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Demonstrativeness in Mathematics and Physics

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Abstract. An approach to the physical language, pointed out by Poincaré in a philosophical work, is to link probability theory to arithmetic and thermodynamics. We follow this program in the part concerning a generalization of probability theory.

DOI:

I have written repeatedly how difficult it is to speak a language which is understandable both for mathematicians and for physicists. However, when speaking with mathematicians, you can present a proof, and then the issue disappears. On the contrary, a mathematical proof is not convincing for physicists (in [14], I have quoted my conversation with E. G. Maksimov who told me: "*You won't convince me by your lemmas*").

Recently I had a correspondence with V. Ivrii, one of the connoisseurs on asymptotic methods in analysis and mathematical physics. I quote it here, and then compare with the controversies which have with theoretical physicists.

Here is my letter.

> I've found (on the Internet) your work, which reads:
>
> "Another approach to Lagrangian Distributions is based on Maslov Canonical Operator; original not completely rigorous papers of Victor Maslov appeared before [2]."
>
> My corresponding work ("Perturbation Theory and Asymptotic Methods" [3]) was thoroughly tested, first, by Grisha Eskin, who, when I asked him why he does not refer to my work, said this particular phrase. I told him: "If you will find a hole which I will not explain in the presence of V. P. Palamodov and S. P. Novikov, then I'll sweep aside all of my claims." Grisha questioned me, and I had given satisfactory answers. You can ask him about it.
>
> Second, it was Ladyzhenskaya, my opponent on the defense of the doctoral thesis. She clarifies more or less the same points and tried to find a hole in the proof of estimates. Everything has been thoroughly tested, and, as it seems to me that this is quite easy to do.
>
> I hoped that physicists will also understand my work. Therefore, in the first paper I omitted the details that are of no interest for physicists.
>
> All my material was presented more thoroughly for mathematicians in the appendix to Heading's book "An Introduction to Phase-Integral Methods" [4] and in our joint work with M. Fedoryuk [5]. Nobody and never found holes or errors in any of my works. However, I found errors in works of Gel'fand, Kostyuchenko, Shilov, and Maurin, as well as in Gel'fand's report on P_+^γ (see [6]).
>
> In 1968, I found a gap in the paper of Levitan and Suvorchenkova which was sent to me for refereeing. This enabled one to find not only sufficient conditions, as was suggested by the authors, but also necessary and sufficient criteria (see my postscript to the paper of the authors in [7])[1].
>
> Lars Hermander wrote that it was hard to find my book, and possibly not everything is completely rigorous, but still he presented me no points that he could regard as unproven ones.

[1] I have already written in [8] that I tried to help Arnold in his election to Corresponding Member of USSR Academy of Sciences. When there was a discussion about Arnol'd's mistake in the problem of three bodies of different masses one of which is of zero mass (the problem with two degrees of freedom) and Bryuno's counterexample to one of Arnol'd's lemmas, Kantorovich suddenly said: " We have here an expert on errors, namely, Maslov, lat us ask him." I joked: "I do not believe in mistakes found on the eve of the election." The question was dropped.

> Thus, in a way, I challenged Grisha Eskin to a mathematical duel for his sentence. To avoid the same with you at friendly relations that linked us during all our life, I urge you to contact Grisha or Victor Palamodov and to exclude statements of this kind in your work. I send you also my article in the "Doklady of the Academy of Sciences of the USSR" which was published in 1967 and in which the Fourier integral operator was introduced[2]. the operator Introduced where is also called the Lax / Maslov operator. You can look on the Internet.
>
> I would also like to add that the "Fourier integral operator" is referred to as the "Lax–Maslov operator" in Dieudonné's textbook in analysis[3], as well as in many other works (you can check it on the Internet). These two terms mean the same. It is possibly also this fact to avoid any confusion in terms.

Viktor Ivrii answered:

> Dear Viktor Pavlovich,
>
> This is not a paper, it is a postgraduate course with three or four participants. However, of course, this is not an excuse. The text is sitting on my server only, and I have a source, and therefore it is easy to correct it.

