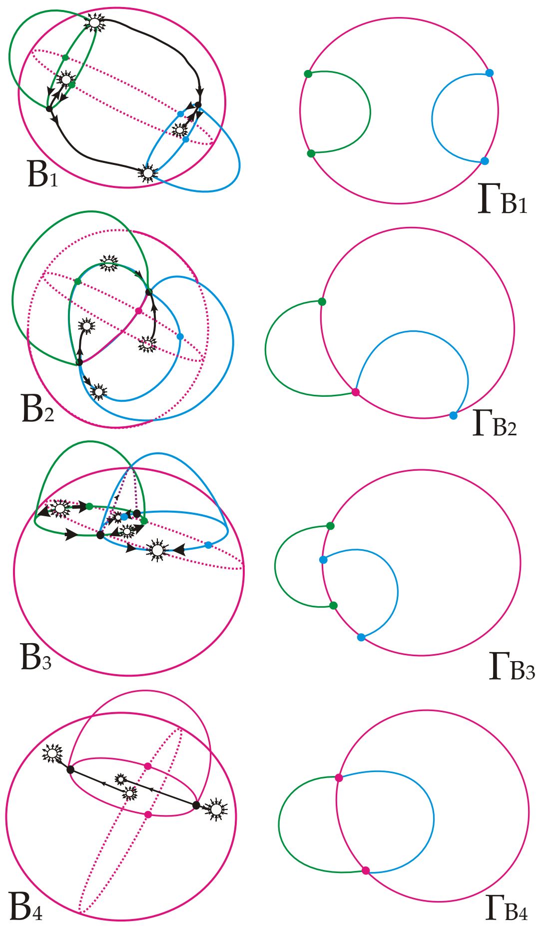

Рис. 2. Magnetic fields and their multi-color graphs

1) the t-vertices are in a one-to-one correspondence with the points of the set X^t;

2) the s-vertices (u-vertices) are in a one-to-one correspondence with the points of the set $X^s \setminus X^t$ ($X^u \setminus X^t$);

3) the t-edges are in a one-to-one correspondence with the connected components of $C \setminus X$ and two vertices of the graph are incident to an t-edge if the corresponding points are boundary points for corresponding connected component;

4) two vertices of the graph are incident to an s-edge (u-edge) if the corresponding points are exactly $F_p \cap C$ for some null $p \in N^s$ ($p \in N^u$).

Theorem 2. *Magnetic fields* $\mathbf{B}, \mathbf{B}'$ *from* $\mathcal{B}$ *are topologically equivalent if and only if their multi-color graphs* Γ_B, $\Gamma_{B'}$ *are isomorphic.*

Theorem 2 motivates to ask the question about the computational complexity of distinguishing two multi-color graphs corresponding to magnetic fields. An algorithm solving the graph isomorphism problem is considered to be efficient if its running time is bounded by a polynomial on the number of vertices of input graphs. This problem can really be solved in polynomial time for the graphs of magnetic fields.

Theorem 3. *Isomorphism of multi-color graphs corresponding to Solar magnetic fields can be recognized in polynomial time.*

2. Necessary and Sufficient conditions for the topological equivalence of magnetic fields from $\mathcal{B}$

To prove the results we compactify the magnetic field lines in the places of point-charge by the bundle of straight lines, such idea was used in [6] for the finding of the separators of magnetic fields in electrically conducting fluids. Then the magnetic lines of the field $\mathbf{B}$ coincide geometrically on M with trajectories of a three-dimensional flow $f^\tau : \mathbb{S}^3 \to \mathbb{S}^3$ with the following properties:

1) the non-wandering set $\Omega(f^\tau)$ of f^τ consists of finite number hyperbolic equilibrium states[2] all of them belong to the photosphere P;

2) all trajectories of f^τ are symmetric with respect the photosphere P and number of sinks coincide with number of sources;

3) the closures of one-dimensional invariant manifolds of different saddle points are disjoint;

4) if two-dimensional invariant manifolds of different saddle points are intersected then they are either coincide, either have contact along one curve on the photosphere or have transversal intersection along two symmetric with respect to the photosphere curves.

