ГОСУДАРСТВЕННЫЙ УНИВЕРСИТЕТ
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A MATHEMATICAL MODEL OF ATHENIAN DEMOCRACY

Препринт WP7/2006/03
Серия WP7
Теория и практика
общественного выбора

Москва
ГУ ВШЭ
2006

1. Introduction

In the contemporary world, democracy is becoming a universal value. During the last decade it has both significantly advanced throughout the world, and expanded from the level of single countries to the level of the world community. Democratic governance facilitates a political dialogue and collaboration between countries with different economic and cultural backgrounds. Democracy is viewed as a right of passage for legitimate international relations. Building democratic governance seems to be the only way to reduce the negative aspects and consequences of globalization.

On the other hand, democratic reforms can fail. Good intentions can turn adverse and result in the election of an authoritarian power. Additionally, democracy may have quite particular manifestations and its realization can deviate considerably from accepted norms.

There are also signs of absolutizing democracy. The notion of democracy tends to become an ideological symbol of superiority of Western countries over the rest of the world. Moreover, political intolerance often stems from opposite viewpoints of democracy, contradicting its very idea. All of this shows that, in spite of its general use, the notion of democracy remains ill defined.

Democracy appeared in Ancient Greece and it is usually associated with Cleisthenes' constitution of 508/507 BC. Its most known innovation was the *ostracism* (= vote-based banishing of unpopular politicians for 10 years), but it also prescribed a systematic participation of all citizens in the political life and an active work in legislative, executive, and juridical bodies. All important questions were discussed in the Popular Assembly, and at least 600 of 700 magistrates (= public offices) were distributed by lot (= lottery) or by rotation. Election by vote was considered an attribute of oligarchy and aristocracy. Elections were avoided whenever possible except for nominating strategs (= military generals).

Democracy in its pure form, or in interaction with other types of governance, had several advantages or disadvantages, depending on one's point of view. After its culmination in Athens and use in Rome together with oligarchic and hereditary power, it was gradually abandoned in favor of other forms of power which seemed more appropriate in the historical context. Christianity played not the least role in this process. The ideal of an active citizen was replaced by the ideal of a true believer (Pocock 1975), having left less initiative to both rulers and ruled. Collegial decisions, election and lot are still practiced sometimes in clergy and politics, but they were neither democratic with regard to the circle of participants, nor with regard to the generality of issues.

The Roman Republic and Medieval republics in Italy combined elements of monarchy, aristocracy, and democracy. For more than 2400 years these three forms of power were distinguished with respect to the way that power was acquired and executed. Democracy was always identified with selecting representatives and magistrates by lot. The word combination "democratic election" would sound contradictory to Montesquieu and Rousseau. The situation changed radically after the French Revolution of 1789. The "oligarchic" election by vote was re-identified as democratic, and the most distinct property of democracy, selection by lot, was abandoned.

To be true, lot did not disappear from social life completely. It survived in juries and, indirectly, in rotation of administrative functions within certain institutions, like dean positions in German universities. However, a recent attempt to reintroduce lot in a broader social context in France (for selecting three quarters of members of the Superior Council of Universities) was prohibited as illegal; see Décision №85—192 DC (1985).

It should be emphasized that neither the French, nor the American revolutions were aimed at reestablishing democracy. They promoted rather republicanism with its remarkable stability and longevity (509 years in Rome and 798 years in Venice). Even Soviet Republics, which emerged after the Russian Revolution of 1917, bared no "democratic" label. The revival of the word "democracy" is largely due to the post-war ideological opposition of communist regimes to bourgeois republics. The former proclaimed their popular nature by adding "democratic" to country names (German Democratic Republic, Popular Democratic Republic of China, etc.). The Western response was the re-identification of "democracy" in its own favor with the emphasis on "democratic elections", human rights, and free press. Now "democracy" serves as a political instrument of the neo-liberal globalization, in particular, in relations between developed and developing countries.

The re-identification of democracy produced a number of inconsistencies. The "oligarchic" election by vote in a broad "democratic" context turned out not to be the universal tool for selecting representatives. Theoretical studies revealed numerous paradoxes, showing that voting results were unacceptable in certain situations. Eliminating such situations from considerations, known in literature as domain restriction, surprisingly resembles Greek opinions that voting is acceptable if the electors share the same values, for example, economic welfare and civil virtue in the case of aristocracy.

These logical problems make clear that in certain circumstances elections and voting are not sufficient to provide a reliable basis for civil consent and political stability. The question is not improving voting procedures for selecting representatives. The problem is rather the general appropriateness of these procedures for involving people into political life and making them satisfied as citizens. This is not a problem of the "rules of the game". The rules can be better or worse, but the game itself can be so designed that it is impossible to win.

In our days, many countries try to adopt democratic governance. For better fitting to national particularities and historical traditions, one has to dispose a certain variety of democratic models. Under these circumstances, updating ideas of the Athenian democracy with a modern mathematical study can make it topical again, especially for countries which look for their own forms of democracy. Finally, it constitutes an alternative to the modern Western model of democracy and thereby can put in question its monopoly as the only norm for all other countries.

We see that the basic principles of the Athenian democracy strikingly differ from its actual understanding. "Why do not we practice lot, and nonetheless call ourselves democrats?" (Manin 1997). On the other hand, why did elections become so popular in our days, contrary to the days of Athens? These questions merit a careful examination. If selection by lot is bad then elections can be considered advantageous. If it is good then there is no evidence in favor of elections.

Our mathematical study confirms the latter which shows that the Athenians were not so naive to adopt lot as instrument of democracy. It is not surprising with regard to the common practice of highly reliable quality control and Gallup polls of public opinions based on random samples. It is proved that lot is fairly good also for selecting appropriate representatives. Moreover, this analogy puts in question finding "best" representatives by elections which is similar to selecting "best" samples for quality control, contrary to its very idea. Thus the question on the superiority of elections over lot remains open.

Section 2, "Outline of Athenian democracy", recalls main historical facts.

Section 3, "Evaluating Athenian politicians in 462 BC", provides a simple example, how to evaluate three political leaders, Pericles, Ephialtes, and Cimon, with regard to several questions at issue which date back to the year 462 BC.

Section 4, "Model", introduces basic definitions. In particular, the indicators of popularity, universality, and goodness of single representatives and decisive bodies are introduced which are later applied to study properties of the Assembly, Council of Five Hundred, juries, and magistrates.

Section 5, "Computing the indicators and their geometrical interpretation", is devoted to the derivation of computational formulas for the indicators of popularity, universality, and goodness for given decisive bodies and for decisive bodies selected by lot. The matrix-vector notation implies a geometric interpretation of decisive bodies.

Section 6, "Quality of decisive bodies selected by lot", shows that the indicators of decisive bodies selected by lot converge to their absolute maxima and their variance vanishes as the size of decisive bodied increases. It proves that a large decisive body selected by lot is likely to be highly representative.

Section 7, "Efficiency of democracy: size of bodies and social stability", considers the properties of decisive bodies with respect to their size and majority-to-minority ratios in the society. In particular, if the society is unstable, having majority-to-minority ratios close to 50:50 for questions of the agenda then decisions made by a single individual are as good as the ones made democratically by a large parliament. In other words, the efficiency of personal power is higher than that of democratic power.

Section 8, "Application to German parliament elections 2002", is additional. It shows that the model of democracy, in particular the indicators of representativeness of decisive bodies, can be useful in a broad political context.

Section 9, "Conclusions", recapitulates the main statements of the paper.

Proofs of four main theorems are collected in Section 10.

2. Outline of Athenian democracy

According to modern views, democracy emerged in Athens in 507 BC, although Athenians officially adopted the already existing word "democracy" somewhat later (Hansen 1986). The preceding history is described in detail by Hansen (1991).

About 100 years before the date mentioned, in the 7th century BC, Athens was governed by magistrates formed by and from *Eupatridai* (= "well born"), that is, leading clans. The economic development resulted in a polarization between rich and poor, and to maintain the social order, in 621 Athens acquired its first written code of laws. These laws compiled by legislative Dracon were so severe that they were said to have been "written not in ink but in blood". Having gained in protection measures, the rich lost their legislative and juridical monopoly, since the laws became obligatory for all citizens.

The Draconic laws were in use for about 30 years with little success. In 594 archon Solon (= one of the nine superior governors) was entitled to find a better social compromise. He was not much over 30 as he announced a general amnesty, abolished enslavement for debt, gave freedom to those so enslaved, and proceeded with political, economic, and juridical reforms. The latter was embodied into "the laws of Solon", which with modifications remained valid until the abolishment of democracy in 322. The election now depended on wealth instead of birth, and the offices could be held exclusively by members of the top property class of four, and in the case of archons, of the top two. The idea was to shift the society from heredity traditions to economic priorities. The major constitutional innovation was the Council of Four Hundred entrusted to prepare the agenda for the People's Assembly, which was the oligarchic prototype of the later democratic Council of Five Hundred.

The Solon's social compromise failed because neither side was completely satisfied. Solon defended himself in verse pamphlets (prose being as yet unknown for literary purposes) which are the first historical reflections of a European statesman. The discouraged Solon voluntarily travelled abroad for 10 years, having visited kings of Lydia and Egypt. Soon after that the society was again split into fractions led by Lykourgos, Pesistratos, and the Alknaionid Megakles. In 561 Pesistratos became tyrant in a coup and ruled in a constant struggle with the opposition until his death in 527, barring two periods of exile.

Pesistratos was succeeded in 527 by his sons Hippais (the elder) and Hipparchus who ruled together. In 514 two young aristocrats, Harmodios and his lover Aristogeiton, attempted a coup, having tried to murder Hippais at the Panathenaic festival. However they only succeeded in killing his brother Hipparchus and were instantly put to death.

Harmodios and Aristogeiton were later proclaimed democratic heroes and freedom-fighters. Their statues were put up in 509 and once again in 477. Their cult was instituted and their descendants were privileged to dine at public expense in the Prytaneion, an archaic town hall east of Akropolis with an ever-burning fire. There, the highest officials regularly met and dined together, and it was the place of reception for state guests, Athenian victors in the Olympic games, and other prominent citizens.

After the coup failure in 514, the Hippais' tyranny lasted for four more years. During this time, the leader of Athenian aristocrats Alkmaionid Cleisthenes (570?—508? BC), with the help of Delphi, induced the Spartans to send a force into Attic under King Kleomenes. In 510 Athens was taken and the tyranny was overthrown. Hippais was forced to go into exile in Sigeion with his family, where he became a honored prisoner of the King of Persia and died in 490.

As Hippais was driven out, a split developed between two aristocratic fractions, one under the newly returned Cleisthenes, and one under Isagoras who had stayed in Athens and had supported Hippais until the Spartans came. As elected archon for 508—507 Isagoras (whose name signified "freedom of speech") managed to get the Spartan King Kleomenes with his army onto his side and sent Cleisthenes into exile. The people rose in revolt, booted Spartans out, recalled Cleisthenes and condemned Isagoras to death in his absence.

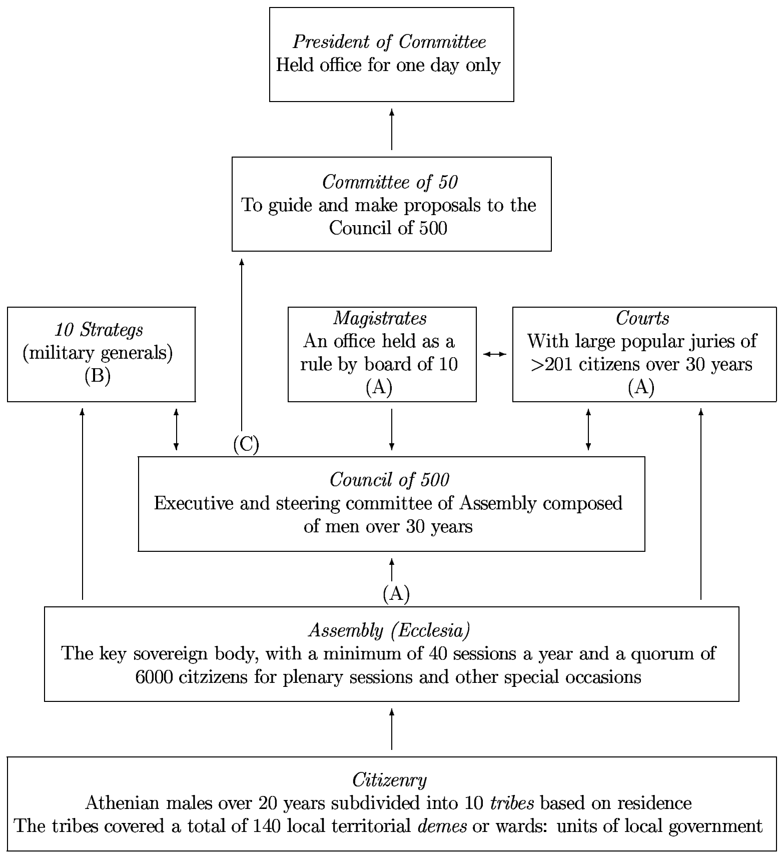
Cleisthenes' most urgent goal was to protect Athens and himself against the political instability which might end in a new coup and a tyranny. In 507 the famous democratic constitution was designed on his initiative (Aristotle 1984, Thucydides 1972); see also (Rodewald 1975, Finley 1973, 1983, Hansen 1987, 1991, Sinclair 1988, Held 1996, Boedeker and Raaflaub 1998). Its greatest innovation was taking the power from a limited circle of changeable aristocrats and diffusing it among common people, basing the individual political responsibility on citizenship rather than on membership of a clan. It met the interests of most Athenian citizens, and, on the other hand, marginalized the intrigues of the unreliable aristocracy.

The legislative body of Athens was the Assembly (*Ecclesia*), the general meeting of all citizens, who were males of Athenian origin over 20; see Figure 1. The meetings were held about 40 times a year in the *Agora* (large open place in Athens north of the *Akropolis*) and later on the *Pnyx*, a hill close to the *Agora*. The quorum was 6000, with the total number of citizens of Athens being estimated from 30 000 to 60 000, see Anderson (1974), Hansen (1991), and Hyland (1995). The people's courts were organized similarly to the Assembly with large juries of 201, 401, and sometimes over 501 jurors selected by lot from the population.

In order to break up the old governing structure, Cleisthenes instituted a new executive body, the Council of Five Hundred based on a new division of *Attica* into 10 *tribes*, 30 *ridings*, and 139 *demes* to be represented in the Council. The latter had to prepare the agenda for the Assembly, to select collective "magistrates" from its members, each position being held by a board of 10, and to select its own supervisor, the Committee of 50 with a president at its top. The Council members held office for one year, and these occasions were restricted to two in a life time with an obligatory interruption. The Committee members served for one month, and the president for a single day. All the appointments have been made by lot or by rotation to equalize everybody's chance for holding office, so that everyone could become a president for one day. The only exception was made for strategists (= military generals) who could be re-elected as many times as the citizenry at the Assembly decided.

In addition to the creation of the Council of Five Hundred, Cleisthenes introduced a new calendar linked to the solar year, and established new cult associations based on the 10 tribes. After 100 years the old lunar calendar was restored, and the new cult organizations

Figure 1. The power structure in Athens, according to the constitution of Cleisthenes of 507 BC (Held 1996)



Methods of election or selection

(A) The ten tribes each sent 50 councillors to the Council, drawn from the *demes*. *Demes* elected candidates in rough proportion to their size to "represent" them in Council and in other offices. The initial choice of candidates was determined by lot. Those selected were put forward into a "pool" of candidates. Finally, the candidates who would actually serve were selected from the pool, again by lot. This method was said to equalize everybody's chance of holding office. The terms of office were short (one year) with typically no provision for immediate re-election. All elected officials were paid for their services, as was attendance of the Assembly at certain times.

(B) These were chosen by the citizenry by direct election and eligible for repeated re-election.

(C) The Committee was made up by rotation from the Council and served for one-tenth of the yearly term of office.

were abandoned, but the redistribution of *Attica*, the Council of Five Hundred together with the Assembly, and the People's Court survived, with modifications, for more than 700 years. A similar fate was inherent in analogous innovations introduced by the French Revolution of 1789: Its new calendar and the new religion were given up, but the National Assembly based on a new division of France into 88 *départements* and 500 districts became permanent.

The best-known of all Cleisthenes innovations is probably the annual procedure of *ostracism*. It was designed to banish unpopular politicians for 10 years, however with no loss of status or property. It was aimed at preventing a new tyranny or a split of the state. The procedure contained two steps. At first, the citizens in the Assembly decided whether they wanted the ostracism this year by a show of hands. If a majority decided to proceed further, then two months later all citizens of Athens went by tribes in the *Agora*. Each scratched the name of the politician he wished to banish on an *ostraka*, which was potsherd. Then the potsherds were counted. If the occurrence of every name was under 6000 (the quorum), nobody was ostracized. The names which occurrence overstepped 6000 were sorted, and the politician, whose name appeared most times was banished. The procedure of ostracism was used for the first time 20 years after it had been established. It was applied about 15 times, mainly in the 480s when military dangers increased the risk of re-establishing the tyranny.

The Athenian democracy culminated under Pericles (495?—429 BC). He was the head of the Democratic Party (if one can apply such a word, see Hansen 1991), 15 times reelected strategist, and the chief of the state in 443—429 BC. During this period Athens achieved the summit of its power and prosperity. Under Pericles Athenian citizens began to be paid for their political participation, even for simply coming to the Assembly. It was caused by growing business activities, so a compensation was needed for losses of working time. At the same time, Pericles restricted the rules for becoming Athenian citizens to filter out unreliable ones in the background of permanent conflicts with neighboring states. This measure reduced the number of citizens and accordingly intensified their political participation.

A deep involvement of citizens into politics met Pericles' views of civil duties. The historian Thucydides (471—400 BC) cited Pericles' Funeral Oration: "We do not say that a man who takes no interest in politics is a man who minds his own business; we say that he has no business here at all." (Thucydides 1972). The total politicalization of society with a subordination of private life to public affairs assumed that citizens were free from other duties which were performed by slaves. Finally it became crucial for democracy itself, since a low efficiency of slave labor limited the economical growth and thereby limited the age of the Athenian democracy.

Although Pericles was convinced that "our city is an education to Greece" (Thucydides 1972), there was a number of objections against the Athenian democracy. From the modern standpoint, it was not really democratic. It was favorable only to native Athenian males who constituted about 10% of the Athenian population, and for others it was a "tyranny of citizens" (Held 1996).

