WEYL n-ALGEBRAS

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Abstract. We introduce Weyl n-algebras and show how their factorization homology may be used to define invariants of manifolds. In the appendix we heuristically explain why these invariants must be perturbative Chern–Simons invariants.

Introduction

The aim of this article is to develop the idea announced in [Mar1]: Chern–Simons perturbative invariants of 3-manifolds introduced in [AS1, AS2, BC] may be defined by means of factorization homology considered in [BD, Lur, Fra, Gin]. To get these invariants one have to calculate factorization homology of Weyl n-algebra, which is an object of independent interest.

An important property of Weyl n-algebras is that their factorization homology on a closed manifold is one-dimensional (Proposition 11). It would be plausible to find some conceptual proof of this statement, perhaps by using some kind of Morita invariance of factorization homology. As far as I know, such arguments are unknown even in the classical situation, when $n = 1$.

Weyl n-algebras may be applied to the differential calculus in the sense of [TT]. For example, the L_∞-morphism from the Lie algebra of polyvector fields on a vector space, which is a Weyl 2-algebra, to the Lie algebra of endomorphisims of differential forms on it (see e. g. [TT]) is given by the map analogous to the one from Proposition 8 for a 2-dimensional cylinder. We hope to discuss this elsewhere.

In the first section we shortly recall definition of operad and module over it, just to introduce notations. We send reader to e. g. [Lur] for a detailed treatment.

In the second section we collect facts about Fulton–MacPherson operad and L_∞ operad we need.

Section 3 is devoted to factorization homology. There is nothing new here, this notion is deeply discussed in [Lur]. We use the Fulton-MacPherson compactification following [Sal] and others. In Subsection 3.4 a connection between Lie algebra homology and factorization homology is described. Proposition 8 interprets this connection in terms of a morphism of right L_∞-modules. The right L_∞ module $C_*(f\mathscr{C}(M)(S))$ is similar to the Goodwillie derivative of the functor $\Sigma^\infty \operatorname{Hom}_{\mathcal{T}op_*}(M, -)$ (see e. g. [AC2] and references therein). This right L_∞ module has an additional structure: it is a pull back of a right e_n module under the map of operads $L_\infty \to e_n$ (compare with KE_L modules from [AC1]). It seems that invariants of manifolds we introduce below reflect this additional structure.

In the Section 4 we introduce Weyl n-algebras. Euler structure on manifold, which we introduce in Subsection 4.3, simplifies definition of factorization complex

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of a Weyl n-algebra on it. I do not know, whether this is just a technical point, or it has some deep relations with [Tur], where the term is taken from.

As was already mentioned, factorization homology of Weyl algebra on a closed manifold is one-dimensional. It is easy to produce the cycle presenting this the only class. A more subtle and interesting question is to find a cocycle representing the class dual to this cycle, which is an element of the dual complex. For $n = 1$, $M = S^1$ and generators of $\mathcal{W}^n$ of zero degrees this question is solved in [FFS].

If such a formula existed for any n and M, it would substantially simplify the last section, where we apply Weyl n-algebras to the calculation of invariants of a manifold. Instead of using of the non-existent aforementioned formula we analyze what happens with factorization homology when we collapse a homological sphere. The formula we get is similar to the one in [AS1] and [BC].

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1. Operads

1.1. **Definition.** Let C be a symmetric monoidal category with product $\otimes$, $\mathbf{Set}_{\hookrightarrow}$ be the category of finite sets and injective morphisms and $\mathbf{Set}_{\simeq}$ be the category of finite sets and isomorphisms.

A *unital operad* $\mathcal{O}$ in C is defined by the following:

- A contravariant functor from the category of finite sets and injective morphisms $\mathbf{Set}_{\hookrightarrow}$ to C, the image of the set of k elements is called *operations of arity* k,
- For any surjective morphism of sets $p\colon S \to S'$ a morphism called *composition* of operation is given
$$mul_p\colon\ \mathcal{O}(S') \otimes \bigotimes_{i\in S'} \mathcal{O}(p^{-1}(i)) \to \mathcal{O}(S),$$
such that
  - it is functorial with respect to injective morphisms $i\colon S_0 \to S$ for which composition $p \circ i$ is surjective
  - for any pair of surjective morphisms $S \xrightarrow{p} S' \xrightarrow{p'} S''$ equality
  $$mul_{p'\circ p} = mul_{p'} \circ \bigotimes_{i\in S'} mul_{p^{-1}i}$$
  holds.

A *non-unital operad* is defined by the same data, but with $\mathbf{Set}_{\hookrightarrow}$ replaced by $\mathbf{Set}_{\simeq}$, the category of finite sets and isomorphisms.

Any unital operad canonically produces a non-unital one by forgetting structure.

With any unital operad $\mathcal{O}$ in C one may associate the monoidal category $\mathcal{O}^{\otimes}$ enriched over C. Its objects are labeled by finite sets. Morphisms between objects is the product
$$\mathrm{Mor}_{\mathcal{O}^{\otimes}}(S, S') = \prod_{m\colon S\to S'} \bigotimes_{i\in S'} \mathcal{O}(m^{-1}(i)),$$
where product is taken by all maps of finite sets. The composition of elements of $\mathrm{Mor}_{\mathcal{O}^{\otimes}}(S, S')$ given by surjective maps of sets is given by the composition of operations, and composition with ones given by injective morphisms is given by

the action of $\mathbf{Set}_{\hookrightarrow}$. For a non-unital operad the construction is the same, but the product is taken only by surjective morphisms.

The operad may be reconstructed from the monoidal category $\mathcal{O}^{\otimes}$ fibered over **Set**.

A *colored operad* with a set of colors B is a generalization of an operad. In the same way it produces a monoidal category with objects numerated by B^S, where S runs finite sets. So operations in a colored operad are numerated by finite set $[\mathbf{n}]$ and a point in B^{n+1} and composition is a morphism from fibered product over B. For details see e. g. [Lur, 2.1.1].

We will consider operads fibered over a topological space, its definition is an obvious modification of the previous one.

1.2. Modules.

Definition 1. A *left (right) module over an operad* $\mathcal{O}$ in a category M is a covariant (contravariant) functor from $\mathcal{O}^{\otimes}$ to M.

Let D be a symmetric monoidal category with unit $\mathbb{1}$ and $\mathcal{O}$ is an operad in a symmetric monoidal category C. Given an object A in D and an element $e\colon \mathbb{1} \to A$ there are natural functors $\mathbf{Set}_{\simeq} \to D$ and $\mathbf{Set}_{\hookrightarrow} \to D$. The first one sends a set S to $A^{\otimes S}$ with the natural action of isomorphisms of S. The second one sends set S to $A^{\otimes S}$ as well and a morphism $S' \hookrightarrow S$ sends to

$$e^{\otimes(S\setminus S')} \otimes \mathrm{id}^{\otimes S}\colon A^{\otimes S'} \to A^{\otimes S}. \tag{1}$$

Definition 2. *An algebra A over a non-unital operad* $\mathcal{O}$ in D is a left module over $\mathcal{O}$ in D such that its restriction $\mathbf{Set}_{\simeq}$ is the functor as above.

A algebra A with unit $e\colon \mathbb{1} \to A$ over a unital operad $\mathcal{O}$ in D is a left module over $\mathcal{O}$ in D such that its restriction on $\mathbf{Set}_{\hookrightarrow}$ is the functor as above.

Denote these modules by $A^{\otimes}$.

Let $\mathcal{O}$ be an dg-operad, that is an operad in the category of $\mathbb{Q}$-complexes.

Definition 3. Let $\mathcal{O}$ be a dg-operad and L and R be a left and a right dg-modules over it. Then the *tensor product* $L \otimes_{\mathcal{O}} R$ of modules over the operad is the tensor product of functors corresponding to modules from $\mathcal{O}^{\otimes}$ to the category of complexes.

The definition works for both unital and non-unital and also colored operads. Given a unital operad $\mathcal{O}$ denote by $\tilde{\mathcal{O}}$ the corresponding non-unital operad. The canonical embedding $\tilde{\mathcal{O}}^{\otimes} \hookrightarrow \mathcal{O}^{\otimes}$ induces $\tilde{\mathcal{O}}$-structure on any left and a right $\mathcal{O}$-modules L and R the canonical map

$$L \otimes_{\tilde{\mathcal{O}}} R \to L \otimes_{\mathcal{O}} R. \tag{2}$$

2. Fulton-MacPherson operad

2.1. **Fulton-MacPherson compactification.** Let $\mathbb{R}^n$ be an affine space. For a finite set S denote by $(\mathbb{R}^n)^S$ the set of ordered S-tuples in $\mathbb{R}^n$. Let $\mathscr{C}^0(\mathbb{R}^n)(S) \subset (\mathbb{R}^n)^S$ be the configuration space of distinct ordered points in $\mathbb{R}^n$ labeled by S. In [GJ, Mar2] (see also [Sal] and [AS1]) the Fulton–MacPherson compactification $\mathscr{C}(\mathbb{R}^n)(S)$ of $\mathscr{C}^0(\mathbb{R}^n)(S)$ is introduced. This is a manifold with corners and a boundary with interior $\imath\colon \mathscr{C}^0(\mathbb{R}^n)(S) \hookrightarrow \mathscr{C}(\mathbb{R}^n)(S)$. There is a projection $\pi\colon \mathscr{C}(\mathbb{R}^n)(S) \to (\mathbb{R}^n)^S$ such that $\pi \circ \imath\colon \mathscr{C}^0(\mathbb{R}^n)(S) \to (\mathbb{R}^n)^S$ is the natural embedding.