Denote by G the set of flows with properties above. By the construction we see the following interrelation between magnetic field $\mathbf{B} \in \mathcal{B}$ and its compactification $f^\tau \in G$:

- the charges coincide with the sink and source equilibrium states,
- the null points coincide with the saddle equilibrium states,
- the fan (spine) of each null coincides with two-dimensional (one-dimensional) invariant manifold of the corresponding saddle,
- the separators coincide with *heteroclinic curves* — connected component of the intersection of two-dimensional invariant manifolds of the saddle points,
- the magnetic lines of $\mathbf{B}$ coincide with the trajectories of f^τ on M
- magnetic fields $\mathbf{B}, \mathbf{B}'$ are equivalent if and only if corresponding flows f^τ, f'^τ are equivalent.

[2] An equilibrium state w of the flow f^τ is called *hyperbolic* if the matrix of the linearization at the equilibrium has no eigenvalues with zero real part. Any hyperbolic equilibrium state w of the flow f^τ possesses invariant manifolds:
stable manifold $W^s_w = \{y \in \mathbb{S}^3 : \lim\limits_{\tau \to +\infty} d(f^\tau(y), w) = 0\}$,
unstable manifold $W^u_w = \{y \in \mathbb{S}^3 : \lim\limits_{\tau \to -\infty} d(f^\tau(y), w) = 0\}$
which are homeomorphic to $\mathbb{R}^{n_s}$, $\mathbb{R}^{n_u}$, where n_s, n_u — the numbers of the eigenvalues with negative and positive real parts, correspondingly, d — a metric on $\mathbb{S}^3$. We will denote by dim $W^s_w = n_s$, dim $W^u_w = n_u$ the dimensions of W^s_w and W^u_w.

Let $f^\tau \in G$ and σ be a saddle point of f^τ with the unstable manifold W^u_σ and the stable manifold W^s_σ. Denote by Ω_1 (Ω_2) the set of saddle points σ of f^τ such that $\dim\ W^u_\sigma =1$ ($\dim\ W^u_\sigma = 2$) and by Ω_0 (Ω_3) the set of sinks (sources). Let us set

$$A = \bigcup_{\sigma\in\Omega_1} cl\ W^u_\sigma, \qquad R = \bigcup_{\sigma\in\Omega_2} cl\ W^s_\sigma.$$

The following proposition is due to [16] (see also [5] for details).

Proposition 1. For each flow $f^\tau \in G$ the following statements hold:

i) $\mathbb{S}^3 = \bigcup\limits_{x\in\Omega(f^\tau)} W^s_x = \bigcup\limits_{x\in\Omega(f^\tau)} W^u_x$ and each invariant manifold W^s_x (W^u_x) is a submanifold[3] of $\mathbb{S}^3$;

ii) $cl\ W^u_x \cap W^u_y \neq \emptyset$ if and only if $W^u_x \cap W^s_y \neq \emptyset$;

iii) the sets A, R are pairwise disjoint and each of them is connected.

Proof of Theorem 1

Theorem 1 follows from lemma below.

Lemma 1. *For each flow $f^\tau \in G$ there is a circle $C \subset P$ which is transversal to the flow $f^\tau|_P$ and such that two-dimensional invariant manifold of each saddle point intersects C at exactly two points.*

Proof. Let us set $\phi^\tau = f^\tau|_P$. It follows from the description of class G that f^τ is a flow on $\mathbb{S}^3$ with finite hyperbolic non-wandering set, then by Lefschetz formula $|\Omega_0|-|\Omega_1|+|\Omega_2|-|\Omega_3| = 0$, where $|\cdot|$ is the cardinality. In the other side ϕ^τ is a flow on $\mathbb{S}^3$ with the same non-wandering set, then $|\Omega_0| - |\Omega_1| - |\Omega_2| + |\Omega_3| = 2$. Thus