Contemporary critics, among them Plato (427?—347 BC), did not pay much attention to this particularity. Instead, they emphasized the inconsistency that emerged from the fact that democracy "treated all men as equal, whether they were equal or not" (Plato, 1974). Plato found that giving powers to mediocrity was socially unfair and even harmful. He illustrated his viewpoint with a metaphor of an unskillful naval captain who brings the crew to a shipwreck. Other contemporaries noted that equality was only declared but not realized, because of the impossibility to equalize all and to ignore the natural superiority of the active and talented. Since Pericles dominated the Assembly as speaker and proposer, Thucydides (1972) characterized Athens as "in name a democracy but in fact under the rule of the first man".

The most criticized issue was the selection of officials by lot practiced in Athens to nominate at least 600 of 700 magistrates.

Magistrates were often collective, so that one position was held by a committee of 10 members who in case of internal disagreement could bring the question to the Assembly. For Aristotle (384—322 BC) it was the most important feature of democracy at all, as he systematized its development into five stages, see Aristotle (1984), Headlam (1933) and Hansen (1990):

621, selection by lot of minor magistrates in Draconic laws;

594, selection by lot of all magistrates from an elected short list in Solon's laws;

6$^{\text{th}}$ **century — 487,** selection of the archons;

487—403, selection by lot of archons from an elected short list;

403—. . . , selection by lot of archons and other magistrates.

Aristotle believed that democracy had been invented by Solon and only reintroduced by Cleisthenes. In everyday life executives had more real power than the Assembly, and the way of their nomination was important. Therefore, the laws of Solon, which prescribed all magistrates to be selected by lot, were recognized as a sign of democracy. It should be noted that Aristotle (1981) himself had quite moderate views and argued for a mixed government, with certain, but not all, officials selected at random.

In our days, neither participation of people, nor the selection of executives by lot are considered the main distinctions of democracy. The focus has shifted from the goal, total participation, towards its means, "individual freedom' and "human rights", which in Athens served as necessary but auxiliary conditions. Consequently, modern scholars distinguish between the lot in the archaic Greece from that in Cleisthenes' constitution. The difference is in the intention: either fatally leaving decisions to gods (Solon), or providing the equality of all and their equal right to rule (Cleisthenes).

Election, as opposed to lot, was regarded in Athens an attribute of oligarchy and aristocracy rather than of democracy:

- Evaluating persons according to their merits instead of treating everybody equally contradicted the very idea of Athenian democracy. Lot gave at least everybody equal chances.

- Unlike oligarchs and aristocrats who had well-established wealth or virtue criteria, common citizens could have quite arbitrary and sometimes socially questionable ones. Lot was at least free from a "wrong" motivation.

- Elections had the tendency to retain the power of the same persons, which thereby gradually constituted a political oligarchy. Lot broke this trend and provided all citizens with an equal access to power.

- Elections gave better chances to professional politicians with an advantage in wealth and popularity. They were however known to change opinions in order to get and to hold the power. Those selected by lot were at least not suspected of insincereness.

Accordingly, the election by vote was whenever possible avoided with the only exception for the ostracism. In the Assembly, an issue was put to a vote if only no consensus could be achieved.

Already at the early stages of democracy some mathematical "models" were used to illustrate its notions. Athenians knew two types of equality,

$$\text{Arithmetical equality:} \quad x_i = a, \quad i = 1, 2, \ldots,$$

implying equal shares x_i (e.g. of power), and

$$\text{Geometrical equality:} \quad x_i/m_i = a, \quad i = 1, 2, \ldots,$$

implying shares x_i proportional to merits m_i. The arithmetical equality was associated with democracy and with the selection by lot, as giving equal chances to everybody. The geometrical equality was associated with oligarchy and elections, as giving chances proportional to merits. Aristotle (1981) proposed a "unified model", where democracy could be regarded a particular case of oligarchy, because the geometrical equality turned to the arithmetical one as $m_i = 1$, that is, when merits played no role, or the only merit was "to be an Athenian citizen".

The democratic period in Athens ended with the Macedonian conquest in 322 BC. In the background of a general division of labor and growing sophistication of economic relations it was getting difficult to cumulate political and business activities. On the other hand, governing functions could be performed more efficiently by professional politicians than by amateurs selected by lot. Democracy as total participation was given up.

3. Evaluating Athenian politicians in 462 BC

Example 1. (Athenian politicians in 462 BC)
In 462 BC Athen's transformation into a radical democracy has been completed. The power of the traditional aristocratic Council (Court) of the Areopagus has been restricted, payments for political participation have been introduced, and the relationships with neighboring Sparta have turned into a new quality. The three most influential politicians of the time (candidates for leader) were

c_1 : Pericles (495—429 BC), democratic party; for a conditional understanding of parties in Athens see Hansen (1991),

c_2 : Ephialtes (495—461 BC), democratic party, and

c_3 : Cimon (510—450 BC), aristocratic party.

In 462 BC the following three questions were on the agenda:

q_1 : *Remove powers from the the Council (Court) of the Areopagus.*

The Court of the Areopagus was an ancient aristocratic institution. It was composed of "men who were of noble birth" who held office for life; see Blackwell (2003a). Ephialtes opposed aristocrats led by Cimon. Together with Pericles he sponsored laws and decrees that removed many powers from the Areoopagus and gave them to the People's Court or the Assembly.

q_2 : *Pay for political participation.*

The payment for public office and attending the Assembly had been adopted on the initiative of Pericles who promoted the idea of total participation of Athenian citizens in politics.

q_3 : *Help Spartans to put down a rebellion.*

In 462 BC Sparta asked Athens for help in putting down the rebellion of helots in the town of Ithome in Messenia; see Blackwell (2003b). Ephialtes opposed sending help to Sparta, but Athenians delegated a military force under Cimon's command. Ephialtes and Pericles took advantage of his absence to limit the power of Areopagus. Spartans did not appreciate democratic reforms, refused to accept the help, and sent Athenians back. The army returned to Athens in rage and took open measures of hostility against the pro-Spartan people, and above all against Cimon who was ostracized for 10 years.

The opinions of the three politicians on the three questions (Yes/No answers) are shown in the upper section of Table 1 by $\pm$. The $\begin{array}{cc} + & + \\ + & + \end{array}$ at the top-left correspond to Pericles' and Ephialtes' democratic views. The $- \, -$ below show Cimon's opposition. The opinions $\begin{array}{c} + \\ - \\ + \end{array}$ on the last question reflect the known Cimon's will to help Spartans, Ephialtes' opposition, and the fact that the troops have been nevertheless sent, consequently, Pericles might have not support Ephialtes but agreed with Cimon.

Table 1. Evaluation of leading Athenian politicians in 462 BC

c	Politician (candidate for leader)	Questions q			Indicators, %	
		1 Remove powers from the Areopagus	2 Pay for political participation	3 Help Sparta to put down a rebellion	P_c Popularity: average representativeness	U_c Universality: frequency of representing a majority
1	Pericles' opinion	+	+	+		
2	Ephialtes' opinion	+	+	−		
3	Cimon's opinion	−	−	+		
Weight of protagonists in the society, %		66.7	66.7	66.7		
1	Pericles' representativeness, %	66.7	66.7	66.7	66.7	100.0
2	Ephialtes' representativeness, %	66.7	66.7	33.3	55.6	66.7
3	Cimon's representativeness, %	33.3	33.3	66.7	44.4	33.3
Average indicator values, %					$\mathsf{P} = 55.6$	$\mathsf{U} = 66.7$

The social opinions on the three questions were positive, otherwise no positive decision could be made. According to Hansen (1991), no information is available on particular voting results. Usually voting has been performed by a show of hands and votes were not counted but just estimated. Neither were voters divided by opinions into groups to see which group was larger, like it was practiced in Rome. Even several thousand *ostrakas* found in Athens are insufficient to reconstruct vote ratios for the cases of ostracism. Therefore, it remains only to imagine a possible situation.

For simplicity assume that the majority-to minority ratio was always 2 : 1 which is reflected by the protagonist weights 66.7% in the middle row "Weight of protagonists in the society" of Table 1. It allows to find the representativeness of each politician c on each question q, which is the weight of majority or minority represented. For instance, Pericles with his positive opinion on the first question represents the protagonists who constitute 66.7% of the society. Cimon with his negative opinion represents the antagonists who constitute 33.3%. The average representativeness along the row, that is the average size of the group represented, characterizes the popularity P_c of the given politician. The frequency of representing a majority along the row characterizes his universality U_c. As one can see the top-ranked is Pericles, the next best is Ephialtes, and Cimon is ranked third.

If the political leader is selected by lot then the expected popularity and the expected universality are the average of the corresponding indicators, as shown in the last row of the table.

4. Model

The main topic to be discussed is: How good are representatives and representative bodies selected by lot? To answer this question we introduce three indicators of representativeness of decisive bodies, popularity, universality, and goodness. The summary of the model is given in Table 2.

Table 2. Summary of the model

Questions / Agenda

Q agenda of General Assembly (Council of 500, magistrates, or jury) with dichotomous questions at issue q (evoking Yes/No answers)

m number of questions

$\boldsymbol{\mu} = \{\mu_q\}$ m-vector of weights μ_q of questions q

Individuals / Citizens

I set of individuals (Athenian citizens)

n number of individuals (Athenian citizens)

$\boldsymbol{\nu} = \{\nu_i\}$ n-vector of individual weights; $\nu_i = 1/n$ mean equal rights or chances

$\boldsymbol{A} = \{a_{iq}\}$ $(n \times m)$-matrix of ± 1 opinions of individuals i on questions q

$\boldsymbol{a} = \boldsymbol{A}'\boldsymbol{\nu} = \{a_q\}$ m-vector of balance of opinions (the only important information on the individuals)

Candidates for further selection

C set of candidates c for Popular Assembly, Council of 500, magistrates, or jury

N number of candidates

$\boldsymbol{\xi} = \{\xi_c\}$ N-vector of candidate weights; $\xi_c = 1/N$ mean equal chances of candidates to be selected by lot

$\boldsymbol{B} = \{b_{cq}\}$ $(N \times m)$-matrix of ± 1 opinions of candidates c on questions q

Decisive bodies: Assembly, Council of 500, Magistrates, and Jury

$P = (c_1, \ldots, c_k)$ decisive body of candidates which operates on the *majority rule* ("Parliament": Assembly, Council of 500, or jury) with k (odd) votes; multiple instances of c mean a multiple vote holder

$b_{Pq} = \pm 1 = \operatorname{sign} \sum_{c \in P} b_{cq}$ opinion of a majority in P on question q ($b_{Pq} \neq 0$ since k is odd)

$M = (c_1, \ldots, c_k)$ decisive body of candidates who make *socially controlled decisions* in specific domains as magistrates (ministers)

$$b_{Mq} = \pm 1 = \begin{cases} \text{opinion of minority of the society on question } q \\ \text{if all } c \in M \text{ represent the minority} \\ \text{opinion of majority of the society on question } q \\ \text{if any } c \in M \text{ represents the majority} \end{cases}$$

D decisive body: Assembly, Council of 500, magistrate, or jury

k size of decisive body ($k = 1$ implies a single representative, in particular, the president)

ξ_D^k probability to select by lot (with replacement) from N candidates a given decisive body D of size k (ordered k-tuple of candidates); under equal chances $\xi_D^k = 1/N^k$

Indicators of representativeness

$r_{Dq} = \sum_{i: a_{qi} = b_{Dq}} \nu_i$ representativeness of decisive body D on question q (percentage of citizens satisfied by the decisive body D on question q)

$\mathsf{P}_D = \sum_q \mu_q r_{Dq}$ popularity of decisive body D (mean percentage of citizens satisfied averaged on the agenda)

$\mathsf{U}_D = \sum_{q: r_{Dq} \geq 0.5} \mu_q = \sum_q \mu_q \,\mathrm{round}[r_{Dq}]$ universality of decisive body D (percentage of decisions which satisfy a majority of citizens)

$\mathsf{G}_D = \sum_q \mu_q \dfrac{r_{Dq}}{\text{weight of majority on question } q}$ goodness of decisive body D (mean representativeness-to-majority ratio averaged on the agenda)

$\mathsf{P}, \mathsf{U}, \mathsf{G} = \sum_D \xi_D^k \mathsf{P}_D($ or U_D, or $\mathsf{G}_D)$ expected popularity (universality, goodness) of a decisive body of size k selected by lot

Questions/Agenda. Let Q be the actual agenda with m dichotomous questions at issue q, that is, which evoke either positive or negative *opinions* (Yes/No answers coded by ± 1, in tables denoted by $\pm$). This condition is not much restrictive, since a response with more than two grades is revealed by several dichotomous questions. For instance, the question "Which public buildings to construct next year? (None/ Only theater/ Only court / Theater and court)" can be replaced by two dichotomous questions "Construct theater? (Yes/No)" and "Construct court? (Yes/No)".

The importance of questions is reflected by weights μ_q which constitute a *probability measure* μ on Q. It assumes

non-negativity

$$\mu_q \geq 0 \quad \text{for every} \quad q \in Q,$$

additivity

$$\mu_X = \sum_{q \in X} \mu_q \quad \text{for every subset} \quad X \subset Q,$$

and *normality*

$$\sum_q \mu_q = 1 \quad \text{(the totality is 100\%)}.$$

The question weights are collected into the column m-vector

$$\boldsymbol{\mu} = \{\mu_q\}.$$

Individuals/Citizens. A *society* is a set I of n *individuals* i (Athenian citizens) with weights ν_i which constitute a probability measure ν on I. The individual weights are collected into the column n-vector

$$\boldsymbol{\nu} = \{\nu_i\} \qquad (\nu_i = 1/n \quad \text{mean equal rights}).$$

A group $Y \subset I$ is called a (non-strict) *majority* if its weight is not less than 50%

$$\nu_Y \geq 0.5,$$

otherwise it is called *minority*. By default a minority is strict and a majority is non-strict. The positive or negative opinions $a_{iq} = \pm 1$

of individuals i on questions q are collected into the $(n \times m)$-*matrix of individual opinions*

$$\boldsymbol{A} = \{a_{iq}\}, \qquad a_{iq} = \pm 1.$$

The m-vector of *balance of opinions in the society* is defined as follows

$$\boldsymbol{a} = \{a_q\} = \boldsymbol{A}'\boldsymbol{\nu}.$$

For example its coordinate $a_q = 0.2$ means that the protagonists, who are positive on question q, prevail over the antagonists by 20%, measured in % of the total population, $a_q = 0$ means a tie opinion, and $a_q = -1$ means a unanimously negative opinion. The model uses the vector $\boldsymbol{a}$ only; the matrix $\boldsymbol{A}$ is not necessary but just rigourously explains how the vector $\boldsymbol{a}$ is defined.

Candidates. Let C be a set of N *candidates* c for selection whose weights ξ_c constitute a probability measure ξ on C. The candidate weights are collected into the column N-vector

$$\boldsymbol{\xi} = \{\xi_c\} \qquad (\xi_c = 1/N \quad \text{mean equal chances}).$$

Their opinions $b_{cq} = \pm 1$ on questions q are collected into the $(N \times m)$-*matrix of candidate opinions*

$$\boldsymbol{B} = \{b_{cq}\}, \qquad b_{cq} = \pm 1.$$

From the viewpoint of the Athenian Assembly, all the citizens are candidates to appear at its next meeting. If one speaks of the Council of 500 then candidates are the citizens who have registered for participation. Candidates can also be non-human objects like political parties, or reform proposals with a number of items at issue. The only essential condition is specifying candidates and individuals with the same dichotomous questions.

It should be emphasized that the strict dichotomy of answers is required from candidates only (a politician must have an opinion on every question). It is not required from individuals, who may have no definite opinion on certain questions. The only requirement is the existence of balance of opinions in the society (vector $\boldsymbol{a}$).

Assembly, Council, Committee, and Juries. Parliament is a general notion for a decisive body which makes decisions by the majority rule. The Athenian Assembly, the Council of 500, or juries are all parliaments.

A *parliament* with k votes (k is odd to avoid tie vote) is a k-tuple of candidates

$$P = (c_1, \ldots, c_k) \in C^k.$$

Multiple instances of a candidate are allowed (cf. with political parties which have several parliament seats with one vote, or stockholders whose votes are proportional to shares). The opinion (decision) $b_{Pq} = \pm 1$ of parliament P on question q is determined by a majority vote (k is odd!):

$$b_{Pq} = \operatorname{sign} \sum_{c \in P} b_{cq} = \pm 1. \tag{1}$$

A majority in the parliament is determined by the number of votes, but not by the weight of their holders. The weight of holders is implemented in their multiple instances (= multiple votes).

Magistrates. Magistrate is a general notion for a decisive body (in the case of Athens with typically a board of 10 members) which is externally controlled by the society (Assembly). In the case of internal disagreement, the magistrate board brought the question to the Assembly, so that "wrong" decisions (= against the majority of the society) could be made only if the cabinet members were unanimously "wrong".

A *magistrate* with k *members* is a k-tuple of candidates

$$M = (c_1, \ldots, c_k),$$

which opinion (decision) $b_{Mq} = \pm 1$ on question q is the same as that of a majority in the society, except for cases when *all* its k members belong to the minority:

$$b_{Mq} = \begin{cases} -\operatorname{sign}\left(\sum_i a_{iq}\right) & \text{if all } c \in M \text{ represent} \\ & \text{the minority:} \\ & \sum_{c \in M} b_{cq} = -k \operatorname{sign}\left(\sum_i a_{iq}\right) \\ \operatorname{sign}\left(\sum_i a_{iq}\right) & \text{if there exists } c \in M \text{ from} \\ & \text{the majority.} \end{cases} \qquad (2)$$

Decisive bodies and single representatives. Both parliaments and magistrates are decisive bodies. Opinions of a decisive body D on m questions are collected into column m-vector

$$\boldsymbol{b}_D = \{b_{Dq}\}.$$

A decisive body with one candidate is a single representative, for instance, a *president.*

Popularity, universality, and goodness of decisive bodies. Given a question q, the society I falls into two complementary groups, of *protagonists* with a positive opinion $a_{iq} = 1$, and of *antagonists* with a negative opinion $a_{iq} = -1$. These groups are redefined for each question.

The *representativeness* of decisive body D on question q is the weight of the social group which opinion is represented

$$r_{Dq} = \sum_{i: a_{iq} = b_{Dq}} \nu_i. \qquad (3)$$

Since the total weight $\sum_i \nu_i = 1$ (= 100%), the representativeness r_{Dq} is measured in the fraction (percentage) of the society.