And he corrected (http://weyl.math.toronto.edu:8888/victor2/preprints/MLAgradcourse.pdf).

We see that, among the mathematicians, the above problem can be solved in a rather simple way. Before turning to these matters in physics, I shall give two referee's reports concerning my papers sent to a physical journal.

Some time ago I received a negative referee's report from a referee physicist to my paper "The mathematical concept of "phenomenological" equilibrium thermodynamics," which is a version of [9] (elaborated for physicists) overlapping with [10].

In the beginning of the report, the referee compliments related to my old papers and points out works on the topic of my paper which I did not read (in his opinion). I miss everything written about my old papers and omit teachings of the referee, because I've carefully studied all works indicated by the referee in my youth.

In conclusion, the referee writes:

" ... I would like to formulate now what are serious disadvantages of this article, in my opinion.

1. It is unclear what are needs of the traditional thermodynamics of the "three laws" and why it is to be reformed.

2. It is unclear under what circumstances the new thermodynamics reduces to the traditional one. In all other revolutions of the past, namely, in the relativistic one,in the quantum one, and in the others, a new theory admitted a passage to the pre-revolutionary limit theory, and it has always been regarded as a proof of ability of the new theory.

3. I have already written about the collision of thermodynamics and kinetics.[4].

The main information I've learned when reading the paper is that V. P. Maslov is interested in fundamentals of thermodynamics and has revolutionary ideas concerning this topic. ... I forbade myself any temper during the preparation of this report, but I am ready to understand a potential

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[2]The text of this article is in the appendix. Every mathematician can compare it with the 1971 Hörmander's definition. All proofs are given in my 1965 paper cited above, where the operator A is to be replaced by $i\partial/\partial\tau$ only. Apparently, for this reason, Hörmander originally wrote in a preprint: " The work of Egorov is actually an application of ideas from Maslov [20]. The author regrets his lack of direct familiarity with this book, which according to a lecture by Dr. Maslov at the International Congress in Nice, actually contains the ideas attributed here to Egorov [7] and Arnold [1] and even a more general and precise operator calculus than the one that we shall describe. However, since the book is highly unaccessible and perhaps not quite rigorous we hope this paper will still prove to be useful". I did not ask Hörmander why he changed this remark later, in the published paper [2]. I'll be very grateful to any mathematician who will find a difference between my 1965 paper presented in the appendix and Hörmander's 1971 paper [2]. I have already discussed this issue in [8].

[3] *Translator's note*: Jean Dieudonné, Treatise on Analysis, Vol. 8, Academic Press, 1993, 356pp., pp. 261, 269, 282, etc.

[4]It has been said above in the report, "The following unusual specific feature of the new thermodynamics is the direct invasion of kinetics in this thermodynamics. According to the author, the viscosity should be explicitly included in thermodynamic expressions." (Cf. the other referee's report below.)

reader if this paper will anger him to the highest degree. I shall not try to predict the exact number of angry readers and their oral and printed response if the journal will print the paper. Undoubtedly, this number will be large.

I definitely recommend to the editors to reject this article."

The Editor-in-Chief received another referee's report (concerning the same paper) from V. S. Vorob'ev, an outstanding expert in modern equilibrium thermodynamics, who was influenced by my explanations and changed the concept of fractals in the paper [11]. He works with me now in the area of gas mixtures.

I present V. S. Vorob'ev's report entirely.

"Most publications in modern thermodynamics can be divided into two groups. The works of the first group basically use the Gibbs distribution for the statistical ensemble or a Bogolyubov chain (BBKI). The most modern approach is related to a computer experiment, in which sufficiently many homogeneous particles interacting according to some empirical potential with random initial conditions are considered. Here diverse versions of the Monte Carlo or molecular dynamics methods with a given interaction potential between the particles are mainly implemented. Finding a correct set of potentials of this kind and their relation to various forms of molecules are the main problems of these methods.