$$|\Omega_0| - |\Omega_1| = 1.$$

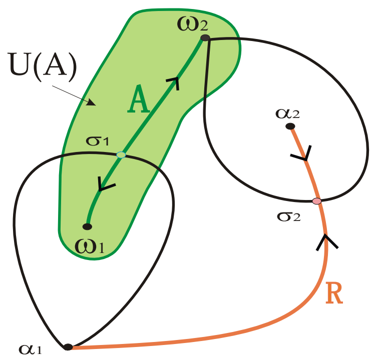

Рис. 3. Neighbourhood $U(A)$

[3]Let $\mu \in \{0,1,2,3\}$. A subset Y of $\mathbb{S}^3$ is said to be its *μ-dimensional submanifold* if for every point y of the set Y there is a neighbourhood U_y of y and a homeomorphism $\psi_y : U_y \to \mathbb{R}^3$ for which $\psi_y(U_y \cap Y) = \mathbb{R}^\mu$ where $\mathbb{R}^\mu \subset \mathbb{R}^3$ is the set of points whose last $(3-\mu)$ coordinates are zero.

Let us choose neighbourhood $U(A)$ of the set A on P such that $\partial U(A)$ is transversal all trajectories in $(W^s_A \setminus A) \cap P$ (see figure 3). Due to item iii) of Proposition 1, $U(A)$ has euler characteristic 1, it means that $U(A)$ is 2-disk. By item i) of Proposition 1, $W^s_A \setminus A = W^u_R \setminus R$. Set $Q = W^s_A \setminus A$ and $C = \partial U(A)$. By item i) of Proposition 1 and symmetry property of f^τ, each two-dimensional manifold of saddle point intersect $Q \cap P$ along exactly two trajectories. Thus C is required photospheric section. $\square$

Proof of Theorem 2

We assign a flow $f^\tau \in G$ for each magnetic field $\mathbf{B} \in \mathcal{B}$, also we have a graph Γ_B corresponding to $\mathbf{B}$. Then theorem 2 follows from the next lemma.

Lemma 2. *Flows f^τ, f'^τ are topologically equivalent if and only if multi-color graphs $\Gamma_B, \Gamma_{B'}$ are isomorphic.*

Proof. **First, we prove necessity.** Suppose that f^τ and f'^τ from G are topologically equivalent, that is, there exists a homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ which sends the trajectories of f^τ to trajectories of f'^τ with preservation of orientation. Let us prove that multi-color graphs $\Gamma_B, \Gamma_{B'}$ are isomorphic. We assume without loss of generality that the graph $\Gamma_{B'}$ was constructed by using the photospheric section $C' = h(C)$. Since the conjugating homeomorphism h takes invariant manifolds of fixed points of f^τ to invariant manifolds of f'^τ with preservation of the stability, it follows that this homeomorphism takes $X^s,\ X^t,\ X^u$ to $X'^s,\ X'^t,\ X'^u$. Then the requaired isomorphism $\xi : \Gamma_B \to \Gamma_{B'}$ is defined by the formula $\xi = \pi_{f'} h \pi_f^{-1}$ where $\pi_f,\ \pi_{f'}$ are one-to-one maps of the set $X,\ X'$ onto the sets of vertices of the graph $\Gamma_B,\ \Gamma_{B'}$, accordingly.

Let us prove sufficiency. Consider the multi-colour graphs $\Gamma_B,\ \Gamma_{B'}$ of the flows $f^\tau, f'^\tau \in G$, respectively. Suppose that there exists an isomorphism ξ between the sets of vertices of $\Gamma_B,\ \Gamma_{B'}$ which preserve the relations of incidence and the color. We construct step by step a homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ conjugating f^τ and f'^τ.

Step 1. Set $V = \mathbb{S}^3 \setminus (A \cup R)$. Similar to proof of Lemma 1, for each flow $f^\tau \in G$ there is a 2-sphere $\Sigma \subset M$ which is transversal to the flow $f^\tau|_V$ and such that two-dimensional invariant manifold of each saddle point intersects Σ at exactly one circles. Moreover, it is possible to construct Σ such that $\Sigma \cap P = C$. Set $C^u = \Sigma \cap \mathcal{F}^u$ ($C^s = \Sigma \cap \mathcal{F}^s$), $C^t = Y^u \cap Y^s$ and do the same for f'^τ.