The representativeness $r = r_{Dq}$ of a decisive body D of size k is a function on the product space

$$C^k \times Q \quad \text{with probability measure} \quad \underbrace{\xi^k}_{\xi\otimes\cdots\otimes\xi} \otimes \mu.$$

Regarded deterministically, r characterizes D with respect to questions q, with ξ^k and μ being measurement/weighting instruments. Under such an interpretation, the indicators defined below are the weighted average of representativeness and of rounded representativeness.

Regarded probabilistically, teams of decisive bodies D are simple events with probability ξ^k, meaning that their participants are selected by lot with probability ξ. The selection is performed with replacement, allowing multiple instances of the same candidate (= multiple-vote holders). Then both r and round$[r]$ are *random variables*, and the indicators below are their mathematical *expectation* with either D being parameter and q being simple event, or both D and q being simple events:

$$\mathsf{P}_D = \sum_q \mu_q \, r_{Dq} \tag{4}$$

(popularity of decisive body D)

$$\mathsf{U}_D = \sum_{q: r_{Dq} \geq 0.5} \mu_q = \sum_q \mu_q \, \text{round}[r_{Dq}] \tag{5}$$

(universality of decisive body D)

$$\mathsf{G}_D = \sum_q \mu_q \, \frac{r_{Dq}}{\text{weight of majority for question } q} \tag{6}$$

(goodness of decisive body D)

$$
\begin{aligned}
\mathsf{P} &= \sum_{D \in C^k} \xi_D^k \, \mathsf{P}_D &(7)\\
&\text{(expected popularity of a decisive body selected by lot)}\\
\mathsf{U} &= \sum_{D \in C^k} \xi_D^k \, \mathsf{U}_D &(8)\\
&\text{(expected universality of a decisive body selected by lot)}\\
\mathsf{G} &= \sum_{D \in C^k} \xi_D^k \, \mathsf{G}_D &(9)\\
&\text{(expected goodness of a decisive body selected by lot).}
\end{aligned}
$$

If required, the type and size of decisive bodies are distinguished by superscript for parliaments, e.g., $\mathsf{U}^{(k)}$, and by subscript for magistrates, e.g., $\mathsf{P}_{(k)}$ for magistrates.

The (degree of) *popularity* of decisive body D is the average size of the group represented, the average being taken over all the questions with their weights. The popularity is measured in fraction (percentage) of the society.

The (degree of) *universality* of decisive body D is the frequency with which it represents a non-strict majority. The universality is measured in the fraction (percentage) of questions with regard to their weights μ_q.

The popularity 51% seems rather low. However, if the majority-to-minority ratio is always 51:49 then no popularity higher than 51% is attainable. Therefore, the effective representativeness of a decisive body should be judged with regard to the majority-to-minority ratio. This is reflected in the indicator of goodness. In a sense, the indicator of goodness is a smoothed version of universality. Indeed, the rounded representativeness and the representativeness-to-majority ratio are equal to 1 as the group represented is a majority. However, the latter has no abrupt leap as the group represented grows and turns from minority to majority.

The *expected* indices of a decisive body selected by lot are the corresponding indices averaged on all the decisive bodies of the given type and size. Under the probability ξ, candidates c with greater weights ξ_c have higher chances to be selected. Equal chances occur if candidate weights are equal, as in the case of Athens.

Thus the indicators of popularity, universality, and goodness reflect spatial, temporal, and relative aspects of representativeness:

- The popularity reflects the **volume** of citizens represented.
- The universality is the **frequency** when a majority is represented.
- The goodness is the **specific** representativeness, that is, reduced to its absolute maximum.

5. Computing the indicators and their geometrical interpretation

Introduce the following notation (all vectors are column vectors!):

$'$ the operation of vector/matrix transpose

$.*$ the operation of element-by-element product of vectors and matrices of the same size, for example, $(1,2).*(3,4) = (3,8)$

$.^k$ the operation of element-by-element k-th power of vectors and matrices, for example, $(2,3).^2 = (4,9)$

$+$ the addition of scalars to matrices or vectors by applying it to all matrix elements, for example, $0.5 + (1,2) = (1.5,\ 2.5)$. Division of scalar by vector is analogous: $\frac{12}{(2,3)} = (6,4)$.

$|\boldsymbol{a}|$ the m-vector *imbalance of opinions* derived from vector $\boldsymbol{a}$ by taking absolute values of its coordinates; then the qth coordinate $|a_q|$ gives the deviation from the tie opinion on question q; the vectors of majority and minority weights are $0.5+0.5|\boldsymbol{a}|$ and $0.5-0.5|\boldsymbol{a}|$, respectively;

diag$\boldsymbol{a}$ the diagonal $(m \times m)$-matrix with elements of vector $\boldsymbol{a}$ on its main diagonal;

sign $\boldsymbol{a}$ the m-vector of *majority opinions* derived from the vector $\boldsymbol{a}$ by applying the sign function to its coordinates

$$\operatorname{sign} a_q = \begin{cases} +1 & \text{if} \quad a_q > 0, \quad \text{i.e. the majority opinion} \\ & \qquad\qquad\quad \text{on question } q \text{ is positive} \\ 0 & \text{if} \quad a_q = 0, \quad \text{i.e. tie opinion on question } q \\ -1 & \text{if} \quad a_q < 0, \quad \text{i.e. the majority opinion} \\ & \qquad\qquad\quad \text{on question } q \text{ is negative,} \end{cases}$$

$\delta_{\boldsymbol{a}} = 1 - \operatorname{abs}(\operatorname{sign}\boldsymbol{a})$ the m-vector of *indicators of tie opinion*, with the qth coordinate being 1 if the opinion on question q is tied, and 0 otherwise; we use this vector to express the total weight of questions with a tie opinion

$$\boldsymbol{\mu}'\delta_{\boldsymbol{a}} = \sum_{q: a_q = 0} \mu_q .$$

$\boldsymbol{b} = \{b_q\} = \boldsymbol{B}'\boldsymbol{\xi}$ the m-vector of *balance of opinions of candidates*. This vector is analogous to $\boldsymbol{a}$.

$I_p(x,y)$ the incomplete beta-function defined as follows

$$I_p(x,y) = \frac{\Gamma(x+y)}{\Gamma(x)\,\Gamma(y)} \int_0^p t^{x-1}(1-t)^{y-1}dt, \qquad 0 \le p \le 1,\ x > 0,\ y > 0. \quad (10)$$

Theorem 1. (Computing the indicators and their geometric interpretation)

$$\underbrace{\mathsf{P}}_{\substack{\text{popularity of}\\ \text{decisive body } D\\ \text{or expected}\\ \text{popularity}\\ \text{of a decisive}\\ \text{body selected}\\ \text{by lot}}} = 0.5 + 0.5 \underbrace{\left(\boldsymbol{\mu}.*\boldsymbol{a}\right)'}_{\substack{=\boldsymbol{p}\\ \boldsymbol{\mu}\text{-weighted}\\ \text{social}\\ m\text{-vector}\\ \text{of balance}\\ \text{of opinions}}} \underbrace{\boldsymbol{d}}_{\substack{m\text{-vector of}\\ \text{opinions of}\\ \text{decisive body } D\\ \text{or characteristic}\\ \text{vector of the}\\ \text{society}}} \tag{11}$$

$$\underbrace{\mathsf{U}}_{\substack{\text{universality of}\\ \text{decisive body } D\\ \text{or expected}\\ \text{universality}\\ \text{of a decisive}\\ \text{body selected}\\ \text{by lot}}} = 0.5 + 0.5 \underbrace{\boldsymbol{\mu}'\delta_{\boldsymbol{a}}}_{\substack{\text{total weight of}\\ \text{questions with}\\ \text{tie opinions}\\ \text{(constant scalar}\\ \text{independent of } D)}} + 0.5 \underbrace{\left(\boldsymbol{\mu}.*\operatorname{sign}\boldsymbol{a}\right)'}_{\substack{=\boldsymbol{u}\\ \boldsymbol{\mu}\text{-weighted}\\ \text{social}\\ m\text{-vector}\\ \text{of majority}\\ \text{opinion}}} \boldsymbol{d} \tag{12}$$

$$\underbrace{\mathsf{G}}_{\substack{\text{goodness of}\\ \text{decisive body } D\\ \text{or expected}\\ \text{goodness}\\ \text{of a decisive}\\ \text{body selected}\\ \text{by lot}}} = \boldsymbol{\mu}'\frac{1}{1+|\boldsymbol{a}|} + \underbrace{\left(\boldsymbol{\mu}.*\frac{1}{1+|\boldsymbol{a}|}.*\boldsymbol{a}\right)'}_{=\boldsymbol{g}} \boldsymbol{d} \tag{13}$$

where $\boldsymbol{d} = \{d_q\}$ is the opinion m-vector of the given decisive body D or the characteristic vector of the society, depending on the type of representative body selected by lot:

$$d_q = \begin{cases} \operatorname{sign} \sum_{c\in P} b_{cq} & \text{for } \textit{given} \text{ parliament } P \\ -\operatorname{sign} a_q & \text{for } \textit{given} \text{ magistrate } M \text{ with no majority representative of the society on question } q \text{, that is, if } \sum_{c\in M} b_{cq} = -k \operatorname{sign} a_q \\ \operatorname{sign} a_q & \text{for } \textit{given} \text{ magistrate } M \text{ otherwise(with a majority representative)} \\ b_{cq} & \text{for } \textit{given} \text{ president } c \\ \operatorname{sign} b_q \, I_{b_q^2}\left(\frac{1}{2}, \frac{k+1}{2}\right) & \text{for parliament } \textit{selected by lot} \\ \operatorname{sign} a_q \left[1 - 2\left(\frac{1-\operatorname{sign}_{a_q} b_q}{2}\right)^k\right] & \text{for magistrate } \textit{selected by lot} \\ b_q & \text{for president } \textit{selected by lot} \end{cases} \tag{14}$$

Thus, the most popular (universal, best) decisive body D has the largest projection of its opinion vector $\boldsymbol{b}_D$ on the $\boldsymbol{\mu}$-weighted social vector of balance of opinions $\boldsymbol{p}$ (respectively, on the $\boldsymbol{\mu}$-weighted vector of majority opinion $\boldsymbol{u}$, or on vector $\boldsymbol{g}$). In case of decisive bodies selected by lot the average characteristic vectors of all candidates are projected on the vectors of the society.

Remark 1. (Analogy with force vectors in physics)
Recall that in mechanics a work is produced by displacements. Accordingly, the only productive constituent of a force vector is its projection on the direction of motion. In our model, the "work for the society" of a decisive body is measured by the projection of its opinion vector on the "main stream", the social vector of balance of opinions, or social vector of majority opinion, or social "goodness

vector". Thus the variety of representatives and representative bodies with numerous opinions on the agenda is projected onto a single line axis, exactly like in the case of physical forces.

Example 2. (Athenian politicians in 462 BC, continued) Return to Example 1 The table above Figure 2 is a modification with extension of Table 1. The opinions and the representatives of the politicians are put together, the column for the goodness indicator and the row for question weights are added, and the bottom section of the table contains now the coordinates of social vectors $\boldsymbol{p}, \boldsymbol{u}$, and $\boldsymbol{g}$ for computing the expected popularity, universality and goodness, respectively.

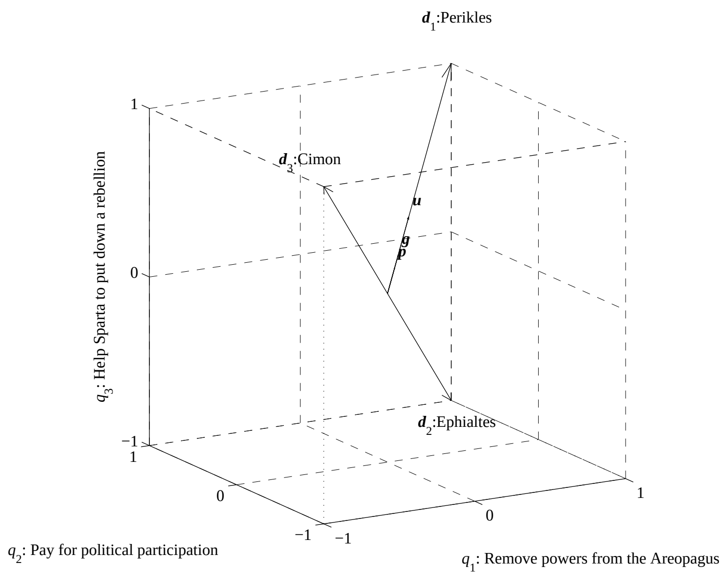
The decision space with vectors of the candidate opinions $\boldsymbol{d}_1$—$\boldsymbol{d}_3$ and characteristic vectors of the society $\boldsymbol{p}, \boldsymbol{u}$, and $\boldsymbol{g}$ are depicted in Figure 2. Note that vectors of politicians composed of ±1-opinions are extended to the cube vertices. The social vectors are shorter because they represent some balance of opinions within the range $[-1;+1]$. Due to the equality of majority-to-minority ratios 2:1 assumed, $|a_1| = |a_2| = |a_3|$ so that the three social vectors are collinear; see how the formulas (11)—(13) depend on $|\boldsymbol{a}|$. Since Pericles always represents a majority, his vector provides the largest possible projection on the social vectors. In the given case of "diagonal" social vectors with equal coordinates it implies the collinearity of Pericles' vector to the social vectors.

6. Quality of decisive bodies selected by lot

Theorem 2. (Deficit of popularity, universality, and goodness)
Consider selection by lot from the individuals, that is, $C = I$, $\boldsymbol{B} = \boldsymbol{A}$, *and* $\boldsymbol{\xi} = \boldsymbol{\nu}$. *Compared to the absolute maxima of the indicators,*

Figure 2. Analysis of political situation in Athens in 462 BC

c	Candidate	Questions q			Indicators, %		
		1	2	3	P_c	U_c	G_c
		Remove powers from the Areopagus	Pay for political participation	Help Sparta to put down a rebellion	Popularity: average representativeness	Universality: frequency of representing a majority	Goodness
1	Pericles	66.7	66.7	66.7	66.7	100.0	100.0
2	Ephialtes	66.7	66.7	33.3	55.6	66.7	83.3
3	Cimon	33.3	33.3	66.7	44.4	33.3	66.7
Weight of protagonists, % $(0.5+0.5\boldsymbol{a})$		66.7	66.7	66.7			
Question weights, % $(\boldsymbol{\mu})$		33.3	33.3	33.3			
$\boldsymbol{p} = \boldsymbol{\mu}.*\boldsymbol{a}$, %		11.1	11.1	11.1	$\mathsf{P} = 55.6$		
$\boldsymbol{u} = \boldsymbol{\mu}.*\operatorname{sign}\boldsymbol{a}$, %		33.3	33.3	33.3		$\mathsf{U} = 66.7$	
$\boldsymbol{g} = \boldsymbol{\mu}.*\frac{1}{1+\lvert\boldsymbol{a}\rvert}.*\boldsymbol{a}$, %		16.7	16.7	16.7			$\mathsf{G} = 83.3$



attainable if a majority could be represented on all the questions

$$\overline{\mathsf{P}} = \sum_q \mu_q \,(0.5 + 0.5|a_q|) = 0.5 + 0.5\boldsymbol{\mu}'|\boldsymbol{a}| \tag{15}$$

$$\overline{\mathsf{U}} = 1, \tag{16}$$

$$\overline{\mathsf{G}} = 1, \tag{17}$$

the expected deficit of the indicators of parliaments and magistrates selected by lot vanishes as their size k increases at a speed of order k^{-1} and 2^{-k}, respectively:

$$\underbrace{\underbrace{\overline{\mathsf{P}}}_{\substack{\text{absolute}\\ \text{limit of}\\ \text{popularity}}} - \underbrace{\mathsf{P}}_{\substack{\text{expected}\\ \text{popularity}\\ \text{of decisive}\\ \text{body of size } k\\ \text{selected by lot}}}}_{\text{expected deficit of popularity}} \leq \begin{cases} \frac{2}{9(k+2)\min\limits_{q:a_q\neq 0}|a_q|} \\ \text{for parliament selected by lot} \\ 2^{-k-2}k^{-1} \\ \text{for magistrate selected by lot} \\ 2^{-3} \\ \text{for president selected by lot} \end{cases} \tag{18}$$

$$\underbrace{\underbrace{1}_{\substack{\text{absolute}\\ \text{limit of}\\ \text{universality}}} - \underbrace{\mathsf{U}}_{\substack{\text{expected}\\ \text{universality}\\ \text{of decisive}\\ \text{body of size } k\\ \text{selected by lot}}}}_{\text{expected deficit of universality}} \leq \begin{cases} \frac{2}{9(k+2)\min\limits_{q:a_q\neq 0} a_q^2} \\ \text{for parliament selected by lot} \\ 2^{-k} \\ \text{for magistrate selected by lot} \\ 2^{-1} \\ \text{for president selected by lot} \end{cases} \tag{19}$$

$$\underbrace{\underbrace{1}_{\substack{\text{absolute}\\ \text{limit of}\\ \text{goodness}}} - \underbrace{\mathsf{G}}_{\substack{\text{expected}\\ \text{goodness}\\ \text{of decisive}\\ \text{body of size } k\\ \text{selected by lot}}}}_{\text{expected deficit of goodness}} \leq \begin{cases} \frac{4}{9(k+2)\min\limits_{q:a_q\neq 0}|a_q|} \\ \text{for parliament selected by lot} \\ 2^{-k-2} \\ \text{for magistrate selected by lot} \\ 2^{-3} \\ \text{for president selected by lot} \end{cases} \tag{20}$$

Theorem 3. (Variance of indicators of decisive bodies selected by lot)
Consider selection by lot from the individuals, that is, $C = I$, $\boldsymbol{B} = \boldsymbol{A}$, and $\boldsymbol{\xi} = \boldsymbol{\nu}$. The variance of indicators of decisive bodies D selected by lot vanishes at the same speed as their expected deficit (for exact formulas see Lemma 2):

$$\mathsf{V}\,\mathsf{P}_D \leq \overline{\mathsf{P}}^2 - \mathsf{P}^2 \leq 2(\overline{\mathsf{P}} - \mathsf{P}) \quad \text{(double deficit of popularity)} \tag{21}$$

$$\mathsf{V}\,\mathsf{U}_D \leq 1 - \mathsf{U}^2 \leq 2(1 - \mathsf{U}) \quad \text{(double deficit of universality)} \tag{22}$$

$$\mathsf{V}\,\mathsf{G}_D \leq 1 - \mathsf{G}^2 \leq 2(1 - \mathsf{G}) \quad \text{(double deficit of goodness)} \tag{23}$$

Remark 2. (Quality of decisive bodies selected by lot)
The main message of Theorems 2 and 3 is that decisive bodies selected by lot are fairly good, being quite representative. Indeed, their expected popularity, universality, and goodness converge to the absolute maxima, and their variance vanishes as the size of decisive body increases. It implies that the probability to select a low representative decisive body is getting negligible as its size k increases. For instance, by (18) for a given $\epsilon > 0$ it holds $\overline{\mathsf{P}} - \mathsf{P}^{(k)} < \epsilon/2$ for all sufficiently large k. Then

$$\mathsf{Prob}\{\underbrace{\overline{\mathsf{P}}-\mathsf{P}_D}_{\substack{\text{deficit of}\\ \text{popularity}\\ \text{of } D}} \geq \epsilon\} \leq \mathsf{Prob}\{|\mathsf{P}_D - \underbrace{\mathsf{P}}_{=\mathsf{E}\,\mathsf{P}_D}| \geq \epsilon/2\} \overset{\substack{\text{Chebychev}\\ \text{inequality}}}{\leq} \frac{\mathsf{V}\,\mathsf{P}_D}{(\epsilon/2)^2} \xrightarrow[k\to\infty]{\text{by (21) and (18)}} 0.$$

Similarly, for other two indicators

$$\mathsf{Prob}\{\text{Indicator deficit} \geq \epsilon\} \leq \frac{\mathsf{V}\,\text{Indicator values}}{(\epsilon/2)^2} \xrightarrow[k\to\infty]{} 0.$$

Remark 3. (Decisive body's performance depends on its size rather than on the size of the society)
The estimates in Theorems 2 and 3 depend on the size k of decisive body but not on the size of the society n. In other words, Monaco needs as large a parliament as China. It is remarkable that the quorum at the Athenian Assembly and for the *ostrakism* required as many as 6000 participants, which were 10—20% of the total number of citizens. Juries were also very numerous, ranging from 201 to 1501 participants.

