OPERATORS OF RANK 1, DISCRETE PATH INTEGRATION AND GRAPH LAPLACIANS

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Abstract. We prove a formula for a characteristic polynomial of an operator expressed as a polynomial of rank 1 operators. The formula uses a discrete analog of path integration and implies a generalization of the Forman–Kenyon's formula for a determinant of the graph Laplacian [6, 4] (which, in its turn, implies the famous matrix-tree theorem by Kirchhoff) as well as its level 2 analog, where the summation is performed over triangulated nodal surfaces with boundary.

Introduction

The primary impulse to write this article was the famous Matrix-tree theorem (MTT), first discovered by Kirchhoff [7] in 1847 and re-proved more than a dozen times since then. The theorem in its simplest form expresses the number of spanning trees in a graph as a determinant of a suitably chosen matrix M. See [3] for a review of existing proofs and generalizations. One of the proofs, given in [1], makes use of the fact that the matrix in question is a weighted sum of special rank 1 matrices (identity minus reflection). This structure of M explains why the MTT appeares sometimes in quite unexpected contexts (see [2] for just an example).

Suppose M be any operator expressed explicitly as a (noncommutative) polynomial of arbitrary operators of rank 1. The article contains a formula (Theorem 1.1) for the characterstic polynomial of M in terms of these operators. Corollaries of the formula include (but are not limited to!) the MTT (cf, Corollary 1.3), the D-analog of the MTT ([1], cf. Corollary 1.4), a formula for the determinant of the Laplacian of a line bundle on a graph [4, 6] (Section 1.2.2; actually, we compute the whole characteristic polynomial), a Pfaffian Hypertree theorem due to Massbaum and Vaintrob [8], and a level 2 analog of the formula from [4] proved in Sections 1.2.3 and 2.3.

The right-hand side of the main formula (1.4) involve summation over the graphs consisting of several cycles and/or several chains. The summand (the weight W_P, see (1.2)) is a function on edges of the graph obtained also by symmation over the set of paths joining the endpoints of the edge. Consequently, corollaries of the main theorem involve summation over various objects, including trees (the MTT), hypertrees (the Massbaum–Vaintrob theorem), cycle-rooted trees (the Forman–Kenyon formula) and nodal surfaces with boundary (the level 2 analog from Section 1.2.3). The standard expression of a determinant via summation over the permutation group also follows from Theorem 1.1 — see Section 1.2.1.

The article is organized as follows. Section 1.1 contains the formulation and the proof of the main theorem (Theorem 1.1). Section 1.2 lists several corollaries of the theorem. Proofs of some corollaries require not only Theorem 1.1 but also some

additional reasoning, which is given in Section 2. Section 2.1 contains technical lemmas (including a duality lemma 2.5 for the angle between two subspaces of a Euclidean space). Section 2.2 contains the proof of the generalization of Forman's formula (Theorem 1.2), and Section 2.3 is devoted to the proof of Theorem 1.9 about the level 2 analog of the graph Laplacian.

The summation is Theorem 1.1 involves a sort of "discrete path integration" and summation over "discrete 1-manifolds". A tempting direction of the future research here is to send dimension to infinity, getting "real" path integration and summation over 2-varieties. The author plans to write a special paper about this.

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1. Sums of operators of rank 1

1.1. **The main theorem.** Let V be a vector space of dimension n with a scalar product $\langle\cdot,\cdot\rangle$. Choose an integer N and fix two sequences of vectors, $e_1,\dots,e_N\in V$ and $\alpha_1,\dots,\alpha_N\in V$. For any i denote by M_i the operator given by $M_i(v)=\langle\alpha_i,v\rangle e_i$; one has $\operatorname{rk} M_i=1$ or $M_i=0$. Consider an operator

$$M=P(M_1,\dots,M_N) \tag{1.1}$$

where

$$P(x_1,\dots,x_N)=\sum_{s=1}^{k}\sum_{1\le i_1,\dots,i_s\le N} p_{i_1,\dots,i_s}x_{i_1}\dots x_{i_s}$$

is a noncommutative polynomial of degree k. This section contains a description of the characteristic polynomial $\operatorname{char}_M(t)$ of the operator M.

Let G be a finite graph with the vertices $1,2,\dots,N$; let a and b be its vertices. Define the weight $W_P(a,b)$ by

$$W_P(a,b)=\sum_{s=1}^{k}\sum_{i_1=a,i_2,\dots,i_s=b} p_{i_1,\dots,i_s}\langle\alpha_{i_2},e_{i_1}\rangle\langle\alpha_{i_3},e_{i_2}\rangle\dots\langle\alpha_{i_s},e_{i_{s-1}}\rangle, \tag{1.2}$$

(the internal summation is taken over the set of paths $i_1,\dots,i_s$ of length s joining the vertices $a=i_1$ and $b=i_s$). Also, denote

$$W_P(G)\stackrel{\text{def}}{=}\prod_{(a,b)\text{ is an edge of }G} W_P(a,b) \tag{1.3}$$

A directed graph G with the vertices $1,2,\dots,N$ is called a discrete oriented one-dimensional manifold with (possibly empty) boundary (abbreviated as DOOMB) if every its connected component is either an oriented chain (a graph with ℓ distinct vectices $i_1,\dots,i_\ell$ and the edges are (i_1,i_2), (i_2,i_3), $\dots$, $(i_{\ell-1},i_\ell)$) or an oriented cycle (the same edges but the vertices are $i_1,\dots,i_{\ell-1}$ and $i_\ell=i_1$).

Theorem 1.1. $\mathrm{char}_M(t) = \sum_{k=0}^{n} (-1)^k \mu_k t^{n+1-k}$ *where*

$$\mu_k = \sum_{G \in \mathcal{D}_{n,k}} W_P(G) \det(\langle \alpha_{d_1^-}, e_{d_2^+} \rangle). \tag{1.4}$$

Here $\mathcal{D}_{n,k}$ is the set of DOOMBs with the vertices $1, 2, \dots, n$ and k edges; d_1 and d_2 run through the set of all edges of the graph G; d^- and d^+ are the initial and the terminal vertex of the directed edge d.

Remark. The main theorem of [1] is a particular case of Theorem 1.1.

Proof. Consider an orthonormal basis $u_1, \dots, u_n \in V$ and fix a sequence $j_1, \dots, j_k$, $1 \le j_1 < \dots < j_k \le N$. Then

$$\begin{aligned} M(u_{j_1} \wedge \dots \wedge u_{j_k}) &= \sum_{s_1,\dots,s_k} \sum_{\substack{1 \le i_m^{(q)} \le N \\ 1 \le m \le s_q, 1 \le q \le k}} \prod_{q=1}^{k} \left(p_{i_1^{(q)},\dots,i_{s_q}^{(q)}} \prod_{r=2}^{s_q} \langle \alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}} \rangle \right) \\ &\quad \times \prod_{q=1}^{k} \langle \alpha_{i_1^{(q)}}, u_{j_q} \rangle \times e_{i_{s_1}^{(1)}} \wedge \dots \wedge e_{i_{s_k}^{(k)}} \\ &= \sum_{s_1,\dots,s_k} \sum_{\substack{1 \le i_m^{(q)} \le N \\ 1 \le m \le s_q, 1 \le q \le k, \\ i_1^{(1)} < \dots < i_1^{(k)}}} \prod_{q=1}^{k} \left(p_{i_1^{(q)},\dots,i_{s_q}^{(q)}} \prod_{r=2}^{s_q} \langle \alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}} \rangle \right) \\ &\quad \times \sum_{\sigma \in S_k} (-1)^\sigma \prod_{q=1}^{k} \langle \alpha_{i_1^{(q)}}, u_{j_{\sigma(q)}} \rangle \times e_{i_{s_1}^{(1)}} \wedge \dots \wedge e_{i_{s_k}^{(k)}} \end{aligned}$$