For any $S' \subset S$ there is the projection map

$$\mathscr{C}(\mathbb{R}^n)(S) \to \mathscr{C}(\mathbb{R}^n)(S'), \tag{3}$$

compatible with the same maps $\mathscr{C}^0(\mathbb{R}^n)(S) \to \mathscr{C}^0(\mathbb{R}^n)(S')$ and $(\mathbb{R}^n)^S \to (\mathbb{R}^n)^{S'}$.

The natural action of the group of affine transformations on $\mathscr{C}^0(\mathbb{R}^n)(S)$ is lifted on $\mathscr{C}(\mathbb{R}^n)(S)$. Denote by $\mathrm{Dil}(n)$ its subgroup consisting of dilatations and shifts. Group $\mathrm{Dil}(n)$ acts freely on $\mathscr{C}(\mathbb{R}^n)(S)$ and the quotient is isomorphic to the fiber $\pi^{-1}(\vec{0})$, where $\vec{0} \in (\mathbb{R}^n)^S$ is S-tuple sitting at the origin. To build this isomorphism consider dilatations with positive coefficients with the center at the origin: $\mathbb{R}_{>0} \times \mathscr{C}^0(\mathbb{R}^n)(S) \to \mathscr{C}^0(\mathbb{R}^n)(S)$. By the construction of the compactification their action is lifted to $r\colon \mathbb{R}_{\geq 0} \times \mathscr{C}(\mathbb{R}^n)(S) \to \mathscr{C}(\mathbb{R}^n)(S)$, which is a fiber bundle. The map $r(0 \times -)$ factors through the quotient by $\mathrm{Dil}(n)$ and its image lies in $\pi^{-1}(\vec{0})$ and is the required isomorphism. It follows that $\pi^{-1}(\vec{0})$ is a retract of $\mathscr{C}(\mathbb{R}^n)(S)$.

As it is just mentioned, manifolds with corners $\mathscr{C}(\mathbb{R}^n)(S)/\mathrm{Dil}(n)$ and $\pi^{-1}(\vec{0})$ are isomorphic. Denote any of these manifolds by $\mathbf{FM}_n^S$. The sequence of manifolds $\mathbf{FM}_n^S$ is a contravariant functor from $\mathbf{Set}_{\hookrightarrow}$ to toplogical spaces: image of an embedding forgets points that are not in its image. The sequence $\mathbf{FM}_n^S$ may be equipped with a structure of a unital operad in the category of topological spaces. This operad is free as an operad of sets and as such is generated by $\mathscr{C}^0(\mathbb{R}^n)(S) \hookrightarrow \mathscr{C}(\mathbb{R}^n)(S)$. The action of k-ary operations $\mathscr{C}^0(\mathbb{R}^n)([\mathbf{k}])/\mathrm{Dil}(n)$ on $\mathscr{C}(\mathbb{R}^n)(S)$ looks as follows. Consider the submanifold of $\mathscr{C}(\mathbb{R}^n)(S)$ for which the image of $\pi\colon \mathscr{C}(\mathbb{R}^n)(S) \to (\mathbb{R}^n)^S$ consists exactly of k different points. This submanifold is isomorphic to $\mathscr{C}^0(\mathbb{R}^n)([\mathbf{k}]) \times \pi^{-1}(\vec{0})$ because fibers of π over any point are isomorphic due to parallel translations. The embedding of this submanifold to $\mathscr{C}(\mathbb{R}^n)(S)$ in composition with the quotient by $\mathrm{Dil}(n)$ gives the map

$$\mathscr{C}(\mathbb{R}^n)(S)/\mathrm{Dil}(n) \times (\mathbf{FM}_n)^{\times k} \to \mathscr{C}(\mathbb{R}^n)(\bullet)/\mathrm{Dil}(n) = \mathbf{FM}_n,$$

which is the desired action.

Definition 4. The sequence of topological spaces $\mathbf{FM}_n^S$ with the unital operad structure as above is called the *Fulton–MacPherson operad*.

2.2. **Chains of Fulton-MacPherson operad.** Given a topological operad, one may produce a dg-operad by taking complexes of chains of its components.

Definition 5. Denote by $\mathfrak{fm}_n$ the operad of $\mathbb{Q}$-chains of $\mathbf{FM}_n$.

By chains we mean the complex of de Rham distributions. Alternatively, one may think about the cooperad of de Rham cochains of $\mathbf{FM}_n$.

Proposition 1. *Operad $\mathfrak{fm}_n$ is weakly homotopy equivalent to e_n, the operad of chains of the little discs operad.*

Proof. See [Sal, Proposition 3.9] and Subsection 3.3 below. □

Spaces $\mathbf{FM}_n^S$ are acted by the general linear group, and, in particular, by its maximal compact subgroup $SO(n)$, we suppose that a scalar product on the space is chosen. This action gives an operad $f\mathbf{FM}_n$ colored by the classifying space $BSO(n)$.

Definition 6. Denote by $f\mathfrak{fm}_n$ the operad of $\mathbb{Q}$-chains of $f\mathbf{FM}_n$.

The closely connected, but not identical object is the operad of framed disks from [Get].

Operations of $f\mathfrak{fm}_n$ form complexes over powers of $BSO(n)^s$. An algebra over $f\mathfrak{fm}_n$ is given by a family of complexes over $BSO(n)^s$. Below we will need only the following restrictive, but simpler class of such algebras.

Definition 7. We say that a dg-algebra A over $\mathfrak{fm}_n$ is *invariant*, if all maps of complexes

$$\mathfrak{fm}_n \otimes A \otimes \cdots \otimes A \to A$$

are invariant under the action of group $SO(n)$.

The important class (and the only class we need, in fact) of invariant e_n-algebras is universal enveloping e_n-algebras, see the end of the next Subsection.

2.3. L_∞ **operad.** A *tree* is an oriented connected graph with three type of vertices: *root* has one incoming edge and no outgoing ones, *leaves* have one outgoing edge and no incoming ones and *internal vertexes* have one outgoing edge and more than one incoming ones. Edges incident to leaves will be called *inputs*, the edge incident to the root will be called the *output* and all other edges will be called *internal edges*. The degenerate tree has one edge and no internal vertexes. Denote by $T_k(S)$ the set of non-degenerate trees with k internal edges and leaves labeled by a set S.

For two trees $t_1 \in T_{k_1}(S_1)$ and $t_2 \in T_{k_2}(S_2)$ and an element $s \in S_1$ the composition of trees $t_1 \circ_s t_2 \in T_{k_1+k_2+1}$ is obtained by identification of the input of t_1 corresponding to s and the output of t_2. Composition of trees is associative and the degenerate tree is the unit. The set of trees with respect to the composition forms an operad.

Call the tree with only one internal vertex the *star*. Any non-degenerate tree with k internal edges may be uniquely presented as a composition of $k+1$ stars.

The operation of *edge splitting* is the following: take a non-degenerate tree, present it as a composition of stars and replace one star with a tree that is a product of two stars and has the same set of inputs . The operation of an edge splitting depends on a internal vertex and a proper subset of incoming edges with more than one element.

For a non-degenerate tree t denote by $\mathrm{Det}(t)$ the one-dimensional $\mathbb{Q}$-vector space that is the determinant of the vector space generated by internal edges. For $s > 1$ consider the complex

$$L(s)\colon \bigoplus_{t\in T_0([\mathbf{s}])} \mathrm{Det}(t) \to \bigoplus_{t\in T_1([\mathbf{s}])} \mathrm{Det}(t) \to \bigoplus_{t\in T_2([\mathbf{s}])} \mathrm{Det}(t) \to \cdots, \tag{4}$$

where $[\mathbf{s}]$ is the set of s elements, the cohomological degree of a tree $t \in T_k([\mathbf{s}])$ is $2-s+k$ and the differential is given by all possible splitting of an edge (see e. g. [GK]). The composition of trees equips the sequence $L(i)\otimes\mathrm{sgn}$ with the structure of a non-unital *dg*-operad, here sgn is the sign representation of the symmetric group.

This operad is called L_∞ *operad*. Denote by $L_\infty[n]$ the *dg*-operad given by the complex $L(s)[n(s-1)] \otimes (\mathrm{sgn})^n$ and refer to it as n-shifted L_∞ operad.