7. Efficiency of democracy: size of bodies and social stability

Theorem 4. (Ordering of the indicators)
Consider selection by lot from the individuals, that is, $C = I$, $\boldsymbol{B} = \boldsymbol{A}$, *and* $\boldsymbol{\xi} = \boldsymbol{\nu}$. *Then the following inequalities hold*

$$\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & & \mathsf{G}^{(k-1)} & \leq & \mathsf{G}^{(k)} & \leq & \mathsf{G}_{(k)} & \leq & \mathsf{G}_{(k+1)} & \leq & \overline{\mathsf{G}} \\
 & & & & & & & & & & \vee\!| & & \vee\!| & & \vee\!| & & \vee\!| & & \| \\
\mathsf{P}^{(k-1)} & \leq & \mathsf{P}^{(k)} & \leq & \mathsf{P}_{(k)} & \leq & \mathsf{P}_{(k+1)} & \leq & \overline{\mathsf{P}} & \overset{(**)}{\leq} & \mathsf{U}^{(k-1)} & \leq & \mathsf{U}^{(k)} & \leq & \mathsf{U}_{(k)} & \leq & \mathsf{U}_{(k+1)} & \leq & \overline{\mathsf{U}} \\
\overset{(*)}{\vee\!|} & & & & & & & & & & \vee & & & & & & & & \| \\
0.5 & & & & & & & & & & 0.5 & & & & & & & & 1
\end{array} \tag{24}$$

where $(*)$ *turns to equality if and only if* $a_q = 0$ *for all questions* q *(= only tie opinions), and* $(**)$ *turns to equality if and only if* $a_q \neq 0$ *for all questions* q *(= no tie opinions). If there is a question* q *with neither unanimity, nor tie opinion, that is,* $|a_q| \neq 0, 1$*, then all the equalities, may be except* $(**)$*, are strict.*

Remark 4. (The larger the decisive body, the higher its representativeness)
According to Theorem 4, *all* decisive bodies and single individuals selected by lot are expected to be rather representative than non-representative. Indeed, all the indicators are superior to 50%; the equality $\mathsf{P} = 0.5$ is attained under tie opinions on every question, when every decision is equally good. Thus the society exhibits the general statistical property that limited samples and even one observation allow to make conclusions about the totality. Certainly, the larger the sample the more reliable conclusion, which is reflected by the indicator growth as the size of decisive body increases.

Remark 5. (Superior performance of magistrates)
The performance of magistrates is higher than that of parliaments of the same size. It is explained by the following reasons. By definition every magistrate decision is made by the member who represents a majority. (Recall that in case of disagreement the magistrate brings the question to the Assembly which solves the question in favor of the majority of the society. Therefore, if the magistrate is not unanimous then the decision finally made is the one as if it would be made by a majority representative.) It assumes fitting certain domains of responsibility to magistrate participants, meaning that two degrees of freedom, members and domains, are reduced to one (members). In other words, magistrates are already "half-optimized" and fitted to specific tasks.

The Assembly (parliament) is not so much determined, since its function is just to reveal the yet unspecific social opinions. Under the lack of information it is neither possible to "optimize" the Assembly, nor to reasonably select its members. It remains but to let decisions be made jointly and the members be selected by lot. Correspondingly, the Assembly indicator values are smaller than that of magistrates of the same size.

Remark 6. (Dependence of indicators on the majority-to-minority ratios)
Another factor which determines the expected performance of decisive bodies selected by lot is the majority-to-minority ratios in the society. As one can see from formulas (43)—(45) for parliaments and (48)—(50) for magistrates, the indicator values depend on the imbalance of opinions $|b_q|$ (the candidates are selected from individuals so that $|a_q| = |b_q|$). Figures 3—5 show the dependence of indicators on the size of minority assumed equal for all questions.

As one can see, the popularity decreases, meaning that the closer to tie opinion, the lower the expected popularity. Indeed, the probability that members of the decisive body belong to the majority decreases, as the size of the majority decreases. Besides, the majority represented is getting smaller.

The universality decreases as well, but the shape of universality curves is different. Note that the universality has an abrupt leap as the tie opinion occurs, because the strict minority turns to non-strict majority, and the rounded representativeness immediately increases from 0 to 1.

The graph of goodness is more remarkable. As already mentioned, the goodness is a kind of smoothed universality, which is also seen while comparing Figures 5 and 4. Unlike popularity and universality, the goodness curve has a minimum at a certain point which is least favorable for decisive bodies selected by lot. For single representatives selected by lot ($k = 1$) this critical point corresponds to the majority-to-minority ratio $\frac{\sqrt{2}/2}{1-\sqrt{2}/2} \approx 71 : 29$. At this ratio the goodness attains its minimum $\approx 82.84\%$.

The minimal goodness depends on the type and size of decisive body. Table 3 shows these minima for Athenian decisive bodies selected by lot and the critical minority size when the minimum

Figure 3. Popularity of assembly and magistrates of size k selected by lot

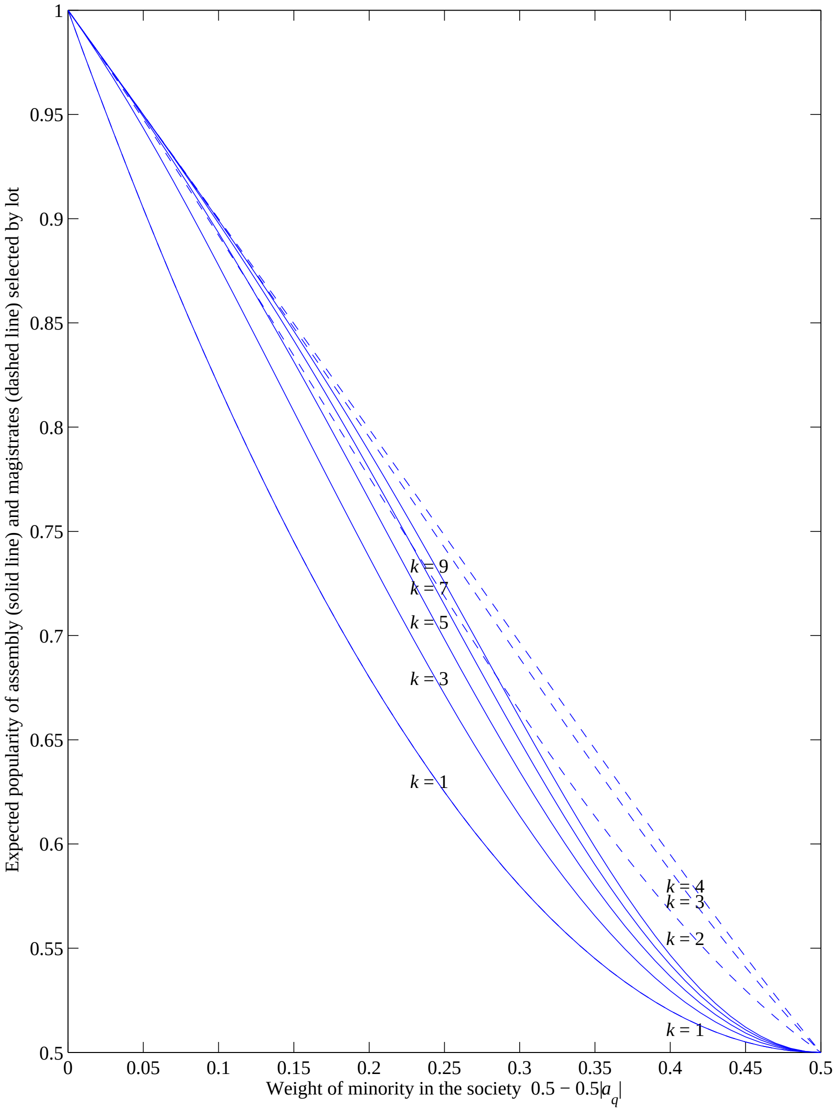


Figure 4. Universality of assembly and magistrates of size k selected by lot

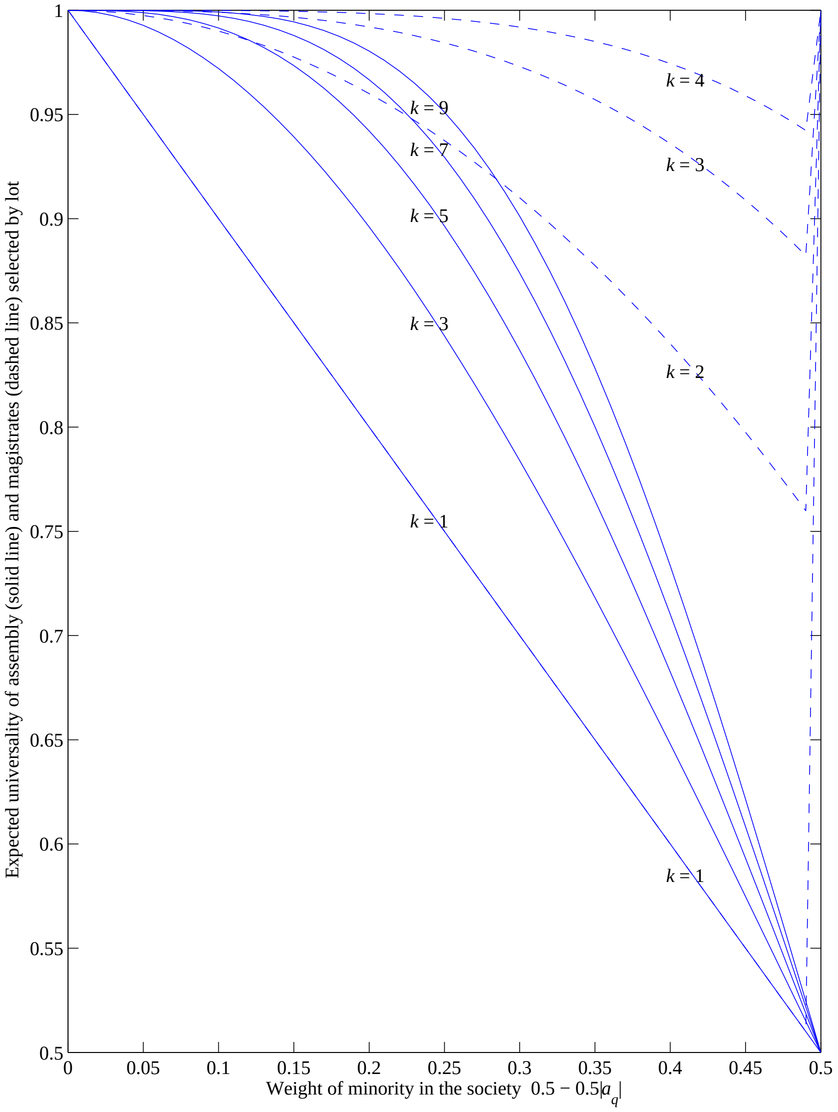


Figure 5. Goodness of assembly and magistrates of size k selected by lot

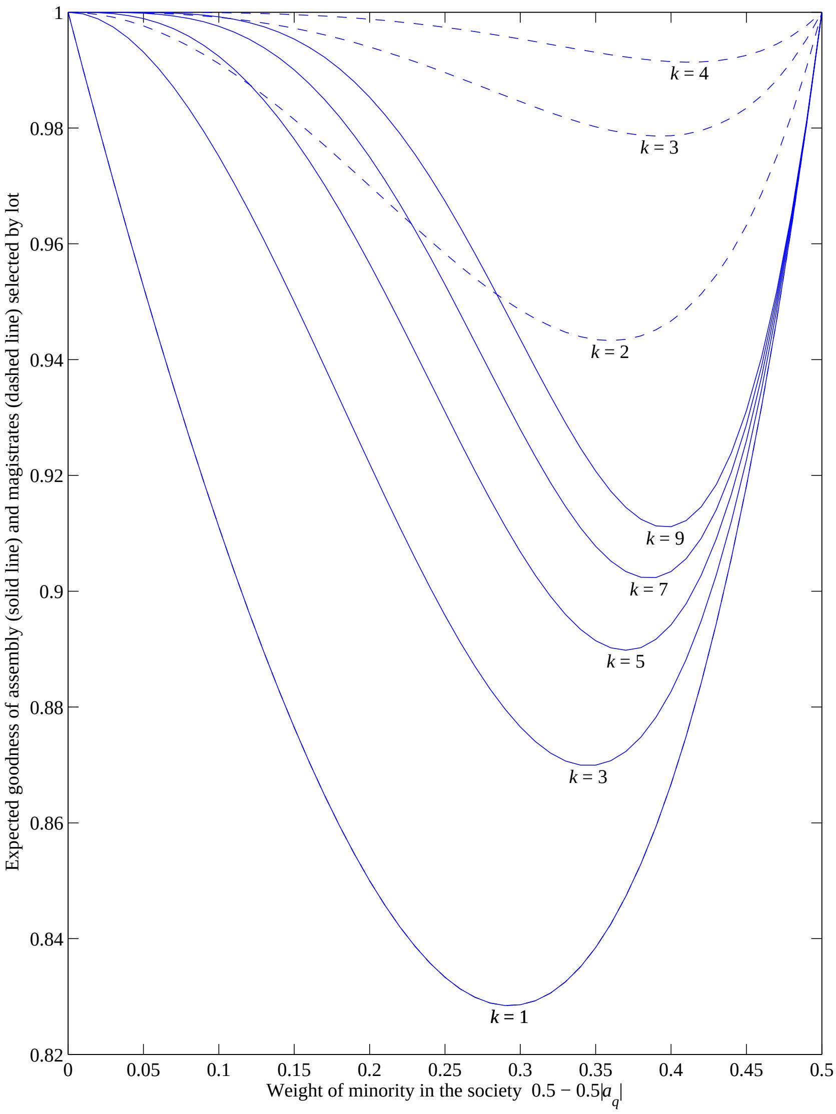


Table 3. Minimal expected goodness of Athenian decisive bodies

Decisive body and its size k	Minimal indicator value, %	Critical minority size, %
President of Committee (1)	82.84	29.29
Committee of 50 (50)	95.67	45.08
Small jury (201)	97.73	47.45
Council of 500 (500)	98.53	48.36
Medium jury (501)	98.53	48.36
Big jury (1501)	99.14	49.04
Assemby quorum (6000)	99.57	49.52
Magistrates (10)	99.99	45.78

is attained. A remarkable intuition of Athenians manifests itself in the matching of representativeness quality of the Assembly with the quorum of 6000 citizens and magistrates with the board of 10 members and the military board with 10 generals. Their goodness is almost equal differing by only 0.4%.

Remark 7. (Inefficiency of democracy in an unstable society)
Political power can be said to be efficient if good results in decision making are achieved by moderate means. If a president is as good as a large Assembly (makes decisions satisfying the same percentage of the population), the efficiency of the president is higher than that of the Assembly.

Figures 3—5 show that personal power can be absolutely efficient when the society is divided by opinions into (almost) equal groups with the majority-to-minority ratio close to 50 : 50%, which is socially unstable, because a small change of individual opinions results in a radical change of the social (majority) decision. In such an unstable society democratic institutions, like parliaments and magistrates, provide the same or just a little better power quality than single representatives selected by lot. This minor advantage is attained at the price of being much larger, meaning the inefficiency of democratic institutions compared to personal power. When the society is stable, i.e. there are dominant groups, large democratic bodies provide a significantly better power quality than single individuals.

To a certain extent this observation explains why tyrants like Pesistratos took the power at the moment of split of the state.

8. Application to German parliament elections 2002

Let us illustrate a certain flexibility of the model. Follow the scenario of Example 1 in application to the German parliament elections held on September 22, 2002.

Parties and their programs. Electors had a choice between five major and several less significant parties. Their comparative programs in a tabular form are given by *Friedrich-Naumann Stiftung* (2002) and Bauer, et al. (2002). In case of contradicting data, we use the data from the first source which is much more comprehensive. Election results are taken from *Der Spiegel* 24.09.2002).

According to the party programs, a number of topical questions have been discussed before the elections. For certain important points, like the participation of Germany in the Iraq crisis, all political parties had similar opinions. On some topics, no specific suggestion has been formulated, so it is difficult to judge about differences between the party programs. Important issues with more or less different specific suggestions fall into the following four categories:

q_1 Radical *employment* policy, e.g. the conceptual program by Peter Hartz (trade union activist, director of personnel at *Volkswagen*),

q_2 *Social cuts*, e.g. an increase in the retirement age.

q_3 An increase in *taxation*, e.g. new ecological taxes.

q_4 Supporting *immigration*, e.g. not conditioning it by economic expediency.

