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Thick attractors of boundary preserving diffeomorphisms

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To the memory of Floris Takens

Abstract

A diffeomorphism is said to have a thick attractor provided that its attractor has positive but not full Lebesgue measure. A set in a functional space is quasiopen, if it may be obtained from an open set by removing a countable number of hypersurfaces. We prove that there exists a quasiopen set in the space of *boundary preserving* diffeomorphisms of a compact manifold with boundary, such that any map in this set has a thick attractor. The meaning of the word "attractor" should be specified. In the above claim an "attractor" is, roughly speaking, a "topologically mixing maximal attractor". We also conjecture that the claim is true for the Milnor attractor of diffeomorphisms and prove the claim for Milnor attractors of mild skew products. We reduce the conjecture above to a general conjecture about Milnor attractors of partially hyperbolic diffeomorphisms.
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1. Introduction

Understanding of the structure of attractors of generic dynamical systems is one of the major goals of the theory of these systems. A vast general program suggested in [20] presents numerous conjectures about this structure. Various particular cases of these conjectures are proved in numerous papers that we do not quote here. New problems and conjectures are presented in [21]. Main part of these investigations is related to diffeomorphisms of closed manifolds.

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Our investigation is in a sense parallel to this direction of research. It is a part of a vast program proclaimed in [8]. We study attractors of *manifolds with boundary onto themselves*. At present, locally generic properties of attractors of such maps are established, that are not yet observed (and plausibly do not hold) for the case of closed manifolds. For instance, an open set of diffeomorphisms of manifolds with boundary onto themselves may have attractors with intermingled basins [17,4,3,15,18].

Another property of this kind is *having thick attractors*. It is a general belief that attractors of typical smooth dynamical systems (diffeomorphisms and flows) on closed manifolds, either coincide with the whole phase space, or have Lebesgue measure zero. In this paper we prove that this is not the fact for diffeomorphisms of manifolds with boundary onto themselves. Namely, in the space of diffeomorphisms of a product $\mathbb{T}^2 \times I$, $I = [0, 1]$, there exists an open set such that any map from a complement of this set to a countable number of hypersurfaces, has a thick attractor: a transitive attractor that has positive Lebesgue measure together with its complement. Note that this is not a formal definition because the word attractor has different meanings.

The problem, whether or not thick attractors exist for locally generic diffeomorphisms of a closed manifold, remains widely open. A closely related problem is studied in the papers [1,22].

1.1. Definition of thick attractors

Consider a homeomorphism F of a measure metric space X into itself. An open domain $U \subset X$ is *absorbing* provided that $F(\bar{U}) \subset U$. The *maximal attractor* of F in U is the intersection

$$A_{\max} = A_{\max}(F, U) = \cap_{n>0} F^n(U). \tag{1}$$

If almost all points of X visit U in future, we say that (1) is the maximal attractor of $F : A_{\max}(F, U) = A_{\max}(F)$. If there is no such absorbing domain, we say that $A_{\max}(F) = X$.

The *Milnor attractor* $A_M(F)$ of F is the minimal closed set that contains ω-limit sets for almost all points of X. Sometimes we write $A_{\max}$, A_M instead of $A_{\max}(F)$, $A_M(F)$.

The maximal attractor does not necessary coincide with the Milnor one. Many examples of this kind may be found in [19]. A simplest example (having codimension infinity) is contained in [2], Section 3.8. An important example of a boundary preserving many-to-one map of an annulus was suggested by Kan [17], see also [4]. The Milnor attractor of this map is the boundary of the annulus, the maximal attractor is the annulus itself. This non-coincidence is generic in the space of boundary preserving maps of the annulus. A sketch of the proof of this statement is given in [3]. A complete proof is presented in [15,18]. Analogous theorem for the boundary preserving diffeomorphism is proved in [18]. A first example of such a diffeomorphism is provided in [13].

In what follows, the phase space is a compact Riemannian manifold with the Riemannian volume also called the Lebesgue measure, if otherwise is not stated.

Definition 1. An attractor is called *thick* if it has positive Lebesgue measure together with its complement.

In general, having thick maximal attractor does not imply having thick Milnor attractor.

Definition 2. A thick attractor is called *almost topologically mixing*, if it is topologically mixing in a subset whose complement in the attractor has measure zero.

Note that a measure zero subset in a thick attractor is in a sense negligible. Such a subset may not belong to the closure of its complement in the attractor.

1.2. Local genericity of thick attractors for boundary preserving diffeomorphisms

A set in a functional space is quasiopen, if it may be obtained from an open set by removing of a countable number of hypersurfaces.

Our main results claim

Theorem 1 (*Main Theorem*)**.** *There exists a quasiopen set in the space of boundary preserving diffeomorphisms of a product of a two torus to a segment that consists of maps with thick almost topologically mixing maximal attractors.*

Conjecture 1. *The same is true for Milnor attractors.*

We reduce this conjecture to the following one.

Conjecture 2. *The Milnor attractor of a partially hyperbolic map with strong unstable fibers of dimension one, is saturated by strongly unstable leaves of this map.*

The first Conjecture is proved for skew product diffeomorphisms.

Theorem 2. *There exists a quasiopen set in the space of boundary preserving skew product diffeomorphisms of a two torus to a segment that consist of maps with a thick almost topologically mixing Milnor attractor.*

Attractors in Theorems 1, 2 and 4 below are proved to be "almost topologically mixing". This means that they contain a topologically mixing invariant subset whose complement in the attractor has measure zero.

The quasiopen set in the Main Theorem is obtained as a set of small perturbations of special skew products, that satisfy some mild restrictions. We believe that some of these restrictions are purely technical, and the quasiopen set in the Main Theorem might be replaced by an open one.

In [24,25] Tsujii introduced *fat attractors*. These are attractors of *many-to-one* maps that have a positive Lebesgue measure together with their complement. At this spot the difference between diffeomorphisms and non-bijective maps is crucial. Dynamical systems with fat attractors are semi-conjugated to diffeomorphisms with a "thin" attractor, that is, having zero Lebesgue measure. The semi-conjugacy decreases the dimension, and maps a "thin" attractor to a fat one. A remarkable property of fat attractors is that they support an SRB measure which is absolutely continuous with respect to the Lebesgue measure, even for an open set of maps [25].

Existence and description of SRB measures for the diffeomorphisms with thick attractors is an open problem.

2. Skew products of class TAT

In this subsection we describe a class of skew products that is much wider than the set for which we prove the existence of thick attractors. We *expect* that all of them have thick Milnor attractors. But we *prove* this property only for those skew products whose map in the base is the Anosov diffeomorphism of the two-torus. Later on we consider skew products over Markov chains.

2.1. Description of class TAT

Let B be a closed manifold, and h be a transitive hyperbolic diffeomorphism $B \to B$ with an SRB measure P. Denote $I = [0, 1]$. Let $X = B \times I$ and $F : X \to X$ be a C^3 skew product over the map h in the base:

$$F : (b, x) \mapsto (h(b), f_b(x)). \tag{2}$$

Here $f_b : I \to I$ is a diffeomorphism called a *fiber map*.

Suppose that F satisfies the following assumptions.

1. The map F is partially hyperbolic. The fibers of the skew product are the central manifolds of F. To gain this property, we assume that the fiber maps of F are close to identity in $C^3(I)$. More precisely, we require

$$L_c := \max_B \{\mathrm{Lip}\, f_b, \mathrm{Lip}\, f_b^{-1}\} \le L_0. \tag{3}$$

The logarithm of the bound L_0 should be small enough, and is chosen below.

2. The fiber maps are onto:

$$f_b(0) = 0, \qquad f_b(1) = 1.$$

3. The boundary component $A_1 := B \times \{1\}$ is strictly repelling, and the component $A_0 := B \times \{0\}$ is "repelling in average":

$$\log f_b'(0) = \varphi(b), \qquad \log f_b'(1) = \psi(b), \quad \psi > 0 \text{ on } B, \int_B \varphi \, dP > 0. \tag{4}$$

Remark 1. It is easy to describe the maximal attractor of a skew product (2) with these properties. Let α be so small that for any $b \in B$, $f_b(1 - \alpha) < 1 - \alpha$. Then

$$A_{\max}(F) = \cap F^n(B \times [0, 1 - \alpha]).$$

As explained in Section 3.3, this attractor is an undergraph of some "boundary function" $\sigma^+ : B \to I$:

$$A_{\max} = \{(b, x) \mid b \in B, x \in [0, \sigma^+(b)]\}. \tag{5}$$

This form of the maximal attractor is used all over the proof, and in particular, in statement of Assumption 7 below.

4. Suppose that B is the closure of the union of two domains

$$\Omega^- = \{\varphi < 0\}, \qquad \Omega^+ = \{\varphi > 0\}.$$

For any $b \in \Omega^-$, the fiber map f_b has only one attractor 0 and only one repeller 1, both hyperbolic. For any $b \in \Omega^+$, f_b has three fixed points: two repellers 0; 1 and one attractor $a(b) \in (0, 1)$, all hyperbolic. The map h has at least two fixed points

$$O^- \in \Omega^-, \qquad O^+ \in \Omega^+.$$

5. There exists a Markov partition of B for h with the following property. The fixed points O^- and O^+ belong to the domains Δ_0, Δ_1 of the partition, and

$$h(\Delta_0) \cap \Delta_1 \ne \emptyset. \tag{6}$$

The unstable manifold $W^u_{O^+}$ of O^+ is dense in B.

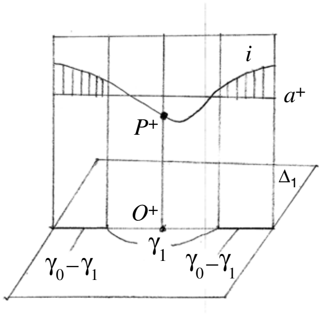

Fig. 1. Assumption 7 for systems of class TAT.

6. Let $f_0 = f_{O^-}$, $f_1 = f_{O^+}$, $\lambda = f_0'(0)$, $\mu = f_1'(0)$, $\tilde{\lambda} = \log\lambda$, $\tilde{\mu} = \log\mu$. Then

$$\frac{\tilde{\lambda}}{\tilde{\mu}} \notin \mathbb{Q}, \tag{7}$$

and

$$\frac{1}{2} < \left|\frac{\tilde{\lambda}}{\tilde{\mu}}\right| < 2, \quad |\tilde{\lambda}| < 1, |\tilde{\mu}| < 1. \tag{8}$$

7. Let γ_0 be the connected component of O^+ in the intersection $W^s_{O^+} \cap \Delta_1$. Let $P^+ = (O^+, a(O^+))$ be the fixed point of F over O^+. Denote by $W^{ss}_{O^+}$ the strongly stable manifold of P^+. It exists because of the partial hyperbolicity of F, belongs to $W^s_{O^+} \times I$ and is transversal to I in $W^s_{O^+} \times I$. We suppose that the manifold $W^{ss}_{P^+}$ lies above the graph Γ of the boundary function over the segment γ_0.

For a given system, it is not easy to see, whether or not this assumption holds. The following condition is sufficient for that.

The connected component of P^+ in the intersection $W^{ss}_{P^+} \cap (\gamma_0 \times I)$ is a graph of a function; denote this function by i. We assume that

$$\sigma^+(b) \le i(b) \quad \forall b \in \gamma_0. \tag{9}$$

This inequality follows from a weaker condition that is easier to check:

$$a^+ := \max_B a < i(b) \quad \forall b \in \gamma_0 \setminus \gamma_1, \gamma_1 = h(\gamma_0), \tag{10}$$

see Fig. 1.

The implication (10) ⇒ (9) is simple. The inequality $\sigma^+ \le a^+$ follows from Assumption 4, as shown in Section 3.3. This implies (9) for any $b \in \gamma_0 \setminus \gamma_1$. Both the graphs of i and $\sigma^+|_{\gamma_0}$ are invariant under F: the image of any two graphs belongs to the corresponding graph. Moreover,

$$\cup_0^\infty F^n(\gamma_0 \setminus \gamma_1) = \gamma_0 \setminus P^+.$$

By monotonicity of the fiber maps, (10) implies (9) for any $b \in \gamma_0 \setminus P^+$. To conclude, note that $i(P^+) = \sigma(P^+) = a(0^+)$. The latter equality follows from Assumption 4.

Skew products that satisfy Assumptions 1–7 are called *skew products of class TAT*. This is an abbreviation for *thick attractors*.

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Theorem 3. *A skew product diffeomorphism of class TAT for which B is a two torus, and h is a linear map (Anosov diffeomorphism), has a thick almost topologically mixing Milnor attractor.*

Theorem 3 implies Theorem 2. We believe that Theorem 3 holds without the assumption that h is an Anosov diffeomorphism of a two-torus, but we have no proof without this assumption.

Remark 2. Requirement (7) in Assumption 6 is technical and probably may be removed. One also may expect that assumption (6) may be replaced by transitivity of h. The set of skew products that satisfy Assumptions 1–6 is an open set in C^r topology, $r \geq 1$, with a countable number of hypersurfaces removed, that is, a quasiopen set. Hence, this set defines both metrically and topologically typical dynamical systems. Assumptions 1–5 and 7 describe an open set in the functional space of skew products with C^r-topology for any $r \geq 1$.

2.2. *Example of a system of class TAT*

Example 1. The following example shows that skew products of class TAT exist. Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ be an Anosov diffeomorphism of a two-torus: $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\mathbb{R}^2 = \{(y, z)\}$. Let

$$\begin{aligned} &F : X \to X, \qquad (b, x) \mapsto (Ab, f_b(x)) \\ &f_b(x) = x + \varepsilon g_b(x)\Omega(b, x), \qquad g_b(x) = x(x-1)(x-\psi(b)), \end{aligned} \tag{11}$$

$\Omega > 0$, $\Omega(b, 1) \equiv 1$, $\psi < 1$, $\varepsilon > 0$ is small.

Some assumptions on $\Omega : X \to \mathbb{R}$ and $\psi : \mathbb{T}^2 \to \mathbb{R}$ are made later. Let us check that for appropriate Ω, ψ, the map (11) is of class TAT.

1. For ε small, the fiber maps are close to identity; hence, the skew product (11) is partially hyperbolic.

2. Property 2 from Section 2.1 follows immediately from (11).

3. The invariant set $A_1 : B \times \{1\}$ is strictly repelling: $f_b'(1) > 1\ \forall b \in B$. Indeed,

$$f_b'(1) = 1 + \varepsilon(1 - \psi(b)) > 1$$

for any $b \in B$.

The invariant set $A_0 : B \times \{0\}$ is repelling in average for appropriate Ω. Indeed, $f_b'(0) = 1 + \varepsilon\psi(b)\Omega(b, 0)$. The sets Ω^-, $(\Omega^+) : \log f_b'(0) < 0$ $(\log f_b'(0) > 0)$ coincide with the sets $\psi < 0$ (respectively, $\psi > 0$). For any ψ such that $\psi(0, 0) > 0$, $\psi(\frac{1}{2}, 0) < 0$, the function $\Omega(\cdot, 0)$ may be so chosen that

$$\int_{\mathbb{T}^2} \log f_b'(0) > 0.$$

4. The map A has two fixed points: $O^+ = (0, 0)$ and $O^- = (\frac{1}{2}, 0)$. For all $b \in \Omega^- = \{b | \psi(b) < 0\}$, the map f_b has only one attractor 0 and one repeller 1. For all $b \in \Omega^+ = \{b | \psi(b) > 0\}$ the map f_b has two repellers 0, 1, one attractor $a(b) = \psi(b)$ and no other fixed points.

5. It is well known that the map $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ has a Markov partition, that satisfies Property 5 of Section 2.1. This property (for A replaced by A^d for some d) may be justified without the reference. The map A is transitive. Hence, for some appropriate d, there exists a Markov partition

$\{\Delta_j\}$ for A^d such that $A^d\Delta_i \cap \Delta_j \neq 0$ for any i, j. This partition (for A replaced by A^d) has Property 5.

6. Assumption 6 holds for every small ε except for a countable set as soon as ψ and Ω are fixed.

7. Assumption 7 requires a special construction that provides ψ and Ω. Let u, v be coordinates in the covering plane over the torus in which the operator A has the diagonal form:

$$A(u, v) = (\lambda_1 u, \lambda_2 v), \quad 0 < \lambda_1 < 1 < \lambda_2.$$

Stable and unstable manifolds of $O^+ = 0$ under A are the lines $v = 0$ and $u = 0$. The component γ_0 is a segment on the u-axis passing through zero.

Let W_0 be the graph of a function i defined on γ_0:

$$i(u) = \frac{1}{2} + \varepsilon u^2.$$

We will define $\Omega|_{W_0}$ and ψ is such a way that Property 7 of systems of class TAT holds. First, the curve W_0 should be invariant under F. This is equivalent to

$$i(\lambda_1 u) = f_b(i(u)) \tag{12}$$

or, for $b \in \gamma_0$, $u = u(b)$:

$$i(\lambda_1 u) = i(u) + \varepsilon g_b(i(u))\Omega(b, i(u)). \tag{13}$$

This implies:

$$\Omega(b, i(u)) = \frac{i(\lambda_1 u) - i(u)}{\varepsilon g_b(i(u))} = \frac{u^2(\lambda_1^2 - 1)}{g_b(i(u))}.$$

The numerator $u^2(\lambda_1^2 - 1)$ takes zero value at 0. Yet Ω should be positive everywhere. To gain this, we need to have the denominator belong to the ideal (u^2). More precisely, let $\psi|_{\gamma_0} = \psi_0 : u \mapsto i(u) - Cu^2$, $C \gg 0$. Then

$$g_b(i(u)) = i(u)(i(u) - 1)Cu^2.$$

Hence, for $b \in \gamma_0$ and $u = u(b)$ we have:

$$\Omega(b, i(u)) = \frac{1 - \lambda_1^2}{i(u)(1 - i(u))C} > 0.$$

Take C so large that $\psi_0 < 0$ at the endpoints of γ_0. Note that $\max_{\gamma_0} \psi_0 = \frac{1}{2}$. Let us now extend ψ_0 to a function ψ on a torus such that $a^+(\varepsilon) := \max \psi = \frac{1}{2}$. Let us extend Ω from $A_0 \cup A_1 \cup W_0$ as a smooth positive function. Let δ be so small that $[-\delta, \delta] \subset A\gamma_0$. Then

$$\min_{u \in \gamma_0 \setminus A\gamma_0} i(u) \geq \frac{1}{2} + \varepsilon\delta^2 > a_+ = \frac{1}{2}.$$

It remains to check that $W_0 \subset W^{ss}_{P^+}$. To do that, note that the point P^+ has a two-dimensional stable manifold, and has a node on this manifold. The eigenvalue of this node corresponding to the central fiber is much closer to one than the other eigenvalue. Hence, all the local invariant curves of F that enter P^+ on its stable manifold, except for $W^{ss}_{P^+}$, tend to the central fiber at P^+. The curve W_0 does not tend to the central fiber at P^+. Hence, it belongs to $W^{ss}_{P^+}$.

This implies Property 7 for the map (11) with ψ and Ω chosen above.

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2.3. Perturbations

Small perturbations of skew products $\mathcal{F}$ described in the previous section in case *when h is an Anosov diffeomorphism of a two-torus, and the fiber maps are close to identity* provide the set whose existence is claimed in Theorem 1.

The Hirsh–Pugh–Shub theory [11] modified for manifolds with boundary, see Section 11.1, claims that under additional normal hyperbolicity assumptions, the perturbed skew product $\mathcal{G}$ is conjugated to some other skew product G over the same base. In more details, there exists a homeomorphism

$$H : X \to X, \qquad (b, x) \mapsto (\beta_b(x), x),$$

where β_b is a family of maps smooth in x and continuous in b, with Lip β_b and difference $\beta_b - b$ small together with the size of the perturbation $\mathcal{F} - \mathcal{G}$. The skew product G has the form

$$G = H^{-1} \circ \mathcal{G} \circ H. \tag{14}$$

Such a skew product will be called *smoothly generated.*

This new product has the same map h in the base and its fiber maps g_b are smooth in x. But they are in no way smooth in b. Yet a recent theorem by Gorodetskii [10] claims that the fiber maps of the new skew product G are *Hölder in b*: there exist C and α such that

$$\|g_b - g_{b'}\|_{C^3} \leq C|b - b'|^{\alpha}. \tag{15}$$

This theorem applies to a wide class of hyperbolic maps h. Skew products that satisfy (15) are called *Hölder skew products.*

In a recent work [16] this result was improved for the case when h is the Anosov diffeomorphism A of the two-torus: *for small perturbations described above and C^3 replaced by C, α in (15) may be taken close to* 1: *the smaller the perturbation is, the smaller is* $1 - \alpha$.

These skew products satisfy Assumptions 1–5 of Section 2.1. For a quasiopen subset of perturbations Assumption 6 holds too. In order to state Theorem 4 below, we need to reformulate Assumption 7. Let us now consider a Hölder skew product G over the Anosov diffeomorphism of a two torus. Let us first describe the lift of G to $\mathbb{R}^2 \times I$.

Consider a universal cover $\mathbb{R}^2$ over $\mathbb{T}^2$ with the projection $\hat{\pi} : \mathbb{R}^2 \to \mathbb{T}^2$. Let $\hat{A}$ be the lift of A to $\mathbb{R}^2 : \hat{A}(0) = 0$. Choose the origin in $\mathbb{R}^2$ in such a way that $\hat{\pi}(0) = O^+$. We identify $O^+ \in \mathbb{T}^2$ and $0 \in \mathbb{R}^2$, and denote the latter point by O^+ too. For any point $b \in \mathbb{T}^2$, $\hat{b}$ is a point from $\hat{\pi}^{-1}(b)$ (not uniquely determined).

Let $\hat{X} = \mathbb{R}^2 \times I$. We can extend $\hat{\pi}$ from $\mathbb{R}^2 \times \{0\}$ to $\hat{X}$, and still denote the extended map by $\hat{\pi} : \hat{X} \to X$, $(\hat{b}, x) \to (b, x) = (\hat{\pi}\hat{b}, x)$. Denote by $\hat{G} : \hat{X} \to \hat{X}$ the map

$$(\hat{b}, x) \to (\hat{A}\hat{b}, \hat{g}_{\hat{b}}(x)), \quad \hat{g}_{\hat{b}}(x) = g_b(x), \text{ for } b = \pi\hat{b}.$$

Let $\tilde{\pi}$ be the natural projection $\hat{X} \to \mathbb{R}^2$ along the fibers. Let P^+ be the fixed point of $\hat{G}$ over O^+. Now we can formulate

Assumption $7'$. Let γ_0 be the same segment in $\mathbb{T}^2$ as in Assumption 7, see Fig. 1, $\hat{\gamma}_0$ be its lift passing through O^+. Let W^u be the unstable manifold of O^+ under $\hat{A}$. We assume that there exist

a neighborhood U of $\hat{\gamma}_0 \cup W^u$ such that $\hat{A}U \subset U$;

a map $i : U \to I$, whose graph W is invariant under $\hat{G} : \hat{G}W \subset W$; a neighborhood $E_0 \subset U$ of γ_0 such that

$$a^+ := \max_B a < i(\hat{b}) \quad \forall \hat{b} \in E_0 \setminus AE_0. \tag{16}$$

Moreover,

$$i(\hat{b}) = \sigma^+(\hat{b}) \quad \forall \hat{b} \in W^u, \tag{17}$$

and i is uniformly continuous along the stable fibers of A. This means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $b' \in \mathbb{R}^2$ that belongs to the same stable fiber of A, $|i(b) - i(b')| < \varepsilon$ provided that $|\hat{b} - \hat{b}'| < \delta$.