By the construction all vertices and all t-edges the multi-color graph form a simple cycle and ξ preserves such cycle with the preserving of the color of the vertices than there exist an orientation-preserving homeomorphism $h_\Sigma : \Sigma \to \Sigma'$ such that $h_\Sigma(C^u) = C'^u$, $h_\Sigma(C^s) = C'^s$ and $h_\Sigma(C^t) = C'^t$. We denote by l_x (l'_x) the trajectory of f^τ (of f'^τ) passing through $x \in \mathbb{S}^3$. According to Proposition 1 there are unique pair of the equilibrium states $\alpha(l_x),\ \omega(l_x)$ ($\alpha(l'_x),\ \omega(l'_x)$) such that $l_x \subset (W^u_{\alpha(l_x)} \cap W^s_{\omega(l_x)})$ ($l'_x \subset (W^u_{\alpha(l'_x)} \cap W^s_{\omega(l'_x)})$). By Proposition 1 we have the following possibilities for point $x \in \Sigma$:

- $\alpha(l_x) \in \Omega_3,\ \omega(l_x) \in \Omega_0$ for $x \in \Sigma \setminus (C^u \cup C^s)$;
- $\alpha(l_x) \in \Omega_2,\ \omega(l_x) \in \Omega_0$ for $x \in (C^u \setminus C^s)$;
- $\alpha(l_x) \in \Omega_3,\ \omega(l_x) \in \Omega_1$ for $x \in (C^s \setminus C^u)$;
- $\alpha(l_x) \in \Omega_2,\ \omega(l_x) \in \Omega_1$ for $x \in C^t$.

For points $y_1, y_2 \in cl\ (l_x)$ denote by $[y_1, y_2]$ the length of arc $[y_1, y_2] \subset l_x$. For each point $y \in l_x$ situated between x and $\alpha(l_x)$ ($\omega(l_x)$) set $\rho(y) = \frac{[x,y]}{[x,\alpha(l_x)]}$ ($\rho(y) = \frac{[x,y]}{[x,\omega(l_x)]}$). Similar situation is for points from Σ'. For any point $x \in \Sigma$, we set $x' = h_\Sigma(x)$. As $h_\Sigma(C^s) =$

C'^s, $h_\Sigma(C^u) = C'^u$ then on the set l_x a homeomorphism $h_{l_x} : l_x \to l'_{x'}$ is well-defined by the formula

$$h_{l_x}(y) = y' \quad where \quad \rho'(y') = \rho(y).$$

Denote by $h_V : V \to V'$ a map composed from $h_{l_x}, x \in \Sigma$. By the construction h_V is a homeomorphism which sends two-dimensional invariant manifolds of the saddle point σ of f^τ to the two-dimensional invariant manifolds of the saddle point σ' of f'^τ. Let us show that $h_V(\omega(l_x)) = \omega(l'_{x'})$ for each $x \in \Sigma$.

Step 2. Denote by $Q \subset \mathbb{S}^3$ compact 3-ball bounded by Σ and containing Ω_0. Then $Q \subset W^s_{\Omega_0 \cup \Omega_1}$ and the set $D_\sigma = W^s_\sigma \cap Q$ is a 2-disk for each $\sigma \in \Omega_1$. Denote by Y a connected component of the set $Q \setminus W^s_{\Omega_1}$. Then there is a unique sink $\omega \in \Omega_0$ such that $\omega \in Y \subset W^s_\omega$. Simultaneously there is a unique connected component K_Y of the set $\Sigma \setminus C^s$ belonging Y and such that $Y \setminus A = \bigcup\limits_{x \in K_Y} (l_x \cap Y) \cup \omega$. Similar situation is for flow f'^τ. Since $h_\Sigma(\Sigma \setminus C^s) = \Sigma' \setminus C'^s$ then $h_\Sigma(K_Y)$ is a connected component of $\Sigma' \setminus C'^s$ belonging to a connected component Y' of the set $Q' \setminus W^s_{\Omega'_1}$ containing a sink $\omega' \in \Omega'_0$. By the construction $h_V(Y \setminus A) = Y' \setminus A'$ and, hence $h_V(\omega(l_x)) = \omega(l'_{x'})$ for each $x \in (\Sigma \setminus C^s)$. By the continuously $h_V(\omega(l_x)) = \omega(l'_{x'})$ for each $x \in C^s$.