The coefficient at $u_{j_1} \wedge \dots \wedge u_{j_k}$ in $M(u_{j_1} \wedge \dots \wedge u_{j_k})$ is then equal to

$$\begin{aligned} &\sum_{s_1,\dots,s_k} \sum_{\substack{1 \le i_m^{(q)} \le N \\ 1 \le m \le s_q, 1 \le q \le k, \\ i_1^{(1)} < \dots < i_1^{(k)}}} \prod_{q=1}^{k} \left(p_{i_1^{(q)},\dots,i_{s_q}^{(q)}} \prod_{r=2}^{s_q} \langle \alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}} \rangle \right) \\ &\quad \times \det(\langle \alpha_{i_1^{(q)}}, u_{j_r} \rangle)_{1 \le q,r \le k} \times \det(\langle u_{j_r}, e_{i_{s_q}^{(q)}} \rangle)_{1 \le q,r \le k}. \end{aligned}$$

Hence

$$\begin{aligned} \mu_k = \mathrm{Tr}\, M^{\wedge k} = \sum_{s_1,\dots,s_k} \sum_{\substack{1 \le i_m^{(q)} \le N \\ 1 \le m \le s_q, 1 \le q \le k, \\ i_1^{(1)} < \dots < i_1^{(k)}}} \prod_{q=1}^{k} \left(p_{i_1^{(q)},\dots,i_{s_q}^{(q)}} \prod_{r=2}^{s_q} \langle \alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}} \rangle \right) \\ \times \det(\langle \alpha_{i_1^{(r)}}, e_{i_{sq}^{(q)}} \rangle)_{1 \le q,r \le k} \end{aligned} \tag{1.5}$$

A multi-index $i_1^{(q)}, \dots, i_{s_q}^{(q)}$, $1 \le q \le k$, can be interpreted as a directed graph G with the edges $(i_1^{(q)}, i_{s_q}^{(q)})$, $1 \le q \le k$, and a path $i_1^{(q)}, \dots, i_{s_q}^{(q)}$ joining endpoints $d_- = i_1^{(q)}$ and $d_+ = i_{s_q}^{(q)}$ of every edge. Conversely, a graph plus collection of paths is just the multi-index. So one can rearrange summation in (1.5) to sum over the

graphs G first, and then over the set of all paths for a given graph, getting $W(G)$ times the determinant. The determinant may be nonzero only if no two $i_1^{(q)}$ and no two $i_{s_q}^{(q)}$ are equal. Thus, in a graph G entering the sum every vertex has at most one outgoing edge and at most one incoming edge. This means that G is a DOOMB. □

1.2. **Graph Laplacians and other examples.** Here are some corollaries of Theorem 1.1.

1.2.1. *Determinants.* It's a funny result demonstrating the nature of Theorem 1.1.

Let $u_1, \dots, u_n$ be an orthonormal basis in $\mathbb{R}^n$. Take $e_{ij} \stackrel{\text{def}}{=} u_i$ and $\alpha_{ij} \stackrel{\text{def}}{=} u_j$ for all $1 \le i, j \le n$. Take

$$P(x_{11}, \dots, x_{nn}) \stackrel{\text{def}}{=} \sum_{i,j=1}^{n} a_{ij} x_{ij}$$

and apply Theorem 1.1. The matrix of the opertor M in the basis $u_1, \dots, u_n$ is (a_{ij}). The polynomial P is linear, so the paths entering equation (1.2) all lave length 1. Consequently, the DOOMB G of (1.4) must be a union of k loops attached to the vertices $(i_1, j_1), \dots, (i_k, j_k)$. So the summation in (1.4) is over the set of unordered k-tuples $(i_1, j_1), \dots, (i_k, j_k)$ with $1 \le i_s, j_s \le N$ for all s. In other words, the summation is over the set of graphs F with the vertices $1, 2, \dots, n$ and k unnumbered directed edges (loops are allowed).

One has $\langle \alpha_{ij}, e_{kl} \rangle = \delta_{jk}$, so the contribution of a graph F into (1.4) is equal to $a_{i_1 j_1} \dots a_{i_k j_k} \det(\delta_{i_p j_q})_{1 \le p \le k}^{1 \le q \le k}$. It is easy to see that the determinant is nonzero only if all the i_p and all the j_q are distinct (else the matrix has identical rows or columns), and for every q there is $p \stackrel{\text{def}}{=} \sigma(q)$ such that $j_q = i_p$ (else a matrix has a zero row). If these conditions are satisfied, the determinant is obviously $(-1)^{\operatorname{sgn}(\sigma)}$ where $\operatorname{sgn}(\sigma)$ is the parity of the permutation σ. Hence, Theorem 1.1 in this case is reduced to the usual formula expressing coefficients of the characteristic polynomial of the operator via its matrix elements.

1.2.2. *Graph Laplacians.* Let F be a directed graph without loops. Following [6], give the following definition:

Definition. A *line bundle with connection on* F is a function attaching a number $\varphi_e \ne 0$ to every directed edge e of the graph. By definition, also take $\varphi_{-e} = \varphi_e^{-1}$ where $-e$ is the edge e with the direction reversed.

To explain the name, attach a one-dimensional space $\mathbb{R}$ (a fiber of the bundle) to every vertex of F and interpret the number φ_e as the 1×1-matrix of the operator of parallel transport along the edge e. For a path $\Lambda = (e_1, \dots, e_k)$ denote $\varphi_\Lambda \stackrel{\text{def}}{=} \varphi_{e_1} \dots \varphi_{e_k}$ (the operator of parallel transport along Λ). If the path Λ is a cycle then φ_Λ is called a holonomy of the cycle.

Suppose now that F has the vertices $1, 2, \dots, n$ and no multiple edges. Supply also every edge (i, j) of H with a weight $c_{ij} = c_{ji}$. Take $N = n(n-1)/2$,

$$P(x_{12}, \dots, x_{n-1,n}) = \sum_{1 \le i < j \le n} c_{ij} x_{ij}, \tag{1.6}$$

and $e_{ij} \stackrel{\text{def}}{=} u_i - \varphi_{ij} u_j$ and $\alpha_{ij} \stackrel{\text{def}}{=} u_i - \varphi_{ji} u_j$), and consider the operator M like in (1.1).

If $v = \sum_{i=1}^n v_i u_i$ then

$$M(v) = \sum_{1\le i<j\le n} c_{ij}(v_i - \varphi_{ji}v_j)(u_i - \varphi_{ij}u_j) = \sum_{i=1}^n u_i \sum_{j\ne i} c_{ij}(v_i - \varphi_{ji}v_j). \tag{1.7}$$

The operator M is called in [6] a Laplacian of the bundle.

Call a graph F a *mixed forest* if every its connected component is either a tree or a graph with one cycle (a connected graph with the number of vertices equal to the number of edges). The graphs where each component contains one cycle are called *cycle-rooted spanning forests (CRSF)* in [6]; hence the name "mixed forest" here.

The following corollary of Theorem 1.1 generalizes the Matrix-CRSF theorem of [4] and [6]:

Theorem 1.2. *The characteristic polynomial of the Laplacian* (1.7) *of a line bundle on a graph is equal to* $\sum_{k=0}^n (-1)^k \mu_k t^k$ *where*

$$\mu_k = \sum_{F\in\mathcal{MF}_{n,k,\ell}} \prod_{\substack{(pq)\ is\\ an\ edge\ of\ F}} c_{pq} \prod_{i=1}^{n-k}(m_i+1) \prod_{j=1}^{\ell}(1-w_j)(1-1/w_j). \tag{1.8}$$

Here $\mathcal{MF}_{n,k,\ell}$ *is the set of mixed forests containing* n *vertices,* k *edges and split into* $n-k$ *tree components and* ℓ *one-cycle componenets;* m_i *is the number of edges in the* i*-th component, and* w_j *is the holonomy of the cycle in the* j*-th component.*