Another group of investigations uses laws of thermodynamical similarity that generalize the van der Waals equation. Among these laws, the most well-known rule is the law of corresponding states, which shows in practice that some group of molecules behaves in the same way in a sense. Recently, another law attracted a definite interest; the law also follows from the van der Waals equation. This law, which is one of the most amazing empirical rules that are confirmed for many substances, claims that the line for the unit compressibility factor (in English-language literature, the *Zeno line*) on the density-temperature plane is represented by a line segment and is characterized by two constants, T_B and ρ_B. These two constants, as well as the parameters for the critical point, are known for many substances both from full-scale experiment experiments and from indirect measurements.

V. P. Maslov's paper is related to the second group. In the first section, V. P. Maslov suggested, for the first time, the idea concerning the relationship between the critical compressibility factor (the dimensionless quantity relating the critical parameters of the real substance, $Z_c = P_c/(T_c\rho_c)$) and a Bose distribution of fractal dimension. The author substantiates in detail that the Bose distribution of fractal dimension corresponds not only to the quantum Bose gas for integer dimensions but also to a classical gas in the presence of macro-measuring device measuring the density of gas in the entire volume.

The problem of interference of a macro-measuring device in the experiment was posed already by the Copenhagen school (Bohr's complementarity principle) and was also treated in a series of similar effects (the Einstein–Podolsky–Rosen effect et al.) The author has shown that the presence of a device measuring the density of gas essentially influences the statistics of classical particles. The author links this fact with Gödel's mathematical theorem.

In our paper (E. M. Apfel'baum, V. S. Vorob'ev, "Correspondence between of the ideal Bose gas in a space of fractional dimension and a dense nonideal gas according to Maslov scheme", Russian J. Math. Phys. 18 (1), 19-25, 2011), we have verified the possibility to use a fractal Bose distribution for real gases. The (fractional) dimension of the space of the ideal Bose gas (BG) was determined from the condition that the compressibility factor of the BG is equal to the compressibility factor of a particular substance. A comparison was conducted for a number of real substances (Ar, Hg, H20, NH3, Cs) and for two model systems, namely, a system described by the van der Waals equation and a system of particles interacting according to the Lennard–Jones potential. In all cases, the critical isotherms calculated from the ideal BG of fractional dimension are in good agreement with tabulated or experimental data.

Note that this method of calculating the interaction potential is not used explicitly. Moreover, the agreement with experimental data is observed over a wide range of pressures, including values which are several times less than the pressure in the case of the ordinary ideal gas, up to a critical point. I would now loke to discuss the "negative pressure" associated with Maslov's amendment to the Bose distribution for dimensions less than two (see, for example, I. A. Molotkov, "Maslov distribution and formulas for the entropy", Russian J. Math. Phys. 17 (4), 476–485, 2010).

It should be noted that, whereas the van der Waals equation well agrees with the Maslov scheme and close to the molecules of mercury, the situation becomes fundamentally changed for negative pressures, because this pressure is not bounded below in the van der Waals equations; however, in the Bose distribution with Maslov's correction, for dimensions less than two, the negative pressure is bounded below indeed, which corresponds to experimental observations. Another problem considered in the paper is to simulate some aspects of thermodynamical behavior when considering the problem of scattering for two particles in a given potential in the presence of an external attractive field. Here a change of variables is used, which reduces the scattering problem at infinity to a finite interval and leads to the disintegration of the Hamiltonian into an attracting part and a repelling part.

The author adopts further a heuristic (physical) reason which modifies the ordinary problem in classical mechanics by assuming that the repulsive part of the potential leads to the creation of a viscosity, like it happens in the Boltzmann equation and its generalizations. In this case, the so-called μ-particle (a pair), which occurred in the potential well by slipping at the ε-distance from the barrier height, eventually falls to the bottom of the well, and the author relates this phenomenon to the stable position of the μ-particle (a dimer). To beat this particle away from the bottom of the well requires an energy equal to the depth of the well. Drawing an analogy of this energy with temperature, the author finds the parameters of the scattering problem that correlate with some of the values of thermodynamical quantities, like the critical temperature and the Boyle temperature.