Thus h_V can be uniquely extended to the sets Ω_0, Ω_1. We keep the notation h_V for the homeomorphism thus obtained and set $p' = H_V(p)$ for each $p \in (\Omega_0 \cup \Omega_1)$.

Step 3. Let $\sigma \in \Omega_1$. Denote A^τ flow in $\mathbb{R}^3$ generated by a system of linear equations

$$\begin{cases} \dot{x} = x, \\ \dot{y} = y, \\ \dot{z} = -z. \end{cases}$$

This flow has a unique equilibrium state — hyperbolic saddle located at the origin O. Stable

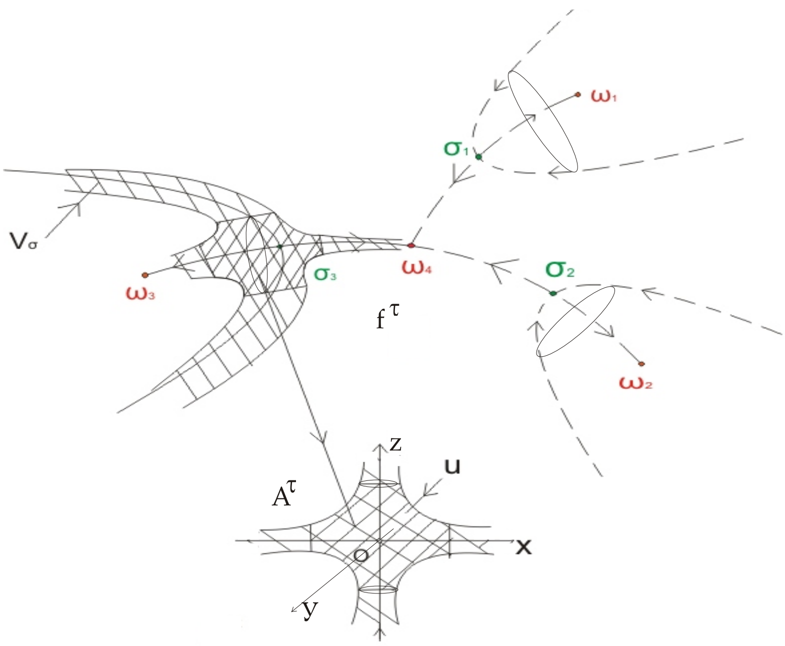

Рис. 4. Linearization of saddle equilibrium state neighborhood

manifold of this saddle is plane XOY, unstable — axis O. Set

$$U = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2)z^2 \leq 1\}.$$

It is immediately verified that U is invariant with respect to the flow A^τ. Due to [13] there is a neighborhood $V_\sigma \subset \mathbb{S}^3$ of the saddle equilibrium state σ and a homeomorphism $H_\sigma : V_\sigma \to U$ such that the homeomorphism sends the trajectories of flow $f^\tau|_{V_\sigma}$ to the trajectories of flow $A^\tau|_U$ (see figure 4). Similar neighborhood $V_{\sigma'}$ and a homeomorphism $H_{\sigma'} : V_{\sigma'} \to U$ exist for flow f'^τ. Set $H_{\sigma,\sigma'} = H^{-1}_{\sigma'} H_\sigma : V_\sigma \to V_{\sigma'}$. Without loss of generality we can assume that homeomorphism $H_{\sigma,\sigma'}$ sends one-dimensional separatrix of σ which contains a sink ω in its closure to one-dimensional separatrix of σ' which contains a sink ω' in its closure (in opposite case we use ζH_σ instead H_σ where $\zeta(x, y, z) = (x, y, -z)$).