A special case of Theorem 1.2 arises if $\varphi_{ij} = 1$ for all i, j. Then (1.7) implies $M = \sum_{1\le p<q\le n} c_{pq}(1-\sigma_{pq})$ where σ_{pq} is a reflection exchanging the p-th and the q-th coordinate: $\sigma_p(u_p) = u_q$, $\sigma_{pq}(u_q) = u_p$ and $\sigma_{pq}(u_i) = u_i$ for $i \ne p, q$. The reflections σ_{pq} generate the Coxeter group A_n. The holonomies here are all equal to 1, so a mixed forest entering a summation in Theorem 1.2 cannot have cycles and should be a "real" forest:

Corollary 1.3. *The characteristic polynomial of the operator* $M = \sum_{1\le p<q\le n-1} c_{pq}(1-\sigma_{pq})$ *is equal to* $\sum_{k=0}^n (-1)^k \mu_k t^k$ *where*

$$\mu_k = \sum_{F\in\mathcal{F}_{n,k}} \prod_{\substack{(pq)\ is\\ an\ edge\ of\ F}} c_{pq} \prod_{i=1}^{n-k}(m_i+1).$$

Here $\mathcal{F}_{n,k}$ *is the set of forests with* n *vertices,* k *edges and* $n-k$ *components;* m_i *is the number of edges in the* i*-th component.*

Apparently, $\det M = 0$ (there are no forests with n vertices and n edges), so the summation is indeed up to $k = n-1$. This corollary follows also from the classical Principal Minors Matrix-Tree Theorem, see e.g. [3] for proofs and related results.

Another possibility is to join every pair of vertices (i, j) with *two* edges: $(i, j)_-$ with $\varphi_{ij}^- = 1$ ("a $-$-edge", because $e_{ij}^- = u_i - u_j$) and $(i, j)_+$ with $\varphi_{ij}^+ = -1$ ("a $+$-edge" because $e_{ij}^+ = u_i + u_j$); the weights are c_{ij}^+ and c_{ij}^-, respectively. The holonomy of a cycle is $w = (-1)^d$ where d is the number of $+$-edges in the cycle. By (1.7), $M = \sum_{1\le p<q\le n} c_{pq}^-(1-\sigma_{pq}) + c_{pq}^+(1-\tau_{pq})$ where σ_{pq} is as before and τ_{pq} is a reflection mapping $\tau_{pq}(u_p) = -u_q$, $\tau_{pq}(u_q) = -u_p$ and $\tau_{pq}(u_i) = u_i$ for $i \ne p, q$; the reflections σ and τ generate a Coxeter group D_n. So one has

Corollary 1.4. *The characteristic polynomial of the operator*

$$M = \sum_{1\le p<q\le n} c^-_{pq}(1-\sigma_{pq}) + c^+_{pq}(1-\tau_{pq})$$

is equal to $\sum_{k=0}^{n}(-1)^k\mu_k t^k$ *where*

$$\mu_k = \sum_{F\in\mathcal{MFD}_{n,k,\ell}} \prod_{\substack{(pq)_s \text{ is} \\ \text{an edge of } F}} c^s_{pq} \prod_{i=1}^{n-k}(m_i+1).$$

Here $\mathcal{MFD}_{n,k,\ell}$ *is the set of mixed forests with* k *edges* $(pq)_s$, $n-k$ *tree components and* ℓ *one-cycle components such that the number of* $+$*-edges in every cycle is odd;* m_i *is the number of edges in the* i*-th components.*

This corollary generalizes [1, Theorem 3.2].

1.2.3. *The level* 2 *Laplacian.* Take up the same setting as in Section 1.2.2 (a line bundle with a connection over a graph F), and take

$$P(x_{12},\dots,x_{n-1,n}) = \sum_{\substack{1\le i\le n,\\ 1\le j<k\le n}} c_{ijk}(x_{ij}x_{ik}-x_{ik}x_{ij}) = \sum_{1\le i,j,k\le n} c_{ijk}x_{ij}x_{ik}; \tag{1.9}$$

here the constants c_{ijk} are defined for all $1\le i,j,k\le n$ and possess the property $c_{ijk}=-c_{ikj}$ for all i,j,k (in particular, $c_{ijj}=0$ for all i,j). Define M by (1.1): $M\overset{\text{def}}{=}P(M_{12},\dots,M_{n-1,n})$ where $M_{ij}(v)=\langle\alpha_{ij},v\rangle e_{ij}$. Explicitly, if $v=\sum_{p=1}^n v_pu_p$ then

$$M_{ij}(v) = (v_i-\varphi_{ji}v_j)(u_i-\varphi_{ji}u_j) = (v_i-\varphi_{ji}v_j)u_i + (v_j-\varphi_{ij}v_i)u_j.$$

and

$$M(v) = \sum_{i\ne j} u_iv_j\left(\varphi_{ij}\sum_{k\ne i,j}(c_{ijk}+c_{jki}) + \sum_{k\ne i,j} c_{kij}\varphi_{ik}\varphi_{kj}\right). \tag{1.10}$$

Remark 1.5. Note that $M_{ij}=M_{ji}$ because $\alpha_{ji}=-\varphi_{ji}\alpha_{ij}$ and $e_{ji}=-\varphi_{ij}e_{ij}$. Nevertheless, since α and e enter the equation (1.4) separately, one has to choose the ordering of indices in every pair (i,j) used; the final result, of course, does not depend on the choice. We will write $\alpha_{[ij]}$ meaning α_{ij} or α_{ji} depending on the choice; the same for e.

Remark 1.6. In particular, if $\varphi_{ij}=1$ for all i,j, then $M(v)=\sum_{1\le i<j<k\le n} w_{ijk}v_iu_j$ where $w_{ijk}\overset{\text{def}}{=}c_{ijk}+c_{jki}+c_{kij}$. Note that w_{ijk} is totally skew-symmetric in all the three indices, and therefore the operator M is skew-symmetric. In [8] a beautiful formula for the Pfaffian of M was proved; see also [1] for a proof of the same formula using a technique close to Theorem 1.1.

We call M the *level* 2 *Laplacian* of the bundle, by an apparent analogy with the operator defined by (1.7). Note that M_{ij} and M_{kl} commute if $\{i,j\}\cap\{k,l\}=\varnothing$ or $\{i,j\}=\{k,l\}$, that's why P contains no terms like $x_{ij}x_{kl}-x_{kl}x_{ij}$.

Application of Theorem 1.1 to the level 2 Laplacian gives the formula similar to (1.8), where the summation is done over the set of triangulated polyhedra of special kind.

A *nodal surface* is obtained from a smooth surface (a 2-manifold, not necessarily connected, possibly with boundary), by gluing a finite number of points. If the surface has boundary then boundary points also can be glued. A boundary of the nodal surface is still well-defined, but unlike the boundary of a smooth surface, it can be any graph, not just collection of circles. Nodal surfaces with boundary attracted much attention in recent years due to their connection with the geometry of moduli spaces of complex structures, see [9].

Depending on the surface, we will speak about nodal disks, annuli, Moebius bands, etc.

A compact 2-dimensional polyhedron H is called reducible if it can be split into a union $H = H_1 \cup \cdots \cup H_\ell$ where the polyhedra H_i are compact and edge-disjoint (but not necessarily vertex-disjoint!). Every reducible polyhedron is a union of irreducible components.

A 2-dimensional polyhedron is called a *cycle polyhedron* if it is ireducible, homeomorphic to a nodal surface, and every its face is a triangle with exactly one side on the boundary. A 2-dimensional polyhedron is called a *chain polyhedron*, if the last condition is satisfied for all the faces except two. Each of these two faces has two sides on the boundary. One face is called an initial face; one of its boundary sides is marked an called an initial side. The other exceptional face is called a terminal one; one of its boundary sides is marked and called a terminal side.

Choosing an orientation of a face of a cycle polyhedron is equivalent to ordering its sides: the first internal side, the second, the boundary side. For a chain polyhedron the rule is the same except for the initial and the terminal face. For the initial face the ordering is: the initial side, the internal side, the second boundary side; for the terminal face: the internal side, the terminal side, the second boundary side. Say that an orientation of two adjacent faces sharing a side a is compatible if a is the first internal side in one face and the second internal side in the other.