This completes Assumption 7′.

Definition 3. A Hölder skew product G over Anosov diffeomorphism of $\mathbb{T}^2$ is of class TAT if it satisfies Assumptions 1–6 and 7′ above.

In Section 11 we prove that a small perturbation of a C^3 skew product diffeomorphism of class TAT is conjugated to a Hölder skew product of class TAT. The following theorem claims existence of thick attractors for such skew products.

Theorem 4. *A Hölder skew product of class TAT over the Anosov diffeomorphism of a two torus with C^3 fiber maps has a thick almost topologically mixing Milnor attractor.*

The difference between Theorems 3 and 4 is two-fold. First, Assumptions 7 and 7′ are different. In Section 11 we prove that Assumption 7 on the unperturbed skew product diffeomorphism F of class TAT implies Assumption 7′ on the Hölder skew product conjugated to a small perturbation of F.

Second, the fiber maps in Theorem 3 are smooth with respect to the base point, and only Hölder continuous in Theorem 4. Smoothness implies Hölder continuity; hence, Theorem 4 implies Theorem 3. Theorem 3, in turn, implies Theorem 2.

We will prove Theorem 4. After that we will deduce Theorem 1 from Theorem 4. To do that, we need to overcome so called "Fubini nightmare". Namely, the conjugacy between the perturbed skew product diffeomorphism of class TAT and a Hölder skew product may be not absolutely continuous. The reason is that the holonomy along the central fibers of the perturbed map may not have this property. This obstacle is bypassed by the tools described in the next subsection.

2.4. Positive measure of maximal attractors of the perturbed maps

To prove that the maximal attractor of the perturbed skew product has positive measure, we use the techniques elaborated in [16] and developed in [15,23,18]. Namely, we will make use of the following *special ergodic theorem* proved in [23].

Let m_k be the k-dimensional Lebesgue measure.

Theorem 5. *Let A be the Anosov diffeomorphism of the two-torus $\mathbb{T}^2$ and φ be a continuous function. Let $\int_{\mathbb{T}^2} \varphi dm_2 > 0$. Then the set of points x, for which the lower time average of φ is negative, has Hausdorff dimension less than 2. In more detail,*

$$\dim_H K < 2 \quad where K = \left\{ b \in \mathbb{T}^2 | \liminf \frac{1}{n} \sum_{k=0}^{n-1} \varphi(A^k b) < 0 \right\}.$$

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Consider now a simpler class of maps called "step skew products". Thick attractors will be observed for an open set of such maps, not only for quasiopen.

3. Step skew products and strategy of the proofs

3.1. Step skew products of class TAT

Consider a skew product (2) in case when B is not a manifold: $B = \Sigma^2$, the map h is a Bernoulli shift. Let

$$F : \Sigma^2 \times I \to \Sigma^2 \times I, \qquad (\omega, x) \mapsto (\sigma\omega, f_{\omega_0} x), \tag{18}$$

such a map is a so called *step skew product*: the fiber map depends on ω_0 only. Such a skew product is a random dynamical system on I with random application of two diffeomorphisms f_0 and f_1. Consider a map (18), where f_0 and f_1 satisfy the following analogs of properties 1–5:

1. Let f_0 have exactly two fixed points: attractor 0 and a repeller 1.
2. Let f_1 have three fixed points: repellers 0, 1 and an attractor $a \in (0, 1)$. The fixed points of f_0, f_1 are hyperbolic.
3. Let $\lambda = f_0'(0)$, $\mu = f_1'(0)$ and $\lambda\mu > 1$.
4. The 2-jets of f_0, f_1 at 0 do not commute under composition.

Such step skew products are, by definition, of class TAT.

Let d be the standard metric of $\Sigma^2 : d(\omega, \omega') = 2^{-n}, n = \min\{|k| | \omega_k \neq \omega_k'\}$. Let P be the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure on Σ^2. Let m_k be the k-dimensional Lebesgue measure. The measure μ on X is the product measure:

$$\mu = P \times m_1;$$

recall that m_1 is the Lebesgue measure on [0,1]. This allows us to speak about Milnor and minimal attractors of homeomorphisms $X \to X$.

The following analog of Theorem 1 holds:

Theorem 6 (*[14]*). *The step skew product (18) of class TAT has a thick Milnor attractor.*

This theorem is easier than Theorem 3 and it is proved before Theorem 3 in Sections 4 and 5. Section 4 deals with Hölder skew products of class TAT, as well as with the step ones. Section 5 is a prototype for the second part of the paper that starts with Section 6.

Step skew products admit numerical simulation described below.

3.2. Numerical experiment

Orbits of a step skew product F, see (18), with special fiber maps were calculated by Denis Volk, then my graduate student. He considered piecewise linear f_0 and f_1, defined as follows:

$$\lambda = \frac{1}{2}, \qquad \mu = 3, \qquad a = \frac{1}{2}, \qquad f_0'(1) = f_1'(1) = 2. \tag{19}$$

These data define the maps f_0 and f_1 in the unique way, provided that the graph of $f_0(f_1)$ has two (three) edges. These functions satisfy Assumptions 1–4, except for smoothness, whose lack does not prevent the existence of the thick attractor.

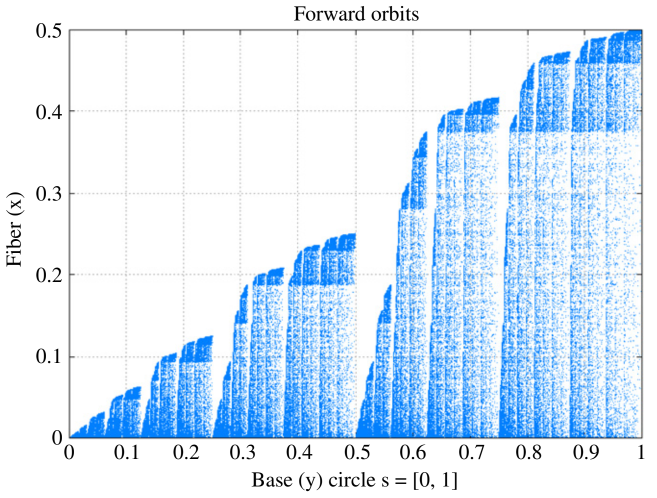

Fig. 2. Milnor attractor of positive Lebesgue measure for the special step skew product.

In the next subsection we prove that there exists a function $\sigma^+ : \Sigma^2 \to I'$ such that

$$A_{\max} = \{(\omega, x) \mid x \in [0, \sigma^+(\omega)]\}. \tag{20}$$

In what follows, we call σ^+ *the boundary function*.

The important feature of the maximal attractor of a step skew product is that the intersection $A_{\max} \cap I_\omega$ depends on the "past part" of ω only. In the figure below Σ^2 is replaced by a segment $[0, 1]$ and any point $\omega = \dots \omega_{-n} \dots \omega_{-1}\omega_0 \dots$ is replaced by

$$y := y(\omega) = \sum_{i=0}^{19} 2^{-(i+1)}\omega_{-(i+1)}.$$

A random sequence $\omega = \omega_0\omega_1 \dots \omega_n$ of 0 and 1 of length $n = 10^6$ was generated, and a random value of $x \in [0, \frac{1}{2}]$ was chosen. Then the future orbit of $p = (\omega, x)$ of length 10^6 was calculated (this orbit is the same, whatever extension $\dots \omega_{-n} \dots \omega_{-1}$ of ω is chosen). The last $5 \cdot 10^5$ points of the orbit where presented, see Fig. 2.

A point $p^k = F^k(p) := (\sigma^k\omega, x_k)$ is projected to (y_k, x_k) with $y_k = y(\sigma^k\omega)$. The shadowed set on the figure is the attractor, the white set is its complement. The picture illustrates the description given below.

3.3. *Brief description of the thick attractor and the plan of the proof of the main results*

In this subsection and the next section we consider Hölder and step skew products simultaneously. Now B, b and h denote either the manifold, its point and the diffeomorphism as in Section 2.1, or Σ^2, ω and σ. The measure μ is either the Lebesgue measure m_3 or the measure $P \times m_1$ introduced above. The skew product F is either (2) or (18).

Step 1. The following proposition describes maximal attractors of the skew products of class TAT.

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Proposition 1. *Let F be a skew product (2) or (18) of class TAT. Then the maximal attractor of F has the form*

$$A_{\max} = \{(b, x) \mid b \in B, x \in [0, \sigma^+(b)]\}, \tag{21}$$

where σ^+ is some function $B \to M$.

The function σ^+ mentioned above is called a *boundary function* because it determines boundary points of the intersections of $A_{\max}$ with the fibers.

Proof. Take $\alpha > 0$ such that $\forall b \; 1 - \alpha > a(b)$. Let $I' = [0, 1 - \alpha]$. The intersection of $A_{\max}$ with $I_b = \{b\} \times I$ is

$$A_{\max} \cap I_b = F^n X \cap I_b = \{b\} \times \bigcap_{n=0}^{\infty} I_{b,n}, \tag{22}$$

where $I_{b,n}$ is a segment:

$$I_{b,n} = f_{h^{-1}b} \circ \cdots \circ f_{h^{-n}b}(I') := [0, \sigma_n^+(b)]. \tag{23}$$

The segments $I_{b,n}$ form a nested sequence. Hence, the functional sequence σ_n^+ monotonically decreases; let σ^+ be its limit:

$$\sigma^+(b) = \lim_{n\to\infty} \sigma_n^+(b). \tag{24}$$

We conclude that for any skew product of class TAT, no matter, step or Hölder one, the maximal attractor has the form (21). $\square$

Step 2. Let us first prove that $\mu A_{\max} > 0$.

The invariant surface $A_0 = B \times \{0\}$ is repelling in average, because $\int \varphi > 0$. Hence, $\sigma^+ > 0$ a.e. (Lemma 1 below). Therefore, $\mu(A_{\max}) > 0$. We should now compare the Milnor and the maximal attractors.

Step 3. "Almost all" of the graph Γ of the boundary function σ^+ belongs to the Milnor attractor (Lemma 2 below). Indeed, all the measure of the complement $X \setminus A_{\max}$ "lands" on Γ under positive iterates of F.

Step 4. The graph Γ is in no way closed. In particular, the set $\{\sigma^+ = 0\}$ is dense in A_0. On the other hand, the Milnor attractor is closed. It appears that there exists a set $E \subset B$, $P(B \setminus E) = 0$ such that

$$\mathrm{Cl}(\Gamma \cap A_M) \supset A_{\max}^E, \quad A_{\max}^E = A_{\max} \cap (E \times I). \tag{25}$$

Together with Lemma 2, this implies that $A_{\max}^E \subset A_M$; hence, A_M is thick.

This completes the proof of Theorem 4. The most difficult part is Step 4: proof of the density property of the graph of the boundary function. The step case (Theorem 5) treated in Section 5 is a model for the Hölder one (Theorem 4). The topologically mixing property for maximal attractors of Hölder skew products is proved in Sections 6–10 and Appendix, where the proof of Theorem 3 is completed.

Step 5. Locally generic diffeomorphisms with thick attractors are obtained from skew products of class TAT via small perturbations. In Section 11 we prove that their maximal attractors are thick. This will prove Theorem 1.

4. Attractors as undergraphs

In this section we prove that the maximal attractor of any skew product of class TAT is an undergraph of some "boundary function". We also prove that this undergraph has positive measure and "almost all" of the graph of the boundary function belongs to the Milnor attractor.

4.1. Structure of the Milnor attractor

The maximal attractor of the system of class TAT is the undergraph of a boundary function $\sigma^+ : B \to I$ defined by (24). This follows from Proposition 1.

Lemma 1. *For skew product diffeomorphisms of class TAT, the boundary function σ^+ is P-a.e. positive. Yet the set $\sigma^+ = 0$ is dense in B.*

Proof. Let us first prove that $\sigma^+ > 0$ P-a.e. To do that, we will prove the following proposition.

Proposition 2. *If the boundary function takes* 0 *value at b, then*

$$\liminf \frac{1}{n} \sum_1^n \varphi(h^{-k} b) \leq 0, \tag{26}$$

where φ is the same as in (4).

Proof. We will prove (26) for φ replaced by $\varphi - \varepsilon := \varphi_\varepsilon$ for any $\varepsilon > 0$. This will imply (26). For any $\varepsilon > 0$ there exists a_0 such that for all $x \in [0; a_0]$, $b \in B$,

$$f_b(x) \geq x \exp \varphi_\varepsilon(b), \tag{27}$$

because B is compact and f_b of class $C^2(I)$.

Fix $\varepsilon > 0$ and the corresponding a_0. We want to prove that for any $m > 0$ there exists $K > m$ such that

$$\sum_1^K \varphi_\varepsilon \circ h^{-k}(b) \leq 0.$$

This will imply (26). Take $L = \max_B \operatorname{Lip} f_b^{-1}$. Let $\delta < L^{-m} a_0$. Fix n such that $\sigma_n^+(b) < \delta$. Recall that $a^+ = \max_{b \in B} a(b)$. Consider the set of points $x_1, \ldots, x_n := a^+, x_k = f_{h^{-k}b}(x_{k+1}), x_1 < \delta$. Obviously,

$$x_k \geq L^{-1} x_{k+1}.$$

Take the smaller K such that

$$x_K \leq a_0 < x_{K+1}.$$

Note that $x_1 \leq \delta < L^{-m} a_0$. Hence, $K > m$. On the other hand, as $x_j \leq a_0$ for $j = 1, \ldots, K$, we have:

$$x_1 \geq x_K \exp \sum_1^K \varphi_\varepsilon(h^{-l} b).$$

Hence, the sum in the exponent is nonpositive, and $K > m$ may be taken arbitrary large. This implies (26) and proves the proposition. □

Proposition 1 implies the first statement of Lemma 1. Indeed, by the Ergodic Theorem and Assumption 2 of Theorem 3,

$$\lim \frac{1}{n} \sum_{1}^{n} \varphi(h^{-k} b) > 0$$

for a.e. $b \in \mathbb{B}$. Hence, by Proposition 1, $\sigma^+ > 0$ a.e.

Let us now prove the second statement for Hölder skew products from Theorem 3. Assumption 4 of Section 2.1 implies that $\sigma^+(O^-) = 0$. The same holds for any $b \in W^u_{O^-}$, the unstable manifold of O^- under h.

By Assumption 4, $W^u_{O^-}$ is dense in B. Hence, in case of Theorem 3, the set $\sigma^+ = 0$ is dense in B.

In case of Theorem 4 we have $B = \Sigma^2$. Let $O^- = (0)$, $O^+ = (1)$. Then $\sigma^+(O^-) = 0$ as well as $\sigma^+|W^u_{O^-}$. Moreover, $W^u_{O^-}$ is dense in Σ^2.

This proves Lemma 1. □

4.2. *Graph of the boundary function and Milnor attractor*

Roughly speaking, almost every point of the graph Γ of the boundary function belongs to the Milnor attractor. More precisely, the following lemma holds:

Lemma 2. *Under Assumptions* 1–3 *of Theorems* 3 *or* 6, *for m_2 or P-a.e. b*

$$(b, \sigma^+(b)) \in A_M.$$

Here and below m_k is the Lebesgue measure of dimension k.

Proof. Consider an "upper basin of attraction"

$$\mathcal{B} = \{(b, x) \mid x \in [\sigma^+(b), 1)\}. \tag{28}$$

We will prove that, under the positive iterates of F, the whole measure of $\mathcal{B}$ "lands" on "almost all of" Γ; recall that Γ is the graph of σ^+. Hence, "almost all of" Γ belongs to the minimal attractor $A_{\min}$ of F, see [7] for the definition of the minimal attractor. The latter belongs to the Milnor attractor. In more detail, for any ε, n let

$$\Omega_{\varepsilon,n} = \{b \in B \mid \sigma_n^+(b) - \sigma^+(b) < \varepsilon\},$$

where σ_n^+ is the same as in (23). The sets $\Omega_{\varepsilon,n}$ are growing:

$$\Omega_{\varepsilon,n} \subset \Omega_{\varepsilon,n+1},$$

and their union is the whole B. Hence, $P(\Omega_{\varepsilon,n}) \to 1$ as $n \to \infty$. Let

$$\mathcal{B}_{\varepsilon,n} = \{(b, x) \mid b \in \Omega_{\varepsilon,n}, x \in [\sigma^+(b), \sigma^+(b) + \varepsilon]\},$$
$$\mathcal{B}(\varepsilon) = \{(b, x) \mid b \in B, x \in [\sigma^+(b), \sigma^+(b) + \varepsilon]\}.$$

Then

$$(F^{n*}\mu)(\mathcal{B}_{\varepsilon,n}) \to \mu(\mathcal{B}), \quad \text{as } n \to +\infty.$$

Hence,

$$(F^{n*}\mu)\mathcal{B}(\varepsilon) \to \mu(B), \quad \text{as } n \to +\infty.$$

Let

$$\mu_n = \frac{1}{n}\sum_0^{n-1} F^{k*}\mu.$$

Then $\mu_n(\mathcal{B}(\varepsilon)) \to \mu(\mathcal{B})$ as $n \to \infty$.

Let μ_∞ be a weak limit point of the sequence (μ_n). Then $\mu_\infty(\mathcal{B}(\varepsilon)) \geq \mu(\mathcal{B})$. Therefore,

$$\mu_\infty(\Gamma) \geq \mu_\infty(\cap_{\varepsilon>0}\mathcal{B}(\varepsilon)) \geq \mu(\mathcal{B}) > 0.$$

Hence, $\mu_\infty(\Gamma \cap A_{\min}) > 0$. Note that for any set $\Omega \subset B$ such that $P(\Omega) = 0$, $\mu_\infty(\Omega \times I) = 0$. Hence,

$$P\{b \in B \mid (b, \sigma^+(b)) \in A_{\min}\} > 0.$$

On the other hand, both A and Γ are invariant under h. Hence, by ergodicity of h,

$$P\{b \in B \mid (b, \sigma^+(b)) \in A_{\min}\} = 1.$$

As $A_{\min}$ belongs to the Milnor attractor, this implies the lemma. □

Thus we completed the first part of the proof of the Main Theorem for skew products of class TAT, both Hölder and step. Namely, we proved that the maximal attractor of such a product has positive measure and is an undergraph of a so called "boundary function". On the other hand, the Milnor attractor of such a product contains "almost all of the graph" of the boundary function. The Milnor attractor is closed. On the contrary, the boundary function is in no way continuous, and its graph is not closed. In fact, this graph is dense in the maximal attractor. For step skew products we will prove:

$$A_{\max} = \mathrm{Cl}(A_M \cap \Gamma), \tag{29}$$

and conclude that $A_{\max} = A_M$. Hence,

$$\mu A_M > 0.$$

For Hölder skew products of class TAT, we will prove (25), and again conclude the thickness of A_M.

These statements are subject of two density lemmas, Lemmas 4 and 6 below. The density lemma for step skew products is much simpler than for the Hölder ones. So we begin with the simpler case.

5. Density properties and Milnor attractors of step skew products of class TAT

In this section we prove the density lemma for step skew products, and conclude the proof of Theorem 6. We turn back to the notations Σ^2, ω, σ and consider in this section step skew products only.

5.1. Density

Lemma 3. *Let f_0, f_1 satisfy Assumptions 1–4 of Theorem 6. Then all orbits of the semigroup $G^+ = G^+(f_0, f_1)$ restricted to the semiinterval $J = (0, a]$ are dense on J. Here a is the attractor of f_1.*

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The idea of the proof of this lemma goes back to [12].

Proof. Take an arbitrary point $b \in J$ and an interval $U \subset J$. Our goal is to find $g \in G^+$ such that $g(b) \in U$.

By Sternberg Theorem, there exists a C^2-chart near zero (denote it still by x) such that f_0 is linear in this chart. This chart may be extended to the whole of $[0, 1)$. An analogous chart y for f_1 may be extended to $[0, a)$.

Let h be the transition function between these two charts: $y = h(x)$, $h'(0) = 1$. Then in the first chart the fiber maps have the form:

$$f_0 : x \mapsto \lambda x, \qquad f_1 : x \mapsto h^{-1} \circ \mu \circ h(x). \tag{30}$$

Nonresonant case: $\frac{\log \lambda}{\log \mu} \notin \mathbb{Q}$.

In this case, the multiplicative semigroup generated by λ, μ is dense in $\mathbb{R}^+$. Let ν be such that $h^{-1}(\nu x) \in U$. We can now choose two sequences $k_n \to +\infty$, $l_n \to +\infty$ such that $\lambda^{k_n} \mu^{l_n} \to \nu$ as $n \to \infty$. Consider the composition

$$g_n = f_1^{l_n} \circ f_0^{k_n}.$$

This composition is well defined on $[0, a]$, because f_0 contracts this segment. Let

$$h(x) = x + R(x), \quad |R(x)| \le Cx^2 \text{ for } x \in \left[0, \frac{a}{2}\right].$$

Then

$$g_n(x) = h^{-1} \circ \mu^{l_n}(\lambda^{k_n} x + R(\lambda^{k_n} x)) \to h^{-1}(\nu x).$$

Indeed, keeping in mind that $\lambda < 1$, $\mu > 1$, we conclude

$$\mu^{l_n} \lambda^{k_n} x \to \nu x, \quad \mu^{l_n} |R(\lambda^{k_n} x)| < C\mu^{l_n} \lambda^{2k_n} x^2 \to 0 \text{ as } n \to \infty.$$

Now, $h^{-1}(\nu x) \in U$ by the choice of ν. Hence, $g_n(b) \in U$ for large n. In the nonresonant case, Lemma 3 is proved.

The ideas of this proof will be used in the proof of Density Lemmas for mild skew products, namely in Section 8 and Appendix.

Resonant case: $\frac{\log \lambda}{\log \mu} = \frac{k}{l} \in Q$.