Step 4. For $\mu \in (0, 1)$ let us set

$$U_\mu = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2)z^2 \leq \mu\}$$

and $V_{\sigma,\mu} = H^{-1}_\sigma(U_\mu)$. Choose μ such that $H_{\sigma,\sigma'}(V_{\sigma,\mu}) \setminus W^u_\sigma \subset H_V(V_\sigma)$. Set $Z = cl\ (V_\sigma \setminus V_{\sigma,\mu})$ and $Z' = cl\ (H_V(V_\sigma) \setminus H_{\sigma,\sigma'}(V_{\sigma,\mu}))$. By the construction the sets Z, Z' consists of two connected components Z_+, Z_-, Z'_+, Z'_- each of them is homeomorphic to $W = \mathbb{S}^1 \times \mathbb{R}^1 \times [0, 1]$. Denote by $H_{Z_+} : Z_+ \to W$, $H_{Z_-} : Z_- \to W$, $H_{Z'_+} : Z'_+ \to W$, $H_{Z'_-} : Z'_- \to W$ corresponding homeomorphisms sending trajectories of flows to lines $\{s\} \times \mathbb{R}^1 \times \{t\}$. For $t \in [0, 1]$, $\delta \in \{+, -\}$ set $W_t = \mathbb{S}^1 \times \mathbb{R}^1 \times \{t\}$ and

$$H_{\delta,0} = H_{Z_\delta} H_{\sigma,\sigma'} H^{-1}_{Z_\delta}|_{W_0} : W_0 \to W_0, \quad H_{\delta,1} = H_{Z_\delta} H_V H^{-1}_{Z_\delta}|_{W_1} : W_1 \to W_1.$$

As H_V and $H_{\sigma,\sigma'}$ send trajectories of f^τ to trajectories of f'^τ then $H_{\delta,0}$, $H_{\delta,1}$ have view

$$H_{\delta,0}(s, r, 0) = (H_{\delta,0,s}(s), H_{\delta,0,r}(r), 0), \quad H_{\delta,1}(s, r, 1) = (H_{\delta,1,s}(s), H_{\delta,1,r}(r), 1).$$

Let us define homeomorphism $H_{\delta,t} : W_t \to W_t$ by formula

$$H_{\delta,t}(s, r, t) = ((1 - t)H_{\delta,0,s}(s) + tH_{\delta,1,s}(s), (1 - t)H_{\delta,0,r}(r) + tH_{\delta,1,r}(r), t).$$

Denote by $H_{Z_\delta,Z'_\delta} : Z_\delta \to Z_{\delta'}$ homeomorphism composed for each $t \in [0, 1]$ by $H^{-1}_{Z'_\delta} H_{\delta,t} H_{Z_\delta}|_{H^{-1}_{Z_\delta}(W_t)}$. Let us define homeomorphism H_{V_σ} by formula

$$H_{V_\sigma}(x) = \begin{cases} H_{Z_\delta,Z'_\delta}(x), \ x \in Z_\delta, \\ H_{\sigma,\sigma'}(x), \ x \in V_{\sigma,\mu}. \end{cases}$$

By similar way we can define homeomorphism H_{V_σ} for each $\sigma \in \Omega_2$. The required homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ is defined by

$$h(x) = \begin{cases} H_V(x), \ x \in \mathbb{S}^3 \setminus (\bigcup\limits_{\sigma\in(\Omega_1\cup\Omega_2)} V_\sigma), \\ H_{V_\sigma}(x), \ x \in V_\sigma, \ \sigma \in (\Omega_1 \cup \Omega_2). \end{cases}$$

□

3. Algorithm to solve the distinguishing problem for multi-color graphs

In this section, we consider the distinguishing problem for multi-color graphs and present an efficient algorithm for its solution. An algorithm to solve the problem is considered to be *efficient* if it occupies polynomial time on the number of vertices of a given graph. The notion of an efficiently solvable problem rises to A. Cobham, who asserts that a problem can be feasibly computed on some computational device only if it can be computed in time, bounded by a polynomial on the length of input data [4]. The complexity status of the general graph isomorphism problem, i.e. for graphs of the general type, is unknown. That is, neither polynomial-time solvability neither intractability was proved for it. The graphs, associated with Solar magnetic fields, have some peculiar combinatorial properties. Namely, they have bounded degrees of vertices. Recall that *degree of a vertex* of a graph is the number of edges incident to it. A finite graph is called *simple* if it does not contain coloured vertices, loops, multiple and directed edges, coloured edges, simultaneously.