Lemma 1.7. *A cycle polyhedron is one of the following:*

(1) *a nodal annulus where all vertices lie in the boundary;*
(2) *a nodal Moebius band with the same property;*
(3) *a disk (smooth) with a vertex in the interior joined by edges with all the other vertices, which lie on the boundary.*

A chain polyhedron is a nodal disk.

For every cycle polyhedron there are exactly two ways to orient all its faces compatibly. For every chain polyhedron (where the initial and terminal sides are chosen) there is exactly one way to orient all its faces compatibly.

See Section 2.3 for proof.

Example 1.8. See Fig. 1. The left-hand side is a cycle polyhedron (a nodal Moebius band with the node c), the right-hand side is a chain polyhedron (a nodal disk with the node c). Solid lines represent internal edges and exceptional edges (ab and ef at the nodal disk); dotted lines represent boundary edges. The graph G is drawn below; the cycle is directed clockwise and a line is directed left to right. The corresponding ordering of edges inside every face of H is shown by the numbers $1, 2, 3$.

Denote by $\mathcal{CP}_{n,k}$ the set of polyhedra with the vertices $1, 2, \ldots, n$, having k faces, such that every its irreducible component is either a cycle polyhedron or a

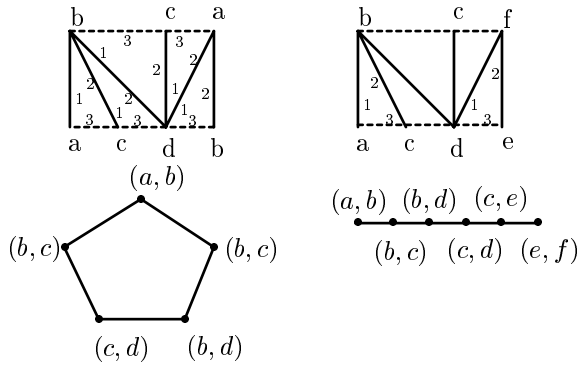

FIGURE 1. A cycle polyhedron and a chain polyhedron

chain polyhedron with compatibly oriented faces. By $\mathrm{ess}_1(H)$ denote the graph formed by the internal edges of H and initial and terminal edges of its chain components. The initial and the terminal edges themselves denote by $I_1(H),\dots,I_s(H)$ and $T_1(H),\dots,T_s(H)$, respectively. Let $H \in \mathcal{CP}_{n,k}$ be such that the graphs $H^I \stackrel{\text{def}}{=} \mathrm{ess}_1(H) \setminus \{I_1(H),\dots,I_s(H)\}$ and $H^T \stackrel{\text{def}}{=} \mathrm{ess}_1(H) \setminus \{T_1(H),\dots,T_s(H)\}$ are mixed forests. Let $H^I_1,\dots,H^I_t$ and $H^T_1,\dots,H^T_t$ be connected components of H^I and H^T, respectively, that are trees (the number of such components $t = m-n-s$, where m is the number of edges in H, is the same for both graphs). Choose in every tree a root $r^\alpha_i \in H^I_i$ and $r^e_j \in H^T_j$ and consider a matrix $M(H)$ such that

$$M(H)^j_i = \sum_{\Lambda \in L_{ij}} \varphi^2_\Lambda n_\Lambda$$

where L_{ij} is the set of paths joining r^e_j with r^α_i, and n_Λ is the number of vertices along the path Λ that belong both to H^I_i and H^T_j.

For an oriented face F of a cycle or a chain polyhedron denote by $s_i(F)$ its i-th side ($i = 1,2,3$); by $v_i(F)$ denote the vertex opposite to the side number i. The internal sides are directed away from their common point; direction of the boundary side is not important.

Theorem 1.9. *Let M be a level 2 Laplacian of a line bundle on a graph F. Then* $\mathrm{char}_M(t) = \sum_{k=0}^n (-1)^k \mu_k t^k$ *where*

$$\mu_k = \sum_{H \in \mathcal{CP}_{n,k}} \prod_{\substack{\Phi \text{ is} \\ \text{a face of } H}} c_{v_3(\Phi)v_1(\Phi)v_2(\Phi)} \langle \alpha_{[s_1(\Phi)]}, e_{[s_2(\Phi)]} \rangle \times \det M(H) \times \prod_{i,j=1}^{s} \prod_{\substack{(pq) \in H \text{ lies} \\ \text{in a path joining} \\ r^\alpha_i \text{ with } r^e_j}} \varphi_{pq} \tag{1.11}$$

See Remark 1.5 for the notation $\alpha_{[s_1]}$, $e_{[s_2]}$.

2. PROOFS

2.1. **Technical lemmas.** Let H be a directed graph without loops or multiple edges.

Definition ([6]). A *line bundle with connection on* H is a function a number $\varphi_{ij} \neq 0$ to every directed edge (i,j) of the graph. By definition, also take $\varphi_{ji} = \varphi_{ij}^{-1}$.

To explain the name, attach a one-dimensional space $\mathbb{R}$ (a fiber of the bundle) to every vertex and interpret the number φ_{ij} as the 1×1-matrix of the operator of parallel transport along the edge (i,j). Consider the space $\mathbb{R}^n$ (which can be interpreted as a total space of the bundle) with the standard basis $u_1, \dots, u_n$. For every i, j consider the vectors $\alpha_{ij} \stackrel{\text{def}}{=} u_i - \varphi_{ij} u_j$ and $e_{ij} \stackrel{\text{def}}{=} u_i - \varphi_{ji} u_j$. Denote by A_H and E_H the sets of vectors α_{ij} or e_{ij}, respectively, where (i,j) runs through the edges of H.

Introduce in $\mathbb{R}^n$ a standard scalar product making $u_1, \dots, u_n$ an orthonormal basis. For two sequences of vectors $M = (\mu_1, \dots, \mu_k)$ and $X = (\xi_1, \dots, \xi_k)$ of the same size k denote by $G(M,X)$ the $k \times k$-matrix where the (i,j)-th entry is equal to the scalar product (μ_i, ξ_j). Call two sequences, M and M', elementarily equivalent, if M' can be obtained from M by a finite number of substitutions $\mu_i \mapsto \mu_i + t\mu_j$ where t is any number. The matrices $G(M,X)$ and $G(M',X)$ are also connected by elementary equivalence (a row is replaced by the sum of itself with a multiple of another row), and therefore $\det G(M,X) = \det G(M',X)$; the same applies to X.

Number the edges of H arbitrarily and consider the matrices $P_H \stackrel{\text{def}}{=} G(A_H, A_H)$, $Q_H \stackrel{\text{def}}{=} G(A_H, E_H)$, $R_H \stackrel{\text{def}}{=} G(E_H, E_H)$. The matrix element of P_H, Q_H and R_H corresponding to the pair of edges s and t depends on their mutual position as follows:

- If s and t do not intersect then the matrix element is zero.
- If s and t have one common vertex b then the matrix element is equal to the product $\psi_{s,a}\psi_{t,a}$ where

$$\psi_{r,a} = \begin{cases} 1, & \text{if the edge } r \text{ points away from the vertex } a, \\ -\varphi_r, & \text{if the edge } r \text{ points towards the vertex } a \\ & \text{and corresponds to the vector } \alpha, \\ -\varphi_r^{-1}, & \text{if the edge } r \text{ points towards the vertex } a \\ & \text{and corresponds to the vector } e. \end{cases}$$

- If the edges have two common vertices a and b then the product is equal to the sum $\psi_{s,a}\psi_{t,a} + \psi_{s,b}\psi_{t,b}$.

The most important cases later will be when H is a tree or a graph with one cycle Describe the determinants of P_H, Q_H and R_H for such H. For a directed path $\Lambda = (\lambda_0, \dots, \lambda_k)$ in the graph denote by φ_Λ the product $\varphi_{\lambda_0\lambda_1} \dots \varphi_{\lambda_{k-1}\lambda_k}$.