In the resonant case the lemma is proved in [14]. We just refer to it here, because it will not be used below. $\square$

5.2. *Closure of the graph Γ*

Lemma 4. *In assumptions of Theorem 6, (29) holds.*

Proof. Let us first prove that

$$A_{\max} = \mathrm{Cl}(\Gamma). \tag{31}$$

Take any $p \in A_{\max}$:

$$p = (\omega, x), \quad x \in [0, \sigma^+(\omega)), \omega = \omega^- | \omega^+, \tag{32}$$

where

$$\omega = \ldots \omega_{-n} \ldots \omega_{-1} | \omega_0 \omega_1 \ldots \omega_n \ldots. \tag{33}$$

Here and below the bar | shows the zero position of the sequence: ω_0 stands next right to the bar.

For any word w with zero position indicated, denote by C_w the corresponding cylinder in Σ^2. Let us take a neighborhood U of p of the form

$$U = C_w \times V, \quad V = [x - \varepsilon, x + \varepsilon] \subset [0, \sigma^+(\omega)], \tag{34}$$

where w is a subword of ω. For any integer a, b let

$$\omega\,|_a^b = \omega_a \ldots \omega_b$$

be a subword of ω. We suppose that for some natural m,

$$w = \omega\,|_{-m}^m, \qquad w = \omega^-(m)|\omega^+(m),$$

where

$$\omega^-(m) = \omega\,|_{-m}^{-1}, \qquad \omega^+(m) = \omega\,|_0^m .$$

For any word w denote by $|w|$ its length. For instance, $|\omega^-(m)| = m$.

For the neighborhood U so chosen, we need to find

$$q = (\omega', \sigma^+(\omega')) \in \Gamma \cap U. \tag{35}$$

The point q belongs to Γ by definition. So we need to find $\omega' \in C_w$ in such a way that $\sigma^+(\omega') \in V$.

For any sequence ω and natural k, denote by $f_{\omega,k}$ the fiber map of F^k over ω:

$$f_{\omega,k} = f_{\omega_{k-1}} \circ \cdots \circ f_{\omega_0}.$$

Similarly, for any word $w = \omega_0 \ldots \omega_{k-1}$, let

$$f_w = f_{\omega_{k-1}} \circ \cdots \circ f_{\omega_0}.$$

Take any sequence $\tilde{\omega}^-$ infinite to the left, and let

$$x_0 = \sigma^+(\tilde{\omega}^-|*).$$

In what follows, it is important that $\tilde{\omega}^-$ is arbitrary. The function σ^+ depends on the past part of the sequence only; hence, x_0 is the same, whatever the right part $*$ is.

We will find the sequence ω' in the form:

$$\omega' = \tilde{\omega}^- \alpha \omega^-(m)|\omega^+(m)*, \tag{36}$$

where $*$ is an extension that does not matter, and α is a word constructed below.

Let V_m be an interval defined by

$$f_{\omega^-(m)}(V_m) = V. \tag{37}$$

Let α be such a word that

$$f_\alpha(x_0) \in V_m. \tag{38}$$

Such a word exists by Lemma 3. Let $|\alpha| = k$. Then

$$y_0 = f_{\sigma^{-(k+m)}\omega', k+m}(x_0) \in V.$$

Indeed,

$$f_{\sigma^{-(k+m)}\omega', k}(x_0) = f_\alpha(x_0) \in V_m.$$
$$f_{\sigma^{-m}\omega', m}(V_m) = f_{\omega^-(m)} V_m = V.$$

On the other hand,

$$x_0 = \sigma^+(\sigma^{-(k+m)}\omega').$$

The graph Γ is F-invariant. Hence, for any l, ω

$$f_{\sigma^{-l}\omega,l}(\sigma^+(\sigma^{-l}\omega)) = \sigma^+(\omega).$$

Therefore,

$$y_0 = \sigma^+(\omega').$$

This proves (35), hence, (31).

Let us now prove that inclusion (35) holds for a set of sequences ω of a positive P-measure. This will imply that $A_M \cap U \neq \emptyset$. Indeed, as proved in Lemma 2,

$$(\omega, \sigma^+(\omega)) \in A_M$$

for a.e. ω. Any set of positive measure has a nonempty intersection with a set of full measure. Hence, if (35) holds for a set of ω having positive measure, this will imply (29).

Let us now find a set S with $P(S) > 0$ such that for any $\omega \in S$, (35) holds. Denote by Σ^- the set of all one-sided sequences infinite to the left. Let P^- be the conditionary measure on Σ^-. Note that $\sigma^+(\omega)$ depends on the past part of ω only. Hence, the function σ^+ is well defined on Σ^-. Take a point x_0 with the following property: for any $\delta > 0$,

$$P^-(S^x_\delta) > 0 \quad \text{where } S^x_\delta = \{\omega \in \Sigma^- | \sigma^+(\omega) \in [x_0 - \delta, x_0 + \delta]\}.$$

Such a point exists. Indeed, take a distribution function Δ for $\sigma^+|_{\Sigma^-}$. It is monotonic. Suppose that for any x there exists δ such that $P^-(S^x_\delta) = 0$. This provides a cover of the range of Δ. Take a finite subcover. The inverse images of the elements of this subcover under Δ have P^- measure zero. Hence, $P^-(\Sigma^-) = 0$, a contradiction.

Take x_0 as above, and let δ be so small that

$$f_\alpha([x_0 - \delta, x_0 + \delta]) \subset V_m.$$

Existence of such a δ follows from (38).

Consider now the set of sequences (36) where $\omega^- \in S^-_\delta$, and $*$ may be replaced by any one-sided sequence infinite to the right. The set S of such sequences has positive P-measure. For these sequences (35) holds.

This proves (29), hence, Lemma 4 and Theorem 6. □

6. Preparations: normal forms and Markov partitions

From now on we prove our main results, Theorems 1 and 2. The main part, Sections 6–10 and Appendix, deals with the skew products; Section 11 considers their perturbations.

We mainly use compositions of maps $f_0 = f_{O^-}$ and $f_1 = f_{O^+}$, see Assumption 6 in Section 2.1. To compare these maps with mere translations, we introduce so called Sternberg logarithmic charts. In Section 6 we study Hölder properties of skew product in these charts, and reduce the smooth skew product to skew products over a Markov chain.

We compare first the fiber map of the iterated skew products over a shifted sequence $\omega^{01} = \dots 0 \dots 01 \dots 1 \dots$, and compositions $f_1^L \circ f_0^K$. In more detail, denote by $g_{\omega,m}$ a fiber map of an iterate G^m over ω. In Section 8 we study the compositions $g_{\sigma^{-K}\omega^{01},K+L}$ for various large K and L. By the way, we need to compare two maps like $g_{\sigma^{-2K}\omega^{01},K}$ and $g_{(0),K} = f_0^K$.

It appears that these maps are close uniformly in K. Moreover, the distance between them tends to 0 as $K \to \infty$. This is guaranteed by so called Distortion Lemmas. These lemmas allow us to compare fiber maps of the iterated skew product over close points in the base. The larger is the number of iterates, the closer points are chosen. In this setting, the fiber maps are close uniformly with respect to the number of iterates; moreover, the distance between the fiber maps tends to zero when the number of iterates tends to infinity, see Section 7.

Density property of the fiber maps over ω^{01} is established in Section 8. Some technical details of this study are postponed to Appendix.

In order to prove the topologically mixing property of the attractor of F, we take two neighborhoods, V and U, of the points of the attractor, and construct a point $q \in V$ that will visit U under some iterate of F.

For this we consider backward iterates of the points from U. In general, these iterates are out of control. Yet there exist points whose backward iterates are well under control. These are the points that lie over the unstable manifold of O^+ under the map h. The problem is to find such points that belong at the same time to the attractor of F and to the neighborhood U. This is done in Section 9.

Section 10 is the central one. Making use of all the tools prepared at the previous sections, we prove that Hölder skew products of class TAT have topologically mixing thick Milnor attractors. Thus we will prove Theorem 4 that implies Theorem 2.

In Section 11 we deduce from here our main result, Theorem 1. To do that, we apply the following recently developed tools: *special ergodic theorem* [23], and *overcoming of the Fubini nightmare* [16], as well as classical results: absolute continuity theorems by Anosov and Pesin.

In Section 6 we state Density Lemmas that imply Theorem 4, and discuss some preliminary results.

6.1. Two density lemmas

Let π be the projection of the phase space onto the fiber along the base:

$$X \to I, \qquad (b, x) \mapsto x.$$

Lemma 5 (*Density Property for Fiber Maps*)**.** *Let F be a skew product of class TAT. Then for any interval $V \subset [0, a(O^+)]$ and any point $x \in (0, a(O^+))$, there exists $b \in B$ and $m \in \mathbb{N}$ such that*

$$\pi F^m(b, x) \in V. \tag{39}$$

Lemma 6. *For Hölder skew products of class TAT, "almost all of the Milnor attractor" belongs to the closure of "almost all of" the graph of the boundary function. In more detail, there exists a set $E \subset B$ such that $P(E) = P(B)$,*

$$\tilde{\Gamma} = \Gamma \cap (E \times I) \subset A_M \quad \text{and} \quad A_{\max} \cap (E \times I) \subset Cl\,\tilde{\Gamma}. \tag{40}$$

Together with the results of the previous section, (40) proves Theorem 4. Indeed,

$$\tilde{\Gamma} \subset A_M \Longrightarrow Cl\,\tilde{\Gamma} \subset A_M \Longrightarrow A_{\max} \cap (E \times I) \subset A_M \Longrightarrow \mathrm{mes}_3 A_M > 0.$$

The latter implication follows from the fact that Γ is a graph of an a.e. positive function.

The main Density Lemma is Lemma 6. To prove it, we use an improvement of Lemma 5, namely, Lemma 11 stated below. This lemma makes use of the Assumptions 1–5 in the definition of the skew products of class TAT only.

6.2. Hölder property in the logarithmic chart

The results of this and the next three subsections are needed for the proof of density lemmas.

Consider a map $\log : (0, 1] \to \mathbb{R}^- = \{\xi \leq 0\}, x \mapsto \log x = \xi$. Any C^2 map $f : [0, 1] \to [0, 1]$ with $f(0) = 0$, $f'(0)$ in the logarithmic chart ξ takes the form:

$$\tilde{f} : \xi \mapsto \log \circ f \circ \exp \xi = \xi + \log f'(0) + O(e^{\xi}). \tag{41}$$

Here and below $O(\cdot)$ is taken near $-\infty$. Hence, in a logarithmic chart any 0-preserving C^2-smooth map becomes a translation modulo an exponentially small remainder term.

Lemma 7. 1. *Let a family of C^2-maps $f_b : I \to I$ be Hölder continuous with respect to b in the C^2-norm:*

$$d_{C^2}(f_b, f_{b'}) \leq C d^{\alpha}(b, b').$$

Let $f_b(0) = 0$ for all $b \in B$. Then the family $\tilde{f}_b$ written in the logarithmic chart is Hölder continuous with respect to b in the C^1-norm.

2. *If $f \in C^3$, then $\tilde{f}$ (the map f written in the logarithmic chart) belongs to C^2.*

This is an analog of the Hadamard Lemma.

Proof. 1. Let $f_b = \nu(b)x(1 + \varphi_b(x))$, $\lambda(b) = \log \nu(b)$. Then, in the logarithmic chart $\xi = \log x$, this map has the form:

$$\tilde{f}_b : \xi \mapsto \xi + \lambda(b) + \log(1 + \varphi_b(e^{\xi})).$$

Hölder continuity of $\tilde{f}_b'$ in b is equivalent to the same property of φ_b'. On the other hand,

$$\frac{f_b(x)}{x} = \int_0^1 f_b'(x\tau) d\tau.$$

Therefore,

$$\varphi_b(x) = \frac{f_b(x)}{\nu(b)x} - 1 = \int_0^1 \frac{f_b'(x\tau) - f_b'(0)}{f_b'(0)} d\tau, \qquad \varphi_b'(x) = \int_0^1 \frac{\tau f_b''(x\tau)}{f_b'(0)} d\tau.$$

Now, Hölder continuity of f_b' and f_b'' in b implies the same property of φ_b and φ_b' respectively.

2. Let $f_b = \nu(b)x(1 + \varphi_b(x)) \in C^3$. Then, by the previous formula,

$$\varphi_b''(x) = \int_0^1 \frac{\tau^2 f_b'''(x\tau)}{f_b'(0)} d\tau.$$

This proves the second statement of the lemma. □

6.3. Logarithmic Sternberg coordinates

The following two maps will be frequently used below:

$$f_0 = f_{O^-}, \qquad f_1 = f_{O^+}. \tag{42}$$

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Recall that the map f_0 has only two fixed points: an attractor 0 and a repeller 1; f_1 has exactly three fixed points: two repellers 0 and 1, and one attractor $a = a(O^+)$. All of them are hyperbolic. Moreover, by the property (8) of the skew products of class TAT,

$$f_0'(0) = \lambda, \qquad f_1'(0) = \mu, \qquad \tilde{\lambda} = \log \lambda, \qquad \tilde{\mu} = \log \mu, \quad \frac{\tilde{\lambda}}{\tilde{\mu}} \notin \mathbb{Q}. \tag{43}$$

Denote by T_α a shift $\mathbb{R} \to \mathbb{R}, \zeta \mapsto \zeta + \alpha$. Note that as $\tilde{\lambda} < 0 < \tilde{\mu}$, and $\tilde{\lambda}/\tilde{\mu} \notin \mathbb{Q}$, the semigroup $G^+ = G^+(T_{\tilde{\lambda}}, T_{\tilde{\mu}})$ is dense in $\mathbb{R}$. Roughly speaking, in what follows we will compare the fiber maps with compositions of $\tilde{f}_0$, $\tilde{f}_1$; these compositions, in turn, will be compared with the elements of the semigroup G^+.

Let us now take the Sternberg parameters for f_0 and f_1 near zero. In more detail, let f_0, f_1 be the same as in (42). Let z (respectively, y) be the linearizing chart for f_0 (respectively, f_1) near zero extended to $[0, 1)$ ($[0, a(O^+))$ respectively), as in Section 5.1. Let h be the transition function between these two charts: $y = h(z)$, $h'(0) = 1$.

Let $\zeta = \log z$, $\eta = \log y$ be the corresponding logarithmic charts. For any map $g : [0, 1] \to [0, 1]$ and any map $f : [0, a(O^+)] \to [0, a(O^+)]$ consider their expressions in ζ and η coordinates respectively:

$$\tilde{g} : \mathbb{R} \to \mathbb{R}, \quad \tilde{g} = \zeta \circ g \circ \zeta^{-1}, \qquad \hat{f} : \mathbb{R} \to \mathbb{R}, \quad \hat{f} = \eta \circ f \circ \eta^{-1}.$$

In the charts ζ and η, the maps f_0 and f_1 become mere translations:

$$\tilde{f}_0 : \zeta \mapsto \zeta + \tilde{\lambda}, \qquad \hat{f}_1 : \eta \mapsto \eta + \tilde{\mu}, \quad \tilde{\lambda} = \log \lambda, \tilde{\mu} = \log \mu. \tag{44}$$

Let ζ and η be the Sternberg logarithmic charts as above. Let $a' \in (0, a(O^+))$, and

$$\mathcal{L} = \mathcal{L}(a') = \zeta(0, a'), \qquad \mathcal{L}' = \mathcal{L}'(a') = \eta(0, a'). \tag{45}$$

The transition function between ζ and η is defined as

$$\tilde{h} : \mathcal{L} \to \mathcal{L}', \qquad \tilde{h} = \eta \circ \zeta^{-1}.$$

As ζ and η are defined up to a translation, we may assume that $\zeta(a') = 0$, $\mathcal{L}(a') = \mathbb{R}^-$, and the transition function $\tilde{h}$ may be taken in the form:

$$\eta = \tilde{h}(\zeta) = \zeta + O(e^\zeta). \tag{46}$$

6.4. *Skew products as topological Markov chains*

The map A (Anosov map of the two torus) in the base of a skew product of class TAT is conjugated to a topological Markov chain. Let $\Delta = (\Delta_1, \dots, \Delta_{k_0})$ be a Markov partition for A, see Assumption 5 of Theorem 3. Let C be the adjacency matrix for Δ:

$$C = (a_{ij}), \quad a_{ij} = \begin{cases} 1, & A(\Delta_i) \cap \Delta_j \neq \emptyset \\ 0, & A(\Delta_i) \cap \Delta_j = \emptyset. \end{cases}$$

Property 5 of class TAT implies that the subwords 00, 11 and 01 are admissible. Let Σ_C be the topological Markov chain with this adjacency matrix. Let $\Phi : B \to \Sigma_C$ be the fate map that attaches a unique sequence $\Phi(b)$ to P-a.e. point in B: $\Phi(b) = \omega = \dots \omega_{-n} \dots \omega_0 \dots \omega_n \dots$; $\omega_n = j \in \{1, \dots, k_0\} \Longleftrightarrow h^n(b) \in \Delta_j$.

From now on we will consider the skew product G over Σ_C conjugated to F:

$$G : (\omega, x) \mapsto (\sigma\omega, g_\omega(x)), \quad g_\omega = f_b \text{ for } b = \Phi^{-1}(\omega). \tag{47}$$

The fiber maps of the iterates of G are expressed in the following way: for any $n \in \mathbb{N}$,

$$G^n(\omega, x) = (\sigma^n\omega, g_{\omega,n}(x)), \qquad G^{-n}(\omega, x) = (\sigma^{-n}\omega, g^-_{\omega,n}(x)),$$

where

$$g_{\omega,n} = g_{\sigma^{n-1}\omega} \circ \cdots \circ g_\omega, \qquad g^-_{\omega,n} = (g_{\sigma^{-1}\omega} \circ \cdots \circ g_{\sigma^{-n}\omega})^{-1}. \tag{48}$$

Note that $\Phi(O^-) = (0)$, $\Phi(O^+) = (1)$, and f_0, f_1 from (42) are at the same time $g_{(0)}$ and $g_{(1)}$; here (0) and (1) are sequences of all zeros and all ones respectively.

By definition, $d(\omega, \omega') = k_0^{-n(\omega,\omega')}$, where $n(\omega, \omega') = \min\{|m| \mid \omega_m \neq \omega_{m'}\}$ and k_0 is the number of symbols in the alphabet of Σ_C. Moreover, the map Φ^{-1} is Hölder continuous: $d(\Phi^{-1}(\omega), \Phi^{-1}(\omega')) \leq C_1 d^{a'}(\omega, \omega')$. Hence, property (15) implies:

$$\|g_\omega - g_{\omega'}\|_{C^3} \leq C_0 d^\beta(\omega, \omega') \quad \text{where } \beta = \alpha\alpha'. \tag{49}$$

6.5. Lebesgue and Markov measures

It is well known that a Markov partition related to an Anosov diffeomorphism of a two torus, provides an isomorphism of this diffeomorphism and some Markov chain with the Markov measure induced by the Lebesgue measure on a torus. Indeed, let $\Delta_1, \ldots, \Delta_{k_0}$ be the rectangles of the Markov partition. Let

$$q_j = \mathrm{mes}\Delta_j, \qquad a_{ij} = \frac{\mathrm{mes}(A\Delta_i \cap \Delta_j)}{\mathrm{mes}\Delta_i}.$$

It is obvious that $\Sigma q_j = 1$ and the matrix $(a_{ij}) = B$ is stochastic. On the other hand, $q = (q_1, \ldots, q_{k_0})$ is a left invariant vector of B with the eigenvalue 1, because A preserves the Lebesgue measure. Consider an adjacency matrix C of 0, 1 such that $c_{ij} = 1$ iff $f(\Delta_i) \cap \Delta_j$ is nonempty. Note that $c_{ij} = \frac{a_{ij}}{|a_{ij}|}$ for $a_{ij} \neq 0$ and $c_{ij} = 0$ for $a_{ij} = 0$. Hence, the matrix B determines C.

Take the topological Markov chain Σ_C and introduce the measure P on it, as to transform it to the Markov chain (B, q), see [5]. Namely, for a cylinder $C_m = C^{n\ldots n+m}_{j_0\cdots j_m}$ let

$$P(C_m) = q_{j_0} \prod_{l=0}^{m-1} a_{j_l j_{l+1}}. \tag{50}$$

Let Φ be the fate (itinerary) map $\mathbb{T}^2 \to \Sigma_C$. Then for any C_m above,

$$P(C_m) = \mathrm{mes}(\Phi^{-1} C_m), \tag{51}$$

see [23] for more details.

Now all the "almost everywhere" statements will be proved in sense of P. Relation (51) allows us to translate them immediately in terms of the Lebesgue measure on $\mathbb{T}^2$.

7. Distortion lemmas

In this section we present Distortion Lemmas that are the main tool for the study of Hölder skew products.

7.1. Middle part

Introduce the following notations:

$$\text{for } \omega = \dots \omega_{-n} \dots |\omega_0 \dots \omega_n \dots, \tag{52}$$

we have:

$$\omega\,|_a^b = \omega_{[a]} \dots \omega_{[b]},$$

where $[a]$ is the integer part of a. Sometimes it is convenient not to take care whether a, b are integer or not; so we consider any real $a < b$ above.

Consider a skew product G of class TAT, and the Sternberg logarithmic chart ζ on its fibers, see 6.3. Recall that the fiber map g written in this chart is denoted by $\tilde{g}$. Let $\tilde{g}_{\omega,n}, \tilde{g}^{-}_{\omega,n}$ be the same as in (48) written in the chart ζ. Let a' and $\mathcal{L}(a')$ be the same as in (45). Assumption 1 of Section 2.1 requires that the fiber maps of G are close to identity in C^3. By Lemma 7, $\tilde{g}_\omega \in C^2(\mathcal{L}(a'))$. Then the following quantity is well defined:

$$\bar{L} = \max\left(\max_{\Sigma_C}(\operatorname{Lip}\, \tilde{g}_\omega^{\pm 1}),\ \max_{\Sigma_C}(\operatorname{Lip}\, \tilde{g}'_\omega) \right).$$

In what follows we suppose that $\bar{L}$ is small. Namely, let $\gamma = \frac{1}{30}$, and β be the same as in (49). We suppose that

$$q_0 := \bar{L} k_0^{-\beta\gamma} < 1, \qquad q_1 := \bar{L} q_0 < 1. \tag{53}$$

Lemma 8 (*Distortion Lemma 1*)**.** *Let G be a Hölder skew product of class TAT over a Markov chain. Then there exists C_1 depending on a' such that for any $\mathcal{L} \subset \mathcal{L}(a')$ and any m the equality*

$$\omega\,|_{-\gamma m}^{(\gamma+1)m} = \omega'\,|_{-\gamma m}^{(\gamma+1)m} \tag{54}$$

implies

$$\|\tilde{g}_{\omega,m} - \tilde{g}_{\omega',m}\|_{C^1(\mathcal{L})} \le C_1 q_1^m := \Delta_m, \tag{55}$$

provided that all the intermediate maps $g_{\omega,k}, g_{\omega',k}$ bring $\mathcal{L}$ into $\mathcal{L}(a')$.