All four questions are assumed equally important, having weights $\mu_q = 25\%$.

Figure 6 displays the pre-election programs coded by +'s and −'s, reflecting a conditional evaluation. For instance, all parties suggest some measures against unemployment, but some are more consistent and specific (they get "+") than others (they get "−"). This code is a (subjective) interpretation of the comparative programs cited. Since the available information concerns five major parties, the minor ones collected under the label "Other" get a "−"-evaluation.

The votes received by a party are interpreted as the percentage of the electorate with the opinions represented by the party. Therefore, the summary table has the weights of candidates attributed to individuals (ν_i but not ξ_c). The row "Weight of protagonists" reflects the balance of *social* opinions on every question derived from the weights ν_i.

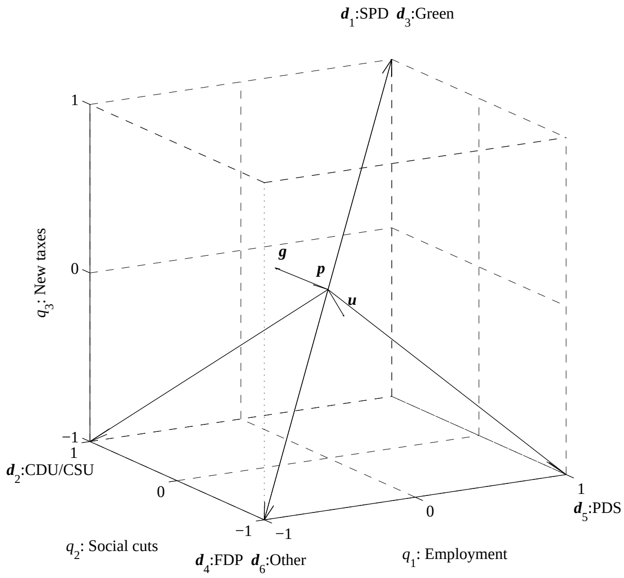
Most popular party. The values of representativeness of parties c on questions q are put next to the party ±-opinions in the summary table in Figure 6.

The most popular parties are SPD and CDU/CSU, having the popularity 66% and 66.9%, meaning that on the average they represent that percentage of the population. The universality of both parties is equal to 75%, meaning that they represent a majority of the population on three of four questions.

The top-ranked party is not necessarily the one with most votes. Although SPD was slightly ahead (which is not seen because of rounding to the first decimal after the point), CDU/CSU has the best both indicators. The two parties have equal universality, but

Figure 6. Analysis of candidate parties in German parliament elections 2002

c Candidate	Weight ν_i	Questions q				Indicators, %		
		1	2	3	4	P_c	U_c	G_c
		Employ-ment	Social cuts	New taxes	Immi-gration	Popula-rity	Univer-sality	Good-ness
1 SPD ($\boldsymbol{d}_1$)	38.5%	+/51.1	+/85.6	+/47.1	−/80.0	66.0	75.0	97.3
2 CDU/CSU ($\boldsymbol{d}_2$)	38.5%	−/48.9	+/85.6	−/52.9	−/80.0	66.9	75.0	98.9
3 Green ($\boldsymbol{d}_3$)	8.6%	+/51.1	+/85.6	+/47.1	+/20.0	51.0	50.0	78.5
4 FDP ($\boldsymbol{d}_4$)	7.4%	−/48.9	−/14.4	−/52.9	+/20.0	34.0	25.0	59.4
5 PDS ($\boldsymbol{d}_5$)	4.0%	+/51.1	−/14.4	−/52.9	+/20.0	34.6	50.0	60.5
6 Other ($\boldsymbol{d}_6$)	3.0%	−/48.9	−/14.4	−/52.9	−/80.0	49.0	50.0	78.1
Weight of protagonists, % $(0.5 + 0.5\boldsymbol{a})$		51.1	85.6	47.1	20.0			
Question weights, % ($\boldsymbol{\mu}$)		25.0	25.0	25.0	25.0			
$\boldsymbol{p} = \boldsymbol{\mu}.*\boldsymbol{a}$, %		0.5	17.8	−1.5	−15.0	$\mathsf{P} = 60.9$		
$\boldsymbol{u} = \boldsymbol{\mu}.*\operatorname{sign}\boldsymbol{a}$, %		25.0	25.0	−25.0	−25.0		$\mathsf{U} = 67.4$	
$\boldsymbol{g} = \boldsymbol{\mu}.*\frac{1}{1+\lvert\boldsymbol{a}\rvert}.*\boldsymbol{a}$, %		0.6	61.8	−1.5	−37.5			$\mathsf{G} = 91.4$



the popularity of CDU/CSU is by 0.9% higher. Taking into account the large number of voters all over Germany, this difference is statistically significant.

Representative parliament with fewest members. The parties have a deficit of universality (no party has the universality 100%), consequently they do not always represent a majority, consequently, they also have a deficit of popularity which decreases in a multi-party parliament. Evaluation of multi-party parliament(s) and finding the best parliaments with fewest voters was performed by formulas of Theorem 1. The results are shown in Table 4, where the best parliaments with $k = 1, 3, 5$ voters are selected (a parliament with $k = 1$ member should be regarded as president who makes decisions alone). The *optimality* in the caption to the table is understood in the sense of Pareto: the optimal parliament is the one, which has no superior with respect to both indices in the class of parliaments with the given number of voters.

Several instances of the same party means that the given party has several votes (= seats in the parliament). For example, the parliament $P = (1, 1, 2, 2, 5)$ consists of two members of the 1st party (SPD), two members of the 2nd party (CDU/CSU), and one member of the 5th party (PDS). The universality of such a parliament is maximal, 100%. Consequently, its popularity 67.4% is maximal as well.

The smallest parliament which provides the absolute maxima of both indicators (which follows from the universality 1) contains three members $1, 2, 5$ (= one from SPD, one from CDU/CSU, and one from PDS).

Representative cabinet with fewest members. Consider cabinets of ministers as magistrates with a board of members. The best cabinets with a few members are shown in Table 5. The smallest cabinet with no deficit of universality and, consequently, with no deficit of popularity contains at least two members. There is a certain choice between two-member cabinets. A cabinet can be formed

Table 4. Optimal parliaments with fewest members

Parliament size k	Parliament votes P	Parliament popularity P_P, %	Parliament universality U_P, %
(∞)	Maximal	$\overline{\mathsf{P}} = 67.4$	$\overline{\mathsf{U}} = 100.0$
1	Average	$\mathsf{P}^{(1)} = 60.9$	$\mathsf{U}^{(1)} = 67.4$
	(2)	66.8	75.0
3	Average	$\mathsf{P}^{(3)} = 63.9$	$\mathsf{U}^{(3)} = 72.5$
	$(1, 2, 5)$	67.4	100.0
5	Average	$\mathsf{P}^{(5)} = 65.2$	$\mathsf{U}^{(5)} = 74.8$
	$(1, 1, 2, 2, 5)$	67.4	100.0
	$(1, 1, 2, 4, 5)$	67.4	100.0
	$(1, 1, 2, 5, 5)$	67.4	100.0
	$(1, 1, 2, 5, 6)$	67.4	100.0
	$(1, 2, 2, 3, 5)$	67.4	100.0
	$(1, 2, 2, 5, 5)$	67.4	100.0
	$(1, 2, 3, 5, 6)$	67.4	100.0

even from representatives of the third party (Green) and of some minor party (from the 6th column "Other" in the summary table in Figure 6). Unlike parliaments, where winning parties must play the leading roles, cabinets can be formed from almost all parties.

At the same time, not every two parties can delegate their members to an optimal cabinet. From 15 cabinets with two members (the number of combinations of two of six elements), only seven are optimal, while others are not. For instance, cabinet $M = (1, 3)$ is not in the list. This means that no optimal cabinet can be formed only from the winner coalition, SPD and Green, so that the participation of the opposition is absolutely necessary. Indeed, as seen from the summary table in Figure 6, neither SPD, nor Green represents a majority for question *New taxes*, so that these parties cannot form a popular cabinet.

In our context, a cabinet with two members does not mean a cabinet with two physical individuals. Since the actual agents in

Table 5. Optimal cabinets with fewest members

Cabinet size k	Cabinet members M	Cabinet popularity P_M, %	Cabinet universality U_M, %
(∞)	Maximal	$\overline{\mathsf{P}} = 67.4$	$\overline{\mathsf{U}} = 100.0$
1	Average	$\mathsf{P}_{(1)} = 60.9$	$\mathsf{U}_{(1)} = 67.4$
	(2)	66.8	75.0
2	Average	$\mathsf{P}_{(2)} = 66.0$	$\mathsf{U}_{(2)} = 87.0$
	$(1,2)$	67.4	100.0
	$(1,4)$	67.4	100.0
	$(1,5)$	67.4	100.0
	$(1,6)$	67.4	100.0
	$(2,3)$	67.4	100.0
	$(2,5)$	67.4	100.0
	$(3,6)$	67.4	100.0

our model are parties, we have to represent parties, which can be done by several physical individuals for each question. For instance, the question *Employment* includes several aspects, requiring a coordination of several ministries, of labor, of technology, of education, etc. All these positions can be given to the members of the corresponding party.

There are also optimal cabinets with more than two members, providing a larger flexibility for appointments. On the other hand, in actuality the number of questions can be large, which restricts the selection of optimal cabinets with few members.

Deficit of popularity and of universality. Table 6 shows the deficit of popularity and universality for parliaments and cabinets selected by lot of size $k = 1, \ldots, 20, \ldots, 601$ computed by the formulas of Theorem 1 for data from Figure 6. (The constitutional number of seats in the *Bundestag* is 598; after elections the proportion of seats is fitted to voting results with the accuracy 1% by adding a few seats; thereby at present there are 603 seats.)

Table 6. Deficit of popularity $\overline{\mathsf{P}} - \mathsf{P}$ and of universality $1 - \mathsf{U}$, %

Size	Parliament				Cabinet			
k	$\mathsf{P}^{(k)}$	$\overline{\mathsf{P}} - \mathsf{P}^{(k)}$	$\mathsf{U}^{(k)}$	$1 - \mathsf{U}^{(k)}$	$\mathsf{P}_{(k)}$	$\overline{\mathsf{P}} - \mathsf{P}_{(k)}$	$\mathsf{U}_{(k)}$	$1 - \mathsf{U}_{(k)}$
1	60.9	6.5	67.4	32.6	60.9	6.5	67.4	32.6
2			70.5	29.5	66.0	1.4	87.0	13.0
3	63.9	3.5	72.5	27.5	67.0	0.4	94.2	5.8
4			73.9	26.1	67.3	0.1	97.3	2.7
5	65.2	2.2	74.8	25.2	67.3	0.1	98.7	1.3
6			75.5	24.5	67.4	0.0	99.4	0.6
7	65.8	1.6	76.1	23.9	67.4	0.0	99.7	0.3
8			76.5	23.5	67.4	0.0	99.9	0.1
9	66.1	1.3	76.8	23.2	67.4	0.0	99.9	0.1
10			77.1	22.9	67.4	0.0	100.0	0.0
11	66.3	1.1	77.4	22.6	67.4	0.0	100.0	0.0
12			77.5	22.5	67.4	0.0	100.0	0.0
13	66.4	1.0	77.7	22.3	67.4	0.0	100.0	0.0
14			77.9	22.1	67.4	0.0	100.0	0.0
15	66.5	0.9	78.0	22.0	67.4	0.0	100.0	0.0
16			78.1	21.9	67.4	0.0	100.0	0.0
17	66.5	0.9	78.2	21.8	67.4	0.0	100.0	0.0
18			78.4	21.6	67.4	0.0	100.0	0.0
19	66.5	0.9	78.5	21.5	67.4	0.0	100.0	0.0
20			78.6	21.4	67.4	0.0	100.0	0.0
101	66.8	0.6	82.7	17.3	67.4	0.0	100.0	0.0
201	66.9	0.5	85.4	14.6	67.4	0.0	100.0	0.0
401	67.0	0.4	88.7	11.3	67.4	0.0	100.0	0.0
601	67.1	0.3	90.7	9.3	67.4	0.0	100.0	0.0

As one can see, the deficit of popularity of a parliament selected by lot with 601 voters is 0.3%. This means that on the average only 0.3% of the population are under-represented, which can be considered quite well. The deficit of universality of such a parliament is about 9.3%. In other words, it makes "wrong" decisions on every 11th question (depreciated by a majority of the society). Since only four questions are considered, "every 11th question" means that approximately every third parliament selected by lot makes a "wrong" decision on one of four questions.

The situation with average cabinets is even more favorable. An average cabinet with already 10 members has no deficit of popularity and of universality. This does not mean that the ministers selected by lot are so perfect, but rather that the domains of their responsibility can be perfectly fitted to already 10 candidates selected by lot.

9. Conclusions

1. Indicators of representativeness, popularity, universality, and goodness are used to evaluate representatives and decisive bodies in a rather broad political context, ranging from Ancient Athens to our days.

2. The model proves the consistent validity of the Athenian democracy with officials selected by lot. The deficit of the indicator values vanishes as the size of decisive bodies increases but it is not dependent on the size of the society.

3. The representative quality of the Assembly with the quorum of 6000 is remarkably well matched to the representative quality of magistrates with 10 board members and to the military board with 10 generals.

4. The model shows that the democratic institutions are inefficient in an unstable society. In such a society personal power is more efficient.

5. The distinction of the model is the emphasis on *calculus* rather than on logic. Indicators can be higher or lower, always allowing to find best compromises, whereas the "yes"—"no" logic restricts solutions exclusively to unobjectionable ones.

10. Proofs

1. Proof of Theorem 1

Lemma 1. (Properties of the incomplete beta-function) *The incomplete beta function (10) increases in p and y and decreases in x. It has the following properties*

$$I_0(x,y) = 0 \tag{25}$$

$$I_1(x,y) = 1 \tag{26}$$

$$I_p(x,y) = 1 - I_{1-p}(y,x) \tag{27}$$

$$I_p(x,x) = 0.5\, I_{1-(1-2p)^2}(x, 0.5), \quad 0 \le p \le 0.5 \tag{28}$$

$$I_p(x, k-x+1) = \sum_{x \le j \le k} \binom{k}{j} p^j (1-p)^{k-j} \quad \text{for integer } 0 < x \le k \tag{29}$$

$$1 - 2I_{0.5-0.5a}(x,x) = I_{a^2}(0.5, x), \quad 0 \le a \le 0.5 \tag{30}$$

$$I_p\left(\tfrac{k+1}{2}, \tfrac{k+1}{2}\right) = \sum_{k/2 < j \le k} \binom{k}{j} p^j (1-p)^{k-j} \quad \text{for odd } k \tag{31}$$

$$I_p\left(\tfrac{k+1}{2}, \tfrac{k+1}{2}\right) \le \frac{1}{18(k+2)(0.5-p)^2}, \quad 0 \le p < 0.5, \text{ for odd } k \tag{32}$$

Proof of Lemma 1. For (25)—(29) see (Abramowitz and Stegun 1972, 26.5.1, 25.5.2, 25.5.14, and 26.5.24).

Formula (30) results from applying (27) and (28).

Formula (31) results from substituting $x = \frac{k+1}{2}$ into (29).

To prove (32) consider the incomplete beta-function as a distribution function of a random variable ξ. Then it has the mathematical expectation and variance (Abramowitz and Stegun 1972, 26.1.33)

$$\mathsf{E}\xi = \frac{x}{x+y} \qquad \mathsf{V}\xi = \frac{xy}{(x+y)^2(x+y+1)}.$$

For $x = y = \frac{k+1}{2}$ we obtain

$$\mathsf{E}\xi = 0.5 \qquad \mathsf{V}\xi = \frac{1}{4(k+2)}.$$

Since in this case the distribution is unimodal and symmetrical with respect to 0.5, we obtain from the strengthened Chebyshev inequality (Abramowitz and Stegun 1972, 26.1.40)

$$\begin{aligned} I_p\left(\frac{k+1}{2}, \frac{k+1}{2}\right) &= \mathsf{Prob}\{\xi \le p\} \qquad (\mathsf{E}\xi = 0.5,\ p < 0.5) \\ &= \frac{1}{2}\mathsf{Prob}\Big\{|\xi - \mathsf{E}\xi| \ge 0.5 - p\} \\ &\le \frac{1}{2}\,\frac{4}{9}\,\frac{\mathsf{V}\xi}{(0.5-p)^2} \\ &= \frac{1}{18(k+2)(0.5-p)^2}, \end{aligned}$$

as required. ■

Proof of Theorem 1. By L_q denote the non-strict minority in the society for question q; if $a_q = 0$ than take protagonists. Its weight is obviously

$$\text{Weight of minority} = \nu_{L_q} = \underbrace{0.5}_{\substack{\text{half}\\ \text{the}\\ \text{society}}} \; -0.5 \; \underbrace{|a_q|}_{\substack{\text{predominance}\\ \text{of protagonists}\\ \text{over antagonists}\\ \text{in the society}}}. \tag{33}$$

Express representativeness, rounded representativeness, and representativeness-to-majority ratio. For given decisive body D and question q, derive from (3)

$$r_{Dq} = \underbrace{0.5}_{\substack{\text{half}\\\text{the}\\\text{society}}} + 0.5 \underbrace{a_q}_{\substack{\text{predominance}\\\text{of protagonists}\\\text{over antagonists}\\\text{in the society}}} \underbrace{b_{Dq}}_{\substack{\text{opinion of}\\\text{decisive}\\\text{body } D}} \tag{34}$$

$$\text{round}[r_{Dq}] = 0.5 + 0.5 \operatorname{sign} a_q \operatorname{sign} b_{Dq} + 0.5 \underbrace{\delta_{a_q}}_{= \begin{cases} 1 & \text{if} \quad a_q = 0 \\ 0 & \text{if} \quad a_q \neq 0 \end{cases}} \tag{35}$$

$$\frac{r_{Dq}}{1 - \nu_{L_q}} \overset{\text{by (33) and (34)}}{=} \frac{0.5 + 0.5 a_q b_{Dq}}{0.5 + 0.5|a_q|} = \frac{1}{1 + |a_q|} + \frac{a_q}{1 + |a_q|} b_{Dq}. \tag{36}$$

To obtain (11)—(13) for a *given* decisive body D, multiply μ_q by each equality (34)—(36) and summarize on q.