Lemma 2.1. *Let H be a rooted tree. Then*

$$\det P_H = \prod_{i,j} \varphi_{ij}^2 \cdot \sum_{\Lambda} \varphi_\Lambda^2 \tag{2.12}$$

where the product is taken over all the edges of H directed away from the root, and and the sum, over all chains Λ joining vertices of the tree with the root. Similarly,

$$\det R_H = \prod_{i,j} \varphi_{ji}^2 \cdot \sum_{\Lambda} \varphi_\Lambda^{-2}$$

in the same notation.

Proof. Clearly, the statements for P_H and R_H are equivalent, so consider the P_H case. Since $\alpha_{ji} = \varphi_{ji}\alpha_{ij}$, changing the direction of an edge $(i,j) \mapsto (j,i)$ multiplies both sides of (2.12) by φ_{ji}^2. So, it is enough to prove the lemma for trees where all the edges are directed away from the root.

Let p be a hanging edge of H and q, its parent. Then $(\alpha_p, \alpha_p) = 1+\varphi_p^2$, $(\alpha_p, \alpha_q) = -\varphi_q$ (the edge q is directed towards p, and p, away from q), and $(\alpha_p, \alpha_x) = 0$ for all the other edges x. Develop the determinant $\det P_H$ by the row p and then by the row q; this gives a relation

$$\det P_H = (1 + 1/\varphi_p^2) \det P_{H'} - 1/\varphi_q^2 \cdot \det P_{H''} \tag{2.13}$$

where H' and H'' are H with p deleted and p and q deleted, respectively.

Suppose by induction that the theorem is proved for H' and H''. Multiply both sides of (2.13) by $\prod_e \varphi_e^2$ where e runs through the set of edges of Γ. It gives

$$\prod_e \varphi_e^2 \det P_H = \sum_{\Lambda'} \varphi_{\Lambda'}^2 + 1/\varphi_p^2 \cdot \sum_{\Lambda'} \varphi_{\Lambda'}^2 - 1/\varphi_p^2 \sum_{\Lambda''} \varphi_{\Lambda''}^2$$

where Λ', Λ'' are chains in H' and H'', respectively, joining vertices with the root. The first sum is over all the chains in Γ not containing p. The second sum is over all the chains containing p, plus the union of p with a chain in Γ'' (because p is hanging). The third sum cancels the second term. □

Lemma 2.2. *If H is a tree with m edges (and $m+1$ vertices) then* $\det Q_H = m+1$.

Proof. Let (i,j) be an edge of H. The graph $H \setminus (i,j)$ is a union of two trees H_1 and H_2 containing vertices i and j, respectively. The matrix Q_H looks like

$$Q_H = \left(\begin{array}{c|ccc|ccc} 2 & 1 & \dots & 1 & -\varphi_{ji} & \dots & -\varphi_{ji} \\ \hline 1 & & & & & & \\ \vdots & & Q_{H_1} & & & 0 & \\ 1 & & & & & & \\ \hline -\varphi_{ij} & & & & & & \\ \vdots & & 0 & & & Q_{H_2} & \\ -\varphi_{ij} & & & & & & \end{array}\right)$$

(we suppose that the edge (i,j) corresponds to the first row and the first column).

Denote the sizes of the first and the second block (i.e. the numbers of edges in H_1 and H_2) by m_1 and m_2 respectively (thus, $m = m_1 + m_2 + 1$). Let $p > m_1 > q$. Then the last m_2 rows of the minor $[Q_H]_{1,p+1}^{1,q+1}$ (deleted columns 1 and $p+1$ and rows 1 and $q+1$) have zeros at the m_1 initial positions; only the last $m-2-m_1 = m_2 - 1$ positions may be nonzero. Hence the rows are linearly dependent, so that $\det[Q_H]_{1,p+1}^{1,q+1} = 0$. The same is true if $p < m_1 < q$. Having $\det Q_H$ decomposed by the first row and then by the first column, one is left only with the terms $(Q_H)_{1,p+1}(Q_H)_{q+1,1} \det[Q_H]_{1,p+1}^{1,q+1}$ where either $p, q \le m_1$ or $p, q \ge m_1 + 1$. In both cases $(Q_H)_{1,p+1}(Q_H)_{q+1,1} = 1$, and therefore $\det Q_H$ does not depend on φ_{ij} (recall that Q_{H_1} and Q_{H_2} contain no φ_{ij} because neither H_1 nor H_2 have an edge (i,j)). Since (i,j) is just an arbitrary edge of H, it proves that $\det Q_H$ does not depend on any φ_{pq} and is a constant.

Suppose now that $\varphi_{pq} = 1$ for all p and q. Then $e_{pr} = u_p - u_r = (u_p - u_q) + (u_q - u_r) = e_{pq} + e_{qr}$, and the same is true for α. Suppose the tree H contains edges pq and qr. Consider a tree H' where the edge qr is replaced by pr; then the systems of

vectors A_H and $A_{H'}$, as well as E_H and $E_{H'}$, differ by an elementary transformation, and therefore $\det Q_H = \det Q_{H'}$. By such operations one can convert H into a line, i.e. a tree with the edges $(p_0, p_1), (p_1, p_2), \dots, (p_{m-1}, p_m)$. The matrix Q_H for such tree is: $(Q_H)_{ii} = 2$, $(Q_H)_{i,i+1} = (Q_H)_{i,i-1} = 1$ for all i. An easy induction shows that $\det Q_H = m + 1$. □

Let now H be a graph with one cycle, i.e. a connected graph with n vertices and n directed edges. It consists of a cycle $p_1, \dots, p_s$ and, possibly, some trees ("antlers") attached to the vertices p_i. The direction of edges in the cycle and in the antlers can be arbitrary. Following [6], call the *holonomy* of the cycle the product $w \stackrel{\text{def}}{=} \varphi_{p_1p_2}^{\pm 1} \dots \varphi_{p_sp_1}^{\pm 1}$ where the i-th exponent is $+1$ if the corresponding edge is directed along the cycle (from p_i to p_{i+1}) and -1 otherwise.

Lemma 2.3. *Let H be a graph with one cycle. Then*

$$\det P_H = (1-w)^2 \prod_{i,j} \varphi_{ij}^2,$$
$$\det R_H = (1-w^{-1})^2 \prod_{i,j} \varphi_{ij}^{-2},$$
$$\det Q_H = -(1-w)(1-w^{-1}),$$

where the product is taken over the set of all the edges in the antlers directed away from the cycle.

Proof. The proofs of all the three statements are similar; here is the proof of the statement about Q_H.

Consider a system of vectors $E_H^{(1)}$ elementarily equivalent to E_H. To obtain $E_H^{(1)}$ from E_H replace every vector $\varepsilon_i \stackrel{\text{def}}{=} e_{p_ip_{i+1}} = u_{p_i} - \varphi_{p_{i+1}p_i} u_{p_{i+1}}$, $i = 1, \dots, s$, with $\varepsilon_i^{(1)}$ where

$$\varepsilon_{s-1}^{(1)} \stackrel{\text{def}}{=} \varepsilon_{s-1} + \varphi_{p_s,p_{s-1}} \varepsilon_s = u_{s-1} - \varphi_{p_s,p_{s-1}} \varphi_{p_1,p_s} u_1,$$
$$\varepsilon_{s-2}^{(1)} \stackrel{\text{def}}{=} \varepsilon_{s-2} + \varphi_{p_{s-1},p_{s-2}} \varepsilon_{s-1}^{(1)} = u_{s-2} - \varphi_{p_{s-1},p_{s-2}} \varphi_{p_s,p_{s-1}} \varphi_{p_1,p_s} u_1,$$
$$\vdots$$
$$\varepsilon_1^{(1)} = u_1(1 - \varphi_{p_2p_1} \dots \varphi_{p_1,p_s}) = (1 - 1/w) u_1.$$

Consider also a system $A_H^{(1)}$ obtained from A_H in a similar way. One has $\det G(E^{(1)}, A_H^{(1)}) = Q_H$ by elementary equivalence.