The words $\omega\,|_0^m = \omega'\,|_0^m$ are called *acting words*. The words $\omega\,|_{-\gamma m}^{-1} = \omega'\,|_{-\gamma m}^{-1}$ and $\omega\,|_{m+1}^{(\gamma+1)m} = \omega'\,|_{m+1}^{(\gamma+1)m}$ are called *marginal*.

This lemma is proved, yet not stated, as a part of the proof of Lemmas on error, 3.1 and 3.2, in [9]. So we reproduce the proof here.

Proof. All the norms in this proof are, by default, in $C(\mathcal{L})$. Equality (54) implies:

$$d(\sigma^k\omega, \sigma^k\omega') \le k_0^{-\gamma m}, \quad 0 \le k \le m.$$

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Then, by the Hölder property (49) and Lemma 7, for the same k,

$$\|\tilde{g}_{\sigma^k\omega} - \tilde{g}_{\sigma^k\omega'}\|_{C^1(\mathcal{L})} \le C_0 k_0^{-\beta\gamma m} := \Delta(m). \tag{56}$$

Let

$$\delta_k = \|\tilde{g}_{\omega,k} - \tilde{g}_{\omega',k}\|, \qquad d_k = \|\tilde{g}'_{\omega,k} - \tilde{g}'_{\omega',k}\|.$$

We will prove that for $k = 0, \dots, m$,

$$\delta_k \le C q_0^k, \qquad d_k \le C q_1^k. \tag{57}$$

This will imply (55). Let us prove the first inequality in (57). For any four diffeomorphisms f, g, f_1, g_1 of $\mathcal{L}(a')$ we have:

$$\begin{aligned} \|f \circ g - f_1 \circ g_1\| &\le \|f \circ g - f_1 \circ g\| + \|f_1 \circ g - f_1 \circ g_1\| \\ &\le \|f - f_1\| + \operatorname{Lip} f_1 \|g - g_1\|, \end{aligned} \tag{58}$$

provided that the compositions above are well defined. These intermediate notations have nothing to do with the fiber map f_1 above. Let us apply this general inequality to

$$f = \tilde{g}_{\sigma^k\omega}, \qquad g = \tilde{g}_{\omega,k}, \qquad f_1 = \tilde{g}_{\sigma^k\omega'}, \qquad g_1 = \tilde{g}_{\omega',k}. \tag{59}$$

In these notations,

$$\tilde{g}_{\omega,k+1} = f \circ g, \qquad \tilde{g}_{\omega',k+1} = f_1 \circ g_1.$$

Moreover,

$$\|f - f_1\| \le \Delta(m).$$

Now, (58) and (59) imply:

$$\delta_{k+1} \le \Delta(m) + \bar{L}\delta_k, \quad 1 \le k \le m-1.$$

By (56), $\delta_1 \le \Delta(m)$. Hence, by induction in k, and (53),

$$\delta_m \le \Delta(m)(1 + \bar{L} + \cdots + \bar{L}^{m-1}) \le C_0 \bar{L}^m k_0^{-\beta\gamma m} = C_0 q_0^m.$$

This proves the first inequality in (57). Let us prove the second inequality. Once again, for general diffeomorphisms f, g, f_1, g_1 we have:

$$\begin{aligned} \|(f \circ g)' - (f_1 \circ g_1)'\| &= \|f' \circ g \cdot g' - f_1' \circ g_1 \cdot g_1'\| \le \|(f' - f_1') \circ g \cdot g'\| \\ &\quad + \|(f_1' \circ g - f_1' \circ g_1) \cdot g'\| + \|f_1' \circ g_1 \cdot (g' - g')\|. \end{aligned} \tag{60}$$

By definition of $\bar{L}$, we have:

$$\operatorname{Lip} \tilde{g}_\omega \le \bar{L}, \qquad \operatorname{Lip} \tilde{g}'_\omega \le \bar{L} \quad \text{for any } \omega.$$

By (59) and (60),

$$\begin{aligned} d_{k+1} = \|\tilde{g}'_{\omega,k+1} - \tilde{g}'_{\omega',k+1}\| &\le \|\tilde{g}'_{\sigma^k\omega} - \tilde{g}'_{\sigma^k\omega'}\| \cdot \bar{L} \\ &\quad + (\operatorname{Lip} \tilde{g}'_{\sigma^k\omega'})\|\tilde{g}_{\omega,k} - \tilde{g}_{\omega',k}\|\bar{L} + \bar{L}\|\tilde{g}'_{\omega,k} - \tilde{g}'_{\omega',k}\|. \end{aligned}$$

On the other hand, by (56),

$$\|\tilde{g}'_{\sigma^k\omega} - \tilde{g}'_{\sigma^k\omega'}\| \le \Delta(m).$$

By the left inequality in (57),

$$\|\tilde{g}_{\omega,k} - \tilde{g}_{\omega',k}\| = \delta_k < C_0 q_0^k.$$

A robust inequality for d_{k+1} is:

$$d_{k+1} \le \bar{L}(\Delta(m) + C_2 q_0^m) + \bar{L} d_k \le C_3 q_0^m + \bar{L} d_k, \qquad d_1 \le \Delta(m).$$

Hence, by induction in k,

$$d_m \le C_3 q_0^m (1 + \cdots + \bar{L}^{m-1}) \le C_4 (q_0 \bar{L})^m = C_1 q_1^m.$$

This proves the right inequality in (57), and, together with it, Lemma 8. $\square$

7.2. Continuity of the boundary function with respect to the right tail

Lemma 9 (*Distortion Lemma 2*)**.** *For any Hölder skew product (47) of class TAT, the boundary function σ^+ is uniformly continuous with respect to the future part of ω in the following sense: for any δ there exists $k^* > 0$ such that the equality*

$$\omega\,|_{-\infty}^{k^*} = \omega'\,|_{-\infty}^{k^*} \tag{61}$$

implies

$$|\zeta \circ \sigma^+(\omega) - \zeta \circ \sigma^+(\omega')| < \delta. \tag{62}$$

Proof. This follows immediately from the formula for σ^+, see (23). Namely, let in the Sternberg logarithmic coordinate the fiber be $\mathbb{R}$ and $b \in \mathbb{R}$ have the property: $\tilde{g}_\omega(b) < b$ for any $\omega \in \Sigma_C$. Then

$$\zeta \circ \sigma^+(\omega) = \lim_{k\to\infty} \tilde{g}_{\sigma^{-1}\omega} \circ \cdots \circ \tilde{g}_{\sigma^{-k}\omega}(b).$$

Under assumption (61) and by Lemma 7,

$$\|\tilde{g}_{\sigma^{-k}\omega} - \tilde{g}_{\sigma^{-k}\omega'}\| \le C_0 k_0^{-\beta(k+k^*)}.$$

Let

$$\delta_k = \|\tilde{g}_{\sigma^{-1}\omega} \circ \cdots \circ \tilde{g}_{\sigma^{-k}\omega} - \tilde{g}_{\sigma^{-1}\omega'} \circ \cdots \circ \tilde{g}_{\sigma^{-k}\omega'}\|.$$

Then

$$\begin{aligned}
\delta_{k+1} \le{}& \|(\tilde{g}_{\sigma^{-1}\omega} \circ \cdots \circ \tilde{g}_{\sigma^{-k}\omega}) \circ \tilde{g}_{\sigma^{-(k+1)}\omega} - (\tilde{g}_{\sigma^{-1}\omega} \circ \cdots \circ \tilde{g}_{\sigma^{-k}\omega}) \circ \tilde{g}_{\sigma^{-(k+1)}\omega'}\| \\
&+ \|(\tilde{g}_{\sigma^{-1}\omega} \circ \cdots \circ \tilde{g}_{\sigma^{-k}\omega}) \circ \tilde{g}_{\sigma^{-(k+1)}\omega'} - \tilde{g}_{\sigma^{-1}\omega'} \circ \cdots \circ \tilde{g}_{\sigma^{-k}\omega'} \circ \tilde{g}_{\sigma^{-(k+1)}\omega'}\| \\
\le{}& L_c^k \cdot C_0 k_0^{-\beta(k+k^*+1)} + \delta_k = \frac{C_0}{L_c} k_0^{-\beta k^*} q_2^{k+1} + \delta_k, \quad q_2 = L_c k_0^{-\beta}.
\end{aligned}$$

Hence,

$$\lim_{k\to\infty} \delta_k \le \frac{C_0}{L_c} \cdot \frac{k_0^{-\beta k^*}}{1 - q_2}. \quad \square \tag{63}$$

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7.3. Fiber maps over the stable and unstable manifolds in the base

Recall that η is the logarithmic Sternberg chart for $f_1, \eta : (0, a((1))) \to \mathbb{R}$.

Denote by $|W|_*$ the length of the segment W in the chart $*$; here $*$ stands for the charts x, y, z, ξ, η, ζ. Another notation is $|*(W)| : |\zeta(W)|, |\eta(W)| \dots$.

Lemma 10 (*Distortion Lemma 3*)**.** 1. *Let G be a skew product (47) of the class TAT, and $\omega^0 \in \Sigma_C$ be a sequence with the right tail of zeros. Then for any $x_0 \in (0, 1)$, there exists*

$$C_0(x_0) := \lim_{n\to\infty} (\zeta \circ \pi(G^n(\omega^0, x_0)) - \tilde{\lambda} n); \tag{64}$$

2. *Similarly, let ω^1 be a sequence with the left tail of ones. Then for any $x_1 \in (0, \sigma^+(\omega^1))$, there exists*

$$C_1(x_1) := \lim_{m\to\infty} (\eta \circ \pi(G^{-m}(\omega^1, x_1)) + \tilde{\mu} m). \tag{65}$$

3. *Moreover, let ω belong to the unstable manifold $\hat{W}$ of the point (1) under $\sigma : \Sigma_C \to \Sigma_C$, that is, ω has a left tail of ones:*

$$\omega = (1)w^-|\omega^+, \tag{66}$$

where w^- and ω^+ are a finite and infinite words respectively. Let V be any interval such that $\eta(\bar{V}) \subset (-\infty, \eta \circ \sigma^+(\omega))$. Then there exists a limit:

$$v_\infty(V) = \lim_{m\to\infty} |V_m|_\eta, \quad V_m = \pi \circ G^{-m}(\{\omega\} \times V). \tag{67}$$

4. *At last, let ω belong to the stable manifold $\hat{W}_s$ of (0) under σ. Let V be any interval in $(0, 1)$. Then there exists a limit*

$$v_\infty^+(V) = \lim_{n\to\infty} |V_n^+|_\zeta, \quad V_n^+ = \pi \circ G^n(\{\omega\} \times V). \tag{68}$$

Moreover, for any $\zeta^* \in \mathbb{R}$, there exists $C = C(\zeta^*)$ such that

$$\frac{v_\infty^+(V)}{|V|} < C \tag{69}$$

provided that $\zeta(V) \subset (-\infty, \zeta^*)$.

Proof. Let us prove the first statement. Take small α such that $1 - \alpha > x_0$, and $1 - \alpha > a^+ = \max_{\omega\in\sum_C} a(\omega)$. Then for any $n > 0$ all the fiber maps of the iterate G^n bring the interval $(0, 1 - \alpha)$ into itself. Consider a ray $\mathcal{L}_\alpha = \zeta(0, 1 - \alpha)$. For large k, the map $g_{\sigma^k\omega}$ is close to $T_{\tilde{\lambda}}$, $\tilde{\lambda} < 0$. Thus it brings the ray $\mathcal{L}$ into itself. Hence, there exists a ray $\mathcal{L}$ such that all the maps $g_{\omega^0,l}$, $l \geq 0$, bring $\mathcal{L}$ to $\mathcal{L}_\alpha$. Hence, the map G satisfies assumptions of Lemma 8 on $\mathcal{L}$. In order to apply this lemma to the fiber maps of G^n on the fiber over ω^0, let us split ω^0 in the following way. Let, as before, $\gamma = \frac{1}{30}$. Take $a_k = \gamma^{-k}$, and let:

$$\omega^0 = \omega^-|\underbrace{w^+}_{s}\underbrace{0}_{1=a_0}\underbrace{0\dots0}_{\gamma^{-1}=a_1}\dots\underbrace{0\dots0}_{\gamma^{-k}=a_k}\dots.$$

Consider the same splitting of (0):

$$(0) = (0)^-|\underbrace{0\dots0}_{s}\underbrace{0}_{1=a_0}\underbrace{0\dots0}_{\gamma^{-1}=a_1}\dots\underbrace{0\dots0}_{\gamma^{-k}=a_k}\dots.$$

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Note that

$$\omega^0\,|_{s+1}^{\infty} = (0)\,|_{s+1}^{\infty}\,.$$

Recall that $\tilde{f}_0 = \mathrm{id} + \tilde{\lambda}$. Let

$$t_0 = s+1, \qquad t_k = t_{k-1} + a_k.$$

Apply Lemma 8 (Distortion Lemma 1) to the compositions $\tilde{g}_{\sigma^{t_{k-1}}\omega^0, a_k}$ and $\tilde{f}_0^{a_k} = id + a_k\tilde{\lambda}$. We have: $\omega^0\,|_{t_{k-2}}^{\infty} = (0)\,|_{t_{k-2}}^{\infty}$. The acting word has length a_k. The right marginal word has length infinity, the left one is of length $a_{k-1} = \gamma a_k$. Hence, Lemma 8 is applicable. Denote by Δ_m the right hand side of (55): $C_1 q_1^m = \Delta_m$. Lemma 8 implies:

$$\|\tilde{g}_{\sigma^{t_{k-1}}\omega^0, a_k} - \tilde{f}_0^{a_k}\|_{C^1(\mathcal{L}(a'))} \le \Delta_{a_k}.$$

Let

$$\zeta_0 = \zeta(x_0), \qquad \zeta_k = \tilde{g}_{\omega^0, t_{k-1}}(\zeta_0).$$

Then

$$\zeta_{k+1} = \tilde{g}_{\sigma^{t_{k-1}}\omega^0, a_k}(\zeta_k).$$

Hence,

$$|\zeta_{k+1} - \zeta_k - a_k\tilde{\lambda}| < \Delta_{a_k}.$$

The series $\sum \Delta_{a_k}$ converges (superexponentially). Hence, by the Cauchy criterion, the series

$$\sum_{k=0}^{\infty}(\zeta_{k+1} - \zeta_k - a_k\tilde{\lambda}) = \lim_{k\to\infty}(\zeta_{k+1} - \zeta_1 - t_k\tilde{\lambda})$$

converges. This implies the first statement of the lemma.

Let us now prove the second statement. Take ω' as in (66), and $x_1 \in (0, \sigma^+(\omega'))$. Let $q = (\omega', x_1)$. By Proposition 7 stated below in 9.4, $\eta_m := \eta(\pi \circ G^{-m}(q)) \to -\infty$ as $m \to +\infty$. Hence, $\tilde{h}^{-1}(\eta_m) - \eta_m \to 0$ as $m \to \infty$. On the other hand, the limit

$$C_2(x_1) = \lim_{m\to+\infty}(\eta_m + \tilde{\mu}m) \tag{70}$$

exists. This is proved exactly as the existence of the limit (64), i.e. statement 1 of Lemma 10. Now

$$\eta \circ \pi \circ G^{-m}(q) + \tilde{\mu}m = \tilde{h}^{-1}(\eta_m) + \tilde{\mu}m = \eta_m + \tilde{\mu}m + o(1).$$

Hence, the limit (65) exists and equals $C_2(x_1)$, see (70).

Let us prove the third statement. For any $a' \in (0, a)$, we have: $V_m \subset (0, a')$ for large m. This follows from the second statement of the lemma.

If $\omega \in \hat{W}$, the unstable manifold of the point (1) under σ, then ω has the form (66). Let $|w^-| = s$. Then, for $m = s + k$, $d(\sigma^{-m}\omega, (1)) < k_0^{-k}$. Recall that $\hat{g}$ denotes the fiber map g written in the Sternberg logarithmic coordinate η, see Section 6.3. Then

$$d_{C^1}(\hat{g}_{\sigma^{-m}\omega}, \hat{f}_1) \le C_0 k_0^{-\beta k}.$$

Therefore,

$$\|(\hat{g}_{\sigma^{-m}\omega})' - 1\|_{C(V)} \le C_0 k_0^{-\beta k}.$$

Note that $\pi \circ G^{-m}(\{\omega\} \times I) = (g_{\sigma^{-m}\omega,m})^{-1}$. By the chain rule, the latter inequality implies that the derivatives $\nu_m(\eta) := ((\hat{g}_{\sigma^{-m}\omega,m})^{-1})'_\eta(\eta)$ converge to a nonzero limit for any $\eta \in \eta(V)$. This proves the third statement of the lemma.

The fourth statement is proved along the same lines as the third one. For any $\zeta \in (-\infty, \zeta^*)$, the derivative $g'_{\omega,n}$ tends to a nonzero limit that is bounded away from zero. The bound and the convergence are uniform in $\zeta \in (-\infty, \zeta^*)$. This is proved as in the previous paragraph. □

8. Transitivity property of iterated fiber maps

In this section we study fiber maps of iterates of skew products of class TAT over a shifted sequence

$$\omega^{01} = \dots 0 \dots 0|1 \dots 1 \dots = (0)^-|(1)^+. \tag{71}$$

8.1. A refinement of the first density lemma

Recall that ζ and η are Sternberg logarithmic charts for the map f_0 and f_1 respectively;

$$\zeta : (0, 1) \to \mathbb{R}, \qquad \eta : (0, a(O^+)) \to \mathbb{R}.$$

These charts are well defined up to a translation. Choose and fix any $a' \in (0, a(O^+))$ such that the map f_1 written in the chart ζ is contracting on the segment $[\zeta(a'), \zeta(a(O^+))]$. Such an a' exists because the point $a(O^+)$ is the attractor of f_1. Choose and fix the charts ζ and η is such a way that in (45), $\mathcal{L}(a') = \mathbb{R}^-$, and (46) holds.

The fiber maps in Section 8 are written in the invariant form: g with subscripts is a fiber map of a skew product G, no matter, what the coordinates are. The same map written in the chart ζ (or η) is denoted by $\tilde{g}$ (respectively, $\hat{g}$). We skip tilde in the notation of the transition function and write $\eta = h(\zeta)$. Hence, $\hat{g} = h \circ \tilde{g} \circ h^{-1}$, $\tilde{g} = h^{-1} \circ \hat{g} \circ h$.

The following lemma implies Lemma 5.

Lemma 11. *Consider a skew product G of class TAT. For any ε and any a' described in the previous paragraph there exists $N \in \mathbb{N}$ with the following properties. For any interval $W \subset (0, a')$ such that $|\eta(W)| > \varepsilon$, and any interval $\Omega \subset (0, a')$ such that $|\zeta(\Omega)| < \frac{\varepsilon}{4}$, and dist $(\zeta(\Omega), \zeta(W)) < 2$, there exist sequences $\bar{\mathcal{L}} = (l_j \in \mathbb{N})$, $\mathcal{K} = (k_j \in \mathbb{N})$ and a number j_0, such that:*

$$l_j \nearrow \infty, k_j \nearrow \infty, \quad k_j \in \left[\frac{l_j}{3}, 3l_j\right], \tag{72}$$
$$l_{j+1} - l_j \le N,$$

and for any $j \in \mathbb{N}$, $j > j_0$,

$$g_{\sigma^{-k_j}\omega^{01},k_j+l_j}(\Omega) \subset W. \tag{73}$$

The left hand side of (73) is defined by formula (48). The lemma is proved in the rest of this section and in the Appendix.

8.2. Heuristic arguments

Begin with a comment to the formula (73). When we consider a map $g_{\sigma^K\omega,m}$, we take a sequence ω and shift the zero position K digits to the right (if $K<0$, then it is $|K|$ digits to the left). After that we take first m digits of the shifted sequence and use them as an acting word.

The main idea of the proof is the following. Let us split k_j and l_j into two terms: $k_j = k+K, l_j = L+l$. The corresponding splitting of the shifted sequence ω^{01} takes the form:

$$\sigma^{-k_j}\omega^{01} = \ldots 0|\underbrace{0\ldots0}_{k}\underbrace{0\ldots0}_{K}\underbrace{1\ldots1}_{L}\underbrace{1\ldots1}_{l}1\ldots. \tag{74}$$

Now g from (73) takes the form

$$g = g_1 \circ g_* \circ g_0,$$

where

$$g_0 = g_{\sigma^{-(k+K)}\omega^{01},k}, \qquad g_1 = g_{\sigma^L\omega^{01},l}, \tag{75}$$

and

$$g_* = g_{\sigma^{-K}\omega^{01},K+L}. \tag{76}$$

The first two maps are well under control. They are compared with f_0^k and f_1^l respectively. Indeed, to study g_0 we compare the iterated fiber maps over the base points (74) and

$$(0) = 0\ldots0|\underbrace{0\ldots0}_{k}\underbrace{0\ldots0}_{K}\ldots.$$

The left marginal word is an infinite sequence of zeros; the right one is $\underbrace{0\ldots0}_{K}$, the acting one is $\underbrace{0\ldots0}_{k}$. We apply the First Distortion lemma to claim that the maps g_0 and f_0^k are close. To do that, we need the inequality: $K>\gamma k$ required by the lemma.

The same argument is applied to the proof of proximity of g_1 and f_1^l.

Note that the maps f_0^k, f_1^l are well defined, and changing k and l we can control the change of the position of $g(\Omega)$.

On the other hand, the map g_* from (76) is beyond any control. We have an insufficient information about the fiber maps like $g_{\omega^{01}}$ to make any precise statement about its image. We resolve this problem in the following way. On one hand, by (41),

$$\tilde{g}_*(\zeta) = \zeta + \kappa + R(\zeta), \quad |R(\zeta)| \le Ce^{\zeta} \tag{77}$$

for some $\kappa \in \mathbb{R}$. The numbers κ and C depend on K and L. The first step is to prove that they are uniformly bounded for arbitrary large K and L, provided that $|K\tilde{\lambda} + L\tilde{\mu}| < 2$. This property is claimed in Proposition 3 below and proved in the Appendix.