Now estimate the expected indicators of a decisive body *selected by lot*. By $\mathcal{L}_q \subset C^k$ denote the set of decisive bodies D of size k which make "wrong" decisions on question q, that is, represent a strict minority of the society; if $a_q = 0$ then $\mathcal{L}_q$ is empty. By $\xi^k_{\mathcal{L}_q}$ denote the total weight of these decisive bodies. Hence,

$$r_{Dq} = \begin{cases} \nu_{L_q} & \text{if} \quad D \in \mathcal{L}_q \\ 1 - \nu_{L_q} & \text{if} \quad D \notin \mathcal{L}_q. \end{cases} \tag{37}$$

Then the expected indices are as follows:

$$\begin{aligned} \mathsf{P} &= \sum_D \xi^k_D \sum_q \mu_q \, r_{Dq} \\ &= \sum_q \mu_q \left[\sum_{D \in \mathcal{L}_q} \xi^k_D \, r_{Dq} + \sum_{D \notin \mathcal{L}_q} \xi^k_D \, r_{Dq} \right] \overset{\text{by (37)}}{\Longrightarrow} \\ &= \sum_q \mu_q \left[\xi^k_{\mathcal{L}_q} \, \nu_{L_q} + \left(1 - \xi^k_{\mathcal{L}_q}\right) \left(1 - \nu_{L_q}\right) \right] \overset{\text{identity}}{\Longrightarrow} \end{aligned}$$

$$= \underbrace{\sum_q \mu_q \Big[1 - \nu_{L_q}}_{\overline{\mathsf{P}}} - \left(1 - 2\nu_{L_q}\right) \xi^k_{\mathcal{L}_q}\Big]$$

$$= \underbrace{\overline{\mathsf{P}}}_{0.5+0.5\sum_q \mu_q |a_q|} - \sum_q \mu_q \underbrace{\left(1 - 2\nu_{L_q}\right)}_{|a_q|} \xi^k_{\mathcal{L}_q} \tag{38}$$

$$\mathsf{U} = \sum_D \xi^k_D \sum_q \mu_q \,\text{round}[r_{Dq}] \quad \overset{\text{by (37)}}{\Longrightarrow}$$

$$= \sum_{q:a_q=0} \mu_q \cdot 1 + \sum_{q:a_q\neq 0} \mu_q \left(\sum_{D\in\mathcal{L}_q} \xi^k_D \cdot 0 + \sum_{D\notin\mathcal{L}_q} \xi^k_D \cdot 1 \right)$$

$$= \underbrace{\sum_{q:a_q=0} \mu_q + \sum_{q:a_q\neq 0} \mu_q \Big(1}_{=1} - \xi^k_{\mathcal{L}_q}\Big)$$

$$= 1 - \sum_{q:a_q\neq 0} \mu_q \, \xi^k_{\mathcal{L}_q} \tag{39}$$

$$\mathsf{G} = \sum_D \xi^k_D \sum_q \mu_q \frac{r_{Dq}}{1 - \nu_{L_q}} \quad \overset{\text{identity}}{\Longrightarrow}$$

$$= \sum_q \mu_q \left(\sum_{D\in\mathcal{L}_q} \xi^k_D \frac{\nu_{L_q}}{1 - \nu_{L_q}} + \sum_{D\notin\mathcal{L}_q} \xi^k_D \cdot 1 \right) \quad \overset{\text{identity}}{\Longrightarrow}$$

$$= \sum_q \mu_q \left(\xi^k_{\mathcal{L}_q} \frac{\nu_{L_q}}{1 - \nu_{L_q}} + 1 - \xi^k_{\mathcal{L}_q} \right) \quad \overset{\text{identity}}{\Longrightarrow}$$

$$= 1 - \sum_q \mu_q \left(\frac{1 - 2\nu_{L_q}}{1 - \nu_{L_q}} \right) \xi^k_{\mathcal{L}_q} \quad \overset{\text{identity}}{\Longrightarrow}$$

$$= 1 - \sum_q \mu_q \left(\frac{|a_q|}{0.5 + 0.5|a_q|} \right) \xi^k_{\mathcal{L}_q} \tag{40}$$

Case of parliament. Find $\xi^k_{\mathcal{L}_q}$ which is the probability to select a parliament with at least half the voters being representatives of the strict-minority on question q.

1. At first suppose that the strict minority of the society for question q is represented by a strict minority of candidates with the total weight

$$\xi_{L_q} = 0.5 - 0.5|b_q|. \tag{41}$$

The probability of selecting at least $\frac{k+1}{2}$ of k voters which represent the minority is given by the binomial sum:

$$\begin{aligned} \xi^k_{\mathcal{L}_q} &= \sum_{\frac{k+1}{2} \le j \le k} \left(\begin{array}{c} k \\ j \end{array} \right) (\xi_{L_q})^j (1 - \xi_{L_q})^{k-j} \overset{\text{by (31)}}{\Longrightarrow} \\ &= I_{\xi_{L_q}} \left(\frac{k+1}{2}, \frac{k+1}{2} \right). \end{aligned} \tag{42}$$

Substitute (42) into (38)–(40), and obtain for parliaments selected by lot

$$\begin{aligned} \mathsf{P}^{(k)} &= 0.5 + 0.5 \sum_q \mu_q |a_q| - \\ &\sum_q \mu_q |a_q| \, I_{0.5-0.5|b_q|} \left(\frac{k+1}{2}, \frac{k+1}{2} \right) = 0.5 + \\ &0.5 \sum_q \mu_q |a_q| \left[1 - 2 I_{0.5-0.5|b_q|} \left(\frac{k+1}{2}, \frac{k+1}{2} \right) \right] \overset{\text{by (30)}}{\Longrightarrow} \\ &= 0.5 + 0.5 \sum_q \mu_q \, |a_q| \, I_{b_q^2} \left(\frac{1}{2}, \frac{k+1}{2} \right) \end{aligned} \tag{43}$$

$$\begin{aligned} \mathsf{U}^{(k)} &= 1 - \\ &\sum_{q: a_q \neq 0} \mu_q \, \frac{|a_q|}{0.5 + 0.5|a_q|} \, I_{0.5-0.5|b_q|} \left(\frac{k+1}{2}, \frac{k+1}{2} \right) \\ &= 0.5 + 0.5 \sum_{q: a_q = 0} \mu_q + 0.5 \sum_{q: a_q \neq 0} \mu_q - \end{aligned}$$

$$\sum_{q:a_q\neq 0} \mu_q\, I_{0.5-0.5|b_q|}\left(\frac{k+1}{2},\frac{k+1}{2}\right) = 0.5 + 0.5\sum_{q:a_q=0}\mu_q +$$

$$0.5\sum_{q:a_q\neq 0}\mu_q\left[1-2I_{0.5-0.5|b_q|}\left(\frac{k+1}{2},\frac{k+1}{2}\right)\right] \overset{\text{by (30)}}{\Longrightarrow}$$

$$= 0.5 + 0.5\boldsymbol{\mu}'\delta_{\boldsymbol{a}} + 0.5\sum_q \mu_q\, I_{b_q^2}\left(\frac{1}{2},\frac{k+1}{2}\right) \quad (44)$$

$$\mathsf{G}^{(k)} = \sum_q \mu_q\left[1-\frac{2|a_q|I_{0.5-0.5|b_q|}\left(\frac{k+1}{2},\frac{k+1}{2}\right)}{1+|a_q|}\right]$$

$$= \sum_q \mu_q \frac{1+|a_q|\left[1-2I_{0.5-0.5|b_q|}\left(\frac{k+1}{2},\frac{k+1}{2}\right)\right]}{1+|a_q|} \overset{\text{by (30)}}{\Longrightarrow}$$

$$= \sum_q \mu_q \frac{1+|a_q|\, I_{b_q^2}\left(\frac{1}{2},\frac{k+1}{2}\right)}{1+|a_q|}$$

$$= \boldsymbol{\mu}'\frac{1}{1+|\boldsymbol{a}|} + \sum_q \mu_q\,\frac{|a_q|}{1+|a_q|} I_{b_q^2}\left(\frac{1}{2},\frac{k+1}{2}\right) \quad (45)$$

2. If the strict minority of the society is represented by a majority of candidates then take the complementary probability

$$\mathsf{P}^{(k)} = 0.5 + 0.5\sum_q \mu_q|a_q| -$$

$$\sum_q \mu_q|a_q|\left[1-I_{0.5-0.5|b_q|}\left(\frac{k+1}{2},\frac{k+1}{2}\right)\right] = 0.5 -$$

$$0.5\sum_q \mu_q|a_q|\left[1-2I_{0.5-0.5|b_q|}\left(\frac{k+1}{2},\frac{k+1}{2}\right)\right] \overset{\text{by (30)}}{\Longrightarrow}$$

$$= 0.5 - 0.5\sum_q \mu_q\,|a_q|\, I_{b_q^2}\left(\frac{1}{2},\frac{k+1}{2}\right) \quad (46)$$

The formulas (43) and (46) are united by inserting the factor $\operatorname{sign} a_q \operatorname{sign} b_q = 0, \pm 1$ for controlling the distinctive sign $\pm$:

$$\begin{aligned} \mathsf{P}^{(k)} &= 0.5 + 0.5 \sum_q \mu_q |a_q| \operatorname{sign} a_q \operatorname{sign} b_q \, I_{b_q^2}\left(\frac{1}{2}, \frac{k+1}{2}\right) \\ &= 0.5 + 0.5(\boldsymbol{\mu} . * \boldsymbol{a})' \left[\operatorname{sign} \boldsymbol{b} . * I_{\boldsymbol{b}^2}\left(\frac{1}{2}, \frac{k+1}{2}\right)\right]. \end{aligned}$$

Similarly, one obtains the general formulas for universality and goodness of parliaments selected by lot:

$$\mathsf{U}^{(k)} = 0.5 + 0.5\boldsymbol{\mu}'\delta_{\boldsymbol{a}} + 0.5(\boldsymbol{\mu} . * \operatorname{sign} \boldsymbol{a})' \left[\operatorname{sign} \boldsymbol{b} . * I_{\boldsymbol{b}^2}\left(\frac{1}{2}, \frac{k+1}{2}\right)\right]$$

$$\mathsf{G}^{(k)} = \boldsymbol{\mu}' \frac{1}{1+|\boldsymbol{a}|} + \left(\boldsymbol{\mu} . * \frac{1}{1+|\boldsymbol{a}|} . * \boldsymbol{a}\right)' \left[\operatorname{sign} \boldsymbol{b} . * I_{\boldsymbol{b}^2}\left(\frac{1}{2}, \frac{k+1}{2}\right)\right],$$

which are (11)–(13) for parliaments selected by lot.

Case of magistrate. By definition a magistrate represents a strict minority of the society if and only if all its k members represent the minority.

1. Suppose that the strict minority of the society for question q is represented by a strict minority of candidates with the total weight (41). The probability that all magistrate members belong to the strict minority of candidates is

$$\xi^k_{\mathcal{L}_q} = \left(\xi_{L_q}\right)^k = (0.5 - 0.5|b_q|)^k = \left(\frac{1-|b_q|}{2}\right)^k. \qquad (47)$$

Substitute (47) into (38)—(40):

$$
\begin{aligned}
\mathsf{P}_{(k)} &= 0.5 + 0.5\sum_q \mu_q|a_q| - \sum_q \mu_q|a_q|\left(\frac{1-|b_q|}{2}\right)^k \\
&= 0.5 + 0.5\sum_q \mu_q|a_q|\left[1 - 2\left(\frac{1-|b_q|}{2}\right)^k\right] \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{U}_{(k)} &= 1 - \sum_{q:a_q\neq 0} \mu_q\left(\frac{1-|b_q|}{2}\right)^k \quad \overset{\text{identity}}{\Longrightarrow} \\
&= 0.5 + 0.5\left[\sum_{q:a_q=0}\mu_q + \sum_{q:a_q\neq 0}\mu_q - 2\sum_q \mu_q\left(\frac{1-|b_q|}{2}\right)^k\right] \\
&= 0.5 + 0.5\boldsymbol{\mu}'\delta_{\boldsymbol{a}} + 0.5\sum_{q:a_q\neq 0}\mu_q\left[1 - 2\left(\frac{1-|b_q|}{2}\right)^k\right] \qquad (49)
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{G}_{(k)} &= 1 - \sum_q \mu_q \frac{2|a_q|}{1+|a_q|}\left(\frac{1-|b_q|}{2}\right)^k \quad \overset{\text{identity}}{\Longrightarrow} \\
&= \sum_q \mu_q\left[1 - \frac{2|a_q|}{1+|a_q|}\left(\frac{1-|b_q|}{2}\right)^k\right] \quad \overset{\text{identity}}{\Longrightarrow} \\
&= \sum_q \mu_q\left[\frac{1}{1+|a_q|} + \frac{2|a_q|}{1+|a_q|}\left(\frac{1-|b_q|}{2}\right)^k\right] \qquad (50)
\end{aligned}
$$

2. Suppose that the strict minority of the society for question q is represented by a majority of candidates with the total weight $0.5 + 0.5|b_q|$. The probability that all magistrate members belong to the majority of candidates is

$$
\xi^k_{\mathcal{L}_q} = \left(\xi_{L_q}\right)^k = (0.5 + 0.5|b_q|)^k = 2^{-k}(1+|b_q|)^k. \qquad (51)
$$

Substitute (51) into (38) and obtain similarly to the Case 1:

$$
\mathsf{P}_{(k)} = 0.5 + 0.5\sum_q \mu_q|a_q|\left[1 - 2\left(\frac{1+|b_q|}{2}\right)^k\right] \qquad (52)
$$

The formulas (48) and (52) are united by inserting the factor $\operatorname{sign} a_q \operatorname{sign} b_q$ for controlling the distinctive sign $\pm$ (and annihilation of the terms with $a_q = 0$):

$$\begin{aligned}\mathsf{P}_{(k)} &= 0.5 + 0.5 \sum_q \mu_q |a_q| \left[1 - 2\left(\frac{1 + \operatorname{sign} a_q \operatorname{sign} b_q |b_q|}{2}\right)^k\right] \\ &= 0.5 + 0.5(\boldsymbol{\mu}. * \boldsymbol{a})' \left\{\operatorname{sign} \boldsymbol{a} . * \left[1 - 2^{1-k}(1 - \operatorname{sign} \boldsymbol{a}. * \boldsymbol{b}).^k\right]\right\}.\end{aligned}$$

Similarly, one obtains the general formulas for universality and goodness of parliaments selected by lot:

$$\begin{aligned}\mathsf{U}_{(k)} = 0.5 + 0.5\boldsymbol{\mu}'\delta_{\boldsymbol{a}} + 0.5\Big(\boldsymbol{\mu}. * \operatorname{sign} \boldsymbol{a}\Big)' \cdot \\ \left\{\operatorname{sign} \boldsymbol{a} . * \left[1 - 2\left(\frac{1 + \operatorname{sign} \boldsymbol{a} . * \boldsymbol{b}}{2}\right).^k\right]\right\} \\ \mathsf{G}_{(k)} = \boldsymbol{\mu}' \frac{1}{1 + |\boldsymbol{a}|} + \left(\boldsymbol{\mu}. * \frac{1}{1 + |\boldsymbol{a}|} . * \boldsymbol{a}\right)' \cdot \\ \left\{\operatorname{sign} \boldsymbol{a} . * \left[1 - 2\left(\frac{1 + \operatorname{sign} \boldsymbol{a} . * \boldsymbol{b}}{2}\right).^k\right]\right\},\end{aligned}$$

which are (11)—(13) for magistrates selected by lot.

Case of president. This is a particular case of magistrate of size $k = 1$. ∎

2. Proof of Theorem 2

PROOF OF THEOREM 2. Since by assumption the selection by lot of candidates is performed from the individuals, $\boldsymbol{\xi} = \boldsymbol{\nu}$, $b_q = a_q$ for all questions q, and the minority in the society $\nu_{L_q} = 0.5 - 0.5|a_q|$ is at the same time the minority in the set of candidates.