If $w = 1$ then $\varepsilon_1^{(1)} = 0$, so $\det Q_H = \det G(E^{(1)}, A_H^{(1)}) = 0$, and the lemma is proved; we suppose from now on that $w \neq 1$.

Consider now the sequence $E_H^{(2)}$, which is $E_H^{(1)}$ with every $\varepsilon_i^{(1)}$ replaced with

$$\varepsilon_i^{(2)} = \varepsilon_i^{(1)} - \varepsilon_1^{(1)}/(1 - 1/w) = -\varphi_{p_i,p_{i-1}} u_i.$$

Consider also $A_H^{(2)}$ obtained from $A_H^{(1)}$ in a similar way. The sequences $E_H^{(2)}$, $A_H^{(2)}$ are elementarily equivalent to $E_H^{(1)}$, $A_H^{(1)}$, so $\det G(E_H^{(2)}, A_H^{(2)}) = Q_H$.

Denote by H_i a subtree of H attached to the vertex p_i, $1 \le i \le s$, and let $\beta_j = e_{q_jr_j}$, $1 \le j \le m_i$ be vectors in E_H (and $E_H^{(2)}$) corresponding to its edges. For all edges attached immediately to p_i (so that $q_j = p_i$) replace $\beta_j \mapsto \beta_j^{(1)} = \beta_j +$

$\varepsilon_i\varphi_{p_{i+1},p_i} = -\varphi_{r_jp_1}u_{r_j}$. Then do the same for all the edges attached to endpoints of $\beta_j^{(1)}$, etc. Having done this for all i, $1 \le i \le s$, one obtaines the system $E'_H = ((1-1/w)u_1, -\varphi_{12}u_2, \dots, -\varphi_{1m}u_m)$ elementarily equivalent to E_H. Similarly, A_H is elementarily equivalent to $A'_H = ((1-w)u_1, -\varphi_{12}u_2, \dots, -\varphi_{m1}u_m)$. Now $\det Q_H = \det G(E'_H, A'_H)$; the matrix $G(E'_H, A'_H)$ is a diagonal matrix with $(1-w)(1-1/w)$ in the corner and 1 in all the other positions on the main diagonal. □

Below we will need two more statements from the general linear algebra.

Lemma 2.4. *Let $e_1, \dots, e_n$ be an orthonormal basis in $\mathbb{R}^n$, and $\mu_i = \sum_{j=1}^n a_{ij}e_j$. Then $\det G(M,M)$ is equal to the sum of squares of all the $k \times k$-minors of the matrix $A = (a_{ij})_{1\le i\le k}^{1\le j\le n}$.*

The lemma is classical, see e.g. [5, IX§5] for proof.

Let now $\mathcal{M}, \mathcal{X} \subset \mathbb{R}^n$ be two linear subspaces of the same dimension k, and $M = (\mu_1, \dots, \mu_k)$ and $X = (\xi_1, \dots, \xi_k)$ be bases in them. Denote

$$\angle(\mathcal{M}, \mathcal{X}) \stackrel{\text{def}}{=} \det G(M,X)^2/(\det G(M,M) \cdot \det G(X,X)).$$

Lemma 2.5. (1) *$\angle(\mathcal{M}, \mathcal{X})$ depends only on the subspaces and not on the choice of the bases M and X in them.*

(2) *$\angle(\mathcal{M}^\perp, \mathcal{X}^\perp) = \angle(\mathcal{M}, \mathcal{X})$.*

Proof. Let $N = (\nu_1, \dots, \nu_k)$ be another basis in $\mathcal{M}$; denote by $A = (a_{ij})$ the transfer matrix: $\nu_i = \sum_{j=1}^k a_{ij}\mu_j$. Then $G(N,X) = AG(M,X)$, and $G(N,N) = AG(M,M)A^*$; so $G(N,X)^2/(\det G(N,N)\cdot\det G(X,X)) = \det A^2 \det G(M,X)^2/(\det A^2 \cdot \det G(M,M) \cdot \det G(X,X)) = \angle(\mathcal{M}, \mathcal{X})$; the same is for X.

Let now $\mathcal{M}_1, \mathcal{M}_2 \subset \mathbb{R}^n$ be two spaces of equal dimension, and let $\mathcal{M} \stackrel{\text{def}}{=} \langle \mathcal{M}_1 \cup \mathcal{M}_2 \rangle$ be their linear hull. Then $\mathcal{M}_i^\perp = \mathcal{M}_i^{\perp,\mathcal{M}} \oplus \mathcal{M}^\perp$, $i = 1, 2$; here $\perp$ means an orthogonal complement in $\mathbb{R}^n$, and $(\perp, \mathcal{M})$ means an orthogonal complement in $\mathcal{M}$. Choose an orthonormal basis $T = (\tau_1, \dots, \tau_q)$ in $\mathcal{M}^\perp$, and bases $\Lambda_i = (\lambda_1^{(i)}, \dots, \lambda_p^{(i)})$ in $\mathcal{M}_i^{\perp,\mathcal{M}}$, $i = 1, 2$, normal to $\mathcal{M}^\perp$. Denote $Y_1 = (\lambda_1^{(1)}, \dots, \lambda_p^{(1)}, \tau_1, \dots, \tau_q)$ and $Y_2 = (\lambda_1^{(2)}, \dots, \lambda_p^{(2)}, \tau_1, \dots, \tau_q)$ are bases in $\mathcal{M}_1^\perp$ and $\mathcal{M}_2^\perp$, respectively. By the first statement of the theorem, $\angle(\mathcal{M}_1^\perp, \mathcal{M}_2^\perp) = \det G(Y_1,Y_2)^2/(\det G(Y_1.Y_1) \cdot \det G(Y_2,Y_2))$. The matrix $G(Y_1,Y_2)$ is block diagonal; its first block is the matrix $G(\Lambda_1,\Lambda_2)$, and the second block is the unit matrix $G(T,T)$; thus $\det G(Y_1,Y_2) = \det G(\Lambda_1,\Lambda_2)$. Similarly, $\det G(Y_1,Y_1) = \det G(\Lambda_1,\Lambda_1)$ and $\det G(Y_2,Y_2) = \det G(\Lambda_2,\Lambda_2)$. Hence, $\angle(\mathcal{M}_1^\perp, \mathcal{M}_2^\perp) = \angle(\mathcal{M}_1^{\perp,\mathcal{M}}, \mathcal{M}_2^{\perp,\mathcal{M}})$.

Let now $\mathcal{X} = \mathcal{M}_1 \cap \mathcal{M}_2 \ne 0$. A similar choice of a basis shows that $\angle(\mathcal{M}_1, \mathcal{M}_2) = \angle(\mathcal{X}_1, \mathcal{X}_2)$ where $\mathcal{X}_i = \mathcal{X}^{\perp,\mathcal{M}_i}$, $i = 1, 2$. So, it is enough to prove the second statement of the theorem for the k-dimensional subspaces $\mathcal{M}_1, \mathcal{M}_2 \subset \mathbb{R}^n$ such that $n = 2k$ and $\mathcal{M}_1 \cap \mathcal{M}_2 = 0$, so that $\mathbb{R}^n = \langle \mathcal{M}_1 \cap \mathcal{M}_2 \rangle$. To do this take an orthonormal basis $e_1, \dots, e_{2k}$ in $\mathbb{R}^n$ such that $X_1 \stackrel{\text{def}}{=} (e_1, \dots, e_k)$ is a basis in $\mathcal{M}_1$, and $X_2 \stackrel{\text{def}}{=} (e_{k+1}, \dots, e_{2k})$ is a basis in $\mathcal{M}_1^\perp$. Let $F = (f_1, \dots, f_k)$ be a basis in $\mathcal{M}_2$, $f_i = \sum_{j=1}^k b_{ij}e_j + c_{ij}e_{j+k}$. Since $\mathcal{M}_1 \cap \mathcal{M}_2 = 0$, the matrix $C = (c_{ij})_{1\le i\le k}^{1\le j\le k}$ is nondegenerate. By elementary transformations of rows one can make C a unit matrix; assume that $C = \mathrm{id}$ from the very beginning, so that $f_i = \sum_{j=1}^k b_{ij}e_j + e_{i+k}$. Then $G(F,X_1) = B$, and therefore $\det G(F,X_1) = \Delta \stackrel{\text{def}}{=} \det B$. By Lemma 2.4, the determinant $\det G(F,F)$ is equal to the sum of squares of all $k \times k$-minors of the

$k \times (2k)$-matrix composed of two blocks, B and the identity $k \times k$-matrix. It is easy to see that the latter sum is the sum of squares of all the minors of B (of all sizes), including Δ^2 and 1 (the square of the empty minor).