After that we consider an auxiliary composition

$$\tilde{f} = \tilde{f}_1^l \circ \tilde{g}_* \circ \tilde{f}_0^k \tag{78}$$

which is close to $\tilde{g}$. This composition in a sense "forgets" the remainder term R, and "remembers" the shift κ only. The reason is that $\tilde{f}_0^k$ and $\tilde{f}_1^l$ are similar to two shifts (precise

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details on similarity are discussed in the rigorous proof). Consider two shifts T_{-A}, T_B with $A \gg 1, B \gg 1$. Then a composition $T_B \circ \tilde{g}_* \circ T_{-A}$ will be exponentially close to a shift on $\mathbb{R}^- = \{\zeta \le 0\}$. Indeed, by (77),

$$T_B \circ \tilde{g}_* \circ T_{-A} = \zeta + B - A + \kappa + \tilde{R}(\zeta), \quad |\tilde{R}(\zeta)| < Ce^{(\zeta - A)} < Ce^{-A}.$$

So we first choose K and L to be large and so that κ in (77) should be "moderate". After that, for any given α (it will be constructed below from the data: Ω and V prescribed), we will take κ from (77) and choose k and l in such a way that

$$|\alpha - \kappa - (\tilde{\lambda} k + \tilde{\mu} l)| < \delta$$

for given δ. For these k and l we take the composition $\tilde{f}$ above, which is close to $h^{-1} \circ T_B \circ \tilde{g}_* \circ T_{-A}$ for $-A = \tilde{\lambda} k$, $B = \tilde{\mu} l$. This composition is close to $\tilde{g}$, the map g written in the logarithmic Sternberg coordinate ζ. This will allow us to control the whole composition $\tilde{g}$. The precise details follow.

8.3. Bounded displacements

Our goal is to choose k_j, l_j so that (73) holds. First, we will need the following.

Proposition 3. 1. *For any skew product of class TAT, there exist C and ζ_*, depending on the parameters in (49) only such that for any positive K, L, the inequality*

$$|K\tilde{\lambda} + L\tilde{\mu}| < 2 \tag{79}$$

implies

$$\left| \log \frac{dg_*}{dx}(0) \right| < C \quad \text{where } g_* = g_{\sigma^{-K}\omega^{01}, K+L}. \tag{80}$$

2. *Moreover, the following inequalities hold uniformly in K and L under assumption (79):*

$$Lip_{(-\infty, -\zeta_*)} \tilde{g}_* \le C, \tag{81}$$

$$|\tilde{g}_*'(\zeta) - 1| \le Ce^{\zeta} \tag{82}$$

for $\zeta < -\zeta_*$.

Note that (80) is equivalent to $\lim_{\zeta \to -\infty} \tilde{g}_*(\zeta) - \zeta = \kappa$, $|\kappa| < C$.

The proof is technical, and we postpone it until the Appendix.

8.4. Accurate approximation

The following elementary proposition deals with the shifts of the real line.

Proposition 4. *For any $\tilde{\lambda} < 0$, $\tilde{\mu} > 0$ that satisfy (7), (8), namely*

$$|\tilde{\lambda}| < 1, \qquad |\tilde{\mu}| < 1, \qquad \frac{\tilde{\lambda}}{\tilde{\mu}} \notin \mathbb{Q}, \qquad \frac{1}{2} < \left| \frac{\tilde{\lambda}}{\tilde{\mu}} \right| < 2, \tag{83}$$

and any $\nu > 0$, there exists N such that for any arc W_0 of length ν there exist two unbounded monotonically increasing sequences $\bar{\mathcal{L}} = (l(j) \in \mathbb{N}), \mathcal{K} = (k(j) \in \mathbb{N})$ with the following properties:

for any $j \in \mathbb{N}$,

$$\tilde{\lambda}k(j) + \tilde{\mu}l(j) \in W_0; \tag{84}$$

$$l(j+1) - l(j) \le N; \tag{85}$$

for j *large enough,*

$$\frac{k(j)}{l(j)} \in \left[\frac{1}{3}, 3\right]. \tag{86}$$

The number N *depends on* $\tilde{\lambda}, \tilde{\mu}, \nu$.

Proof. This proposition follows from the well known arguments that prove density in $\mathbb{R}$ of the semigroup $G^+(\tilde{\lambda}, \tilde{\mu})$ with the operation addition. Let ζ_0 be the midpoint of W_0, $\zeta_0 < 0$. The case $\zeta_0 > 0$ is treated in the same way. Take N_1 such that

$$\tilde{\lambda}N_1 \le \zeta_0 < \tilde{\lambda}(N_1 - 1).$$

Now take any $k_* \in \mathbb{N}$, $l_* \in \mathbb{N}$ such that

$$\left|\frac{\tilde{\mu}}{\tilde{\lambda}} + \frac{k_*}{l_*}\right| < \frac{1}{l_*^2} \quad \text{and} \quad \frac{1}{l_*} < \frac{\nu}{2}.$$

Then

$$|\tilde{\lambda}k_* + \tilde{\mu}l_*| < \frac{\nu}{2}.$$

Let $\nu_* = \tilde{\lambda}k_* + \tilde{\mu}l_*$. Then $|\nu_*| < \frac{\nu}{2}$. Let

$$N_* = \left[\frac{\zeta_0 - \tilde{\lambda}N_1}{\nu_*}\right], \qquad \zeta_1 = \tilde{\lambda}N_1 + N_2\nu_*.$$

Then

$$|\zeta_0 - \zeta_1| < \nu_* < \frac{\nu}{2}.$$

Hence, $\zeta_1 \in W_0$. The base of induction is completed:

$$W \ni \zeta_1 = k(1)\tilde{\lambda} + l(1)\tilde{\mu}, \quad k(1) = N_1 + N_2k_*, l(1) = N_2l_*.$$

Let us now proceed the induction step from j to $j+1$. Let $\zeta_j = k(j)\tilde{\lambda} + l(j)\tilde{\mu} \in W_0$. Then either $\zeta'_{j+1} := \zeta_j + N_*(\tilde{\lambda}k_* + \tilde{\mu}l_*) - \tilde{\mu} \in W_0$ or $\zeta''_{j+1} := \zeta'_{j+1} + (\tilde{\lambda}k_* + \tilde{\mu}l_*) \in W_0$. Indeed, $\zeta_j \in W_0$, $\zeta'_{j+1} \le \zeta_j < \zeta''_{j+1}$, and $\zeta''_{j+1} - \zeta'_{j+1} < \nu$. If $\zeta'_{j+1} \notin W_0$, then $\zeta''_{j+1} \in W_0$.

If $\zeta'_{j+1} \in W_0$, take $\zeta_{j+1} = \zeta'_{j+1}$ and

$$k(j+1) = k(j) + N_*k_*, \qquad l(j+1) = l(j) + N_*l_* - 1.$$

If $\zeta''_{j+1} \in W_0$, take $\zeta_{j+1} = \zeta''_{j+1}$ and

$$k(j+1) = k(j) + (N_*+1)k_*, \qquad l(j+1) = l(j) + (N_*+1)l_* - 1.$$

For $k(j+1), l(j+1)$ so chosen, inclusion (84) holds.

Moreover,

$$l(j+1) - l(j) \le (N_*+1)l_* := N.$$

This is N whose existence is claimed in Proposition 4. For this N, inequality (85) holds.

Inequality (86) follows from (83) to (84) for j large enough. □

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8.5. Construction of sequences $\mathcal{K}$ and $\mathcal{L}$ and two auxiliary maps

Let W and Ω be the same intervals as in Lemma 11. Namely, let a' be the same as in the paragraph before Lemma 11, ε be preassigned, $W \subset (0, a')$, $|\eta(W)| > \varepsilon$, $\Omega \subset (0, a')$, $|\zeta(\Omega)| < \frac{\varepsilon}{4}$.

In the construction of sequences $\mathcal{K}$ and $\mathcal{L}$ we will find k_j, l_j as sums

$$k_j = k(j) + K(j), \qquad l_j = l(j) + L(j), \tag{87}$$

whose terms are defined as follows. Let

$$\nu = \frac{\varepsilon}{8}, \tag{88}$$

and N_0 be the number provided by Proposition 4 for $\tilde{\lambda}, \tilde{\mu}$ from (83). Take

$$L(j) = N_0 j \tag{89}$$

for $j > j_0$, j_0 to be chosen later. Take $K(j)$ such that

$$|K(j)\tilde{\lambda} + L(j)\tilde{\mu}| < 2. \tag{90}$$

As $\frac{\tilde{\lambda}}{\tilde{\mu}} \in [\frac{1}{2}, 2]$, for j large enough we have

$$K(j) \in \left[\frac{L(j)}{3}, 3L(j)\right]. \tag{91}$$

Consider

$$\tilde{g}_{*j} = \tilde{g}_{\sigma^{-K(j)}\omega^{01}, \quad K(j)+L(j)}.$$

We have:

$$\tilde{g}_{*j}(\zeta) = \zeta + \kappa_j + O(e^{\zeta})$$

as $\zeta \to -\infty$. By Proposition 3, (90) implies that there exists C independent of j such that $|\kappa_j| < C$. Let us now apply Proposition 4 to $\nu = \frac{\varepsilon}{8}$ and the arc W_0 constructed as follows. Let ζ_0 (and η_0) be the middle-point of the arc $\zeta(\Omega)$ (respectively, $\eta(W)$). Let W_0 be the interval of radius ν centered at $\eta_0 - \zeta_0 - \kappa_j$. By Proposition 4, there exists $l(j), k(j)$ such that

$$l(j) \in [L(j), L(j) + N_0] \tag{92}$$

and (84), (86) hold. Relation (84) implies:

$$|\tilde{\lambda}k(j) + \tilde{\mu}l(j) - (\eta_0 - \zeta_0 - \kappa_j)| < \frac{\varepsilon}{8}. \tag{93}$$

Take k_j, l_j from (87), again with $j > j_0$, j_0 to be chosen later.

By definition of $l(j), L(j)$, we have: $0 \leq l_{j+1} - l_j \leq 3N_0 := N$. This is N, whose existence is claimed in Lemma 9. We have now to prove (73). Fix any $j > j_0$ and replace $\kappa_j, k(j), l(j), K(j), L(j)$ by κ, k, l, K, L respectively. We will prove (73) in the form

$$\tilde{g}_{\sigma^{-k-K}\omega^{01}, k+K+L+l}(\zeta(\Omega)) \subset \zeta(W) \tag{94}$$

for j_0 large enough; let us choose it now. Denote by T_a a translation:

$$\zeta \mapsto \zeta + a, \quad a = \tilde{\lambda}k + \tilde{\mu}l + \kappa.$$

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We have:

$$T_a(\zeta_0) \in Q(\nu, \eta_0), \tag{95}$$

where $Q(c, r)$ is an interval on the real axis centered at c with the radius r.

Let ζ_* and C be the same as in (82). Let $\mathbf{L}$ be the maximum of two Lipschitz constants:

$$\mathbf{L} = \max(\mathrm{Lip} h|_{\mathcal{L}(a')}, \mathrm{Lip} h^{-1}|_{\mathcal{L}'(a')}). \tag{96}$$

Take

$$\delta = \frac{\varepsilon}{8\mathbf{L}(1 + C\mathbf{L}^2)}. \tag{97}$$

Take j_0 so large that: in (55),

$$\Delta_{K+L} < \delta; \tag{98}$$

$$\tilde{\lambda} k \le -(\zeta_* + 1), \tag{99}$$

$$C e^{\tilde{\lambda} k} < \frac{\nu}{2\mathbf{L}}, \tag{100}$$

$$|h(\zeta) - \zeta| < \frac{\nu}{2} \quad \text{for } \zeta < \tilde{\lambda} k, \tag{101}$$

where ζ_* is from Proposition 3, ν is from (88). Inequality (101) for k large enough follows from (46).

The choice of j_0 completes the construction of the sequences $(k_j), (l_j)$ in Lemma 11. It remains to prove (73) in the form (94).

8.6. Proof of the main inclusion in the form (94)

We have:

$$T_a \zeta(\Omega) \subset \eta(W) = h(\zeta(W)).$$

Moreover, for $\tilde{f}$ from (78), $h \circ \tilde{f}$ is close to T_a, $\tilde{f}$ is close to $\tilde{g}$. We have to formalize these arguments to prove (94).

Step 1. *Comparison of $h \circ \tilde{f}$ and T_a.*

As $\Omega = Q(\zeta_0, \nu)$, we have:

$$T_a(\Omega) \subset W(\eta_0, 2\nu). \tag{102}$$

Now, let us prove that

$$h \circ \tilde{f}(\Omega) \subset W(\eta_0, 3\nu). \tag{103}$$

Indeed, $\tilde{f}_1 = h^{-1} \circ T_{\tilde{\mu}} \circ h$. Hence, by (78),

$$h \circ \tilde{f} = T_{l\tilde{\mu}} \circ h \circ \tilde{g}_* \circ T_{k\tilde{\lambda}}.$$

Let us prove that

$$\|h \circ \tilde{f} - T_a\|_{C(\mathbb{R}^-)} < \nu, \quad \text{for } a = \tilde{\lambda} k + \tilde{\mu} l + \kappa. \tag{104}$$

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By (99), $T_{k\tilde{\lambda}}(\mathbb{R}^-) \subset (-\infty, -(\zeta_* + 1))$. By (82), for any $\zeta < -\zeta_*$, $|\tilde{g}_*(\zeta) - T_\kappa| < Ce^{\zeta}$. Hence,

$$\|\tilde{g}_* \circ T_{\tilde{\lambda}k} - T_{\tilde{\lambda}k+\kappa}\|_{C(\mathbb{R}^-)} < \frac{\nu}{2}. \tag{105}$$

Therefore,

$$\begin{aligned}\|h \circ \tilde{f} - T_a\|_{C(\mathbb{R}^-)} &= \|T_{\tilde{\mu}l} \circ h \circ g_* \circ T_{\tilde{\lambda}k} - T_a\|_{C(\mathbb{R}^-)} \\ &= \|h \circ g_* \circ T_{\tilde{\lambda}k} - T_{\tilde{\lambda}k+\kappa}\|_{C(\mathbb{R}^-)} \\ &\le \|h - id\|_{C(-\infty,\tilde{\lambda}k+1)} + \|\tilde{g}_* \circ T_{\tilde{\lambda}k} - T_{\tilde{\lambda}k+\kappa}\|_{C(\mathbb{R}^-)} \le \nu.\end{aligned}$$

The last inequality makes use of (99), (101) and (105). This proves (104). Together with (102) this implies (103).

Let us now compare $\tilde{g}$ and $\tilde{f}$. We will apply two times Distortion Lemma 1. Recall that $\mathcal{L}(a') = (-\infty, 0)$.

Step 2. *First application.*

Let $\tilde{g}_0$ and ν be the same as in (75) and (88). We will prove that

$$\|\tilde{g}_0 - \tilde{f}_0^k\|_{C^1(\mathcal{L}(a'))} \le \delta. \tag{106}$$

Consider two sequences

$$\sigma^{-(k+k(j))}\omega^{01} = \cdots 0 \cdots 0 | \underbrace{0 \cdots 0}_{k} \underbrace{0 \cdots 0}_{K} 1 \cdots 1 \cdots$$

$$(0) = \cdots 0 \cdots 0 | \underbrace{0 \cdots 0}_{k} \underbrace{0 \cdots 0}_{K} 0 \cdots 0 \cdots.$$

Note that $\tilde{f}_0^k = \tilde{g}_{(0),k}$. We have:

$$\sigma^{-(k+K)}\omega^{01}|_{-\infty}^{k+K} = (0)|_{-\infty}^{k+K}.$$

For two fiber maps of G^k considered in (106), the acting word $\underbrace{0 \cdots 0}_{k}$ is of length k, and the marginal words are of length infinity and K. Recall that $\gamma = \frac{1}{30}$. We have: $K \ge \frac{L}{3} \ge \frac{l}{6} \ge \frac{k}{18} > \gamma k$. Hence, the Distortion Lemma 1 is applicable. It implies (106).

Step 3. *Second application.*

In the same way, let us compare $\tilde{g}_1$, see (75), and $\tilde{f}_1^l = \tilde{g}_{(1),l}$. We will prove that

$$\|\tilde{g}_1 - \tilde{f}_1^l\|_{C^1(\mathcal{L}(a'))} \le \delta. \tag{107}$$

Consider two sequences:

$$\sigma^L\omega^{01} = \cdots 1 \cdots 1 \cdots 0 \cdots 0 \underbrace{1 \cdots 1}_{L} | \underbrace{1 \cdots 1}_{l} 1 \cdots 1 \cdots$$

and $(1) = \cdots 1 \cdots 1 \cdots$. For them

$$\sigma^L\omega^{01}|_{-L}^{\infty} = (1)|_{-L}^{\infty}.$$

For $\tilde{g}_1$, that is, the fiber maps of G^l considered in (107), and $f_1^l = g_{(1),l}$, the acting word has the length l, and the marginal words have lengths L and infinity. We have: $L \ge \frac{l}{2} > \gamma l$. Hence, Distortion Lemma 1 is applicable, and implies (107).

Step 4. *Main inclusion.*

Let us turn to the proof of (94). Inclusion (103) implies

$$\tilde{f}(\zeta(\Omega)) \subset h^{-1}(W(\eta_0, 3\nu)) := W_1.$$

We have:

$$W(\eta_0, 3\nu) \subset \eta(W),$$

and

$$\text{dist}\,(W(\eta_0, 3\nu), \partial\eta(W)) \geq \nu.$$

Hence, by (96),

$$W_1 \subset \zeta(W), \quad \text{dist}\,(W_1, \partial\zeta(W)) \geq \frac{\nu}{\mathbf{L}}. \tag{108}$$

Let us now prove that

$$\|\tilde{g} - \tilde{f}\|_{C(\Omega)} \leq \frac{\nu}{\mathbf{L}}; \tag{109}$$

here and below in the proof all the norms are in $C(\Omega)$, and the subscript $C(\Omega)$ is dropped. Together with (108) this will imply (73).

Decompose the left hand side of (109) in two terms: $\|\tilde{g} - \tilde{f}\| \leq T_1 + T_2$, where

$$T_1 = \|(\tilde{g}_1 - \tilde{f}_1^l) \circ \tilde{g}_* \circ \tilde{g}_0\|, \qquad T_2 = \|\tilde{f}_1^l \circ \tilde{g}_* \circ \tilde{g}_0 - \tilde{f}_1^l \circ \tilde{g}_* \circ \tilde{f}_0^k\|.$$

The map $\tilde{g}_* \circ \tilde{g}_0$ brings $\mathcal{L}(a')$ into $\mathcal{L}(a')$. Such a right composition factor does not change the C-norm. Hence, $T_1 < \delta$ by (107).

Let us estimate the second term. We have: all the images of Ω under the maps $\tilde{f}_0^{k'}, 0 \leq k' \leq k, \tilde{g}_* \circ \tilde{f}_0^k, \tilde{f}_1^{l'} \circ \tilde{g}_* \circ \tilde{f}_0^k, 0 \leq l' \leq l$ belong to $\mathcal{L}(a')$. Moreover, by (99), $\tilde{f}_0^k(\Omega) \subset (-\infty, -(\zeta_* + 1)), \tilde{g}_0(\Omega) \subset (-\infty, -\zeta_*)$. We have:

$$\tilde{f}_1^l \circ \tilde{g}_* = h^{-1} \circ T_{l\tilde{\mu}} \circ h \circ \tilde{g}_*.$$

The Lipschitz constant of the translation $T_{l\tilde{\mu}}$ equals 1 for any l. The map $\tilde{g}_*$ depends on K and L. The Lipschitz constant of this map is estimated from above *uniformly in* K *and* L, see Proposition 3: Lip $_{(-\infty,-\zeta_*)}\tilde{g}_* \leq C$. By (96),

$$\text{Lip}\,\tilde{f}_1^l \leq \mathbf{L}^2.$$

Hence, by the choice of δ, see (97), we have:

$$\|\tilde{g} - \tilde{f}\| \leq T_1 + T_2 \leq (1 + \mathbf{L}^2 C)\delta \leq \frac{\varepsilon}{8\mathbf{L}} = \frac{\nu}{\mathbf{L}}.$$

This proves (109) and, together with it, (94), hence, (73). Lemma 11 is proved modulo Proposition 3.

9. Graph of the boundary function over the unstable manifold of O^+ and nearby

Lemma 11 provides the first step in the proof of our main results. Namely, it allows us to control fiber maps of iterates of a skew product G of class TAT over a shifted sequence $\omega^{01} = \cdots 0 \cdots 0|1 \cdots 1 \cdots$. It is an analog of Lemma 3 from 5.1. On the other hand, given a point $p \in A_{\max}(G)$, and a neighborhood U of p, we need to find a point p' on the graph of the boundary function in $U = C_w \times V$. It is done in the same way as Lemma 4 is proved; yet our arguments are more involved. Let $p = (\omega, x)$, $x \in (0, \sigma^+(\omega))$. We want to consider a neighborhood V_m similar to (37). To do that, we need to replace ω by a nearby sequence with a left tail of ones.

But the boundary function is drastically discontinuous with respect to the past part of ω. We want to replace ω by ω' in such a way that still $x \in (0, \sigma^+(\omega'))$. This construction heavily relies upon Assumption 7 in the definition of skew products of class TAT. It is carried on in the current section.

In this section we mainly deal with skew product diffeomorphisms.

9.1. Upper semicontinuity lemma

Lemma 12. *Let F be a skew product diffeomorphism or a Hölder skew product of class TAT, with the Anosov diffeomorphism of the torus in the base, $h = A : \mathbb{T}^2 \to \mathbb{T}^2$, A having at least two fixed points. Let O^+ be the fixed point, over which the fiber map has a repeller. Then there exists $m = m(F) \in \mathbb{N}$ with the following property. Let $b \in \mathbb{T}^2$ be the point whose past fate has an infinite number of clusters of ones of the length $2m + 1$. Then for any $x \in (0, \sigma^+(b))$ and any ε there exists $b' \in W^u$ such that*

$$\sigma^+(b') > x, \qquad |b' - b| < \varepsilon. \tag{110}$$

This lemma is proved in the next three sections.

9.2. A cover surface and semicontinuity property of the boundary function

Proof. Consider first the case when F is a skew product diffeomorphism of class TAT.

Let us pass to the universal cover $\mathbb{R}^2$ over $\mathbb{T}^2$ with the projection $\hat{\pi} : \mathbb{R}^2 \to \mathbb{T}^2$, as in Section 2.3 above. For any $\hat{b} \in \mathbb{R}^2$, let $b = \hat{\pi}\hat{b} \in \mathbb{T}^2$ be its projection. Fix an arbitrary lift of O^+; denote it by O^+ also. This allows us to lift to $\mathbb{R}^2$ in a unique way various simply connected subsets of $\mathbb{T}^2$ that contain O^+. The covering map $\hat{F} : \hat{X} \to \hat{X}$ is defined in Section 2.3.