As $\mathbf{FM}_n$ is freely generated by $\mathscr{C}^0(\mathbb{R}^n)(S)/\mathrm{Dil}(n)$ as the operad of sets, there is a map μ from it to the free operad with one generator in each arity, which sends generators to generators. Elements of the latter operad a numerated by rooted trees. The map above sends $\mathscr{C}^0_{[\mathbf{k}]}(\mathbb{R}^n)/\mathrm{Dil}(n)$ to the star tree with k leaves. For a tree $t \in T(S)$ denote by $[\mu^{-1}(t)] \in C_*(F_n(S))$ the chain presented by its preimage under μ.

Proposition 2. *Map $[\mu^{-1}(\cdot)]$ as above gives a morphism from shifted L_∞ operad $L(s)[s(1-n)]$ to the dg-operad $\mathfrak{fm}_n$ of rational chains of the Fulton–MacPherson operad. The last operad here is treated as a non-unital one.*

Proof. To see that the map commutes with the differential note, that two strata given by μ with dimensions differing by 1 are incident if and only if one of the corresponding trees is obtained from another by edge splitting. In this way we get a basis in the conormal bundle to a stratum labeled by the internal edges. It follows the consistency of the map from the statement with signs. □

It follows that there is a morphism of *dg*-operads

$$L_\infty[1-n] \to \mathfrak{fm}_n \tag{5}$$

Definition 8. For a $\mathfrak{fm}_n$-algebra A call its pull-back under (5) the *associated L_∞-algebra* and denote it by $L(A)$.

Since the operad $\mathfrak{fm}_n$ is weakly homotopy equivalent to e_n (Proposition 1), it gives a homotopy morphism of operads $L_\infty[1-n] \to e_n$.

This morphism of operads produces the map the category of e_n-algebras to L_∞-algebras. This functor has the left adjoint, which is called the universal enveloping e_n-algebra. The important example of the latter is the rational homology of an iterated loop space $\Omega^n X$, which is an universal enveloping e_n-algebra of the homotopy groups Lie algebra $\pi_{*-1}(X)$, for more details see e. g. [Fra, Section 5]. Note, that $\Omega^n X$ is equipped with a natural $SO(n)$ action. This is in good agreement with the fact that any universal enveloping e_n-algebra is invariant.

3. Factorization homology

3.1. **Factorization complex.** Let M be a n-dimensional oriented topological manifold. In the same way as for $\mathbb{R}^n$ there is the Fulton–MacPherson compactification $\mathscr{C}(M)(S)$ of the space $\mathscr{C}^0(M)(S)$ of ordered pairwise distinct points in M labeled by S. Locally it is the same thing. Inclusion $\mathscr{C}^0(M)(S) \hookrightarrow \mathscr{C}(M)(S)$ is a homotopy equivalence, there is a projection $\mathscr{C}(M)(S) \xrightarrow{\pi} M^S$.

Recall that a point in the Fulton–MacPherson compactification $\mathscr{C}(\mathbb{R}^n)(S)$ of the configuration space of $\mathbb{R}^n$ looks like a configuration from the configuration space $\mathscr{C}^0(\mathbb{R}^n)(S')$ with elements of $\mathbf{FM}_n$ sitting at each points of the configuration. It follows space $\mathscr{C}(\mathbb{R}^n)(\bullet)$ is a right module over $\mathbf{FM}_n$ and as a set it is freely generated by $\mathscr{C}^0(\mathbb{R}^n)(\bullet)$. The same is nearly true for the Fulton–MacPherson compactification of any oriented manifold M. But to define such an action one needs to choose coordinates at the tangent space of any point of configuration of $\mathscr{C}(M)(S)$. To fix it one have to consider either only framed manifolds or introduce framed configuration space.

Definition 9. Framed Fulton–MacPherson compactification $f\mathscr{C}(M)(S)$ is the principal $SO(n)^S$ bundle over $\mathscr{C}(M)(S)$, which is the pull back of product of principal bundles associated with the tangent bundles to each point under the projection map $\pi\colon \mathscr{C}(M)(S) \to M^S$.

The eqiuvariant chain complex $C_*(f\mathscr{C}(M)(S))$ over $BSO(n)^S$ is naturally a right module over operad $f\mathfrak{fm}_n$ (see Definition 6).

Definition 10. For an algebra A over $f\mathfrak{fm}_n$ and an oriented manifold M the *factorization complex* $\int_M A$ is the tensor product (Definition 3) of the left $f\mathfrak{fm}_n$-module $A^{\otimes}$ and the right $f\mathfrak{fm}_n$-module $C_*(f\mathscr{C}(M)(S))$.

The homology of $\int_M A$ is called *factorization homology* of A on M.

The definition of factorization complex may be rephrased as follows.

Proposition 3. *For an equivariant unital $f\mathfrak{fm}_n$-algebra A and an oriented manifold M the factorization homology $\int_M A$ is the complex given by a colimit of the diagram*

$$\begin{array}{c} \bigoplus_{S'} C_*(\mathscr{C}(M)(S')) \underset{Aut(S')}{\otimes} A^{\otimes S'} \\ \uparrow \\ \bigoplus_{i\colon S'\to S} (C_*(f\mathscr{C}^0(M)(S)) \underset{SO(n)^S}{\otimes} \bigotimes_{s\in S} (f\mathfrak{fm}_n(i^{-1}s) \underset{SO(n)^{(i^{-1}s)}\rtimes Aut(i^{-1}s)}{\otimes} A^{\otimes(i^{-1}s)}) \\ \downarrow \\ \bigoplus_{S} C_*(\mathscr{C}^0(M)(S)) \underset{Aut(S)}{\otimes} A^{\otimes S} \end{array} \tag{6}$$

where the summation in the middle is taken by two finite sets and a morphism between, the downwards arrow is given by the left action of $\mathfrak{fm}_n$ on A for $\operatorname{Im} i$ *and the unit for $S\setminus \operatorname{Im} i$ and the upwards arrow is given by the right action of $f\mathfrak{fm}_n$ on $\mathscr{C}(M)(\bullet)$.*

Proof. The formula is the direct interpretation of Definition 10. □

If the manifold is framed, that is its tangent bundle is trivialized, the definition may be simplified: one should substitute the $\mathfrak{fm}_n$ instead of $f\mathfrak{fm}_n$ and take the usual tensor products, not over the classifying space.

As the upwards arrow in (6) is surjective, for any class of a coequalizers above there is a unique chain, which is in interior of the Fulton–MacPherson compactification, that is in a configuration space of distinct points. Thus on the complex (6) (that calculates the factorization homology) there is an increasing filtration by the number of points of the configuration space and the associated graded object is $\bigoplus_S C_*(\mathscr{C}^0(M)(S)) \otimes A^{\otimes S}$.

Note, that this filtration splits as a filtration of vector spaces. Thus any morphism from or to the factorization complex may be presented as the one for all graded pieces of the filtration consistent in a proper way.

The definition above may be again rephrased as follows. Denote by $Ran(M)$ the *Ran space* of M, that is the set of finite subsets of M with the natural topology. There is the natural map $M^{\times i} \to Ran(M)$, which sends a set of points to its support. Denote the composite map $\mathscr{C}(M)([\mathbf{i}]) \to M^{\times i} \to Ran(M)$ by ϖ_i. The fibers of this map is the product of some copies of the Fulton–MacPherson operad. Take a $\mathfrak{fm}_n$ algebra A and consider chains $\bigoplus_i C_*(\mathscr{C}(M)([\mathbf{i}]))\otimes_{\Sigma_i} A^{\otimes i})$ factorized by relations (6). As all relations respect ϖ_*, for any open subset of the complex of these chains factorized by relations is defined; being restricted $\mathscr{C}^0(M)([\mathbf{i}]) \hookrightarrow Ran(M)$ this complex equals to $C_*(\mathscr{C}^0(M)([\mathbf{i}])) \otimes_{\Sigma_i} \otimes A^{\otimes i}$. The way these complexes are glued together defines a cosheaf (see e. g. [Cur]) on the Ran space. The factorization homology is homology of this cosheaf, for details see [Lur].

3.2. **Polynomial algebra.** Any commutative algebra canonically is an invariant $\mathfrak{fm}_n$ algebra. Let A be the polynomial algebra $k[V]$ generated by a super vector space V over the base field k of characteristic zero. Its factorization homology $\int_M A$ is a commutative algebra because any commutative algebra is a commutative algebra in the category of commutative algebras.