Further, the author also takes into account the attractive thermal potential. In this problem, the potential has a minimum and a maximum. The line along which the minima and maxima are the same is an analog of the unit line of the compressibility factor, which is well known in thermodynamics (the Zeno line). It is straight up to within 3%. The quantity which is an analog of the compressibility factor at the critical point coincides with the critical values for monatomic gases.

Further, using the fact that the compressibility factor is equal to one on the Zeno line, as in the case of the ordinary ideal gas, the author derives differential equations for a nonideal gas and obtaiins fairly good agreement with isotherms of supercritical fluids.

I shall now discuss the section concerning the "Wiener quantization." The author draws attention to the formal similarity between the Schrödinger equation and the heat equation when written in the variables $\varphi = \exp(iS/h)$ and $u = \exp(S/\nu)$, where ν stands for the viscosity; in the quantum case, the role of "viscosity" is played by the quantity i/h. Asymptotic solutions for these equations, in the presence of infinitesimal viscosity, are well known, and are fundamentally distinct from solutions in which the value of viscosity is set to be zero from the very beginning. This enables us to look anew to the distinction between the classical and nonclassical critical exponents. Here the author also draws attention to the fact that a similar problem occurs when using the semiclassical approaches in ordinary quantum mechanics when studying turning points and inflection points. This topic has been neglected in the literature, although it is rather clear both for the turning point (when using the Airy functions) and to the point of inflection (when using the Weber functions). It is advisable to consider this section in more detail, in order that a wider circle of readers learn the idea of Wiener quantization.

The Wiener quantization, as well as the consideration of scattering of μ-particles, is related to introducing an infinitesimal viscosity. In other words, one introduces the viscosity which affects the thermodynamical problem in such a way that the thermodynamical problem is substantially modified as the viscosity tends to zero. A similar fact occurs in hydrodynamics when describing the emergence of a shock wave.

It is stressed in the paper that, in some cases, the experiment is better explained by means of Wiener quantization than by using the scaling hypothesis, although the Wiener quantization is also a conjecture. However, this conjecture fits well into the overall context of the paper. As a new conjecture, which explains the difference between the values of the critical indices (exponents) and the classical ones, the Wiener quantization can be of particular interest for a group of experts in thermodynamics who questions some conclusions of scaling theory. In the paper, some critical exponents arising from the renormalization-group theory are compared with the predictions of the paper under consideration. The analysis is based on experiments of several well-known experts (W. Wagner, M. S. Green, D. Yu. Ivanov, etc.)

It should also be noted that rather complicated Maslov's formulas for mixtures unexpectedly lead to an almost linear law, the so-called Kay's rule. In particular, for the air, these formulas give coincidences up to within 0.5%.

The section on mixtures concluded the paper. This is quite natural, because, if the preceding sections were based on experiments that are already known, the computer-aided approaches of molecular dynamics, when applied to mixtures that are characterized by several interaction potentials, become dozens and dozens of times more complicated. This increases the practical role of applying the methods suggested by the author to mixtures."

I do not cite the benevolent conclusion of this report.

These reports show that, to speak a language which can be understood by physicists, means to induce associations which are consonant to them and cause no rejection. Moreover, for different physicists, the consonant associations may be different. Therefore, it is difficult to please everyone at once.

For example, the association related to fractals helped me to convince E. M. Apfel'baum and V. S. Vorob'ev, outstanding experts in modern thermodynamics, but caused a natural rejection of a first-rate theoretical physicist, since the fractional dimension is related in essence to the frequency spectrum of molecules. The last remark has convinced some physicists when they referred to these molecules as "quasiparticles" (see [9]).

On the other hand, fractional values of the number of degrees of freedom, which are associated with the spectrum density of the molecule, were accepted by the same theoretical physicist without objection. This is possibly the most natural physical explanation for the introduction of a new parameter in thermodynamics, since it seems to evoke an association with reasonings presented by physicists for the one-dimensional, two-dimensional, and multi-dimensional numbers of degrees of freedom for molecules in the old ideal gas.