Consider now the vectors $h_i = -e_i + \sum_{j=1}^{k} b_{ji} e_{j+k}$, $i = 1, \dots, k$. Apparently, $H = (h_1, \dots, h_k)$ is a basis in $\mathcal{M}_2^{\perp}$; also $G(H, X_2) = B^T = G(F, X_1)^T$, and also $\det G(H, H) = \det G(F, F)$ by Lemma 2.4. This proves the lemma. □

2.2. **Proof of Theorem 1.2.** To apply Theorem 1.1 note that the polynomial P of (1.6) has degree 1, so every path involved should contain one vertex only. Therefore, the DOOMB G must be a union of k loops attached to the vertices $(i_1, j_1), \dots, (i_k, j_k)$. So the summation is over the set of unordered k-tuples $(i_1, j_1), \dots, (i_k, j_k)$ with $1 \le i_s, j_s \le N$ for all s. In other words, the summation is over the set of graphs F with the vertices $1, 2, \dots, n$ and k unnumbered directed edges.

Let $F_1, \dots, F_\ell$ be connected components of F. If the edges d_1 and d_2 belong to different components then $\langle \alpha_{d_1}, e_{d_2} \rangle = 0$. So the matrix Q_F is block diagonal, and $\det Q_F = \det Q_{F_1} \dots \det Q_{F_\ell}$. If F is a connected graph with m edges $(i_1, j_1), \dots, (i_m, j_m)$ and $\mu < m$ vertices then the vectors $e_{i_1,j_1}, \dots, e_{i_m,j_m}$ belong to a vector space of dimension μ spanned by the corresponding basis elements u_i. Therefore they are linearly dependent, so that $\det Q_F = 0$. Thus, if $\det Q_F \ne 0$ then every connected component F_i of F should be either a tree (with $\mu = m + 1$) or a graph with one cycle ($\mu = m$). So, F is a mixed forest.

Theorem 1.2 now follows from Theorem 1.1, Lemma 2.2 and Lemma 2.3.

2.3. **Proof of Theorem 1.9.** Apply Theorem 1.1 to the operator M of (1.10). Like in Section 2.2, vertices of the graph G are pairs (i, j), $1 \le i < j \le n$, that is, edges of a directed graph with the vertices $1, \dots, n$. The polynomial P contains only terms $x_{ij} x_{ik}$; therefore if (a, b) is an edge of G then the pairs $a = \{i, j\}$ and $b = \{i, k\}$ have exactly one common element. Represent such edge by a triangle ijk. Color its sides ij and ik (corresponding to the vertices of G) black, and the third side jk, red (they are shown as sold and the dashed lines, respectively, in Fig. 1). Thus a graph G is represented by 2-dimensional polyhedron (call it H) with triangular faces and edges colored red and black so that every face has two black sides and one red side. The black edges of H form a graph denoted by $\mathrm{ess}_1(H)$ and called an essential 1-skeleton of H.

If G consists of connected components $G_1, \dots, G_s$ then the corresponding subpolyhedra $H_1, \dots, H_s$ of H are edge-disjoint but not necessarily vertex-disjoint; thus, H is reducible, and $H_1, \dots, H_s$ are its irreducible components. By Theorem 1.1 every G_i is either an oriented cycle or an oriented chain.

Let first G_i be an oriented cycle. In the corresponding H_i every black edge belongs to two faces and every red edge, to one face. Hence H_i is a nodal surface with boundary where every face has two internal sides and one boundary side. An orientation of an edge of G_i gives rise to an orientation of the corresponding face of H_i; since the orientations of the edges in a cycle G_i are compatible, so are orientations of the faces in H_i. The graph G_i is connected, so the polyhedron H_i is irreducible. Hence, H_i is a cycle polyhedron. Conversely, if H_i is a cycle polyhedron, then G_i is connected and every its vertex has valency 2 — hence, G_i is a cycle.

In a similar manner one proves that G_i is an oriented chain if and only if H_i is a chain polyhedron.

Proof of Lemma 1.7. Let H be a cycle polyhedron with the vertices $1, 2, \dots, n$. Build the graph G such that the vertices of G are internal edges of H (elements of $\mathrm{ess}_1(H)$), and two vertices are joined by an edge if the corresponding edges belong to a face.

As proved before, the graph G is an oriented cycle where every vertex is marked by two indices (i, j), and the neighbouring vertices have exactly one common index. For every index i denote by K_i the set of vertices in G containing i as one of the indices. If for some i the corresponding K_i contains all the vertices, then the second indices j in all the vertices are pairwise distinct, and we have a disk described in clause 3 of the lemma. From now on suppose that for every index i there is at least one vertex of G *not* containing it.

Every K_i is a union of several "solid arcs" $K_{i,1}, \dots, K_{i,s_i}$ (a solid arc is one vertex or a sequence of several successive vertices in a cycle). Consider an auxiliary graph G' where for all i and $p = 1, \dots, s_i$ the index i in the vertices in $K_{i,p}$ is renamed into a new index i_p. Thus, the polyhedron H is obtained from the polyhedron H' corresponding to G' by identification of some vertices. Thus it is enough to prove that if every K_i is one solid arc (but not the whole cycle) then the polyhedron H is an annulus or a Moebius band.

Let G contain m vertices. Consider an auxiliary circle S^1 with m equidistant points $a_1, \dots, a_m$ on it. Let K_i be a solid arc covering the vertices $p, p+1, \dots, p+q$. Define a subset $L_i \subset S^1$ as an arc from a_p to a_{p+q}, including both endpoints. Every two neighbouring vertices in G have a common index i, so every point of every arc $[a_p, a_{p+1}]$ belongs to some L_i, hence $\bigcup L_i = S^1$. Triple intersections of different L_is are all empty; the intersections $L_i \cap L_j$ are either empty or one point. There are exactly m pairs with nonempty intersection, because L_i are arcs. Thus, the Euler characteristics is

$$0 = \chi(S^1) = \chi(\bigcup L_i) = \sum_i \chi(L_i) - \sum_{i,j} \chi(L_i \cap L_j).$$

One has $\chi(L_i) = 1$ for all i and $\chi(L_i \cap L_j) = 1$ for m pairs i, j where the intersection is nonempty, and $\chi(L_i \cap L_j) = 0$ otherwise. So, the number of indices i, that is, the number of vertices of the polyhedron H, is equal to m. The total number of its faces is the number of edges in G, that is, m. Also H contains m red edges (one per face) and m black edges (corresponding to the vertices of G). Thus $\chi(H) = m - 2m + m = 0$, and H is either an annulus or a Moebius band.

The proof in the chain case is similar. □

So, summation in the right-hand side of Theorem 1.1 for the $\mathrm{char}_M(t)$ is performed over the set of nodal sufraces with boundary $H = H_1 \cup \dots \cup H_m$ where the irreducible components $H_1, \dots, H_s$ are either cycle polyhedra or chain polyherda.

The weight $W_P(H)$ of a polyhedron is equal to the product of the weights of the faces. The weight of a triangular face pqr where pq is the first internal edge, pr, the second, and qr, a boundary edge, is equal to $c_{pqr}\langle \alpha_{[pq]}, e_{[pr]} \rangle$. So, the second factor is equal to 1, $-\varphi_{pq}$, $-\varphi_{pr}$ or $\varphi_{pq}\varphi_{pr}$ depending on how the edges pq and pr are directed. The weight of H is $W_P(H) = W_P(H_1) \dots W_P(H_m)$.