Proposition 4 (see [BD, Ch. 4.6], [GTZ])**.** *$\int_M A = k[H_*(M) \otimes V]$, where $H_*(M)$ is the integer homology groups of M negatively graded.*

Proof. Choose a homogeneous basis of V numerated by a set B. The action of $\mathfrak{fm}_n$ on a commutative algebra factorizes through the augmentation map $\mathfrak{fm}_n(\bullet) \to k$. It means, that the complex $\oplus A^{\otimes i} \otimes C_*(\mathscr{C}(M)([\mathbf{i}]))$ factorized by relations (6) equals to $\oplus A^{\otimes i} \otimes \overline{C}_*(\mathscr{C}(M)([\mathbf{i}]))/\sim$, where $\sim$ are relations given by the unit and $\overline{C}_*(\mathscr{C}(M)([\mathbf{i}]))$ is the chain complex of the Fulton–MacPherson compactification with all border components shrunk to points. The latter space is simply the power $M^{\times i}$. Thus taking into account relations $\sim$ we see that $\int_M A$ is the homology of space of finite subsets of M labeled by B, that is the direct sum of homology of $M^{\times i_1} \times \cdots \times M^{\times i_{|B|}}$ factorized by the action of product of symmetric groups $\Sigma_{i_1} \times \cdots \times \Sigma_{i_{|B|}}$, which is given by permutations for components that corresponds to elements of the basis of even degree and by permutation multiplied by the sign representation for odd degrees. The multiplication on this space is obviously defined. $\square$

3.3. **Disk operad.** In this Subsection we sketch a connection between our definition (which follows [Sal] and others) of factorization homology and the one given in [Lur, Gin, Fra].

Given a $\mathfrak{fm}_n$-algebra A let us calculate its factorization homology on the disk $D = \{x \in \mathbb{R}^n | |x| < 1\}$.

Proposition 5. *For a $\mathfrak{fm}_n$-algebra A the factorization complex $\int_D A$ is homotopy equivalent to A.*

Proof. Define a morphism $A \to \int_D A$ as $a \mapsto [O] \otimes a$, where $a \in A$ and $[O]$ is the 0-cycle presented by the origin of coordinates. To define the morphism in the opposite direction recall, that operations of the Fulton–MacPherson operad is given by quotients $\mathscr{C}(\mathbb{R}^n)(S)/\mathrm{Dil}(n)$. Define morphism from the factorization complex $\int_D A$ to A as the composite map

$$C_*(\mathscr{C}(\mathbb{R}^n)(S)) \otimes A^{\otimes S} \to C_*(\mathscr{C}(\mathbb{R}^n)(S)/\mathrm{Dil}(n)) \otimes A^{\otimes S} = \mathfrak{fm}_n(S) \otimes A^{\otimes S} \to A,$$

where the first arrow is given by the projection and the last arrow is the action of operad. We have to show that composition of this map with the previous one is homotopic to the identity map. To build the homotopy consider a retraction of the disk to the origin of coordinates. Arguments as in the beginning of Subsection 2.1 shows that it induces the homotopy we need. $\square$

Embedding of disks into a bigger disk induces a map from tensor powers of $\int_D A$ to $\int_D A$ parametrized by the space of disks embedding. This produces action on $\int_D A$ of nerve of disks operad $N(Disk)$ in the sense of [Lur]. Moreover, the [Lur, Definition 5.3.2.6] of factorization homology $N(Disk)$-algebra being applied to $\int_D A$ gives the same result as the definition we use for factorization homology of A.

3.4. **Factorization homology and Lie algebra homology.** Following the definition of a tree from the beginning of Subsection 2.3, we say that a *bush* is an oriented connected graph with three type of vertices: *root* has no outgoing ones, *leaves* have one outgoing edge and no incoming ones and *internal vertexes* have one outgoing edge and more than one incoming ones. That is the only difference is that the root may have many incoming edges. The composition of bushes is not defined, but one may compose a tree and a bush by identification of an input of the bush and the output of the tree. Thus bushes form a right module over the operad of trees. Denote by $B_k(S)$ the set of bushes with k edges not incident to leaves and leaves labeled by a set S.

Continuing on the same lines, define the operation of *edge splitting* in the same way as for trees: we choose a vertex and a subset of incoming edges with more than one element, then we cut off trees that grow from the chosen edges, then glue an incoming edge to the vertex we choose and then glue trees we cut to the input of the glued edge. Note that an edge splitting for a bush may be done not only for an internal vertex, but for a root as well. But for an internal edge the subset of edges must be proper and for the root it may be the whole set.

For a bush b denote by $\mathrm{Det}(b)$ the one-dimensional $\mathbb{Q}$-vector space that is the determinant of the vector space generated by internal edges. For $s > 0$ consider the complex

$$B(s)\colon \bigoplus_{b\in B_0([\mathbf{s}])} \mathrm{Det}(b) \to \bigoplus_{b\in B_1([\mathbf{s}])} \mathrm{Det}(b) \to \bigoplus_{b\in B_2([\mathbf{s}])} \mathrm{Det}(b) \to \cdots, \tag{7}$$

where $[\mathbf{s}]$ is the set of s elements, the cohomological degree of a bush $B \in B_k([\mathbf{s}])$ is $k-s$ and the differential is given by all possible splitting of an edge. The composition of a tree and a bush is compatible with differentials on complexes (4) and (7) and thus equips the complex with a structure of right module over the operad L_∞.

Given a L_∞-algebra $\mathfrak{g}$ its homology (with trivial coefficients) may be calculated by means of the homological Chevalley–Eilenberg complex. Its n-th term is the symmetric power $S^n(\mathfrak{g}[1])$ and the differential is the coderivation defined by the operations $l_i\colon S^i(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ corresponding to star trees (for the definition of the latter see Subsection 2.3).

This definition may be nicely formulated in terms of modules over operads as follows.

Proposition 6. *For a L_∞-algebra $\mathfrak{g}$ the product $\mathfrak{g}^{\otimes} \otimes_{L_\infty} B(\bullet)$ is isomorphic to the Chevalley–Eilenberg complex calculating homology of $\mathfrak{g}$ with trivial coefficients modulo the zero-degree component.*

Proof. The proof is straightforward. For a more conceptual treatment see [Bal]. □

The homology of a L_∞-algebra with coefficients in the adjoint module is calculated by the complex with n-th term $S^n(\mathfrak{g}[1]) \otimes \mathfrak{g}$. The differential is a sum of the Chevalley–Eilenberg differential and the coderivation $d_{ad}\colon \mathfrak{g} \otimes S^n(\mathfrak{g}[1]) \to \bigoplus_i S^i(\mathfrak{g}[1])$ given by the adjoint action. A light modification of the foregoing allows us to define it in terms of modules over operads.

A *marked bush* is a bush with one of edges incoming to root marked. Denote by $B'_k(S)$ the set of marked bushes with k non-marked edges not incidental to leaves and leaves labeled by a set S. The edge splitting for marked bushes is defined in the same way, if the root vertex is chosen then the inserted edge is marked if the chosen subset of edges contains the marked edge and is not marked otherwise.

As before, for a bush b denote by $\mathrm{Det}(b)$ the one-dimensional $\mathbb{Q}$-vector space that is the determinant of the vector space generated by not marked edges. For $s > 0$ consider the complex

$$B'(s)\colon \bigoplus_{b\in B'_0([\mathbf{s}])} \mathrm{Det}(b) \to \bigoplus_{b\in B'_1([\mathbf{s}])} \mathrm{Det}(b) \to \bigoplus_{b\in B'_2([\mathbf{s}])} \mathrm{Det}(b) \to \cdots, \tag{8}$$

where $[\mathbf{s}]$ is the set of s elements, the cohomological degree of a bush $B \in B_k([\mathbf{s}])$ is $k - s$ and the differential is given by all possible splitting of an edge. The composition of a tree and a bush again equips the complex with a structure of right module over the operad L_∞.

On the analogy of Proposition 6 we have the following.

Proposition 7. *For a L_∞-algebra $\mathfrak{g}$ the product $\mathfrak{g}^\otimes \otimes_{L_\infty} B'(\bullet)$ is isomorphic to the Chevalley–Eilenberg complex calculating homology of $\mathfrak{g}$ in the adjoint module.*

Proof. The proof is straightforward. For a more conceptual treatment see [Bal]. □

In Subsection 2.3 we have defined a morphism from operad L_∞ to $\mathfrak{fm}_n$. Applying this morphism to the right $\mathfrak{fm}_n$-module $C_*(f\mathscr{C}(M)(S))$ introduced in Subsection 3.1 we get the right L_∞ action on $C_*(f\mathscr{C}(M)(S))$. A morphism from the right L_∞-module given by complexes (7) and (8) generated by bushes to this right L_∞-module produces morphisms from Chevalley–Eilenberg complexes to the factorization complex. It may be formulated as follows.