From a physical point of view, it is more natural to regard D as the number of degrees of freedom rather than as the dimension. The new ideal gas consists of very many molecules each of which has its own degrees of freedom at a given temperature, i.e., different degrees of freedom can be " frozen" for different molecules.

The average degree of freedom at a given temperature, which corresponds to the average speed of molecules, is a fractional, a continuous, a "fractal" value.

Moreover, the formulas (5)–(10) in [10], which we obtain from number theory, need not be multiplied by the volume V in general. They can be multiplied by V^{α} or, when viewed in a dimensionless form, by $(V/V_0)^{\alpha}$, where α depends on γ only and is preserved for $T \leqslant T_c$. In this case, the compressibility factor Z is multiplied by α, and this multiplier enables us to compare our results in a more natural way with the old ideal gas, which is obtained as $\rho \to 0$, $P \to 0$, and $Z \to 1$.

However, as is pointed out in the textbook by Landau and Lifshitz [12], the Poisson adiabat, in the case of $D = 3$ (in the three-dimensional case) and for an ideal Bose gas, coincides with the adiabat of the monatomic ideal gas (see L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics,* Vol. 5: *Statistical Physics* (Pergamon Press, Oxford, 1968), § Bose and Fermi gases). As we saw in [9], this is not occasional. For $T > T_c$ (in the fluid region), the quantity γ can depend on the density and on the other thermodynamic quantities.

The physicist believes that we obtain a new state by changing the gas particles according to Boltzmann. However the instrument measuring the density by the Rayleigh interferometer or by using the refractive index (which is proportional to the density) does not know this. This instrument will give the same value of density after changing the indexing of two particles.

The professor of thermodynamics can drive this idea to students. However, it is impossible to drive this to a device. It does not feel the new state.

" We must apply a more sensitive instrument," says the theoretical physicist. However, an experimental physicist denies the suggestion. It is very difficult to apply such a device to a gas which is so dense, and this is not necessary.

Another approach to the physical language, as Poincaré had pointed out in his philosophical work, is to link probability theory to arithmetic and thermodynamics. We shall follow this program in the part concerning a generalization of probability theory.

Example. Random processes are self-similar (like a random walk). Their fractal dimension for the Wiener process is equal to 1/2.

Unlimited Probability Theory

Let N be the number of trials. Consider a discrete order statistics: an ordered set of sample values $\xi_1 \leqslant \xi_2 \leqslant \dots \leqslant \xi_n \leqslant \xi_{n+1} \leqslant \dots \leqslant \xi_N \cdots$ of a random variable ξ (a variational series).

The sequence of positive integers,

$$\xi_i = i, \tag{1}$$

and also the sequence

$$\xi_i = i^{D/2}, \tag{2}$$

are special cases of such a sequence ξ_n; here we use the letter D to denote the dimension and refer to the variational series (2) as the *basis series* of dimension D.

Further, let n_i be the value of the outcome corresponding to ξ_i.

The sample value (an estimate) of the expectation is equal to

$$\frac{\sum n_i \xi_i}{N} = E.$$

Assume that $E \to \infty$ as $N \to \infty$.

It is also convenient to introduce the parameter γ by the formula

$$D = 2(\gamma + 1), \qquad \gamma = \frac{D}{2} - 1.$$

Let us consider the one-parameter family of entropies S_γ as the logarithm of the number of solutions of the system of relations

$$\sum_{i=1}^{\infty} n_i = N, \qquad \sum_{i=1}^{\infty} n_i \xi_i \leqslant NE. \tag{3}$$

We refer to the asymptotic relation $N(E)$ between N and E as $N \to \infty$ at which the maximal number of solutions of the system (1) is attained (up to $o(N)$) as the point of maximal accumulation of the estimate (the sample value) of the expectation and say that the families of solutions $\{n_i\}/N$ of the equations (3) are typical; the value N at which the maximal accumulation is attained is denoted by N_{c}.

The binary logarithm of the number of solutions of equation (1) is the Hartley entropy S.