Let W^u and W^s be the unstable and stable manifolds of O^+ under $\hat{A}$; these are two lines in $\mathbb{R}^2$, see Fig. 3. Let $\hat{\Gamma}$ be the lift of the graph Γ to $\hat{X}$. Let P^+ be the fixed point of $\hat{F}$ that lies over O^+ strictly inside $\hat{X} : x(P^+) \in (0, 1)$. The point $x(P^+)$ is the unique attractor of f_1. Let W^u_{P+} be the strongly unstable manifold of P^+ under $\hat{F}$. This is a curve that lies over W^u.

In Section 9.4 we will prove that

$$W^u_{P+} \subset \hat{\Gamma}. \tag{111}$$

Now we prove Lemma 12 using the inclusion (111).

The following proposition is the first step in the proof of Lemma 12.

Proposition 5. *The boundary function lifted to $\mathbb{R}^2$ is upper semicontinuous on the unstable manifold W^u of O^+.*

Remark 3. Inclusion (111) implies that the restriction of the boundary function to W^u is even smooth. The proposition claims that when we approach to W^u from outside, the upper limit of the boundary function is no greater than its value on W^u.

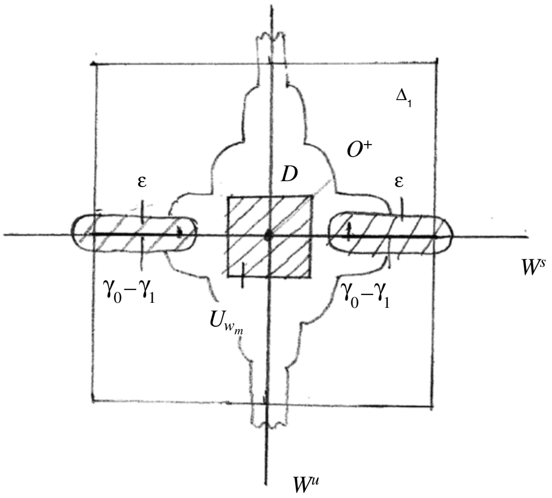

Fig. 3. A special neighborhood of the unstable manifold of O^+.

Proof. Let us construct a surface $W \subset \hat{X}$ that coincides with $\hat{\Gamma}$ over W^u and lies over $\hat{\Gamma}$ in some neighborhood of W^u. Let W^u_{P+} be the same as above. For any $p \in W^u_{P+}$, let W^s_p be the leaf of the strongly stable foliation of $\hat{F}$ through p. Then

$$W = \cup_{p \in W^u_{P+}} W^s_p$$

is the desired surface, see Fig. 4. This surface is continuous.

By Assumption 7 in the definition of the class TAT, see Section 2.3, the surface W lies above the level plane $x = a^+ \geq \sigma^+$ over the difference $\gamma_0 \setminus \gamma_1$. By continuity of W, it lies above the plane $x = a^+$ in some neighborhood $\mathcal{E}$ of $\gamma_0 \setminus \gamma_1$. The union

$$\cup_{l \geq 0} \hat{A}^l(\mathcal{E}) \cup W^u = D$$

is a neighborhood of W^u, see Fig. 3. Over $\mathcal{E}$, the set W lies over $\hat{\Gamma}$. Both sets, W and $\hat{\Gamma}$, are $\hat{F}$-invariant, and $\hat{F}$ is monotonic in x. Hence, W lies above $\hat{\Gamma}$ over all the neighborhood D; this is proved modulo (111). This concludes the proof of the proposition. □

Denote by i a function $\mathbb{R}^2 \to I$ for which W is the graph. We proved that

$$\sigma^+(\hat{b}) < i(\hat{b}) \quad \forall \hat{b} \in D \setminus W^u.$$

By (111), we have:

$$\sigma^+(\hat{b}) = i(\hat{b}) \quad \forall \hat{b} \in W^u.$$

Thus Proposition 5, together with (111), implies:

$$\sigma^+(\hat{b}) \leq i(\hat{b}) \quad \forall \hat{b} \in D. \tag{112}$$

9.3. Approximation of the values of the boundary function

Here we prove Lemma 12, modulo (111).

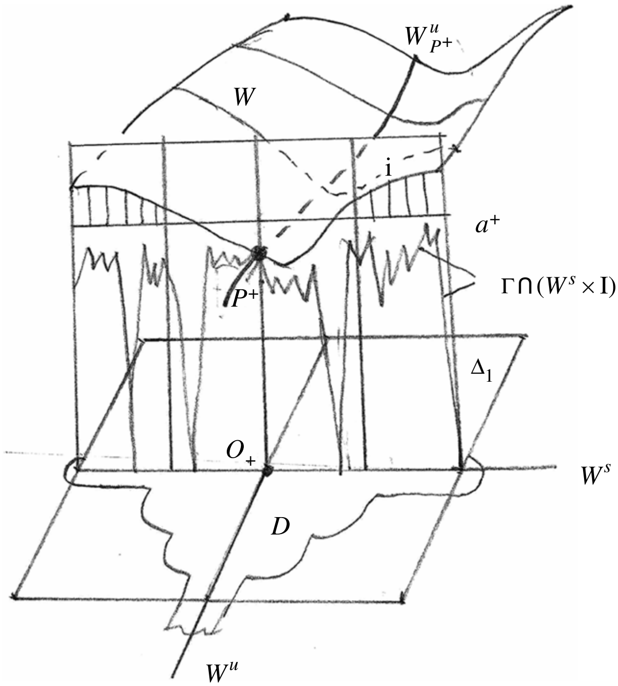

Fig. 4. Skew product of class TAT.

Let us first choose m. For any word w admissible for Σ_C, let C_w be the corresponding cylinder in Σ_C, and $U_w^0 = \Phi^{-1}(C_w) \subset \mathbb{T}^2$ be the corresponding subset of the torus. If $U_w^0 \ni O^+$, let U_w be U_w^0 lifted to $\mathbb{R}^2$ by the lift fixed above.

Denote by w_m the word $\underbrace{1\cdots 1}_{m} | \underbrace{1\cdots 1}_{m+1}$. Take and fix m in such a way that

$$U_{w_m} \subset D, \tag{113}$$

see Fig. 3. This completes the choice of m.

Now take any $b \in \mathbb{T}^2$ that satisfies assumptions of Lemma 12 with m chosen above. Consider two cases.

Case 1: $b \in \hat{\pi} W^u$. Then the lemma becomes a tautology: we take $b' = b$ and gain (110).

Case 2: $b \notin \hat{\pi} W^u$. The curve $\hat{\pi} W^u$ is dense in $\mathbb{T}^2$. We may take a special arc of this curve passing close to b, and project b on this arc along the stable manifold of A. In fact, we will make a similar construction on the lift $\mathbb{R}^2$ for the map $\hat{A}$. Instead of considering different arcs of $\hat{\pi} W^u$, we will consider different lifts of the point b.

Denote by π_s the projection $\mathbb{R}^2 \to W^u$ along W^s.

Proposition 6. *Let m and w_m be the same as above. For any $\varepsilon > 0$ and $b_0 \in \mathbb{T}^2$ such that the past fate $\omega(b_0)$ has an infinite number of subwords w_m there exists a lift $\hat{b}$ of b such that*

$$\hat{b} \in D, |\hat{b} - \pi_s \hat{b}| < \varepsilon. \tag{114}$$

Proof. Let us take N so large that

$$\forall \hat{b} \in U_{w_m}, \quad l > N, \qquad |\hat{A}^l \hat{b} - \pi_s \hat{A} \hat{b}| < \varepsilon. \tag{115}$$

This is possible because $\hat{A}$ is hyperbolic. Let us now take a subword w_m in the sequence $\omega = \Phi(b)$ that lies to the left of the position $-N$. It exists by the assumption on b. Then for some $l > N$,

$$A^{-l} b \in \Phi^{-1}(C_{w_m}) = U^0_{w_m}. \tag{116}$$

Let us take the lift $\hat{b}^l$ of $A^{-l}b$ that belongs to U_{w_m}. The point $\hat{b} := \hat{A}^l \hat{b}_l$ is a lift of b. By Definition of D, $\hat{b} \in D$. By (115), it satisfies (114). □

Let us now complete the proof of Lemma 12. By partial hyperbolicity of $\hat{F}$, the slopes of all the strongly stable leaves are uniformly bounded; in particular, there exists C such that $\forall \hat{b} \in D$,

$$|i(\hat{b}) - i(\pi_s(\hat{b}))| < C|\hat{b} - \pi_s \hat{b}|. \tag{117}$$

Let b, x and ε be the same as in the lemma. Let $\hat{b}$ be the lift of b such that

$$\hat{b} \in D, \tag{118}$$

$$|\hat{b} - \pi_s \hat{b}| < \min\left(\varepsilon, \frac{\sigma^+(\hat{b}) - x}{C}\right). \tag{119}$$

Such a lift exists by Proposition 6. Take $\hat{b}' = \pi_s \hat{b}$, $b' = \pi \hat{b}'$. Then $|b' - b| < \varepsilon$ as required in the right part of (110).

Let us check that $\sigma^+(b') > x$. By (114) and (118), $i(\hat{b}) > \sigma^+(\hat{b})$. By (117) and (119), $i(\hat{b}') > i(\hat{b}) - C|\hat{b} - \hat{b}'| > \sigma^+(\hat{b}) - (\hat{\sigma}(\hat{b}) - x) = x$. By (111), $i(\hat{b}') = \sigma^+(\hat{b}') = \sigma^+(b')$. This proves the left part of (110). □

9.4. Proof of (111)

In the following proposition, the first statement is equivalent to (111). The second statement was used in the proof of the second statement of Lemma 10. Both are easy consequences of the Grobman–Hartman theorem.

Proposition 7. *For any $\hat{b} \in W^u_{O^+}$,*

$$i(\hat{b}) = \sigma^+(\hat{b}). \tag{120}$$

Moreover, for this $\hat{b}$ and any $x \in (0, i(\hat{b}))$, the fiber coordinate of the point $\hat{F}^{-n}(\hat{b}, x)$ tends to 0 as $n \to +\infty$.

Recall that $\pi_B : X \to \mathbb{T}^2$ is the projection on the base along the fiber. Consider a restriction of $\hat{F}$ to the surface $S = \pi_B^{-1} W^u_{O^+}$. This map has a hyperbolic fixed point P^+ with the one-dimensional unstable manifold $W^u_{P^+}$ and stable manifold $\{O^+\} \times (0, 1)$. This map is locally topologically conjugated to a linear map $A_0 : \mathbb{R}^2 \to \mathbb{R}^2$, $(u, v) \mapsto (2u, \frac{u}{2})$. In more detail, there exist a neighborhood U of P^+ on S, a neighborhood V of 0 on $\mathbb{R}^2$ and a homeomorphism $H : V \to U$ that conjugates A_0 and $\hat{F}|_U$, see Fig. 5. This conjugacy may be extended to a neighborhood of $xy = 0$ of the form $V_C : |xy| < C$ for C small: as usually, the conjugacy is

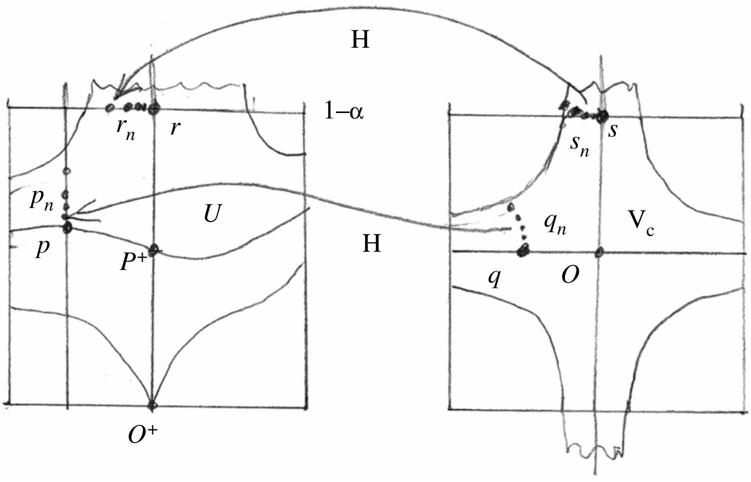

Fig. 5. Grobman–Hartman conjugacy in Proposition 7.

transported to a wider neighborhood by dynamics. The homeomorphism H takes $\{x = 0\}$ to $\{O^+\} \times (0, 1)$, and $\{y = 0\}$ to W^u_{P+}.

Recall that the boundary function σ^+ is defined in (24). To calculate $\sigma^+(b)$ for $b \in W^u$, fix the point $p \in W^u_{P+}$ over $b : \pi_B p = b$, and small α. Take a point $q = H^{-1}(p) = (x_0, 0)$ and a point $s \in \{x = 0\}$ such that $r := H(s) = (O^+, 1 - \alpha)$. Let $r_n = (A^{-n}b, 1 - \alpha)$. Then $r_n \to r$ because $A^{-n}b \to O^+$ as $n \to +\infty$. Then $s_n := H^{-1}(r_n) \to s$. For the linear map A_0, $q_n := A_0^n(s_n) \to q$ as $n \to +\infty$. Therefore, $p_n := F^n(r_n) \to p$. By (22) this implies the first statement of the proposition.

Let us prove the second one. Take any $b \in W^u_{O+}$ and any $x_0 \in (0, \sigma^+(b))$. Take $y \in (x_0, \sigma^+(b))$ so close to $\sigma^+(b)$ that $p := (b, y) = H(q)$ for some $q \in V_C$. As $y < \sigma^+(b)$, we conclude: $u(q) < 0$. Then $u \circ F_0^{-n}(q) \to -\infty$ as $n \to +\infty$. This implies that $x \circ F^{-n}(p) \to 0$ as $n \to +\infty$. By monotonicity of the fiber maps, $x \circ F^{-n}(b, x_0) \to 0$ as $n \to +\infty$.

9.5. *Case of Hölder skew products*

In this subsection we prove Lemma 12 for Hölder skew products of class TAT. The difference with the case of diffeomorphisms of class TAT is twofold. On the one hand, the Hölder skew product is not a diffeomorphism, but rather satisfies assumption (15). On the other hand, Property 7 from Section 2.1 is replaced by property 7′ from Section 2.3.

The latter property provides the surface W over a neighborhood U of $\hat{\gamma}_0 \cup W^u$, such that $\hat{F}W \subset W$ and (16) holds. This implies (112). Uniform continuity of i over u is also required by Assumption 7′. This continuity and (112) imply (110) quite in the same way as in Section 9.3. This proves Lemma 12 for Hölder skew products of class TAT.

10. Closure of the graph of the boundary function

In this section we complete the proof of Theorem 4, hence 2 and 3.

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10.1. The closure lemma

In this subsection we consider smoothly generated Hölder skew products of class TAT over Markov chains, see (47) and 2.1 and 2.3. For any such skew product G let $m' = m(G)$ be the constant provided by Lemma 12. We write m' instead of m from Lemma 12, and reserve m for future purposes. Let $E = E(G)$ be the set of all sequences whose left tail has an infinite number of clusters $\underbrace{1\cdots 1}_{2m'+1}$. This set has the full P-measure. The following lemma implies Lemma 6:

Lemma 13. *For any skew product G of class TAT and the set $E = E(G)$ mentioned above,*

$$Cl(\Gamma \cap A_M) \supset (E \times I) \cap A_{\max} =: A^E_{\max}; \quad A_{\max} = A_{\max}(G), A_M = A_M(G). \tag{121}$$

The lemma implies that the Milnor attractor of G is thick. Below we deduce the lemma from the following

Proposition 8. 1. *For any Hölder skew product G of class TAT, let E and $A^E_{\max}$ be the same as in Lemma 13. Then for any point $p \in A^E_{\max}$ and any neighborhood U of p there exists $\omega'' \in E$ such that*

$$(\omega'', \sigma^+(\omega'')) \subset U. \tag{122}$$

2. *Moreover, for any preassigned left and right tails ω_0^- and ω_0^+, the sequence ω'' in (122) may be so taken that it has these tails starting at appropriate positions (ω_0^- is shifted to the left, ω_0^+ to the right).*

3. *There exists a set of positive P-measure E^U such that for any $\omega'' \in E^U$, (122) holds.*

The second statement of Proposition 8 will be used in the proof of the third one.

Let us deduce Lemma 13 from Proposition 8. By Lemma 2, there exist a set E^0 of the full P-measure such that $\Gamma \cap (E^0 \times I) \subset A_M$. For any set E^U of positive P-measure, $E^0 \cap E^U \neq \emptyset$. Hence, the point $(\omega'', \sigma^+(\omega''))$ in (122) may be taken from A_M; thus, Proposition 8 implies Lemma 13.

In parallel to Proposition 8, we will prove the following

Lemma 14. *Maximal attractor (may be with measure zero set subtracted) of a smoothly generated Hölder skew product of class TAT is topologically mixing.*

This lemma, together with Lemma 13, implies Theorem 4.

In contrast to the notation (52), we will denote by ω_0, ω_1 sequences, rather than digits. This should not bring to a confusion.

Proposition 8 and Lemma 14 follow from

Proposition 9. *In assumption of Proposition 8, for any $p \in A^E_{\max}$ and any neighborhood U of p, there exists $\kappa > 0$ with the following property. For any $\omega_0 \in W^s_{(0)}$, $(0) = \ldots 0 \ldots 0 \ldots$, any interval $\Omega_0 \subset \mathbb{R}^-$ with $|\Omega_0| < \kappa$ and any $t > 0$, there exist a sequence $\omega_1 \in \Sigma_C$ and $T > 0$ such that:*

$$\omega_0 \mid^t_{-\infty} = \omega_1 \mid^t_{-\infty}, \tag{123}$$

$$G^T(\{\omega_1\} \times \Omega_0) \subset U. \tag{124}$$

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This proposition is proved in the next three subsections. Statement 3 of Proposition 8 and Lemma 14 are deduced from it in Sections 10.5 and 10.6 respectively. Here we deduce from it Statements 1 and 2 of Proposition 8.

Take the preassigned left tail ω_0^-, and (for this proof only) let

$$\omega_0 = \omega_0^- | 0 \ldots 0 \ldots .$$

Take κ from Proposition 9. Let Ω_0 be the $\kappa/2$ neighborhood of $\sigma^+(\omega_0)$. Take t so large that for any ω_1 that satisfies (123), the following inequality holds:

$$|\sigma^+(\omega_1) - \sigma^+(\omega_0)| < \frac{\kappa}{2}.$$

Such a t exists by Distortion Lemma 2. Now take

$$r = (\omega_1, \sigma^+(\omega_1)).$$

Then

$$r \in \{\omega_1\} \times \Omega_0.$$

Hence, by (124), $G^T(r) \in U$. But the graph Γ is invariant under G. Hence,

$$G^T(r) = (\sigma^T \omega_1, \sigma^+(\sigma^T \omega_1)).$$

Taking $\omega'' = \sigma^T \omega_1$, we obtain (122).

This proves the first statement of Proposition 8, and a "half of the second one": the sequence ω' found above has the preassigned left tail. Let us now complete the proof of the second statement. By Distortion Lemma 2, there exists k^* so large that relations:

$$(\omega'', \sigma^+(\omega'')) \in U, \qquad \omega''' |_{-\infty}^{k^*} = \omega'' |_{-\infty}^{k^*} \tag{125}$$

imply $(\omega''', \sigma^+(\omega''')) \in U$. Now take ω''' for which (125) holds, and the right tail starting from the $k^* + 1$ position is preassigned, equal to ω_0^+. For this ω''' (substituted instead of ω'') (122) holds. This proves Statement 2 of Proposition 8.

Let us now switch to the proof of Proposition 9.

10.2. Idea and the first step of the proof

In the statement of Proposition 9, let $p = (\omega, \zeta) \in A_{\max}^E$. By assumption, $\omega \in E$, that is, satisfies the hypothesis of Lemma 12. Hence, by this lemma, there exists

$$\omega' \in W_{(1)}^u$$

such that

$$(\omega', \zeta) \in U \cap A_{\max}^E.$$

Without loss of generality, we may now consider U to be a neighborhood of the point (ω', ζ) above.

So in assumptions of Proposition 9 we have two sequences: ω_0, with the right tail of zeros, and ω', with the left tail of ones. Let

$$\omega_0 = \omega_0^- | w^+ 0 \ldots 0 \ldots$$

and

$$\omega' = \dots 1 \dots 1 w^- | \omega^+.$$

Here ω_0^- and ω^+ are the left and the right tails of ω_0 and ω' starting with positions -1 and 0 respectively, w^+ and w^- are finite words.

Given two sequences ω_0, ω', and the neighborhood U, we find a number κ, as required in Proposition 9. After that we will fix an interval $\Omega_0 \subset \mathbb{R}^-$, $|\Omega_0| < \kappa$. Given $\Omega_0, \omega_0, \omega'$ and U, we will construct the sequence ω_1. In order to do that, we will glue together the left part of ω_0 and the right part of ω':

$$\omega_1 = \omega_0^- | w^+ \underbrace{0 \dots 0}_{m_0} \underbrace{1 \dots 1}_{m_1} w^- \omega^+. \tag{126}$$

Let

$$|w^-| = s, \qquad |w^+| = t, \qquad T = t + m_0 + m_1 + s.$$

The only thing to do is to find appropriate values of m_0 and m_1 in such a way that (124) holds with ω_0, Ω_0, U preassigned, and ω_1, T constructed.

The important feature of the sequence ω_1 is the following: various segments of ω_1 resemble some segments of the sequences of ω_0, ω' and

$$\omega^{01} = \dots 0 \dots 0 | 1 \dots 1 \dots.$$

So fiber maps of some iterates of G over ω_1 resemble the fiber maps of the same iterates over properly shifted sequences $\omega_0, \omega^{01}, \omega'$. This proximity follows from Distortion Lemma 1. This will allow us to obtain (124).

Let us now pass to detailed constructions.

10.3. Choice of κ and ω_1

Step 1. *Choice of κ.*

Let us begin with notations.

For any interval V and any $\delta > 0$ denote by V^δ and $V^{-\delta}$ the following intervals: V^δ is the δ-neighborhood of V in the ζ-chart; $V^{-\delta}$ is an interval whose δ-neighborhood in the chart ζ coincides with V. For any V the set $V^{-\delta}$ is nonempty for δ small enough. By default, all the lengths below are taken in the ζ-chart:

$$|V| = |V|_\zeta = |\zeta(V)|.$$

Let $p = (\omega, \zeta)$ be the same as in Proposition 9. The neighborhood U of p may be chosen in the form

$$U = C_w \times V,$$

where w is a subword of ω, and V a neighborhood of ζ, $V \in \mathbb{R}^-$.