Proposition 8. *Let A be an invariant $\mathfrak{fm}_n$-algebra. Let $C_{Ch} = (S^*(L(A)[1]), d_{Ch})$ $C^{ad}_{Ch} = (S^*(L(A)[1]) \otimes L(A), d_{Ch} + d_{ad})$ be the Chevalley–Eilenberg complexes calculating the homology of L_∞-algebra $L(A)$ with trivial coefficients and in the adjoint module correspondingly. Let M be a closed manifold and $p \in M$ is a point. Then morphisms*

$$\begin{aligned} a_1 \otimes \cdots \otimes a_i &\mapsto [\mathscr{C}^0(M)([\mathbf{i}])] \otimes_{\Sigma_i} (a_1 \otimes \cdots \otimes a_i) \\ a_1 \otimes \cdots \otimes a_i \otimes a_0 &\mapsto [\mathscr{C}^0(M \setminus p)([\mathbf{i}])] \otimes_{\Sigma_i} (a_1 \otimes \cdots \otimes a_i) \otimes a_0 \end{aligned}$$

define maps from complexes $C_{Ch}(L(A))$ and $C^{ad}_{Ch}(L(A))$ respectively to the factorization complex $\int_M A$, where $[\mathscr{C}^0(M)(S)]$, $[\mathscr{C}^0(M \setminus p)(S)]$ and $[p]$ are cycles in $C_*(\mathscr{C}(M)(S))$ presented by the configuration space of distinct points, distinct points different from p and the point p.

Proof. These morphisms are given by morphisms of right modules over $\mathfrak{fm}_n$ operad, see the discussion before the Proposition. □

The first map above was introduced in [Mar1] in a more explicit form.

4. Weyl n-algebra

4.1. **Definition.** The usual Weyl algebra is a deformation of the polynomial algebra. We have seen that a commutative algebra is an algebra over operad $\mathfrak{fm}_n$ for any n. The analogous deformation of a commutative algebra in the category of $\mathfrak{fm}_n$-algebras gives us what we call the Weyl n-algebra.

Let V be a super (that is $\mathbb{Z}$-graded) vector space over the base field k of characteristic zero equipped with a non-degenerate super skew-symmetric pairing $\omega\colon V \otimes V \to k$ of degree $1 - n$. Let $k[V]$ be the polynomial algebra generated by V

and $k[[h]]$ be the ring of formal series and $k[[h]][V]$ is the polynomial algebra over it. Denote by

$$\partial_\omega \colon k[V] \otimes k[V] \to k[V] \otimes k[V] \tag{9}$$

the differential operator that is a derivation in each factor and acts on generators as ω.

Consider $\mathbf{FM}_n([\mathbf{2}])$, the space of 2-ary operations of the Fulton–MacPherson operad. This is $(n-1)$-dimensional sphere. Denote by v the standard $SO(n)$-invariant $(n-1)$-differential form on it. For any two-element subset $\{i,j\} \subset S$ denote by $p_{ij} \colon \mathbf{FM}_n(S) \to \mathbf{FM}_n([\mathbf{2}])$ the map that forgets all points except ones marked by i and by j. Denote by v_{ij} the pullback of v under projection p_{ij}. Let α be an element of endomorphisms of $k[V]^{\otimes S} \underset{Aut(S)}{\otimes} C^*(\mathbf{FM}_n(S))$ (where $C^*(-)$ is the de Rham complex) given by

$$\alpha = \sum_{i,j\in S} \partial_\omega^{ij} \wedge v_{ij},$$

where ∂_ω^{ij} is the operator ∂_ω applied to the i-th and j-th factors.

Proposition 9. *Formula*

$$k[V]^{\otimes S} \xrightarrow{\exp(h\alpha)} k[[h]][V]^{\otimes S} \otimes C^*(\mathbf{FM}_n(S)) \xrightarrow{\mu} k[[h]][V] \otimes C^*(\mathbf{FM}_n(S)),$$

where μ is the product in the polynomial algebra, defines an $k[[h]]$-algebra over the operad $\mathfrak{fm}_n$ with the underlying space $k[[h]][V]$.

Proof. This is a simple check. □

The algebra defined in this way is obviously invariant under action of $SO(n)$.

Definition 11. For a pair (V,ω) as above the invariant $\mathfrak{fm}_n$-algebra given by Proposition 9 is called the *Weyl $\mathfrak{fm}_n$-algebra*. Denote it by $\mathcal{W}^n{}_h(V)$.

Note that construction from Subsection 3.3 provides us with the *Weyl e_n-algebra*.

One may give an alternative definition of the Weyl algebra as the universal enveloping of the Heisenberg Lie algebra, compare with [BD, 3.8.1].

Example 1. For $n = 1$ and a vector space of degree 0 one get the Moyal product.

Denote by $\mathcal{W}^n(V)$ the algebra over Laurent formal series, which is the localization $\mathcal{W}^n{}_h(V) \otimes_{k[h]} k[h^{-1}, h]]$. Both of algebras $\mathcal{W}^n{}_h(V)$ and $\mathcal{W}^n(V)$ are equipped with increasing filtration: the degree of an element is the degree both in h and V of the polynomial that presents it.

Consider the L_∞-algebra $L(\mathcal{W}^n{}_h(V))$ associated with the Weyl algebra . One may see that this is a super Lie algebra, all higher operations vanish. This Lie algebra $L(\mathcal{W}^n{}_h(V))$ is a deformation of the Abelian one. The first order deformation gives the *Poisson Lie algebra*: the underlying space is the supercommutative algebra $k[V]$, the bracket is defined by $h\omega \colon V \otimes V \to k[[h]]$ on generators and satisfies the Leibniz rule. For the classical one-dimensional Weyl algebra it is known, that higher terms of the deformation are non-trivial: $L(\mathcal{W}^1_h(V))$ differs from the Poisson Lie algebra ([Vey]). But for $n > 1$ the situation is simpler.

Proposition 10. *For $n > 1$ Lie algebra $L(\mathcal{W}^n)$ is isomorphic to the Poisson Lie algebra of $(V \otimes k[h^{-1}, h]], \omega)$ over $k[h^{-1}, h]]$, the definition of the latter is as above.*

Proof. Obvious, because for $n > 1$ the square of the de Rham cochain v is zero. $\square$

4.2. **Factorization homology of $\mathcal{W}^n$.** Weyl n-algebra is a deformation of a commutative algebra. From Subsection 3.2 we know factorization homology of a commutative algebra. Below we use deformation arguments to calculate factorization homology of the Weyl algebra on a closed manifold M.

Proposition 11. *Let V be a super vector space with a skew-symmetric pairing of degree $1-n$ and $V = \oplus_i V_i$ is its decomposition by degrees. Let M be a n-dimensional closed oriented manifold and b_i its rational Betti numbers. Then factorization homology $H_*(\int_M \mathcal{W}^n(V))$ is one-dimensional $k[h^{-1}, h]]$-module of total degree*

$$\sum_{\substack{\{i,j\}\\ i+j \ odd}} (-i+j) b_i \dim V_j$$

Proof. Consider the filtration of $\int_M \mathcal{W}^n{}_h(V)$ by powers of h and the corresponding spectral sequence. The associated graded complex calculates the factorization homology of the commutative polynomial algebra $\mathcal{W}^n{}_{h=0}(V)$, the result is given by Proposition 4. Let us calculate the 0-th differential. Since the action of the $\mathfrak{fm}_n$ operad at the first order by h is given by a differential operator of degree 2, the 0-th differential is a differential operator of degree 2 as well. Thus it is enough to calculate the differential on the degree 2 part of algebra $\mathcal{W}^n{}_{h=0}$.

By Proposition 3.2, it is equal to homology of pairs of points of M marked by elements of a basis of V. To get the differential of a given homology class one need to present it by a cycle, lift this cycle to the complex, calculating $\int$ and take the differential there. Present a given class $[c] \in H_*(M^2)$ by a cycle c that intersects the diagonal $\delta\colon M_\delta \hookrightarrow M^2$ transversally. Then one may see, that differential of the lifted cycle is $c \cap M_\delta \cdot \omega(x, y)$, where x and y are element of the basis marking the points. In other words, it is equal to $\delta_* \delta^*[c] \cdot \omega(x, y)$.

Thus the 0-th differential is given by the differential operator of degree two given by the non-degenerate degree 1 pairing on $H_*(M) \otimes V$. The resulting complex is the Koszul complex, which has the only homology class and consequently the spectral sequence degenerates at the first term. This only class is presented by the top degree power of the odd part of vector space $H_*(M) \otimes V$, which gives the formula from the Proposition. $\square$

Example 2. Let $n = 1$ and V is concentrated in degree 0. Then by Example 1, $\mathcal{W}^n(V)$ is the usual Weyl algebra. For $M = S^1$ the factorization homology is the Hochschild homology and Proposition 11 matches with the well-known fact about Weyl algebra:

$$\dim HH_i(\mathcal{W}^1(V)) = \begin{cases} 1, & i = \dim V, \\ 0, & \text{otherwise}, \end{cases}$$

see e. g. [FT].

The proof of Proposition 11 allows to produce an explicit cycle presenting the only non-trivial class in factorization homology of the Weyl algebra on a closed manifold similarly to the example. Below we consider the simplest case, leaving the general one to the reader.