Let α_1, α_2 be the Lagrange multipliers for finding the maximum of S_γ with the conditions (3),

$$dS = \alpha_1\, dN + \alpha_2\, d(EN), \tag{4}$$

$$d(EN) = \frac{1}{\alpha_2}\, dS - \frac{\alpha_1}{\alpha_2}\, dN. \tag{5}$$

We denote the quantity $1/\alpha_2$ by T and the quantity α_1/α_2 by $\widetilde{\mu}$. We refer to T as the mean value of a random variable, $T >> 1$, and to $\widetilde{\mu}$ as the disaccumulation coefficient. We also write $b = 1/T$. To the value $N = N_{\mathrm{c}}$ there corresponds $T = T_{\mathrm{c}}$.

For a given E, the maximum of accumulation is attained at the value $N(E)$, which we have denoted above by N_c.

The entropy S_γ, for $\gamma > 0$, is of the form

$$S = N\Big[(2+\gamma) + \frac{\widetilde{\mu}}{T}\Big] \tag{6}$$

.

If $\widetilde{\mu} = 0$ and for $N > N_{\mathrm{c}}$, the entropy S can be extended continuously by a constant (this is the phenomenon which is called by the physicists (in the special case $D = 3$) the “Bose–Einstein condensate”).

Introduce a four-dimensional phase space, taking the value $-S$ as the momentum p_1, the value T as the coordinate q_1, the value $\widetilde{\mu}$ as the coordinate q_2, the value of N as the momentum p_2, and NE as the action. Then relation (5) defines a two-dimensional Lagrangian manifold in the four-dimensional phase space.

Passing to integrals by the Euler–Maclaurin formulas, for a fractional dimension $D > 2$, we obtain a one-parameter basis family [13–15],

$$EN = T_r^{2+\gamma} \int_0^\infty t^{1+\gamma}\, dt\Big(\frac{1}{\exp(t+\widetilde{\mu}/T_r)-1} - \frac{N}{\exp(t+\widetilde{\mu}/T_r)-1}\Big), \tag{7}$$

$T_r = T/T_c$.

We also have

$$N = T_r^{1+\gamma} \int_0^\infty t^{\gamma}\, dt\Big(\frac{1}{\exp(t+\widetilde{\mu}/T_r)-1} - \frac{N}{\exp(t+\widetilde{\mu}/T_r)-1}\Big), \tag{8}$$

or, by virtue of

$$\mathrm{Li}_s(x) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{(e^t/x)-1}, \tag{9}$$

we have

$$EN = T_r^{2+\gamma}\Big(\mathrm{Li}_{2+\gamma}(z) - \frac{1}{N^{\gamma+1}}\,\mathrm{Li}_{2+\gamma}(z^N)\Big), \tag{10}$$

where $z = e^{-\widetilde{\mu}/T_r}$ and

$$N = T_r^{1+\gamma}\Big(\mathrm{Li}_{1+\gamma}(z) - \frac{1}{N^{\gamma}}\,\mathrm{Li}_{1+\gamma}(z^N)\Big). \tag{11}$$

In the case of $D \leqslant 0$ ($\gamma \leqslant 0$) and in the case of $\widetilde{\mu} \to 0$ ($z \to 1$), the second term in (11) and (8) becomes significant.

In the paper [10], the case of $\mu = -O(\frac{1}{\log N})$ was calculated (in thermodynamics, $\mu = -\tilde{\mu}$). However, if $\widetilde{\mu} > \delta$, where $\delta > 0$ does not depend on N, then the second integral in formula (11) can be neglected.

For $\mu \equiv 0$, the formula can be simplified considerably.

In the paper [13] cited above, in particular, we have introduced the notation

$$F(\xi) = \Big(\frac{1}{\xi} - \frac{1}{e^\xi - 1}\Big)$$

and

$$C(\gamma) = \int_0^\infty F(\xi)\xi^\gamma\, d\xi, \qquad -1 < \gamma < 0.$$

V. E. Nazaikinskii noticed that the Euler–Maclaurin transition to the integrals is not applicable here, whereas, in the discrete case, thr constant can be expressed quite simply by and using the function $F(\xi)$.