Proof of Theorem 3

It is known that for some concrete constant c^* and function $f(\cdot)$ the isomorphism problem can be solved in $O(f(\Delta)n^{c^*\Delta ln(\Delta)})$ time for simple n-vertex graphs with maximum degree Δ [9]. For each fixed k, this result gives a polynomial-time algorithm to solve the isomorphism problem in the class of all simple graphs having degrees of all vertices at most k. This observation and the facts that the graphs of Solar magnetic fields have degrees of all vertices at most three, the three colors are used to color their vertices and edges lead to the following idea. By the graphs Γ_{B_1} and Γ_{B_2} of magnetic fields B_1 and B_2, we construct simple graphs Γ'_{B_1} and Γ'_{B_2} such that Γ_{B_1} and Γ_{B_2} are isomorphic if and only if Γ'_{B_1} and Γ'_{B_2} are isomorphic. The graphs Γ'_{B_1} and Γ'_{B_2} will have degrees of all vertices at most 9, which implies polynomial complexity of their distinguishing, by the result of Luks.

Recall that a *multi-color graph* is a graph Γ, equipped by two functions $c_1 : V(\Gamma) \longrightarrow \{1, 2, \ldots, k_1\}$ and $c_2 : E(\Gamma) \longrightarrow \{1, 2, \ldots, k_2\}$. Let $\Delta(\Gamma)$ be the maximum degree of vertices of the graph Γ. By Γ, we construct a simple graph Γ' as follows. An *s-star implantation into an edge* (a, b) of a graph is to delete the edge from the graph, add vertices $c, c_1, \ldots, c_s$ and the edges $(a, c), (c, b), (c, c_1), (c, c_2), \ldots, (c, c_s)$. *Inscribing an s-cycle in a vertex* v of a graph is to add vertices $v_1, v_2, \ldots, v_{s-1}$ and the edges $(v, v_1), (v_1, v_2,), \ldots, (v_{s-2}, v_{s-1}), (v_s, v)$ to the graph. For each $v \in V(\Gamma)$, we inscribe a $c_1(v) + 2$-cycle in v. For each $e \in E(\Gamma)$, we implant a $c_2(e)+\Delta(\Gamma)$-star into e. Clearly, the number of vertices of Γ' is at most $(k_1+2)|V(\Gamma)|+(k_2+\Delta(\Gamma)+1)|E(\Gamma)|$ and degrees of all its vertices are at most $k_2+\Delta(\Gamma)+2$. As the sum of degrees of vertices of Γ is equal to $2|E(\Gamma)|$, $|E(\Gamma)| \leq \frac{1}{2}\Delta(\Gamma)|V(\Gamma)|$. Hence, $|V(\Gamma')| \leq \frac{1}{2}((k_2 + \Delta(\Gamma) + 1)\Delta(\Gamma) + 2k_1 + 4)|V(\Gamma)|$. Given Γ', one can uniquely restore Γ as follows. All vertices of Γ' having degrees at least $\Delta(\Gamma) + 3$ are the central vertices of the implanted stars. This observation permits to restore all edges of Γ with their colors. Deleting all vertices of all stars from Γ' produces a disjoint sum of $|V(\Gamma)|$ simple cycles. The number of vertices in each of the cycles determines the color of the corresponding vertex of Γ. Therefore, two multi-color graphs Γ_1 and Γ_2 are isomorphic if and only if the corresponding simple graphs Γ'_1 and Γ'_2 are isomorphic. We may consider that $|V(\Gamma_1)| = |V(\Gamma_2)| = |V|$ and $\Delta(\Gamma_1) = \Delta(\Gamma_2) = \Delta$, $c_1 : V(\Gamma_i) \longrightarrow \{1, 2, \ldots, k_1\}$ and $c_2 : E(\Gamma_i) \longrightarrow \{1, 2, \ldots, k_2\}$ for each $i = 1, 2$, otherwise Γ_1 and Γ_2 are not isomorphic. Therefore, isomorphism of Γ_1 and Γ_2 can be tested in $O(f(k_2 + \Delta + 2)(\frac{1}{2}(\Delta(k_2 + \Delta + 1) + 2k_1 + 4))^{c^*(k_2+\Delta+2)ln(k_2+\Delta+2)}|V|^{c^*(k_2+\Delta+2)ln(k_2+\Delta+2)})$ time. For the graphs of magnetic fields, $\Delta = k_1 = k_2 = 3$.

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