Case of parliament. To estimate the expected deficit of the indicators of a parliament selected by lot, substitute (42) into (38)–

(40), taking into account that if $a_q = 0$ then there is no strict minority and, consequently, $\xi^k_{\mathcal{L}_q} = \nu^k_{\mathcal{L}_q} = 0$:

$$
\begin{aligned}
\overline{\mathsf{P}} - \mathsf{P}^{(k)} &= \sum_{q:a_q\neq 0} \mu_q \left(1 - 2\nu_{L_q}\right) I_{\nu_{L_q}}\left(\frac{k+1}{2}, \frac{k+1}{2}\right) \overset{\text{by (32)}}{\Longrightarrow} \\
&\leq \sum_{q:a_q\neq 0} \mu_q \frac{1 - 2\nu_{L_q}}{18(k+2)(0.5 - \nu_{L_q})^2} \overset{\text{since } \nu_{L_q} = 0.5 - 0.5|a_q|}{\Longrightarrow} \\
&= \sum_{q:a_q\neq 0} \mu_q \frac{2}{9(k+2)|a_q|} \\
&\leq \frac{2}{9(k+2)\min_{q:a_q\neq 0}|a_q|} \\
1 - \mathsf{U}^{(k)} &= \sum_{q:a_q\neq 0} \mu_q I_{\nu_{L_q}}\left(\frac{k+1}{2}, \frac{k+1}{2}\right) \overset{\text{by (32)}}{\Longrightarrow} \\
&\leq \sum_{q:a_q\neq 0} \mu_q \frac{1}{18(k+2)(0.5 - \nu_{L_q})^2} \overset{\text{since } \nu_{L_q} = 0.5 - 0.5|a_q|}{\Longrightarrow} \\
&\leq \frac{2}{9(k+2)\min_{q:a_q\neq 0} a_q^2}
\end{aligned}
$$

$$
1 - \mathsf{G}^{(k)} = \sum_{q:a_q\neq 0} \mu_q \left(\frac{1 - 2\nu_{L_q}}{1 - \nu_{L_q}}\right) I_{\nu_{L_q}}\left(\frac{k+1}{2}, \frac{k+1}{2}\right) \overset{\text{by (32)}}{\Longrightarrow}
$$

$$
\leq \sum_{q:a_q\neq 0} \mu_q \left(\frac{1 - 2\nu_{L_q}}{1 - \nu_{L_q}}\right) \frac{1}{18(k+2)(0.5 - \nu_{L_q})^2} \overset{\text{since } \nu_{L_q} = 0.5 - 0.5|a_q|}{\Longrightarrow}
$$

$$
\leq \frac{4}{9(k+2)\min_{q:a_q\neq 0}|a_q|}.
$$

Case of magistrate. Estimate the expected deficit of indicators for a magistrate selected by lot. First of all find the maximum of $f(a) = a\,(1-a)^k$, $0 \leq a \leq 1$. For this purpose solve $f'(a) = 0$, which gives $a^* = \frac{1}{k+1}$. Hence,

$$
\begin{aligned}
f(a^*) &= \frac{k^k}{(k+1)^{k+1}} \\
&= k^{-1} \underbrace{\left(1 - \frac{1}{k+1}\right)^{k+1}}_{\to e^{-1}+ \ \leq \frac{1}{4}} . \qquad (53)
\end{aligned}
$$

Substitute (47) into (38)—(40), taking into account that if $a_q = 0$ then $\nu^k_{\mathcal{L}_q} = 0$:

$$
\begin{aligned}
\overline{\mathsf{P}} - \mathsf{P}_{(k)} &= \sum_{q:a_q \neq 0} \mu_q \left(1 - 2\nu_{L_q}\right) \left(\nu_{L_q}\right)^k \quad \overset{\text{since } \nu_{L_q} = 0.5 - 0.5|a_q|}{\Longrightarrow} \\
&= 2^{-k} \sum_{q:a_q \neq 0} \mu_q \, |a_q|(1 - |a_q|)^k \\
&\leq 2^{-k} \max_{0 \leq a \leq 1} a(1-a)^k \overset{\text{by (53)}}{\Longrightarrow} \leq 2^{-k-2} k^{-1} \\
1 - \mathsf{U}_{(k)} &= 2^{-k} \sum_{q:a_q \neq 0} \mu_q \left(\nu_{L_q}\right)^k \quad \overset{\text{since } \nu_{L_q} = 0.5 - 0.5|a_q|}{\Longrightarrow} \\
&= 2^{-k} \sum_{q:a_q \neq 0} \mu_q \underbrace{\left(1 - |a_q|\right)^k}_{<1} < 2^{-k}
\end{aligned}
$$

$$
\begin{aligned}
1 - \mathsf{G}_{(k)} = 1 - \\
\sum_{q:a_q \neq 0} \mu_q \left(\frac{1 - 2\nu_{L_q}}{1 - \nu_{L_q}}\right) \left(\nu_{L_q}\right)^k \overset{\text{by (32) and since } \nu_{L_q} = 0.5 - 0.5|a_q|}{\Longrightarrow} \\
\leq \sum_{q:a_q \neq 0} \mu_q \frac{|a_q|}{0.5 + 0.5|a_q|} \cdot \frac{1}{18(k+2)0.5^2|a_q|^2} \\
\leq \frac{4}{9(k+2)\min_{q:a_q \neq 0} |a_q|} .
\end{aligned}
$$

Case of president This is a particular case of magistrate of size $k = 1$. ∎

3. Proof of Theorem 3

Lemma 2. (Variance of indicators of decisive bodies selected by lot)
The variance of popularity and of universality of a decisive body D of size k selected by lot from the individuals is as follows:

$$\mathsf{VP}_D = \overline{\mathsf{P}}^2 - \mathsf{P}^2 - \sum_{p,q} \mu_p\mu_q \Big[x_{pq}|a_p| + y_{pq}|a_p|\Big(1+|a_q|\Big)\Big] \quad (54)$$

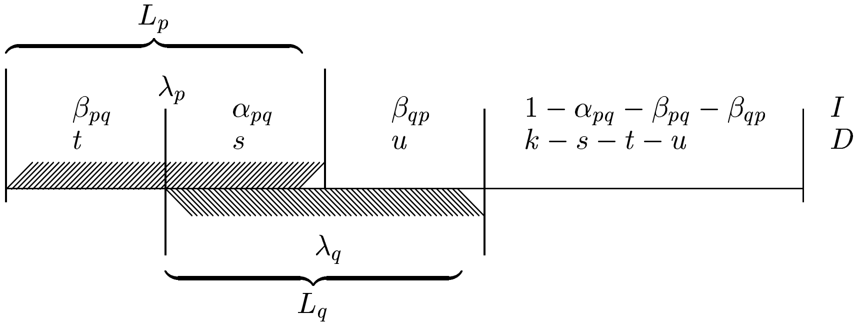
$$\mathsf{VU}_D = 1 - \mathsf{U}^2 - 2(1-\mathsf{U})\boldsymbol{\mu}'\delta_{\boldsymbol{a}} - \sum_{\substack{p\,:\,a_p \neq 0\\ q\,:\,a_q \neq 0}} \mu_p\mu_q\left(x_{pq} + 2y_{pq}\right) \quad (55)$$

$$\mathsf{VG}_D = 1 - \mathsf{G}^2 - 4\sum_{p,q} \mu_p\mu_q \frac{|a_p|}{1+|a_q|}\left(\frac{x_{pq}}{1+|a_p|} + y_{pq}\right), \quad (56)$$

where

$$x_{pq} = \begin{cases} \displaystyle\sum_{\substack{0 \leq \quad s,t,u \\ \quad s+t+u \leq k \\ k/2 \leq \quad s+t \quad \leq k \\ k/2 \leq \quad s+v \quad \leq k}} \frac{k!\,\alpha_{pq}^s\beta_{pq}^t\beta_{qp}^u(1-\alpha_{pq}-\beta_{pq}-\beta_{qp})^{k-s-t-u}}{s!\,t!\,u!\,(k-s-t-u)!} \\ \text{for parliaments} \\ \alpha_{pq}^k \\ \text{for magistrates} \end{cases}$$

Figure 7. Subdivision of the society and allocation of members of a representative body



$$y_{pq} = \begin{cases} \displaystyle\sum_{\substack{0 \le \; s,t,u \\ s+t+u \le k \\ k/2 \le \; s+t \; \le k \\ 0 \le \; s+u \; < k/2}} \frac{k!\, \alpha_{pq}^{s} \beta_{pq}^{t} \beta_{qp}^{u} (1-\alpha_{pq}-\beta_{pq}-\beta_{qp})^{k-s-t-u}}{s!\, t!\, u!\, (k-s-t-u)!} \\ \text{for parliaments} \\ 2^{-k}(1-|a_p|)^k - \alpha_{pq}^k \\ \text{for magistrates} \end{cases}$$

$$\{\alpha_{pq}\} = 0.25\,[1 - \boldsymbol{A}\ \mathrm{diag}(\mathrm{Sign}\,\boldsymbol{a})]\ \mathrm{diag}\,\boldsymbol{\nu}\,[1 - \boldsymbol{A}\ \mathrm{diag}(\mathrm{Sign}\,\boldsymbol{a})]' \quad (\alpha_{pq} = \alpha_{qp}) \tag{57}$$

$$\{\beta_{pq}\} = 0.25\,[1 - \boldsymbol{A}\ \mathrm{diag}(\mathrm{Sign}\,\boldsymbol{a})]\ \mathrm{diag}\,\boldsymbol{\nu}\,[1 + \boldsymbol{A}\ \mathrm{diag}(\mathrm{Sign}\,\boldsymbol{a})]' \tag{58}$$

$$\mathrm{sign}\,x = \begin{cases} 1 & \text{if} \quad x \ge 0 \\ -1 & \text{if} \quad x < 0 \end{cases} \tag{59}$$

Proof of Lemma 2. By L_p and L_q denote the non-strict minorities with respect to questions p and q with weights $\lambda_p = 0.5 - 0.5|a_p|$ and $\lambda_q = 0.5 - 0.5|a_q|$, respectively; if the society is divided into equally large groups, take the antagonists. Consider a decisive body D of size k and introduce the following notation (see Figure 7):

$\alpha_{pq} = \alpha_{qp} = \nu\,(L_p \cap L_q)$, the total weight of individuals who belong both to L_p and L_q,

$\beta_{pq} = \nu(L_p \setminus L_q)$, the total weight of individuals who belong to L_p but not L_q,

$x_{pq} = \nu^k(\mathcal{X}_{pq})$, $\mathcal{X}_{pq} = \{D \in I^n : D$ represents L_p on question p and L_q on question $q\}$, that is, x_{pq} is the probability to select a decisive body D which represents L_p on question p, and L_q on question q (then $r_{Dp} = \lambda_p$ and $r_{Dq} = \lambda_q$),

$y_{pq} = \nu^k(\mathcal{Y}_{pq})$, $\mathcal{Y}_{pq} = \{D \in I^n : D$ represents L_p on question p and $I^k \setminus L_q$ on question $q\}$, that is, y_{pq} is the probability to select a decisive body D which represents L_p on question p, and $I^k \setminus L_q$ on question q (then $r_{Dp} = \lambda_p$ and $r_{Dq} = 1 - \lambda_q$).

For the second moment of popularity we have

$$\mathsf{E}\,\mathsf{P}_D^2 = \sum_D \nu_D^k \mathsf{P}_D^2 \quad \overset{\text{by (5)}}{\Longrightarrow}$$

$$= \sum_D \nu_D^k \left(\sum_q \mu_q r_{Dq}\right)^2 \quad \overset{\text{expand}}{\Longrightarrow}$$

$$= \sum_D \nu_D^k \sum_{p,q} \mu_p \mu_q r_{Dp} r_{Dq} \quad \overset{\text{change summation order}}{\Longrightarrow}$$

$$= \sum_{p,q} \mu_p \mu_q \sum_D \nu_D^k r_{Dp} r_{Dq} \quad \overset{\text{make four sums}}{\Longrightarrow}$$

$$= \sum_{p,q} \mu_p \mu_q \left(\sum_{D \in \mathcal{X}_{pq}} \nu_D^k r_{Dp} r_{Dq} + \sum_{D \in \mathcal{Y}_{pq}} \nu_D^k r_{Dp} r_{Dq} + \sum_{D \in \mathcal{Y}_{qp}} \nu_D^k r_{Dp} r_{Dq} \right.$$

$$\left. + \sum_{D \in I^k \setminus \mathcal{X}_{pq} \setminus \mathcal{Y}_{pq} \setminus \mathcal{Y}_{qp}} \nu_D^k r_{Dp} r_{Dq} \right) \quad \overset{\text{see Lemma's notation}}{\Longrightarrow}$$

$$
\begin{aligned}
&= \sum_{p,q} \mu_p \mu_q \Big[x_{pq}\lambda_p\lambda_q + y_{pq}\lambda_p(1-\lambda_q) + y_{qp}(1-\lambda_p)\lambda_q \\
&\quad +(1-x_{pq}-y_{pq}-y_{qp})(1-\lambda_p)(1-\lambda_q)\Big] \stackrel{\text{regroup w.r.t. } x_{pq},\, y_{pq},\, y_{qp}}{\Longrightarrow} \\
&= \underbrace{\sum_{p,q} \mu_p\mu_q(1-\lambda_p)(1-\lambda_q)}_{\left[\sum_q \mu_q(1-\lambda_q)\right]^2} - \sum_{p,q}\mu_p\mu_q x_{pq} + \sum_{p,q}\mu_p\mu_q x_{pq}\lambda_p + \\
&\quad \underbrace{\sum_{p,q}\mu_p\mu_q x_{pq}\lambda_q}_{=\sum_{p,q}\mu_p\mu_q x_{pq}\lambda_p} - \sum_{p,q}\mu_p\mu_q y_{pq}(1-2\lambda_p)(1-\lambda_q) - \\
&\quad \underbrace{\sum_{p,q}\mu_p\mu_q y_{qp}(1-\lambda_p)(1-2\lambda_q)}_{=\sum_{p,q}\mu_p\mu_q y_{pq}(1-2\lambda_p)(1-\lambda_q)} \qquad \stackrel{\text{identity}}{\Longrightarrow} \\
&= \underbrace{\Big[\sum_q \mu_q \underbrace{(1-\lambda_q)}_{\text{majority}}\Big]^2}_{\overline{\mathsf{P}}^2 \text{ by (15)}} - \\
&\quad \sum_{p,q}\mu_p\mu_q\Big[x_{pq}(1-2\lambda_p) + 2y_{pq}(1-2\lambda_p)(1-\lambda_q)\Big]. \qquad (60)
\end{aligned}
$$

Since $\mathsf{E}\,\mathsf{P}_B = \mathsf{P}$ by (7), we obtain (54):

$$
\begin{aligned}
\mathsf{V}\,\mathsf{P}_D &= \mathsf{E}\,\mathsf{P}_D^2 - (\mathsf{E}\,\mathsf{P}_D)^2 \stackrel{\text{use (60) and substitute } \lambda_p = 0.5-0.5|a_p|}{\Longrightarrow} \\
&= \overline{\mathsf{P}}^2 - \sum_{p,q}\mu_p\mu_q\Big[x_{pq}|a_p| + y_{pq}|a_p|\Big(1+|a_q|\Big)\Big] - \mathsf{P}^2.
\end{aligned}
$$

For the second moment of universality we have:

$$\mathsf{E}\,\mathsf{U}_D^2 = \sum_D \nu_D^k \mathsf{U}_D^2 \quad \overset{\text{by (5)}}{\Longrightarrow}$$

$$= \sum_D \nu_D^k \left(\sum_q \mu_q \operatorname{round}[r_{Dq}] \right)^2 \quad \overset{\text{make two sums}}{\Longrightarrow}$$

$$= \sum_D \nu_D^k \Bigg(\underbrace{\sum_{q:a_q=0} \mu_q \underbrace{\operatorname{round}[r_{Dq}]}_{=1}}_{=\boldsymbol{\mu}'\delta_{\boldsymbol{a}}} + \sum_{q:a_q\neq 0} \mu_q \operatorname{round}[r_{Dq}] \Bigg)^2 \overset{\text{expand}}{\Longrightarrow}$$

$$= (\boldsymbol{\mu}'\delta_{\boldsymbol{a}})^2 + 2\boldsymbol{\mu}'\delta_{\boldsymbol{a}} \underbrace{\left(\sum_D \nu_D^k \sum_{q:a_q\neq 0} \mu_q \operatorname{round}[r_{Dq}] \right)}_{\mathsf{U} - \boldsymbol{\mu}'\delta_{\boldsymbol{a}}}$$

$$+ \sum_D \nu_D^k \sum_{\substack{p\,:\,a_p \neq 0 \\ q\,:\,a_q \neq 0}} \mu_p \mu_q \operatorname{round}[r_{Dp}] \operatorname{round}[r_{Dq}] \overset{\text{see Lemma's notation}}{\Longrightarrow}$$

$$= (\boldsymbol{\mu}'\delta_{\boldsymbol{a}})^2 + 2(\mathsf{U} - \boldsymbol{\mu}'\delta_{\boldsymbol{a}}) + \sum_{\substack{p\,:\,a_p \neq 0 \\ q\,:\,a_q \neq 0}} \mu_p \mu_q \Bigg(\sum_{D\in\mathcal{X}_{pq}} \nu_D^k \cdot 0$$

$$+ \sum_{D\in\mathcal{Y}_{pq}} \nu_D^k \cdot 0 + \sum_{D\in\mathcal{Y}_{qp}} \nu_D^k \cdot 0 + \sum_{D\notin\mathcal{X}_{pq},\mathcal{Y}_{pq},\mathcal{Y}_{qp}} \nu_D^k \cdot 1 \Bigg) \quad \overset{\text{regroup}}{\Longrightarrow}$$

$$= 2\mathsf{U}\boldsymbol{\mu}'\delta_{\boldsymbol{a}} - (\boldsymbol{\mu}'\delta_{\boldsymbol{a}})^2 + \underbrace{\sum_{\substack{p\,:\,a_p \neq 0 \\ q\,:\,a_q \neq 0}}}_{(1-\boldsymbol{\mu}'\delta_{\boldsymbol{a}})^2} \mu_p \mu_q (1 - x_{pq} - y_{pq} - y_{qp}) \overset{\text{identity}}{\Longrightarrow}$$

$$= 1 - 2\boldsymbol{\mu}'\delta_{\boldsymbol{a}} + 2\mathsf{U}\boldsymbol{\mu}'\delta_{\boldsymbol{a}} - \sum_{\substack{p\,:\,a_p \neq 0 \\ q\,:\,a_q \neq 0}} \mu_p \mu_q (x_{pq} + 2y_{pq}). \quad (61)$$

Since $\mathsf{E}\,\mathsf{U}_D = \mathsf{U}$ by (8), we obtain (55):

$$\begin{aligned}
\mathsf{V}\,\mathsf{U}_D &= \mathsf{E}\mathsf{U}_D^2 - (\mathsf{E}\,\mathsf{U}_D)^2 \\
&= \mathsf{E}\mathsf{U}_D^2 - \mathsf{U}^2 \quad \overset{\text{by (61)}}{\Longrightarrow} \\
&= 1 - 2\boldsymbol{\mu}'\delta_{\boldsymbol{a}} + 2\mathsf{U}\boldsymbol{\mu}'\delta_{\boldsymbol{a}} \\
&\quad - \sum_{\substack{p\,:\,a \neq 0 \\ q\,:\,a_q \neq 0}} \mu_p\mu_q(x_{pq} + 2y_{pq}) - \mathsf{U}^2 \quad \overset{\text{identity}}{\Longrightarrow} \\
&= 1 - \mathsf{U}^2 - 2\boldsymbol{\mu}'\delta_{\boldsymbol{a}}(1 - \mathsf{U}) - \sum_{\substack{p\,:\,a_p \neq 0 \\ q\,:\,a_q \neq 0}} \mu_p\mu_q(x_{pq} + 2y_{pq}).
\end{aligned}$$