Let $H_1, \dots, H_s$ be chain polyhedra with the initial edges $I_1(H), \dots, I_s(H)$ and the terminal edges $T_1(H), \dots, T_s(H)$, and $H_{s+1}, \dots, H_m$ be cycle polyhedra. The determinant in (1.4) is equal to $\det G(A_{H^I}, E_{H^T})$ where $H^I \stackrel{\mathrm{def}}{=} \mathrm{ess}_1(H) \backslash \{I_1(H), \dots, I_s(H)\}$ and $H^T \stackrel{\mathrm{def}}{=} \mathrm{ess}_1(H) \setminus \{T_1(H), \dots, T_s(H)\}$. In particular, if the determinant is

nonzero, then $e_i, i \neq I_1(H), \dots, I_s(H)$, as well as $\alpha_j, j \neq T_1(H), \dots, T_s(H)$, are linearly independent. According to Section 2.2, this implies that the graphs H^I and H^T are mixed forests.

Thus, equation (1.4) for the case considered looks as follows: if M is the level 2 Laplacian then $\operatorname{char}_M(t) = \sum_{k=0}^{n}(-1)^k \mu_k t^k$ where

(2.14)
$$\mu_k = \sum_{\substack{H \in \mathcal{CP}_{n,k} \\ H^I \text{ and } H^T \text{ are} \\ \text{mixed forests}}} \prod_{\substack{\Phi \text{ is} \\ \text{a face of } H,}} c_{v_3(\Phi)v_1(\Phi)v_2(\Phi)} \langle \alpha_{[s_1(\Phi)]}, e_{[s_2(\Phi)]} \rangle \times \det G(A_{H^I}, E_{H^T})$$

The determinantal term in (2.14) can be simplified using Lemma 2.5:

$$\begin{aligned}(\det G(A_{H^I}, E_{H^T}))^2 &= \angle(\langle A_{H^I}\rangle, \langle E_{H^T}\rangle) \cdot \det G(A_{H^I}, A_{H^I}) \det(E_{H^T}, E_{H^T}) \\ &= \angle(\langle A_{H^I}\rangle^\perp, \langle E_{H^T}\rangle^\perp) \cdot \det G(A_{H^I}, A_{H^I}) \det(E_{H^T}, E_{H^T}) \\ &= \frac{\det G(A'_{H^I}, E'_{H^T})^2}{\det G(A'_{H^I}, A'_{H^I}) \det G(E'_{H^T}, E'_{H^T})} \\ &\qquad \times \det G(A_{H^I}, A_{H^I}) \det(E_{H^T}, E_{H^T}); \end{aligned} \tag{2.15}$$

here $\langle X\rangle$ mean the subspace in $\mathbb{R}^n$ spanned by X, $^\perp$ means an orthogonal complement, and A'_{H^I}, E'_{H^T} are bases in $\langle A_{H^I}\rangle^\perp$ and $\langle E_{H^T}\rangle^\perp$, respectively.

By Lemma 2.5 the formula above is true for any choice of the bases A'_{H^I}, E'_{H^T}; below we describe orthogonal bases the most convenient for our purposes. For any graph G denote by $\mathcal{C}(G)$ the set of its connected components. Let $\mathcal{C}(H^I) = \{H^I_1, \dots, H^I_s\}$ and $\mathcal{C}(H^T) = \{H^T_1, \dots, H^T_t\}$; denote by $V^I_i \subset \mathbb{R}^n$ and $V^T_i \subset \mathbb{R}^n$ the subspaces spanned by the vertices of H^I_i and H^T_i, respectively. Then

$$\langle A_{H^I}\rangle^\perp = \bigoplus_{i=1}^{s} \langle A_{H^I_i}\rangle^{\perp, V^I_i},$$

where the summands are pairwise orthogonal; the same is true for E_{H^T}.

Every H^I_i and H^T_i is either a tree or a graph with one cycle; choose a root in every tree component and denote it by r^α_i and r^e_i, respectively.

Lemma 2.6. *If H is a tree with a root r, then the spaces $\langle A_H\rangle^\perp$ and $\langle E_H\rangle^\perp$ have dimension 1 and are spanned by vectors $b_H \stackrel{def}{=} \sum_u u\varphi_{ur}$ and $f_H \stackrel{def}{=} \sum_u u\varphi_{ru}$, respectively, where the summation is over the set of vertices of H, and ur, ru are paths joining u with r and r with u.*

If H is a graph with one cycle with monodromy $w \neq 1$ then $\langle A_H\rangle = \langle E_H\rangle = \mathbb{R}^n$.

Proof. Let H be a tree. The spaces $\langle A_H\rangle$ and $\langle E_H\rangle$ are spanned by $n-1$ vectors in $\mathbb{R}^n$; the vectors are linearly independent by Lemma 2.2. Hence, $\dim\langle A_H\rangle^\perp = \dim\langle E_H\rangle^\perp = 1$. Apparently, $(b_H, \alpha_{ij}) = 0$ and $(f_H, e_{ij}) = 0$ for any edge ij of H, and the first statement follows.

The second statement follows from Lemma 2.3. □

The number of rows and columns of the matrix $G(A'_{H^I}, E'_{H^T})$ is equal to the number of tree components of the graphs H^I and H^T. If H^I_i is a tree component of H^I and H^T_j is a tree component of H^T, then, obviously, $G(A'_{H^I}, E'_{H^T})^j_i = (b_{H^I_i}, f_{H^T_j}) = \sum_{\Lambda \in L_{ij}} \varphi^2_\Lambda n_\Lambda = M(H)^j_i$ (recall that L_{ij} is the set of paths joining r^e_j

with r_i^α and n_Λ is the number of vertices along the path Λ that belong both to H_i^I and H_j^T).

Lemmas 2.6 and 2.1 imply now that

$$\det G(A_{H^I}, E_{H^T}) = \det M(H) \times \prod_{i=1}^{s} \prod_{\substack{(pq)\in H_i^I \text{ looks} \\ \text{away from } r_i^\alpha}} \varphi_{pq} \times \prod_{j=1}^{s} \prod_{\substack{(pq)\in H_j^T \text{ looks} \\ \text{away from } r_j^e}} \varphi_{qp}$$

$$= \det M(H) \times \prod_{i,j=1}^{s} \prod_{\substack{(pq)\in H \text{ lies} \\ \text{in a path joining} \\ r_i^\alpha \text{ with } r_j^e}} \varphi_{pq},$$

and Theorem 1.9 is proved.

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References

[1] Yu. Burman, A. Ploskonosov, A. Trofimova, *Higher matrix-tree theorems*, preprint arXiv:1109.6625 (math.CO).

[2] Yu.Burman, D.Zvonkine, Cycle factorizations and 1-faced graph embeddings, *European Journal of Combinatorics*, **31** (2010) no. 1. pp. 129–144.

[3] S. Chaiken, A combinatorial proof of the all minors matrix tree theorem, *SIAM J. Alg. Disc. Meth.*, **3** (1982), no. 3, pp. 319–329.

[4] R. Forman, Determinants of Laplacians on graphs, *Topology* **32** (1993), no. 1, pp. 35–46.

[5] F.R.Gantmacher, *The theory of matrices*, AMS Chelsea publishing, 2000.

[6] R. Kenyon, *The Laplacian on planar graphs and graphs on surfaces*, preprint arXiv:1203.1256 (math.PR, math.CO).

[7] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung det linearen Verteilung galvanischer Ströme gefurht wird, *Ann. Phys. Chem.*, **72** (1847), pp. 497–508.

[8] G. Masbaum, A. Vaintrob, A new matrix-tree theorem, *Internat. Math. Res. Notices*, **27** (2002), pp. 1397–1426.

[9] D.J. LaFountain, R.C. Penner, *Cell decomposition and odd cycles on compactified Riemann's moduli space*, preprint arXiv:1112.3915 (math.GT, math.AG).

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