Proposition 12. *Let M be an odd-dimensional rational homology sphere and the super-vector space V has only odd-degree components. Then the only non-trivial cycle in homology of $\int_M \mathcal{W}^n(V)$ is presented by a cycle in $C_0(M)$ given by a point marked by an element $S^{\mathrm{top}}V$ of top degree in the symmetric power of V, since V lies in the odd degree the latter makes sense.*

Proof. This obviously a cycle and it presents a non-trivial class at the first leaf of the spectral sequence from the proof of Proposition 11. Since the spectral sequence degenerates at the first sheet, this cycle survives. □

4.3. **Euler structures.** As it was mentioned after Proposition 3, framing on a manifold simplifies the definition of the factorization complex. For Weyl n-algebra a weaker structure is sufficient.

For a manifold M and a map of finite sets $S' \to S$ denote by $\mathscr{C}(M)(S' \to S)$ the fiber product

$$\begin{array}{ccc} & & \mathscr{C}(M)(S') \\ & & \big\downarrow \\ \mathscr{C}^0(M)(S) & \longrightarrow & M^{S'} \end{array} \tag{10}$$

where the horizontal map is composition of the embedding $\mathscr{C}^0(M)(S) \hookrightarrow M^S$ and the map $M^S \to M^{S'}$ induced by the map $S' \to S$, and the vertical map is the projection. Space $\mathscr{C}(M)(S' \to S)$ is equipped with the projection

$$\pi\colon \mathscr{C}(M)(S' \to S) \to \mathscr{C}^0(M)(S).$$

For the only map from $[\mathbf{2}]$ to $[\mathbf{1}]$ the space $\mathscr{C}(M)([\mathbf{2}] \to [\mathbf{1}])$ is the total space of the sphere bundle associated with the tangent bundle.

Definition 12. The *Euler structure* on a n-manifold M is a closed differential form $\mathfrak{v}$ on $\mathscr{C}(M)([\mathbf{2}] \to [\mathbf{1}])$ such that its restriction on any fiber of the projection $\mathscr{C}(M)([\mathbf{2}] \to [\mathbf{1}]) \to M$ is the standard volume form on the sphere.

The only obstruction to existence of the Euler structure is the rational Euler class. In particular, on odd-dimensional manifolds an Euler structure always exists.

Fix an Euler structure on M given by a form $\mathfrak{v}$ on $\mathscr{C}(M)([\mathbf{2}] \to [\mathbf{1}])$. For any morphism of arrows from $[\mathbf{2}] \to [\mathbf{1}]$ to $S' \to S$ the natural map

$$\mathscr{C}(M)(S' \to S) \to \mathscr{C}(M)([\mathbf{2}] \to [\mathbf{1}])$$

is defined. Denote by $\mathfrak{v}_{ij}$ the pull back of $\mathfrak{v}$ under this map.

Let V be a super vector space equipped with a non-degenerate skew-symmetric pairing $\omega\colon V \otimes V \to k$ of degree $1-n$. Let $k[V]$ be the polynomial algebra generated by V. As before let A be an element of endomorphisms of $k[V]^{\otimes S} \underset{Aut(S')}{\otimes} C^*(\mathscr{C}(M)(S' \to S))$ given by

$$\mathrm{A} = \sum_{\{i,j\}} \partial_\omega^{ij} \wedge \mathfrak{v}_{ij},$$

where the sum is taken by all morphisms of arrows from $[\mathbf{2}] \to [\mathbf{1}]$ to $S' \to S$ and ∂_ω^{ij} is the operator ∂_ω applied to the i-th and j-th factors, where ∂_ω is defined by

(9). The exponent of $h\mathrm{A}$ in composition with the cup product gives endomorphism of $k[[h]][V]^{\otimes S'} \underset{Aut(S')}{\otimes} C_*(\mathscr{C}(M)(S' \to S))$. Consider the composite map

$$
\begin{array}{c}
k[[h]][V]^{\otimes S'} \underset{Aut(S)}{\otimes} C_*(\mathscr{C}(M)(S' \to S)) \\
\Big\downarrow{\scriptstyle \exp(h\mathrm{A})} \\
k[[h]][V]^{\otimes S'} \underset{Aut(S')}{\otimes} C_*(\mathscr{C}(M)(S' \to S)) \\
\Big\downarrow{\scriptstyle \mu\otimes\pi_*} \\
k[[h]][V]^{\otimes S} \underset{Aut(S)}{\otimes} C_*(\mathscr{C}^0(M)(S)),
\end{array} \tag{11}
$$

where μ is action of morphism in the category $Comm^{\otimes}$.

Proposition 13. *Let V be a super vector space equipped with a non-degenerate super skew-symmetric pairing $\omega\colon V \otimes V \to k$ of degree $1-n$, $A = \mathcal{W}^n{}_h(V)$ be the corresponding Weyl algebra and M be a closed manifold with Euler structure. Then the factorization complex $\int_M \mathcal{W}^n{}_h(V)$ is the colimit of the diagram*

$$
\begin{array}{c}
\bigoplus_{i\colon S' \to S} (C_*(\mathscr{C}(M)(S' \to S))) \underset{Aut(S')}{\otimes} A^{\otimes S'} \\
\Big\uparrow \\
\bigoplus_{S'} C_*(\mathscr{C}(M)(S')) \underset{Aut(S')}{\otimes} A^{\otimes S'} \\
\Big\downarrow \\
\bigoplus_{S} C_*(\mathscr{C}^0(M)(S)) \underset{Aut(S)}{\otimes} A^{\otimes S}
\end{array} \tag{12}
$$

where the downwards arrow is the composite map (11) and the upwards arrow is induced by the natural embedding.

Proof. The statement is local along $\mathscr{C}^0(M)(S)$. And locally it directly follows from Proposition 6 and the definition of Weyl algebra. □

5. Perturbative invariants

5.1. **Propagator.** Let M be a rational homological sphere of dimension n. Denote by $\tilde{M}$ the complement in M to the interior of a little ball around a point $p \in M$.

Below we will need Fulton–MacPherson compactification of manifolds with boundary. Let X be such a manifold and $X \hookrightarrow X'$ be its closed embedding in a manifold of the same dimension, for example, X' is obtained from X by gluing a collar. Then denote by $\mathscr{C}(\tilde{X})(S)$ the fiber product

$$
\begin{array}{ccc}
 & & \mathscr{C}(X')(S) \\
 & & \Big\downarrow \\
\tilde{X}^S & \longrightarrow & X'^S
\end{array}
$$

where the upwards arrow is the embedding and the vertical one is the projection.

Consider the differential $(n-1)$-form on $\mathscr{C}^0(\mathbb{R}^n)([\mathbf{2}])$ which is pull back of the standard form on the sphere under the map $(x, y) \mapsto (x-y)/|x-y|$ and continue it on $\mathscr{C}(\mathbb{R}^n)([\mathbf{2}])$ straightforwardly (in Subsection 4.1 it was denoted by v). Consider the subset of $\mathscr{C}(\mathbb{R}^n)([\mathbf{2}])$ where the both points lie on the unit sphere and restrict the form as above on it. Call the result the *standard form*.

The following Proposition stays, that on the 2-point Fulton–MacPherson configuration space of the "fake disk" $\tilde{M}$ there is a differential $(n-1)$-form similar to the standard form on the configuration space of the real disk.

Proposition 14. *For a rational homological sphere M choose a point O in the interior of its complement $\tilde{M}$ to a little disk. Then on manifold with corners $\mathscr{C}(\tilde{M})([\mathbf{2}])$ as above there exists a differential $(n-1)$-form such that*

1. *it is smooth and closed;*
2. *its restriction to any fiber of $\pi\colon \mathscr{C}(\tilde{M})([\mathbf{2}]) \to \tilde{M}^2$ over any point on the diagonal, which is a sphere, is equal to the standard form on the sphere;*
3. *its restriction to the subset where both points of the configuration lie on the boundary equals to the standard form;*
4. *its restriction to $O \times \partial\tilde{M}$ and $\partial\tilde{M} \times O$ equals to the standard form on the sphere.*

Proof. It follows from elementary considerations with Mayer–Vietoris sequence, see for example [AS1]. □

Definition 13. We call the $(n-1)$-form as above on $\mathscr{C}(\tilde{M})([\mathbf{2}])$ a *propagator* and denote it by ν.

Note, that our definition of propagator differs slightly from the one given in [AS1], [BC].

5.2. **Collapse.** Let M and M' be any closed n-manifolds. Choose a point in each manifolds and cut off small open balls around them. We get two manifolds $\tilde{M}$ and $\tilde{M}'$ with boundaries S^{n-1}. Denote their interiors by M_0 and M_0'. The connected sum $M\#M'$ is a result of gluing together of these two manifolds by their boundaries. Call the continuous map $\mathfrak{Col}\colon M\#M' \to M'$ that shrinks M to a point $p \in M'$ by the *collapse map*.

In general the collapse map does not produce any map between factorization homologies of $M\#M'$ and M. There are two cases when it obviously does.