Let $\delta_0 = \frac{|V|}{4}$, and

$$V_m = \pi \circ G^{-(s+m)}(\{w'\} \times V^{-\delta_0}). \tag{127}$$

In other words,

$$\tilde{g}_{\sigma^{-(m+s)}\omega', m+s}(V_m) = V^{-\delta_0}.$$

The sequence ω' has a left tail of ones. Hence, Distortion Lemma 3 is applicable. By Statement 3 of this lemma, there exists a limit:

$$v_\infty = \lim_{n\to\infty} |V_m|.$$

Take

$$\varepsilon = \frac{v_\infty}{4}.$$

Let m_* be such that

$$|V_m| > 2\varepsilon \quad \forall m > m_*. \tag{128}$$

Now consider the sequence ω_0. It has a right tail of zeros. Hence, Distortion Lemma 3 is applicable. By Statement 4 of this lemma, there exists C such that for any interval $\Omega_0 \subset \mathbb{R}^-$, any $n > 0$, and

$$\Omega_n = \pi \circ G^{t+n}(\{\omega_0\} \times \Omega_0) \tag{129}$$

we have:

$$|\Omega_n| < C|\Omega_0|.$$

Let us now take $\kappa = \frac{\varepsilon}{16C}$. This completes the choice of κ depending on U in the proof of Proposition 9.

Step 2. *Plan of the further construction.*

For κ chosen above, take arbitrary interval $\Omega_0 \subset \mathbb{R}^-$, $|\Omega_0| \le \kappa$.

In Steps 3 and 4, we will find m_0, m_1 in the form: $m_0 = n + k$, $m_1 = l + m$. In (124) we will take $T = t + m_0 + m_1 + s$. Let

$$\delta = \min\left(\delta_0, \frac{\varepsilon}{8}\right).$$

We will construct intermediate intervals Ω and W such that :

$$\pi \circ G^{t+n}(\{\omega_0\} \times \Omega_0) = \Omega^{-\delta}, \tag{130}$$

$$\pi \circ G^{k+l}(\{\sigma^{-k}\omega^{01}\} \times \Omega) \subset W^{-\delta}, \tag{131}$$

$$\pi \circ G^{m+s}(\{\sigma^{-(m+s)}\omega'\} \times W) = V^{-\delta}. \tag{132}$$

We will take m, n, k, l so large that in (55), $\Delta_{k+l} < \delta$,

$$\Delta_m < \delta, \qquad \Delta_n < \delta \tag{133}$$

and

$$|h - id| < \delta \quad \text{on } V_m. \tag{134}$$

Here, as before, h is the transition function between the charts η and ζ: $\eta = h(\zeta)$. By (46) and the first statement of Distortion Lemma 3, there exists m_* such that for any $m > m_*$, (134) holds.

Let N be the constant from Lemma 11, and $s = |w^-|$, $t = |w^+|$, as before. We will chose m, n for which:

$$m > \max(m_*, N, s), \qquad n > 10t. \tag{135}$$

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Some other assumptions on m, n, k, l will be stated later.

Step 3. *Going down; rough tuning.*

Proposition 10. *Let ω_0 and ω' be two sequences, the first one with the right tail of zeros, the second one with the left tail of ones. Let p_0 be an arbitrary point in $(0, 1)$, $q_0 \in (0, \sigma^+(\omega'))$. Then there exist arbitrary large n and m such that*

$$|\tilde{g}_{\omega_0,n}(p_0) - (\tilde{g}_{\omega',m})^{-1}(q_0)| < 1, \quad n \in \left[\frac{m}{3}, 3m\right]. \tag{136}$$

Proof. This proposition is an immediate corollary of the first two statements of Lemma 10 and Assumption 6 of Section 2.1. □

We will apply this proposition to p_0, q_0 that are the middle points of Ω_0 in the ζ-chart and of $V^{-\delta}$ in the η-chart respectively. Let us choose m, n such that (136), (135), (134) and (133) hold. Moreover, take m so large that for l, k satisfying (137) below, the following holds: $l > l_{j_0}, k > k_{j_0}$. Here j_0, l_{j_0}, k_{j_0} are the same as in Lemma 11.

Fix these m and n. Let $\Omega^{-\delta}$ be the same as in (130). Let W be the same as in (132). After the choice of m and n, it is well defined. These definitions end up the rough tuning.

Step 4. *Fine tuning: choice of k and l.*

The neighborhoods Ω and $W^{-\delta}$ satisfy all the assumptions of Lemma 11: dist $(\Omega, W^{-\delta}) < 2$ by (136), $|W^\delta| > \varepsilon$, $|\Omega| < \frac{\varepsilon}{4}$. Recall that we work in the chart ζ, hence identify Ω and $\zeta(\Omega)$.

By Lemma 11, there exist k and l such that

$$l \in [m, m + N), \qquad k \in \left[\frac{l}{3}, 3l\right], \tag{137}$$

and (131) holds. The choice of k and l is over. Now take $m_0 = n + k$, $m_1 = l + m$ in (126). The construction of ω_1 given by (126) is over. The only remaining part is to prove (124).

Note that

$$\Delta_{k+l} < \delta \tag{138}$$

because $k + l > m$, and (133) holds.

10.4. Proof of the main inclusion

Inclusion (124) follows from Distortion Lemma 1, applied three times. We have to replace $\omega^0, \omega^{01}, \omega'$ in (130), (131), (132) by appropriate shifts of ω_1, replace $\Omega^{-\delta}, W^{-\delta}, V^{-\delta}$ by Ω, W, V respectively, and prove the resulting inclusions. This will be done in three steps.

First inclusion

$$\pi \circ G^{t+n}(\{\sigma^{-T}\omega_1\} \times \Omega_0) \subset \Omega. \tag{139}$$

This follows from Distortion Lemma 1 applied to two sequences:

$$\omega_0 = \omega_0^-|w^+\underbrace{0\ldots0}_{n}\underbrace{0\ldots0}_{k}\ldots$$

$$\sigma^{-T}\omega_1 = \omega_0^-|w^+\underbrace{0\ldots0}_{n}\underbrace{0\ldots0}_{k}\ldots.$$

Note that for these sequences

$$\omega_0 \,|_{-\infty}^{t+n+k} = \sigma^{-T} \omega_1 \,|_{-\infty}^{t+n+k} .$$

Let us prove that

$$\| \tilde{g}_{\omega_0, t+n} - \tilde{g}_{\sigma^{-T}\omega_1, t+n} \| < \delta. \tag{140}$$

All the norms below are taken in $C(\mathbb{R}^-)$. Together with (130), (133) this will imply (139). For two maps considered in (140), the acting word has length $t+n$, the right and left marginal words have lengths k and infinity respectively. The inequality: $k > \gamma(n+t)$ holds, $\gamma = \frac{1}{30}$. Indeed, by (137), (135) and (136),

$$k \geq \frac{l}{3} \geq \frac{m}{3} \geq \frac{n}{9} \geq \frac{n+t}{10} \Rightarrow k > \frac{n+t}{30}.$$

Therefore, Distortion Lemma 1 is applicable, it implies (140), hence, (139).

Second inclusion

We want to prove that

$$\pi \circ G^{k+l}(\{\sigma^{-T(k+m_1+s)}\omega_1\} \times \Omega_0) \subset W. \tag{141}$$

This follows from Distortion Lemma 1 applied to two sequences:

$$\sigma^{-k}\omega^{01} = \ldots 0 \underbrace{0\ldots0}_{n} | \underbrace{0\ldots0}_{k} \underbrace{1\ldots1}_{l} 1 \ldots,$$

$$\sigma^{-k+l+m+s}\omega_1 = \omega_0^- w^+ \underbrace{0\ldots0}_{n} | \underbrace{0\ldots0}_{k} \underbrace{1\ldots1}_{l} \underbrace{1\ldots1}_{m} w^- * .$$

For these sequences

$$\sigma^{-k}\omega^{01} \,|_{-n}^{k+l+m} = \sigma^{-(k+l+m+s)}\omega_1 \,|_{-n}^{k+l+m} .$$

Let us prove that

$$\| \tilde{g}_{\sigma^{-k}\omega^{0,1}, k+l} - \tilde{g}_{\sigma^{-(k+l+m+s)}\omega_1, k+l} \| < \delta. \tag{142}$$

Together with (131) and (138) this will imply (141). For two maps considered in (142), the acting word is of length $k+l$, and the marginal words are of lengths m and n respectively. We have to check that

$$m \geq \gamma(k+l), \qquad n \geq \gamma(k+l), \tag{143}$$

then apply (55). We have: $l \leq m+N$, $k \leq 3l$, $m \geq N$, $\gamma = \frac{1}{30}$. Hence,

$$\gamma(k+l) \leq 4\gamma(m+N) \leq 8\gamma m < m.$$

On the other hand, $n > \frac{m}{3}$. Hence,

$$\gamma(k+l) \leq 24\gamma n < n.$$

Together with Distortion Lemma 1, this implies (142), hence, (141).

Third inclusion

We want to prove that

$$\pi \circ G^{m+s}(\{\sigma^{-(m+s)}\omega_1\} \times W) \subset V. \tag{144}$$

This follows from Distortion Lemma 1 applied to two sequences:

$$\sigma^{-(m+s)}\omega' = \dots 1\,\underbrace{1\dots 1}_{l}\,|\,\underbrace{1\dots 1}_{m}\,\underbrace{w^-}_{s}\,*,$$

$$\sigma^{-(m+s)}\omega_1 = *\,\underbrace{1\dots 1}_{l}\,|\,\underbrace{1\dots 1}_{m}\,\underbrace{w^-}_{s}\,*.$$

For these sequences

$$\sigma^{-(m+s)}\omega' \,|_{-l}^{\infty} = \sigma^{-(m+s)}\omega_1 \,|_{-l}^{\infty}.$$

Let us prove that

$$\|\tilde{g}_{\sigma^{-(m+s)}\omega', m+s} - \tilde{g}_{\sigma^{-(m+s)}\omega_1, m+s}\| < \delta. \tag{145}$$

Together with (132) and (133) this will imply (144). For two maps considered in (145), the acting word is of length $m+s$, and the marginal words are of lengths l and ∞. Let us check that

$$l \geq \gamma(m+s).$$

Indeed, $l \geq m \geq s, \gamma > \frac{1}{2}$. Hence, Distortion Lemma 1 is applicable and implies (145).

Composition

Note that

$$G^T = G^{m+s} \circ G^{k+l} \circ G^{n+t}.$$

Hence (139), (141), (144) imply (124). This proves Proposition 9.

10.5. From one sequence to a set of positive measure

The arguments of this section resemble those of 5.2.

Begin with some preparations. Denote by Σ_C^- (Σ_C^+) the set of all C-admissible left (respectively, right) one sided sequences.

Consider the probabilistic measure P^- (P^+) on Σ_C^-, (respectively, Σ_C^+) determined by Eq. (50) with the additional assumption: in the definition of C_m, $n+m < 0$ ($n \geq 0$ respectively). Denote $S = \Sigma_C^- \times \Sigma_C^+$, and let $P_S = P^- \times P^+$ be the probability measure on it.

Remark 4. Consider a subset $S' \subset S$ of positive P_S-measure. Fix any C-admissible word w. Then the set of sequences

$$\Sigma(w) = \{\omega \in \Sigma_C \mid \omega = \omega_0^- w \omega_0^+, (\omega_0^-, \omega_0^+) \in S'\}$$

has the positive P-measure. Indeed, the measure P on $\Sigma(w)$ is a pullback of the measure P_S on S under the natural map $\omega_0^- w \omega_0^+ \mapsto (\omega_0^-, \omega_0^+)$ up to a multiplication by a positive number that depends on the word w.

Now let us pass to the proof of the third statement of Proposition 8. The first two statements of the proposition, already proved, imply that for any C-admissible word w_0, any $(\omega_0^-, \omega_0^+) \in S$ and any neighborhood of the form $U = C_{w_0} \times V$ that intersects $A_{\max}^E$, there exists a sequence $\tilde{\omega} = \omega_0^- w \omega_0^+$ such that

$$(\tilde{\omega}, \sigma^+(\tilde{\omega})) \in U, \tag{146}$$

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the word $w = w(\omega_0^-, \omega_0^+)$ extends w_0:

$$w = *w_0 * .$$

For any C-admissible word w, denote by $S(w)$ the set $S(w) = \{(\omega_0^-, \omega_0^+)|w(\omega_0^-, \omega_0^+) = w\}$. The union of all the sets $S(w)$ over all the finite words w coincides with S. Hence, at least one of these sets has the positive P_S measure. By the remark above, the corresponding set

$$\Sigma(w) = \{\tilde{\omega} = \omega_0^- w \omega_0^+ |(\omega_0^-, \omega_0^+) \in S_w\}$$

has the positive P-measure. For all such sequences $\tilde{\omega}$ inclusion (146) holds.

This concludes the proof of Proposition 8, hence, Lemma 13.

10.6. Topologically mixing property of maximal attractors of skew products of class TAT

Here we deduce Lemma 14 from Proposition 9. The key argument is the following lemma.

Lemma 15. *Maximal attractor of a diffeomorphism $\mathcal{G}$ of class TAT is saturated by strongly unstable leaves of this diffeomorphism.*

Proof. Say that a point p has a complete orbit under a map $\mathcal{G}$ if all the iterates $\mathcal{G}^k(p)$, $k \in \mathbb{Z}$, are well defined. It is well known that the maximal attractor in its absorbing domain is the union of all the complete orbits lying in this domain. If a point of a partially hyperbolic diffeomorphism in an absorbing domain has a complete orbit, then all the points in its strongly unstable leaf also have complete orbits. $\square$

Remark 5. Strongly unstable leaves mentioned in Lemma 15 have bounded slopes by the partial hyperbolicity of $\mathcal{G}$.

Proof of Lemma 14. Let G be a smoothly generated skew product of class TAT. That is, G is topologically conjugated to $\mathcal{G}$ by a homeomorphism H, see (14). The maximal attractor of G is saturated by the images of strongly unstable leaves of $\mathcal{G}$. We call them strongly unstable fibers of G. These images are graphs of functions that bring strongly unstable leaves of the Markov shift in the base into the fiber I. These functions depend on the initial condition as a parameter. They are uniformly continuous both with respect to the metric in the base, and to the parameter.

For $q = (\omega_0, \zeta_0)$ denote by $W^u = W^u_{\omega_0}$ the unstable leaf of the map of the base passing through ω_0. Let i_q be a map $W^u \to I$ whose graph is the strongly unstable fiber of G passing through q.

Let now U and U' be two open sets whose intersections with $A^E_{\max}$ are non-empty. We need to find $r \in A^E_{\max} \cap U'$ and $T > 0$ such that

$$G^T(r) \in A^E_{\max} \cap U.$$

It is sufficient to establish that $G^T(r) \in U$, because $A^E_{\max}$ is G-invariant.

Let us take a point q' in a nonempty intersection $A^E_{\max} \cap U'$, $q' = (\omega', \zeta')$. Then, by Distortion Lemma 2, there exists $q = (\omega_0, \zeta') \in U'$ such that $\omega_0 \in W^s_{(0)}$. Indeed, $\sigma^+(\omega') > \zeta'$. By Distortion Lemma 2, $\omega_0 \in W^s_{(0)} \cap E$ may be found so close to ω' that still $\sigma^+(\omega_0) > \zeta'$. This proves the existence of q required.

Let us take κ from Proposition 9, and let Ω_0 be the $\frac{\kappa}{2}$ neighborhood of ζ'. Note that for any $t \geq 0$ and ω_1 that satisfy (123), we have: $\omega_1 \in W^u_{\omega_0}$, because the left tails of ω_0 and ω_1 coincide.

Hence, i_q is well defined at ω_1. Take t so large that for any ω_1 that satisfy (123), the following holds:

$$|i_q(\omega_1) - i_q(\omega_0)| \leq \frac{\kappa}{2}. \tag{147}$$

The existence of such t follows from the uniform continuity of the function i_q mentioned above.

Let us now take ω_1 and T for these Ω_0 and t such that (123), (124) hold. We will take $r = (\omega_1, i_q(\omega_1))$. By (147), $r \in \{\omega_1\} \times \Omega_0$. By (124), $G^T(r) \in U$. By Lemma 15, $r \in A^E_{\max}$. This proves Lemma 14. □

The lemma, in turn, implies Theorem 4, hence 2 and 3.

We are now prepared to gain thick attractors for small perturbations of skew products.

11. Perturbations

In this section we deduce Theorem 1 from Theorem 4. Namely, we consider a skew product of class TAT whose fiber maps are C^3-close to identity. A small neighborhood of this diffeomorphism with a countable number of hypersurfaces deleted will be the quasiopen set mentioned in Theorem 1.

In two phrases, the strategy of the proof is the following. First, we prove that the perturbed map $\mathcal{G}$ is topologically equivalent to a Hölder skew product G of class TAT, whose maximal attractor has a transitive invariant subset $A^E_{\max}(G)$ of positive Lebesgue measure. Second, we prove that the set

$$A^E_{\max}(\mathcal{G}) = H(A^E_{\max}(G)),$$

where H is the conjugacy mentioned above, has positive Lebesgue measure though H may be not absolutely continuous; note that the topologically mixing property is preserved by H.

Let us give more details. In Section 11.1 we prove the conjugacy of $\mathcal{G}$ to a Hölder skew product G. In Section 11.2 we check that G is of class TAT. Most part of assumptions in the definition of class TAT persist under small perturbations. The most tricky is Assumption 7. After perturbation, it transforms to Assumption 7′.

By Theorem 4, $A^E_{\max}(G)$ has positive measure. To prove the same for $A^E_{\max}(\mathcal{G})$, we use special ergodic theorem [23], Falconer Lemma [6] and Anosov and Pesin Theorems about absolute continuity. This is done in Sections 11.3 and 11.4.

11.1. Hölder continuity of central fibers for boundary preserving maps

Consider a partially hyperbolic skew product $F : B \times I \to B \times I$ whose central fibers satisfy the dominated splitting condition with the exponent r. We will not recall this condition here. We only mention that it follows from the proximity of the fiber maps to the identity, and implies the C^r-smoothness of the central fibers of the perturbed map. Let us embed I in S^1, X in $\tilde{X} = B \times S^1$. Let us continue the fiber maps from X to $\tilde{X}$ as C^r-smooth functions. Denote the extended fiber maps by $\bar{f}_b$. Under this extension, the dominated splitting condition mentioned above may be preserved.

Denote the extended map by $\mathcal{F}$ and let $\mathcal{G}$ be its C^r-small perturbation. Suppose that under this perturbation, $\mathcal{G}(A_j) = A_j$; recall that $A_j = B \times \{j\}$, $j = 0, 1$.

By [11], the perturbed map has an invariant central foliation with the leaves diffeomorphic to circles, and the following diagram commutes:

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\mathcal{G}} & \hat{X} \\ \downarrow p & & \downarrow p. \\ B & \xrightarrow{h} & B \end{array}$$

The fibers of p are C^r-smooth and are graphs of functions:

$$M_b = P^{-1}(b) = \text{graph}\ (\beta_b : S^1 \to B).$$

Moreover,

$$\|\beta_b\|_{C^r} \to 0 \quad \text{as } \|\mathcal{F} - \mathcal{G}\|_{C^r} \to 0. \tag{148}$$

Theorem 7 ([10]). *The maps β_b are Hölder continuous in the C^r-norm:*

$$\|\beta_b - \beta_{b'}\|_{C^r} \le C|b - b'|^{\alpha_r}. \tag{149}$$

Theorem 8 ([16]). *For C^r-small perturbation $\mathcal{G}$ of $\mathcal{F}$, $r \ge 2$, and for h being the Anosov map of $B = \mathbb{T}^2$, the Hölder exponent above is close to* 1 *in the C-norm:*

$$\|\beta_b - \beta_{b'}\|_C \le C|b - b'|^{\alpha};$$

the smaller the perturbation is, the smaller is $1 - \alpha$.

11.2. Perturbed maps as Hölder skew products of class TAT

For the same $\tilde{X}, \mathcal{G}$ and the family β_b, as in the previous subsection, consider a homeomorphism

$$H : \tilde{X} \to \tilde{X}, \qquad (b, x) \mapsto (\beta_b(x), x).$$

The map H^{-1} rectifies the central fibers M_b of $\mathcal{G}$: $H^{-1}(M_b) = \{b\} \times S^1$. Consider a skew product

$$G : X \to X, \qquad (b, x) \mapsto H^{-1} \circ \mathcal{G} \circ H|_X,$$

see (14). By (149), this is a boundary preserving Hölder skew product. After constructing the skew product G, we turn back to X, and abandon $\tilde{X}$.

Let us now check that G is a Hölder skew product of class TAT, that is, let us check Assumptions 2–6 of Section 2.1 and $1'$, $7'$ of Section 2.3.

Suppose that $\mathcal{G}$ is a C^3-small perturbation of $\mathcal{F}$:

$$\|\mathcal{F} - \mathcal{G}\|_{C^3} \le \rho.$$

Then G will be also a C^3-small perturbation of $\mathcal{F}$:

$$\|f_b - g_b\|_{C^3} \le C(\rho) \to 0 \quad \text{as } \rho \to 0. \tag{150}$$

The maps g_b are Hölder in b. Together with (150), this proves that Assumptions 2–6 of Section 2.1 hold for G as soon as they hold for $\mathcal{F}$, provided that ρ is small enough.

Assumption $1'$ requires that the fiber maps g_b have Lipschitz constant close to 1. This also holds for G if it holds for $\mathcal{F}$ and ρ is small.

Let us now check Assumption 7′. The map $\mathcal{G}$ has a fixed point $P^+(\mathcal{G})$ close to P^+ of $\mathcal{F}$, see Section 9. Consider the lifted map of $\hat{\mathcal{G}} : \hat{X} \to \hat{X}$. Recall that $\hat{X} = \mathbb{R}^2 \times I$. The lifted map $\hat{\mathcal{G}}$ has a fixed point over $P^+(\mathcal{G})$, a special lift of $P^+(\mathcal{G})$, that we denote by $P^+(\hat{\mathcal{G}})$.

Let $W^u_{P^+(\hat{\mathcal{G}})}$ be its one-dimensional unstable manifold. For any $p \in W^u_{P^+(\hat{\mathcal{G}})}$ let $W^s_p(\hat{\mathcal{G}})$ be its strongly stable manifold. Let

$$W(\hat{\mathcal{G}}) = \cup_{p \in W^u_{P^+(\hat{\mathcal{G}})}} W^s_p(\hat{\mathcal{G}}).$$

Consider the corresponding surface $W(\hat{G}) = H^{-1}(W(\hat{\mathcal{G}}))$. We will prove that it is a cover surface mentioned in Assumption 7′, see 2.3.