For the second moment of goodness we have

$$\mathsf{E}\,\mathsf{G}_D^2 = \sum_{D \in I^k} \nu_D^k\,\mathsf{G}_D^2 = \sum_{D \in I^k} \nu_D^k \left(\sum_q \mu_q \frac{r_{Dq}}{1 - \lambda_q} \right)^2 \quad \overset{\text{expand}}{\Longrightarrow}$$

$$= \sum_{D \in I^k} \nu_D^k \sum_{p,q} \mu_p\mu_q \frac{r_{Dp}\,r_{Dq}}{(1 - \lambda_p)(1 - \lambda_q)} \quad \overset{\text{change summation order}}{\Longrightarrow}$$

$$= \sum_{p,q} \mu_p\mu_q \sum_D \nu_D^k \frac{r_{Dp}\,r_{Dq}}{(1 - \lambda_p)(1 - \lambda_q)} \quad \overset{\substack{\text{make four sums and then} \\ \text{use Lemma's notation}}}{\Longrightarrow}$$

$$= \sum_{p,q} \mu_p\mu_q \left[\sum_{D \in \mathcal{X}_{pq}} \nu_D^k \frac{r_{Dp}\,r_{Dq}}{(1 - \lambda_p)(1 - \lambda_q)} + \sum_{D \in \mathcal{Y}_{pq}} \nu_D^k \frac{r_{Dp}\,r_{Dq}}{(1 - \lambda_p)(1 - \lambda_q)} \right.$$

$$\left. + \sum_{D \in \mathcal{Y}_{qp}} \nu_D^k \frac{r_{Dp}\,r_{Dq}}{(1 - \lambda_p)(1 - \lambda_q)} + \sum_{D \in I^k \setminus \mathcal{X}_{pq} \setminus \mathcal{Y}_{pq} \setminus \mathcal{Y}_{qp}} \nu_D^k \frac{r_{Dp}\,r_{Dq}}{(1 - \lambda_p)(1 - \lambda_q)} \right]$$

$$= \sum_{p,q} \mu_p \mu_q \left[x_{pq} \frac{\lambda_p \lambda_q}{(1-\lambda_p)(1-\lambda_q)} + y_{pq} \frac{\lambda_p (1-\lambda_q)}{(1-\lambda_p)(1-\lambda_q)} \right.$$

$$+ y_{qp} \frac{(1-\lambda_p)\lambda_q}{(1-\lambda_p)(1-\lambda_q)}$$

$$\left. +(1 - x_{pq} - y_{pq} - y_{qp}) \frac{(1-\lambda_p)(1-\lambda_q)}{(1-\lambda_p)(1-\lambda_q)} \right] \stackrel{\text{regroup w.r.t. } x_{pq}, y_{pq}, y_{qp}}{\Longrightarrow}$$

$$= \underbrace{\sum_{p,q} \mu_p \mu_q \frac{(1-\lambda_p)(1-\lambda_q)}{(1-\lambda_p)(1-\lambda_q)}}_{=1} - \sum_{p,q} \mu_p \mu_q x_{pq} \frac{1}{(1-\lambda_p)(1-\lambda_q)}$$

$$+ \sum_{p,q} \mu_p \mu_q x_{pq} \frac{\lambda_p}{(1-\lambda_p)(1-\lambda_q)} + \underbrace{\sum_{p,q} \mu_p \mu_q x_{pq} \frac{\lambda_q}{(1-\lambda_p)(1-\lambda_q)}}_{=\sum_{p,q} \mu_p \mu_q x_{pq} \frac{\lambda_p}{(1-\lambda_p)(1-\lambda_q)}}$$

$$- \sum_{p,q} \mu_p \mu_q y_{pq} \frac{(1-2\lambda_p)(1-\lambda_q)}{(1-\lambda_p)(1-\lambda_q)} -$$

$$\underbrace{\sum_{p,q} \mu_p \mu_q y_{qp} \frac{(1-\lambda_p)(1-2\lambda_q)}{(1-\lambda_p)(1-\lambda_q)}}_{=\sum_{p,q} \mu_p \mu_q y_{pq} \frac{(1-2\lambda_p)(1-\lambda_q)}{(1-\lambda_p)(1-\lambda_q)}} \stackrel{\text{identity}}{\Longrightarrow}$$

$$= 1 - \sum_{p,q} \mu_p \mu_q \left[x_{pq} \frac{(1-2\lambda_p)}{(1-\lambda_p)(1-\lambda_q)} + 2 y_{pq} \frac{(1-2\lambda_p)}{(1-\lambda_p)} \right]. \quad (62)$$

Since $\mathsf{E}\,\mathsf{G}_D = \mathsf{G}$ by (9), we obtain (56):

$$\mathsf{V}\,\mathsf{G}_D = \mathsf{E}\,\mathsf{G}_D^2 - (\mathsf{E}\,\mathsf{G}_D)^2 \stackrel{\text{use (62) and substitute } \lambda_p = 0.5 - 0.5|a_p|}{\Longrightarrow}$$

$$= 1 - \sum_{p,q} \mu_p \mu_q \left(x_{pq} \frac{4|a_p|}{(1+|a_p|)(1+|a_q|)} + y_{pq} \frac{4|a_p|}{1+|a_q|} \right) - \mathsf{G}^2.$$

Case of parliament. Find the probabilities x_{pq}, y_{pq} of events $\mathcal{X}_{pq}, \mathcal{Y}_{pq}$ for parliaments. Denote (Figure 7)

s, the number of participants of D from α_{pq} (= from L_p and L_q),

t, the number of participants of D from β_{pq} (= from L_p but not L_q),

u, the number of participants of D from β_{qp} (= from L_q but not L_p).

A parliament belongs to $\mathcal{X}_{pq}$ if it shares the opinion of minorities L_p and L_q on questions p and q. For this purpose, at least its $k/2$ voters must belong to L_p, that is, $k/2 \leq s+t \leq k$, and at least its $k/2$ voters must belong to L_q, that is, $k/2 \leq s+u \leq k$. By the known formulas for the multinomial (= polynomial) distribution, the probability of this event is

$$x_{pq} = \sum_{\substack{0 \leq \quad s,t,u \\ s+t+u \leq k \\ k/2 \leq \quad s+t \quad \leq k \\ k/2 \leq \quad s+u \quad \leq k}} \frac{k!\, \alpha_{pq}^s \beta_{pq}^t \beta_{qp}^u (1-\alpha_{pq}-\beta_{pq}-\beta_{qp})^{k-s-t-u}}{s!\, t!\, u!\, (k-s-t-u)!}.$$

A parliament belongs to $\mathcal{Y}_{pq}$ if it shares the opinion of the minority L_p on question p, and the opinion of majority $I^k \setminus L_q$ on question q. For this purpose, at least its $k/2$ voters must belong to L_p, that is, $k/2 \leq s+t \leq k$, but fewer than $k/2$ voters must belong to L_q, that is, $0 \leq s+u \leq k/2$. We have

$$y_{pq} = \sum_{\substack{0 \leq \quad s,t,u \\ s+t+u \leq k \\ k/2 \leq \quad s+t \quad \leq k \\ 0 \leq \quad s+u \quad < k/2}} \frac{k!\, \alpha_{pq}^s \beta_{pq}^t \beta_{qp}^u (1-\alpha_{pq}-\beta_{pq}-\beta_{qp})^{k-s-t-u}}{s!\, t!\, u!\, (k-s-t-u)!}.$$

Case of magistrate. Find the probabilities x_{pq}, y_{pq} of events $\mathcal{X}_{pq}, \mathcal{Y}_{pq}$ for magistrates. A magistrate belongs to $\mathcal{X}_{pq}$ if it has the opinion of the minorities on both questions p and q. For this purpose, all its k members must belong to L_p and at the same time to L_q, which gives the condition $s = k$, implying $t = 0$ and $u = 0$. By the known formulas for the multinomial distribution, the probability of this event is

$$x_{pq} = \sum_{\substack{s = k \\ t = u = 0}} \frac{k!\, \alpha_{pq}^s \beta_{pq}^t \beta_{qp}^u (1 - \alpha_{pq} - \beta_{pq} - \beta_{qp})^{k-s-t-u}}{s!\, t!\, u!\, (k - s - t - u)!} = \alpha_{pq}^k.$$

A magistrate belongs to $\mathcal{Y}_{pq}$ if it has the opinion of the minority on question p, and of the majority on question q. For this purpose, all its k members must belong to L_p, that is, $s + t = k$, $u = 0$, but at least one member must not belong to L_q, that is, $1 \leq t \leq k$. We obtain

$$\begin{aligned} y_{pq} &= \sum_{\substack{s + t = k \\ 1 \leq t \leq k \\ u \;\;\, = 0}} \frac{k!\, \alpha_{pq}^s \beta_{pq}^t \beta_{qp}^u (1 - \alpha_{pq} - \beta_{pq} - \beta_{qp})^{k-s-t-u}}{s!\, t!\, u!\, (k - s - t - u)!} \\ &= \sum_{1 \leq t \leq k} \frac{k!\, \alpha_{pq}^{k-t} \beta_{pq}^t}{t!\, (k - t)!} \\ &= (\alpha_{pq} + \beta_{pq})^k - \alpha_{pq}^k \\ &= \lambda_p^k - \alpha_{pq}^k \quad \overset{\lambda_p = 0.5 - 0.5|a_p|}{\Longrightarrow} \\ &= 2^{-k}(1 - |a_p|)^k - \alpha_{pq}^k. \end{aligned}$$

It remains to compute α_{pq} and β_{pq}. By definition (59) the function "Sign" is not 0-sensitive. Hence, $\boldsymbol{A}\,\mathrm{diag}(\mathrm{Sign}\boldsymbol{a})$ is the logical matrix obtained from $\boldsymbol{A}$ by replacing opinions a_{iq} of majorities (in case of tie opinion, of protagonists) by 1's and opinions of strict minorities by -1's. Consequently,

$$\boldsymbol{L} = 0.5[1 - \boldsymbol{A}\,\mathrm{diag}(\mathrm{Sign}\boldsymbol{a})]$$

is the matrix of logical indicators (0 or 1) of non-strict minority opinions. Then the pqth element of the product $\boldsymbol{L}\ \mathrm{diag}\boldsymbol{\nu}\, \boldsymbol{L}'$ is the total weight of the individuals who simultaneously belong to minorities L_p and L_q. It implies (57).

The logical complement to matrix $\boldsymbol{L}$ is the matrix

$$\overline{\boldsymbol{L}} = 0.5[1 + \boldsymbol{A}\,\mathrm{diag}(\mathrm{Sign}\boldsymbol{a}]$$

which consists of indicators of majority opinions. Then the pqth element of the product $\boldsymbol{L}\ \mathrm{diag}\boldsymbol{\nu}\, \overline{\boldsymbol{L}}'$ is the total weight of the individuals who simultaneously belong to minority L_p and to majority $I \setminus L_q$. It implies (58). ■

PROOF OF THEOREM 3. The right-hand sums in (54)—(56) are non-negative, because the probabilities $x_{pq}, y_{pq} \geq 0$. Omitting the sums, we obtain the first column of inequalities (21)—(23). The second column of inequalities (21)–(23) follows immediately, for example,

$$\overline{\mathsf{P}}^2 - \mathsf{P}^2 = (\overline{\mathsf{P}} - \mathsf{P})\underbrace{(\overline{\mathsf{P}} + \mathsf{P})}_{\leq 2} \leq 2(\overline{\mathsf{P}} - \mathsf{P}),$$

as required. ■

4. Proof of Theorem 4

PROOF OF THEOREM 4. As follows from (11) for president selected by lot from individuals, $\mathsf{P}^{(1)} = 0.5 + 0.5\mu'(\boldsymbol{a}.^2)$. The latter term is always positive, except for the case when all $a_q = 0$. It gives the inequality $(*)$ of the theorem.

As follows from (15) and (13) for president selected by lot from individuals,

$$\begin{aligned}\overline{\mathsf{P}} &= 0.5 + 0.5\boldsymbol{\mu}'|\boldsymbol{a}| \\ \mathsf{U}^{(1)} &= 0.5 + 0.5\boldsymbol{\mu}'\delta_{\boldsymbol{a}} + 0.5\boldsymbol{\mu}'|\boldsymbol{a}|.\end{aligned}$$

Consequently, $\overline{\mathsf{P}} \leq \mathsf{U}^{(1)}$, and the inequality turns in equality if and only if all $a_q \neq 0$. It is the inequality $(**)$ of the theorem.

By (63) $\mathsf{U}^{(1)} > 0.5$, since by definition $\boldsymbol{\mu}'\delta_{\boldsymbol{a}} = 0$ implies $\boldsymbol{\mu}'|\boldsymbol{a}| > 0$, and $\boldsymbol{\mu}'|\boldsymbol{a}| = 0$ implies $\boldsymbol{\mu}'\delta_{\boldsymbol{a}} > 0$.

The superiority of larger parliaments over smaller ones $\mathsf{P}^{(k-1)} \leq \mathsf{P}^{(k)}$, $\mathsf{U}^{(k-1)} \leq \mathsf{U}^{(k)}$, and $\mathsf{G}^{(k-1)} \leq \mathsf{G}^{(k)}$ follows from (11)—(14) for parliaments *selected by lot* and the fact that $d_q = I_{a_q^2}\left(\frac{1}{2}, \frac{k+1}{2}\right)$ strictly increases in k by Lemma 1 if only there exists a question q with neither unanimity, nor tie opinion, that is, $|a_q| \neq 0, 1$ increase in the second argument.

The superiority of larger magistrates over smaller ones $\mathsf{P}_{(k)} \leq \mathsf{P}_{(k+1)}$, $\mathsf{U}_{(k)} \leq \mathsf{U}_{(k+1)}$, and $\mathsf{G}_{(k)} \leq \mathsf{G}_{(k+1)}$ follows from (11)—(14) for magistrates *selected by lot* and the fact that $\left[1 - 2^{1-k}(1 - \operatorname{sign} a_q\, b_q)^k\right]$ strictly increases in k if only there exists a question q with neither unanimity, nor tie opinion, that is, $|a_q| \neq 0, 1$.

The superiority of magistrates over parliaments of the same size $\mathsf{P}^{(k)} \leq \mathsf{P}_{(k)}$, $\mathsf{U}^{(k)} \leq \mathsf{U}_{(k)}$, and $\mathsf{G}^{(k)} \leq \mathsf{G}_{(k)}$ is due to the following observation. To represent a majority in the society, a parliament needs half the voters who represent the majority, and a magistrate suffices only one such a representative . Therefore, for every question the representativeness of a magistrate is greater or equal than that of the same team organized as a parliament.

The superiority of goodness over the universality, $\mathsf{U}^{(k)} \leq \mathsf{G}^{(k)}$ and $\mathsf{U}_{(k)} \leq \mathsf{G}_{(k)}$ follows from the inequality

$$\text{round}[r_{Dq}] \leq \frac{r_{Dq}}{0.5 + 0.5|a_q|} \quad \text{for every question} \quad q.$$

In fact, if a majority is represented then $\text{round}[r_{Dq}] = \frac{r_{Dq}}{0.5+0.5|a_q|} = 1$. In case of minority, $\text{round}[r_{Dq}] = 0$, but $\frac{r_{Dq}}{0.5+0.5|a_q|}$ can be positive. ■

References

[1] Abramowitz M., Stegun I. Handbook of Mathematical Functions. N.Y.: Dover, 1972.

[2] Anderson P. Passages from Antiquity to Feudalism. L.: New Left Books, 1974.

[3] Aristotle. The Politics. Harmondsworth: Penguin, 1981.

[4] Aristotle. The Constitution of Athens. Harmondsworth: Penguin, 1984.

[5] Bauer G., Brüning N., Krumrey H., Opitz O. Wechselspannung // Focus, 2002. Vol. 39. P. 16—20.

[6] Blackwell Ch.W. Ephialtes // Demos. Classical Athenian Democracy. 2003. http://doxa.stoa.org/cocoon/demos/home.

[7] Blackwell Ch.W. Cimon // Demos. Classical Athenian Democracy. 2003. http://doxa.stoa.org/cocoon/demos/home.

[8] Boedeker D., Raaflaub K.A. (eds.). Democracy, Empire, and the Arts in Fifth-Century Athens. Cambridge, MA: Harvard University Press, 1998.

[9] Décision №85-192 DC. Loi portant diverses dispositions d'ordre social. 1985.
http://www.conseil-constitutionnel.fr
/decision/1985/85192dc.htm.

[10] Finley M.I. Democracy, Ancient and Modern. L.: Chatto & Windus, 1973.

[11] Finley M.I. Politics in the Ancient World. Cambridge: Cambridge University Press, 1983.

[12] Friedrich Nauman Stiftung. Die Wahlprogramme der Parteien — eine vergleichende Gegenüberstellung nach Schwerpunkten (Bearbeiter: Dr. Horst Werner, et al.). 2002. http://www.fnst.org/pdf/wahlprogramme.pdf.

[13] Hansen M.H. The Origin of the Word Demokratia // Liverpool Classical Monthly. 1986. Vol. 11. P. 35—36.

[14] Hansen M.H. The Athenian Assembly in the Age of Demosthenes. Oxford: Blackwell, 1987.

[15] Hansen M.H. When Was Selection by Lot of Magistrates Introduced in Athens? // Classica et Medievalia. 1990. Vol. 41. P. 51—54.

[16] Hansen M.H. The Athenian Democracy in the Age of Demosthenes: Structure, Principles and Ideology. Oxford: Blackwell, 1991.

[17] Headlam J.W. Election by Lot at Athens. Cambridge: Cambridge University Press, 1933.

[18] Held D. Models of Democracy. 2nd ed. Cambridge: Polity Press, 1996.

[19] Hyland J.L. Democratic Theory. Manchester: Manchester University Press, 1995.

[20] Manin B. The Principes of Representative Government. Cambridge: Cambridge University Press, 1997.

[21] Plato. The Republic. Harmondsworth: Penguin, 1974.

[22] Pocock J.G. The Machiavellian Moment: Florentine Political Thought and the Atlantic Republican Tradition. Princeton, NJ: Princeton University Press, 1975.

[23] Rodewald C. (Ed.). Democracy: Ideals and Realities. L.: Dent, 1975.

[24] Sinclair R.K. Democracy and Participation in Athens. Cambridge: Cambridge University Press, 1988.

[25] Der Spiegel: Wahlsonderheft'02. 24.09.2002.

[26] Thucydides. History of the Peloponnesian War. Harmondsworth: Penguin, 1972.

Препринт WP7/2006/03
Серия WP7
Теория и практика общественного выбора

A. Tangian

A mathematical model of Athenian democracy

Публикуется в авторской редакции

Зав. редакцией *А.В. Заиченко*

ЛР № 020832 от 15 октября 1993 г.
Формат 60×84 1/16. Бумага офсетная. Печать трафаретная.
Тираж 150 экз. Уч.-изд. л. 4,5. Усл. печ. л. 4,4
Заказ № . Изд. № 634

ГУ ВШЭ. 125319, Москва, Кочновский проезд, 3
Типография ГУ ВШЭ. 125319, Москва, Кочновский проезд, 3