The first case is when the algebra is commutative. The factorization homology is given by homology of the powers of the space and the morphism is given by the direct image on homology of the powers.

The second case is when $M = S^n$. Then $M\#M' = M'$. To build the morphism one need loosely speaking to take everything sitting in M, multiply it and put the result to the point p. Arguments as in Proposition 5 shows that this is an isomorphism.

There is another case when such morphism exists: when M is an odd-dimensional homology sphere and the algebra in hand is the Weyl algebra. Its construction occupies the rest of this Subsection.

The morphism factorizes through an intermediate object we are going to define.

Let M be a rational homology odd-dimensional sphere and M' be any closed n-manifold of the same dimension. Choose Euler structures on M and M', this is possible, because they are odd-dimensional. By the same reason, Euler structures on M and M' naturally define the one on $M\#M'$.

For a surjective morphism of manifolds $f\colon X' \to X$ and a map of sets $S' \to S$ define space $\mathscr{C}(X'/X)(S' \to S)$ as the fiber product

$$\begin{array}{ccc} & & \mathscr{C}(X')(S') \\ & & \downarrow \\ \mathscr{C}^0(X)(S) & \longrightarrow & X^{S'} \end{array} \tag{13}$$

where the vertical arrow is the composition of projection $\mathscr{C}(X')(S') \to {X'}^{S'}$ with $f^{S'}$ and the horizontal arrow is composition of the embedding $\mathscr{C}^0(M)(S) \hookrightarrow X^S$ and the map $X^S \to X^{S'}$ induced by the map $S' \to S$. Space $\mathscr{C}(X'/X)(S' \to S)$ is equipped with the projection

$$\pi\colon \mathscr{C}(X'/X)(S' \to S) \to \mathscr{C}^0(X)(S).$$

For the collapse map $M\#M' \to M'$ consider space $\mathscr{C}(M\#M'/M')([\mathbf{2}] \to [\mathbf{1}])$. This space contains $\mathscr{C}(M')([\mathbf{2}] \to [\mathbf{1}])$ and M_0^2 as subspaces. On the first one the Euler structure gives a differential $(n-1)$-form and on the second one choose a propagator (Definition 13). Property 3 of propagator (Proposition 14) allows to glue it in a global differential $(n-1)$-form on $\mathscr{C}(M\#M'/M')([\mathbf{2}] \to [\mathbf{1}])$. Denote it by $\mathcal{V}$. Although the space is not manifold, it is glued up from manifolds and $\mathcal{V}$ is a well-defined cochain of the corresponding relative complex.

Let V be a super vector space equipped with a non-degenerate skew-symmetric pairing $\omega\colon V \otimes V \to k$ of degree $1-n$. Let $k[V]$ be the polynomial algebra generated by V. Mimicking construction from Subsection 4.3 let $\mathcal{A}$ be an element of endomorphisms of $k[V]^{\otimes S} \underset{Aut(S')}{\otimes} C^*(\mathscr{C}(M\#M'/M')(S' \to S))$ given by

$$\mathrm{A} = \sum_{\{i,j\}} \partial_\omega^{ij} \wedge \mathcal{V}_{ij},$$

where the sum is taken by all morphisms of arrows from $[\mathbf{2}] \to [\mathbf{1}]$ to $S' \to S$ and ∂_ω^{ij} is the operator ∂_ω applied to the i-th and j-th factors, where ∂_ω is defined by (9). The exponent of $h\mathcal{A}$ in composition with the cup product gives endomorphism of $k[[h]][V]^{\otimes S'} \underset{Aut(S')}{\otimes} C_*(\mathscr{C}(M\#M'/M')(S' \to S))$. Consider the composite map

$$\begin{array}{c} k[[h]][V]^{\otimes S'} \underset{Aut(S)}{\otimes} C_*(\mathscr{C}(M\#M'/M')(S' \to S)) \\ \downarrow{\scriptstyle \exp(h\mathcal{A})} \\ k[[h]][V]^{\otimes S'} \underset{Aut(S')}{\otimes} C_*(\mathscr{C}(M\#M'/M')(S' \to S)) \\ \downarrow{\scriptstyle \mu\otimes\pi_*} \\ k[[h]][V]^{\otimes S} \underset{Aut(S)}{\otimes} C_*(\mathscr{C}^0(M')(S)), \end{array} \tag{14}$$

where μ is action of morphism in the category $Comm^{\otimes}$.

By analogy with (12) consider the diagram

$$
\begin{array}{c}
\bigoplus_{S'} C_*(\mathscr{C}(M\#M')(S')) \underset{Aut(S')}{\otimes} A^{\otimes S'} \\
\uparrow \\
\bigoplus_{i\colon S'\to S} (C_*(\mathscr{C}(M\#M'/M')(S'\to S))) \underset{Aut(S')}{\otimes} A^{\otimes S'} \\
\downarrow \\
\bigoplus_{S} C_*(\mathscr{C}^0(M')(S)) \underset{Aut(S)}{\otimes} A^{\otimes S}
\end{array}
\tag{15}
$$

where the downwards arrow is the composite map (14) and the upwards arrow is induced by the natural embedding.

The desired intermediate object is colimit of diagram (15). Property 2 of propagator (Proposition 14) supplies us with a natural map from the diagram presenting $\int_{M\#M'} \mathcal{W}^n{}_h(V)$ by Proposition 13 to (15), thus with a map from $\int_{M\#M'} \mathcal{W}^n{}_h(V)$ to colimit of (15).

The following Proposition completes the construction.

Proposition 15. *Colimit of (15) is isomorphic to* $\int_{M'} \mathcal{W}^n{}_h(V)$.

Proof. As it was discussed after Proposition 6, the factorization complex is equipped with an increasing filtration by the number of points of the configuration space of distinct points. Introduce slightly different filtration on $\int_{M'} \mathcal{W}^n{}_h(V)$ fixing an element of the algebra sitting at point p, if there nothing at this point, we assume that it is 1. The associated graded object of this filtration is $\mathcal{W}^n{}_h(V)_p \otimes \bigoplus_S C_*(\mathscr{C}^0(M'\setminus p)(S)) \underset{Aut(S)}{\otimes} \mathcal{W}^n{}_h(V)^{\otimes S'}$. For the same reason, because the horizontal arrow in (15)is surjective, colimit of (15) is also filtrated with the same quotients.

To prove the statement we are going to define a map from $\int_{M'} \mathcal{W}^n{}_h(V)$ to colimit of (15). As it was already discussed after Proposition 6, every chain in $\int_{M'} \mathcal{W}^n{}_h(V)$ may be presented as a chain of $\mathcal{W}^n{}_h(V)_p \otimes \bigoplus_S C_*(\mathscr{C}^0(M'\setminus p)(S)) \underset{Aut(S)}{\otimes} \mathcal{W}^n{}_h(V)^{\otimes S'}$. Take such a representative $a_0 \otimes c \otimes a_1 \otimes \cdots \otimes a_s$, where $c \in C_*(\mathscr{C}^0(M' \setminus p)([\mathbf{s}]))$ and $a_i \in \mathcal{W}^n{}_h(V)$, and send it to $a_0 \otimes a_1 \otimes \cdots \otimes a_s \otimes \imath_{O*}c$, where map

$$\imath_O \colon \mathscr{C}^0(S)(M') \hookrightarrow \mathscr{C}(M\#M'/M')((S+1) \xrightarrow{id} (S+1))$$

embeds configuration and adds point O to it.

One may see that this map is a map of complexes due to property 4 of the propagator (Proposition 14) and an isomorphism on the associated graded object. Consequently, it gives an isomorphism of complexes. □

Call the morphism $\mathfrak{col}\colon \int_{M\#M'} \mathcal{W}^n{}_h(V) \to \int_{M'} \mathcal{W}^n{}_h(V)$ just constructed by the *collapse morphism*.

The proof of this Proposition may be interpreted by means of cosheaves in spirit of the discussion at the end of Subsection 3.1. Indeed, colimit of digram (15) gives a cosheaf on the Ran space of M'. The Proposition 15 states that it is isomorphic to the one given by the Weyl algebra.

Note finally, that Proposition 15 may be reformulated as follows: for a homological sphere M the factorization complex $\int_{\tilde{M}} \mathcal{W}^n{}_h(V)$ is isomorphic to $\mathcal{W}^n{}_h(V)$

as $\int_{[0,1]\times S^{n-1}} \mathcal{W}^n{}_h(V)$-module (about the module structure on the factorization complex of a manifold with boundary see e. g. [Gin] and references therein).

5.3. **Invariants.** Factorization homology of Weyl n-algebras may be used to construct invariants of manifolds. Let M be a closed n-manifold and V be a super vector space with a non-degenerate pairing of degree $1-n$. By Proposition 11 the factorization homology of $\mathcal{W}^n{}_h(V)$ on M is one-dimensional. The idea of invariant we are going to build is to construct somehow a cycle in $\int_M \mathcal{W}^n{}_h(V)$ and calculate the class represented by it. As the homology group is one-dimensional, this class is a multiple of the standard one. The number we get this way is the invariant of the manifold.