Let us prove that $W(\hat{G})$ is a graph of a function $i(\hat{G}) : \mathbb{R}^2 \to I$. Indeed, center stable fibers of $\mathcal{G}$ have dimension two and are smooth. The map $\mathcal{G}$ is semiconjugated to A: the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathcal{G}} & X \\ \downarrow p & & \downarrow p \\ \mathbb{T}^2 & \xrightarrow{A} & \mathbb{T}^2 \end{array}$$

commutes. Here p is the projection along the central fibers of the perturbed map $\mathcal{G}$.

The diagram above may be lifted to $\hat{X}$; $\hat{p}$ is the lift of p: the diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{\mathcal{G}}} & \hat{X} \\ \downarrow \hat{p} & & \downarrow \hat{p} \\ \mathbb{R}^2 & \xrightarrow{\hat{A}} & \mathbb{R}^2 \end{array}$$

commutes.

Note that $W(\hat{\mathcal{G}})$ is a continuous submanifold of $\hat{X}$. We use here the fact that strongly stable leafs depend continuously on the initial condition, hence the surface W is continuous. On the central-stable leaves of $\hat{\mathcal{G}}$, the central fibers are transversal to the strongly stable ones. Hence, $\hat{p}$ is a local homeomorphism of $W(\hat{\mathcal{G}})$ to $\mathbb{R}^2$. Hence, it is a global homeomorphism.

Let $W(\hat{G}) = H^{-1}W(\hat{\mathcal{G}})$. Then, the projection of $W(\hat{G})$ to $\mathbb{R}^2$ along the second factor of the product $\bar{X} = \mathbb{R}^2 \times I$ is a homeomorphism too. Hence, $W(\hat{G})$ is a graph of a function $i = i(\hat{G}) : \mathbb{R}^2 \to I$ indeed.

The angle between strongly stable and central fibers is bounded from zero. Hence, the function i is absolutely continuous and even Lipschitz above the stable fibers of h. This proves that the function i is defined and absolutely continuous as required in Assumption 7′.

Let us check the requirement (17) of Assumption 7′. The restriction of $\hat{G}$ to the central-unstable invariant manifold of P^+ is smooth. Hence, the Grobman–Hartman theorem may be applied to this restriction. In the very same way as in the proof of Proposition 7, we obtain that $i|_{W^u} = \sigma^+|_{W^u}$.

Let us check (16). Consider the number m chosen for $\hat{\mathcal{F}}$ in (113), (112). Denote by $i(\hat{\mathcal{F}})$ the function i from 9.2. The function $i(\hat{G}) : \mathbb{R}^2 \to I$ depends continuously on $\hat{\mathcal{G}}$ in a sense that it is close to $i(\hat{\mathcal{F}})$ if the perturbation $\hat{\mathcal{G}} - \hat{\mathcal{F}}$ is small in C^3. Hence, (112), thus (16), holds for $i(\hat{G})$. A neighborhood D is $\cup_{-m}^{\infty} \hat{A}^l U_{w_m}$.

This implies Assumption 7′ for G. Hence, Theorem 4 is applicable. It claims that the Milnor attractor of G is thick: $m_3(A_M(G)) > 0$.

We have to prove that

$$m_3(A_M(\mathcal{G})) > 0. \tag{151}$$

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We have: $A_M(\mathcal{G}) = H(A_M(G))$. But H is not, in general, absolutely continuous. Hence, (151) does not immediately follow from Theorem 4 applied to G. We need more arguments presented in the next two subsections.

11.3. Zeros of the boundary function

As mentioned before, the skew product G is still of class TAT.

Hence, Theorem 4 is applicable to G. Let $m = m(G)$ and $E = E(G)$ be the same as at the beginning of Section 10.1. Theorem 4 claims that the Milnor attractor of G is thick. It follows from (121) that

$$\mathrm{Cl}(\Gamma \cap A_M(G)) \supset A_{\max}^E = (E \times I) \cap A_{\max}(G).$$

Moreover, $\mathrm{mes}_3 A_{\max}^E > 0$.

Let $K_0 = \{\sigma^+ = 0\}$, $K_1 = \mathbb{T}^2 \setminus E$, $K = K_0 \cup K_1$. Let $H_0 = H|_{\mathbb{T}^2 \times \{0\}}$.

Lemma 16. *The set $H_0(K) \subset \mathbb{T}^2$ has Lebesgue measure zero.*

Proof. Let us first prove that $\dim_H(K_0) < 2$. Let $\varphi(b) = g_b'(0)$. Then φ is continuous and close to $\tilde{\varphi}(b) = f_b'(0)$. Hence, $\int \varphi > 0$. Lemma 1 implies that

$$K_0 \subset K' := \left\{ b \mid \liminf \frac{1}{n} \sum_0^{n-1} \varphi \circ A^{-k}(b) \le 0 \right\}. \tag{152}$$

By the *special ergodic theorem*, [23],

$$\dim_H K_0 = d < 2.$$

By [16], the map H_0 is Hölder with an exponent α close to one. More precisely, one can take the perturbation so small that $\alpha > \frac{d}{2}$. By the Falconer lemma, [6],

$$\dim_H H_0(K_0) \le \frac{d}{\alpha} < 2.$$

Hence,

$$\mathrm{mes}_2 H_0(K_0) = 0. \tag{153}$$

Consider now the set K_1. From the definition of the set E, it easily follows that

$$\dim_H(K_1) < 2.$$

By the same reason,

$$\mathrm{mes}_2 H_0(K_1) = 0.$$

This proves the lemma. □

Remark 6. By the same tools we may prove that the saturation of the set $H_0(K)$ by the central fibers of the perturbed map $\mathcal{G}$ has m_3-measure zero. We will not use this statement, but we wish to mention it here.

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11.4. Positive measure of the maximal attractor

Here we prove that the maximal attractor of the perturbed map $\mathcal{G}$ has positive Lebesgue measure.

Lemma 17. *Let $\mathcal{G}$ be a small perturbation of a skew product diffeomorphism of class TAT, and let $K \subset \mathbb{T}^2$ be the set defined in the previous subsection. Then $H_0(K)$ is saturated by strongly unstable manifolds of $\mathcal{G}$.*

Proof. Note first that $A_0 = \mathbb{T}^2 \times \{0\}$ is invariant under $\mathcal{G}$, and the restriction $\mathcal{G}_0 = \mathcal{G}|_{A_0}$ is hyperbolic and close to A. The homeomorphism H_0 conjugates the maps A and $\mathcal{G}_0$. Hence, it brings the unstable fibers of the first maps to those of the second one. Hence, it is sufficient to prove that the set K is saturated by the unstable leaves of A. This holds for K_1 by definition. For K_0 this follows from (152). $\square$

Corollary 1. *Let $H_0(K)$ be the same subset of $\mathbb{T}^2$ as in Lemma 16. Then every cross-section γ transversal to the unstable direction of $\mathcal{G}_0$ in A_0 (lower part of the boundary ∂X) intersects the set $\mathbb{T}^2 \setminus H_0(K)$ by a subset of full Lebesgue measure m_1.*

Proof. By Lemma 16, $m_2(H_0(K)) = 0$. Hence, for any open set $U \subset \mathbb{T}^2$,

$$m_2(U \setminus H_0(K)) = m_2(U).$$

Note that, by Lemma 17, the set $U \setminus H_0(K)$ is saturated by the unstable manifolds of the hyperbolic map $\mathcal{G}_0$. But the holonomy along the unstable manifolds of $\mathcal{G}_0$ is absolutely continuous, by a renowned Anosov theorem. If we suppose that $m_1(\gamma \cap K) > 0$, then we conclude, by the Fubini theorem, that $m_2(U \cap H_0(K)) > 0$, a contradiction. $\square$

Lemma 18. *The maximal attractor of a slightly perturbed skew product diffeomorphism of class TAT has positive measure. More precisely,*

$$m_3(H(A^E_{\max}(G))) > 0.$$

Proof. The lower boundary A_0 of X is an invariant closed set of a partially hyperbolic map $\mathcal{G}$, and has nonzero Lyapunov exponents for m_2 almost all orbits. Hence, the Pesin theory is applicable. Consider the same γ and K as in the previous lemma. Near $\gamma \setminus H_0(K)$, the set $H(A^E_{\max}(G))$ is a saturation of $\gamma \setminus H_0(K)$ by the two-dimensional local unstable manifolds of $\mathcal{G}$. These manifolds are pieces of center-unstable manifolds of $\mathcal{G}$ passing through the points of the lower boundary A_0. The holonomy along these manifolds is absolutely continuous, by the renowned Pesin theorem. By Fubini theorem again, the set $H(A^E_{\max}(G)) \subset A_{\max}(\mathcal{G})$ has a positive Lebesgue measure m_3, because $m_1(\gamma \setminus K) > 0$. This proves the lemma. $\square$

In Section 10 we completed the proof of topologically mixing property of the maximal attractor of a Hölder skew product of class TAT. This is a topological property. It is preserved by the homeomorphism H. The image of this homeomorphism, the maximal attractor of the perturbed map $\mathcal{G}$, has therefore topologically mixing property too.

We proved that this attractor is thick. This completes the deduction of the Main Theorem from Theorem 4; the latter one is proved modulo Proposition 3.

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Appendix. A priori estimates of displacements

Here we prove Proposition 3. This proposition is sort of an a priori estimate. The problem is that we know almost nothing about $g_{\omega^{01}}$. This problem is resolved in the following way. In the composition $\tilde{g}_{\sigma^{-K}\omega^{01},K+L}$, the fiber maps $\tilde{g}_{\sigma^{-k}\omega^{01}}$ are in a sense close to $\tilde{f}_0$ for $k > k^*$,where k^* is large enough. Similarly, the maps $\hat{g}_{\sigma^k\omega^{01}}$ are close to $\hat{f}_1$ for $k > k^*$. The "middle" maps $\tilde{g}_{\sigma^k\omega^{01}}$ for $|k| \le k^*$ are not under control. This means that we cannot distinguish them from other fiber maps. But they are subject for restrictions that hold for any fiber map. This is the strategy of the proof below.

Recall that k_0 is the number of symbols in the sequences of the space Σ_C of the Markov chain in the base.

A.1. *Estimate of the multipliers*

Proof of Proposition 3. Let us prove the first statement. For any $k > 0, l > 0$ we have:

$$d_{\Sigma_C}(\sigma^{-k}\omega^{01}, (0)) \le C_0 k_0^{-k} \tag{154}$$

$$d_{\Sigma_C}(\sigma^{l}\omega^{01}, (1)) \le C_0 k_0^{-l}. \tag{155}$$

Hence, by (15),

$$\left|\log g'_{\sigma^{-k}\omega^{01}}(0) - \tilde{\lambda}\right| \le C_1 k_0^{-\beta k}$$

$$\left|\log g'_{\sigma^{l}\omega^{01}}(0) - \tilde{\mu}\right| \le C_1 k_0^{-\beta l}$$

for some $\beta > 0$, $C_1 > 0$. By the chain rule,

$$\left|\log g'_{\sigma^{-K}\omega^{01},K+L}(0) - \left(K\tilde{\lambda} + L\tilde{\mu}\right)\right| \le C_1\left(\Sigma_0^K k_0^{-\beta k} + \Sigma_0^L k_0^{-\beta l}\right) \le \frac{2C_1}{1-k_0^{-\beta}} := M.$$

Together with assumption (79), this proves (80), hence, statement 1 of Proposition 3. □

A.2. *Comparison with translations*

Let us now prove the second statement. The proposition below provides the first uniform estimate we need.

Proposition 11. *There exist K_0, L_0, D such that for $K > K_0$, $L > L_0$*

$$\|\tilde{g}_{\sigma^{-K}\omega^{01},K} - T_{K\tilde{\lambda}}\|_{C(\mathcal{L}(a'))} < D, \tag{156}$$

$$\|\left(\hat{g}_{\omega^{01},L}\right)^{-1} + T_{L\mu}\|_{C(\mathcal{L}'(a'))} < D. \tag{157}$$

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Proof. Denote by $g_k, \tilde{g}_k$ and $\hat{g}_k$ the fiber map $g_{\sigma^{-K+k}\omega^{01}}$ written in the charts x, ζ and η respectively. In these notations,

$$g_{\sigma^{-K}\omega^{01},K} = g_{K-1} \cdots \circ g_0, \qquad g_{\omega^{01},L} = g_{K+L-1} \circ \cdots \circ g_K.$$

Note that the maps $\tilde{g}_k$ for $k \ll K$ are exponentially close to $\tilde{f}_0$, that is to the translation $T_{\tilde{\lambda}}$ on $\Lambda(a')$. The maps $\hat{g}_l$ for $l \gg K$ are exponentially close to $T_{\tilde{\mu}}$ on $\Lambda'(a')$. For k, l close to K we know only that the corresponding fiber maps have bounded displacement

$$\|\tilde{g}_k - id\|_{C(\mathcal{L}(a'))} \le d, \qquad \|\hat{g}_l - id\|_{C(\mathcal{L}'(a'))} \le d. \tag{158}$$

In fact, this holds for all $k, l = -K, \ldots, L-1$.

Let us now take k_* in such a way that for $0 \le k \le K - k_*$, $K + k_* \le l < K + L$,

$$\|\tilde{g}_k - T_{\tilde{\lambda}}\|_{C(\mathcal{L}(a'))} \le \frac{|\tilde{\lambda}|}{2}, \qquad \|\hat{g}_l - T_{\tilde{\mu}}\|_{C(\mathcal{L}'(a'))} \le \frac{|\tilde{\mu}|}{2}. \tag{159}$$

Take K_0, L_0 so large, that

$$(K_0 - k_*)\,\tilde{\lambda} + k_* d < 0, \qquad (k_* - L_0)\,\tilde{\mu} + k_* d < 0.$$

Then for any $K > K_0$, $L > L_0$, we have:

$$\tilde{g}_{K-1} \circ \cdots \circ \tilde{g}_0(\mathcal{L}(a')) \subset \mathcal{L}(a'), \qquad \left(\hat{g}_{K+L-1} \circ \cdots \circ \hat{g}_K\right)^{-1}\left(\mathcal{L}'(a')\right) \subset \mathcal{L}'(a').$$

The deviations of the factors from corresponding translations decreases exponentially by the Hölder property of G:

$$\|\tilde{g}_{K-k} - T_{\tilde{\lambda}}\|_{C(\mathcal{L}(a'))} \le Cq^k, \quad q \in (0, 1),$$
$$\|\hat{g}^{-1}_{K+l} + T_{\tilde{\mu}}\|_{C(\mathcal{L}'(a'))} \le Cq^l.$$

Hence, the total deviation of the compositions above from the translations $T_{K\tilde{\lambda}}, T_{-L\tilde{\mu}}$ remains bounded. This proves (156), (157). □

A.3. *Robust a priori estimate*

The estimate mentioned above is the subject of the following

Proposition 12. *There exists D_* and ζ_* such that for any $\zeta < -\zeta_*$ and K, L large enough that satisfy*

$$|K\tilde{\lambda} + L\tilde{\mu}| < 2, \tag{160}$$

the following inequality holds:

$$|\tilde{g}_{\sigma^{-K}\omega^{01},K+L}(\zeta) - \zeta| < D_*. \tag{161}$$

Proof. As mentioned above, we may identify $\mathcal{L}(a')$ with $\mathbb{R}^- = \{\zeta < 0\}$.

Let $\zeta_0 = \zeta$, $\zeta_k = \tilde{g}_{\sigma^{-K}\omega^{01},k}(\zeta_0)$, $k = 1, \ldots, K+L$.

Step 1. As it is usually done in the proof of the a priori estimates, suppose first, that all the points ζ_k belong to $\mathcal{L}(a')$ for $k = 0, \ldots, K+L$. Let $\eta_l = h(\zeta_{K+L-l})$, $\zeta^* = \zeta_K$, $\eta^* = h(\zeta_K)$. Then

$$\zeta^* = \tilde{g}_{\sigma^{-K}\omega^{01},K}(\zeta_0),$$

and

$$\eta^* = \left(\hat{g}_{\omega^{01},L}\right)^{-1}(\eta_0).$$

By (156) and (157),

$$|\zeta_0 - \zeta^* + K\tilde{\lambda}| < D, \qquad |\eta^* - \eta_0 + L\tilde{\mu}| < D.$$

Let

$$\|h - id\|_{\mathcal{L}(a')} < h_0, \qquad \|h^{-1} - id\|_{\mathcal{L}'(a')} < h_0. \tag{162}$$

Then, making use of (160) and (162), we get:

$$\begin{aligned} |\zeta_{K+L} - \zeta_0| &= |\zeta_{K+L} - \eta_0 + \eta_0 - \eta^* + \eta^* - \zeta^* + \zeta^* - \zeta_0 \\ &\quad + (K\tilde{\lambda} + L\tilde{\mu}) - (K\tilde{\lambda} + L\tilde{\mu})| \\ &\le |\zeta_{K+L} - \eta_0| + |\eta_0 - \eta^* - L\tilde{\mu}| + |\zeta^* - \zeta_0 - L\tilde{\lambda}| \\ &\quad + |\eta^* - \zeta^*| + |K\tilde{\lambda} + L\tilde{\mu}| \\ &< 2D + 2h_0 + 2 := D_*. \end{aligned} \tag{163}$$

Step 2. Take $\zeta_* = D_* + (k_* + 1)d$, where D_* is from (163), d is from (158). By (159), for $\zeta < 0$,

$$\zeta_k < \zeta + \frac{k\tilde{\lambda}}{2} \quad \text{for } k = 0, \ldots, K - k_*. \tag{164}$$

Hence, for such k, $\zeta_k < \zeta_*$. Now, by (158), for $K - k_* < k < K$, we have: $\zeta_k < \zeta_{K-k_*} + dk_* < -D_* - d$. Hence, $\zeta_k \in \mathcal{L}(a')$ for all $k = 0, \ldots, K$.

Let us prove that all the points ζ_k defined above belong to $\mathcal{L}(a')$. Suppose that this is not the fact. All the points ζ_k for $k = 0, \ldots, K$ belong to $\mathcal{L}(a')$, as established above. By the controversy assumption, there exists a point $\zeta_{K+l} \in \mathcal{L}(a'), l > 0$, such that $\zeta_{K+l+1} \notin \mathcal{L}(a')$. Take k such that $|k\tilde{\lambda} + l\tilde{\mu}| < 2$. This is possible because $-1 < \tilde{\lambda} < 0 < \tilde{\mu} < 1$. Then, all the points $\zeta_{K-k}, \zeta_{K-k+1}, \ldots, \zeta_{K+l}$ belong to $\mathcal{L}(a')$. By the statement proved in Step 1, $|\zeta_{K-k} - \zeta_{K+l}| < D_*$. But $\zeta_{K-k} < -D_* - d$. Hence, $\zeta_{K+l+1} < \zeta_{K+l} + d < \zeta_{K-k} + D_* + d < 0$, a contradiction.

Hence, all the points ζ_k defined above belong to $\mathcal{L}(a')$. Application of Step 1 proves the proposition. □

A.4. Final a priori estimate

We now turn directly to the proof of the second statement of Proposition 3. It consists of two inequalities, (81) and (82). The first of them is an immediate corollary of the second one. So we prove (82) in the form: there exists $C > 0$ such that for ζ_* chosen in Proposition 12 and any $\zeta < -\zeta_*$:

$$|\tilde{g}'_*(\zeta) - 1| \le Ce^{\zeta}.$$

Let us summarize the results of the previous two subsections. There exists ζ_* such that for any K and L large enough and any $\zeta < -\zeta_*$, the following finite sequence

$$\zeta_0 = \zeta, \qquad \zeta_k = \tilde{g}_{\sigma^{-K}\omega^{01},k}(\zeta_0), \quad k = 0, \ldots, K + L$$

satisfies the assumption:

all $\zeta_k \in \mathcal{L}(a')$, that is, $\zeta_k < 0$.

Moreover, there exists k_* depending on the skew product G only such that

$$\zeta_{K+L-l} < \zeta - \frac{l\tilde{\mu}}{2} + 2h_0 \quad \text{for } l = 0, \ldots, L - k_*, \tag{165}$$

and (164) holds. What about the middle terms ζ_k, $K - k_* < k < K + k_*$, they are obtained from ζ_{K-k_*} by no more than $2k_*$ applications of the fiber maps. All $\zeta_k \in \mathcal{L}(a')$. Any fiber map has a displacement no greater than d on $\mathcal{L}(a')$. Hence,

$$\zeta_k < \zeta_{K-k_*} + 2k_* d, \quad K - k_* < k < K + k_*. \tag{166}$$

The fiber maps g_ω of the skew product from Proposition 3 are uniformly bounded in $C^3(I)$. Hence, in the Sternberg logarithmic chart, uniformly in ω,

$$\begin{aligned} &\tilde{g}_\omega(\zeta) = \zeta + \kappa(\omega) + R_0(\zeta, \omega)e^\zeta, \\ &\tilde{g}'_\omega(\zeta) = 1 + R_1(\zeta, \omega)\, e^\zeta, \qquad |R_1(\zeta, \omega)| < C_1 \quad \text{for } \zeta \in \mathcal{L}(a'). \end{aligned} \tag{167}$$

That is, R_1 is uniformly bounded in ω for $\zeta \in \mathcal{L}(a')$. This is proved like Lemma 7.

Recall that $\tilde{g}_k = \tilde{g}_{\sigma^{-K+k}\omega^{01}}$.

We have: $\tilde{g}_* = \tilde{g}_{K+L-1} \circ \cdots \circ \tilde{g}_0$. By the chain rule,

$$\log \tilde{g}'_*(\zeta_0) = \sum_0^{K+L-1} \log \tilde{g}'_k(\zeta_k). \tag{168}$$

By (164) and (167), for some $C > 0$,

$$|\log \tilde{g}'_k(\zeta_k)| < Ce^{\zeta + \frac{k\tilde{\lambda}}{2}} \quad \text{for } k = 0, \ldots, K - k_*. \tag{169}$$

By (165) and (167), for some $C > 0$,

$$|\log \tilde{g}'_{K+L-l}(\zeta_{K+L-l})| < Ce^{\zeta - \frac{l\tilde{\mu}}{2}}, \quad l = 0, \ldots, L - k_*. \tag{170}$$

By (142) and (167),

$$|\log \tilde{g}'(\zeta_k)| < Ce^\zeta, \qquad K - k_* < k < K + k_*. \tag{171}$$

The series with the terms from the right hand sides of (169), (170) converge. The number of terms from (168) estimated in (171) equals $2k_*$. Hence, there exists C_1, such that

$$\sum_0^{K+L-1} |\log g'_*(\zeta_k)| < C_1 e^{\zeta_0} = C_1 e^\zeta.$$

This implies (82). This completes the proof of Proposition 3, hence of Theorem 4. □

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