Let us restrict ourselves with the following conditions: M is a rational homology sphere of odd dimension n and V be a super vector space, which has only odd-dimensional components. Under these conditions due to Proposition 12 the only class in the factorization homology is presented by an especially simple cycle, just an element of the top degree power of V sitting at a point, call this cycle the standard one.

To produce a different cycle we shall resort to the morphism given by Proposition 8. It sends the Chavelley–Eilenberg complex of the Lie algebra $L(\mathcal{W}^n(V))$ associated with the Weyl algebra $\mathcal{W}^n(V)$ to factorization homology of $\mathcal{W}^n(V)$.

By Proposition 10 $L(\mathcal{W}^n(V))$ for $n>1$ is super Poisson Lie algebra. Suppose that $\dim V \geq 3$ and denote by $L(\mathcal{W}^n(V)^{\geq 3})$ the Lie subalgebra of polynomials of degree not less than 3. One may see that element $S^{top}V$ is in the center of $L(\mathcal{W}^n(V)^{\geq 3})$. Thus the map

$$k \to L(\mathcal{W}^n(V)^{\geq 3}),$$

which sends the generator to a non-zero element from $S^{top}V$ is a morphism from the trivial $L(\mathcal{W}^n(V)^{\geq 3})$-module to the adjoint one. Consider the induced map

$$C_{Ch}(L(\mathcal{W}^n(V)^{\geq 3})) \to C^{ad}_{Ch}(L(\mathcal{W}^n(V)^{\geq 3}))$$

and combine it with map

$$C^{ad}_{Ch}(L(\mathcal{W}^n(V)^{\geq 3})) \to \int_M \mathcal{W}^n(V)$$

given by Proposition 8. The composite map

$$C_{Ch}(L(\mathcal{W}^n(V)^{\geq 3})) \to \int_M \mathcal{W}^n(V) \tag{16}$$

is the desired invariant. In other words, the invariant is a cohomology class of total degree zero of super Lie algebra $L(\mathcal{W}^n(V)^{\geq 3})$. To get a number one may substitute a homology class of this Lie algebra in it.

As it was already mentioned in the Introduction, a cocycle of the complex linear dual to the factorization complex $\int_M \mathcal{W}^n(V)$ which does not vanish on the standard cycle would make this invariant more explicit. As such a cocycle is unavailable, we shall make of use the collapse morphism from the previous Subsection.

Proposition 16. *If M and M' are both rational homology odd-dimensional spheres and V has only odd-degree components then the collapse morphism*

$$\mathsf{col}\colon \int_{M\#M'} \mathcal{W}^n(V) \to \int_{M'} \mathcal{W}^n(V)$$

induces isomorphism on homologies.

Proof. By Proposition 12 the non-trivial class in homology of $\int_{M\#M'}\mathcal{W}^n(V)$ is presented by a cycle which is an element of the algebra sitting at a point. As it follows from its definition, the collapse morphism sends it to the same cycle in $\int_{M'}\mathcal{W}^n(V)$, which is non-trivial by Proposition 12. □

Assuming $M' = S^n$ in the Proposition above we get an isomorphism $\int_M \mathcal{W}^n(V) \to \int_{S^n}\mathcal{W}^n(V)$. In composition with (16) we get a morphism

$$C_{Ch}(L(\mathcal{W}^n(V)^{\geq 3})) \to \int_{S^n}\mathcal{W}^n(V),$$

which is better than (16), because the target does not depend on M.

Unwinding the definition of the collapse morphism one may see that this cocylce of $L(\mathcal{W}^n(V)^{\geq 3})$ taking values in $\int_{S^n}\mathcal{W}^n(V)$ is a sort of cocyle given by the graph complex, see [Kon1, Kon2, QZ]. It is known ([AS1, AS2]), that perturbative Chern–Simons invariants also give classes in the graph complex in same way, by integration powers of the propagator. It makes us believe that our invariants coincide with the perturbative Chern–Simons ones. Perhaps, some good choice of the propagator will lead to a more explicit formula.

Finally let $n = 3$, V be a super vector space of dimension more than 2 concentrated in degree 1 with skew-symmetric pairing of degree -2, that is $V[1]$ is equipped with a symmetric pairing. In this case for dimensional reasons the cocycle is given by trivalent graphs. If $V[1]$ is the underlying space of a Lie algebra with non-degenerate pairing, then the element in $S^3V[1]$, which is composition of the Lie bracket and the pairing, is a Maurer–Cartan element in $L(\mathcal{W}^n(V)^{\geq 3})$. Its power gives a homology class. Values of the cocycle on it must be the perturbative invariants associated with given Lie algebra. More about this case the reader may find in the Appendix.

Appendix

The physical definition of perturbative Chern–Simons invariant is based on the asymptotic series of the oscillating integral $\int e^{iS}$ taken by the space of all G-connection A on M, where $S = \frac{k}{4\pi}\int_M \operatorname{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$ is the Chern–Simons functional, M is a 3-manifold and G is a semi-simple Lie group. The aim of this appendix is to demonstrate speculatively how to interpret the calculation of such an integral in terms of the factorization homology.

Thus we have the infinite-dimensional space of connections, a function S on it and we want to calculate asymptotic series in $1/k$ of the oscillating integral. If M is a homology sphere, then function S has a non-degenerate critical point at th origin. Thus the free term of the series in hand is the Gaussian integral by an infinite-dimensional space and is unavailable by algebraic methods. But after dividing by this term the series may be calculated by means the method of Feynman diagrams.

To explain how this method works consider an abstract situation, the reader can find more at [JF]. Let V be an Euclidean vector space and f be a smooth function on it such that its Taylor series at the origin start with terms of degree at least three. Choose a volume form on V and consider the integral $\int e^{(-|x|^2+tf)}$. Consider the twisted de Rham complex of polynomial forms Ω^*_t given by differential forms on V with differential $d_{dR} - 2(\mathbf{x}, d\mathbf{x}) + t\, df$, where d_{dR} is the de Rham differential. One may see that complex $\Omega_t^* \otimes \mathbb{R}[[t]]$ has only top degree cohomology, which is one-dimensional over $\mathbb{R}[[t]]$. This one-dimensional vector bundle over the t-line has the Gauß–Manin connection and a section given by the chosen volume form on V. Their quotient is a series in t up to a constant factor and one may show that this is the asymptotic expansion of the oscillating integral up to a constant.

We are now going to construct a $\mathfrak{fm}_3$-algebra (which by Proposition 1 is the same as e_3-algebra) factorization complex of which on M resembles the twisted de Rham complex as above. Let g be a Lie algebra with a non-degenerate invariant bilinear form. The desired $\mathfrak{fm}_3$-algebra is a deformation of the Chevalley–Eilenberg commutative dg-algebra $C^\bullet_{Ch}(g)$ in the class of $\mathfrak{fm}_3$-algebras. Namely, the deformation is the same as in the definition of the Weyl algebra (Definition 11), where the polynomial algebra is the exterior algebra of $g^\vee$ and the pairing is the invariant bilinear form. It is easy to check that this deformation respects the differential. Note, that this e_3-algebra is the algebra of Ext's from the unit to itself in e_2-category of representations of the quantum group. Denote it by $Ch^\bullet_h(g)$.

Alternatively, this $\mathfrak{fm}_3$-algebra may be defined as follows. Start with super vector space $g^\vee[1]$ with the paring of degree -2 given by the invariant scalar product and build the Weyl algebra $\mathcal{W}^3_h(g^\vee[1])$. Then define differential on it as $\frac{1}{h}\{\cdot, q\}$, where $\{,\}$ is image of the Lie bracket under (5) and $q \in S^3(g^\vee[1])$ is composition of the Lie bracket on g and the scalar product. One may show, that similarly to Hochschild homology (see e. g. [Lod, Propositon 1.3.3]) factorization homology on a closed manifold is invariant under inner deformations. It follows by Proposition 11 that the homology of $\int_M Ch^\bullet_h \otimes k[[h^{-1}, h]]$ is free $k[[h^{-1}, h]]$-module of rank 1. And moreover, this homology is equipped with a connection along h-line.

To fulfill the analogy (note, that t corresponds to $1/h$) we have to present a section of this one-dimensional vector bundle and compare it with a constant under this connection one. Formula (16) produces elements in the factorization complex of $Ch^\bullet_h(g)$. One may see, that 1 goes to a cycle (in fact, this is the cycle given by Proposition 12) and this is analog of the section given by the volume form on V in the example above. On the other hand, one may see that a cycle flat with respect to the connection is the image under (12) of the cycle $\sum_i \frac{h^{-i}}{i!} \underbrace{q \wedge \cdots \wedge q}_{i}$.

The quotient of these two sections is an analog of the asymptotic series and is given by invariants as in Subsection 5.3.

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