Namely, in the discrete case,

$$N_c = \sum_{j=1}^\infty \frac{j^\gamma}{e^{bj}-1} = \sum_{j=1}^\infty j^\gamma \frac{1}{bj} - \sum_{j=1}^\infty j^\gamma F(bj). \tag{12}$$

Since the function $f(x) = x^\gamma F(bx)$ is monotonically decreasing, it follows that

$$\sum_{j=1}^\infty j^\gamma F(bj) = \sum_{j=1}^\infty f(j) \leqslant \int_0^\infty f(x)\, dx = \int_0^\infty x^\gamma F(bx)\, dx = b^{-\gamma-1}\int_0^\infty x^\gamma F(x)\, dx. \tag{13}$$

Thus,

$$N_c = b^{-1}\zeta(1-\gamma) + O(b^{-1-\gamma}), \qquad \gamma < 0, \qquad b = 1/T \quad (\text{for} \quad \widetilde{\mu} = 0 \quad \text{and} \quad T = T_{\rm c}),$$

where ζ stands for the Riemann zeta function.

This means that, under the transition from $\widetilde{\mu} \equiv 0$ to $\widetilde{\mu} + O(1/\log N)$, we obtain a jump. Further, for $\widetilde{\mu} = \delta > 0$, where δ does not depend on N, we obtain another jump to the expression (11) without the second term in the brackets. These jumps correspond to the instability jumps when the value $\widetilde{\mu}$ becomes positive under a small perturbation.

Events are said to be independent if their entropies are added. Let us consider the conditional probability for independent events.

As is known, by the *conditional probability* of an event one means the probability of the simultaneous occurrence of a pair which consists of the event and the condition, where the probability is normalized with respect to the probability of the condition, i.e., $P(A|B) = P(A \cap B)/P(B)$, where A stands for the event and B for the condition.

The formula

$$P(B) = \sum_{i=1}^{k} P(B \bigcap A_i) = \sum_{i=1}^{k} P(B|A_i)P(A_i), \qquad \coprod A_i = \Omega, \tag{14}$$

for disjoint events $\{A_i\}$ is referred to as the formula of *full* probability.

A similar formula holds in the new conception of probability. It suffices to obtain the formula for $k = 2$. First, this is an equation for the entropies with regard to the conditional probability with α and $\beta = 1 - \alpha$. Namely, if $\widetilde{\mu} = 0$, then

$$Z_{\rm c}(\gamma_{\rm c} + 2) = \alpha(\gamma_{1,{\rm c}} + 2)Z_{1,{\rm c}} + \beta(\gamma_{2,{\rm c}} + 2)Z_{2,{\rm c}}, \tag{15}$$

where $Z_{\rm c} = \zeta(\gamma + 2)/\zeta(\gamma + 1)$ and $\alpha + \beta = 1$.

Further, it is clear that, for the expectations multiplied by the number of trials for each of them, we have the relation

$$(\gamma_{\rm c} + 1)Z_{\rm c}T_{\rm c} = \alpha(\gamma_{1,{\rm c}} + 1)Z_{1.{\rm c}}T_{1,{\rm c}} + \beta(\gamma_{2,{\rm c}} + 1)Z_{2,{\rm c}}T_{2,{\rm c}}. \tag{16}$$

It is of interest that, as a result, a rather complicated system of equations in thermodynamics almost coincides with the Kay empirical rule [16].

We can obtain the values $T_{\rm c}$ and $\gamma_{\rm c}$ from these formulas for “mixtures” of type (14), using the basis variational series.

Remark. The Erdös theorem implies a bound for the expectation of the random number ξ taking the values in the positive integers, $i = 1, 2, \dots, n, n+1, \dots$, namely,

$$E(\xi) \leqslant N/(c^2 \log^2 N), \qquad c = \sqrt{3/2}/\pi.$$

This additional condition on the set of positive integers can hinder the proof of Gödel’s theorem.

If we have reached the value of $N_{\rm c}$ trials and then begin to increase the disaccumulation coefficient $\widetilde{\mu}$, then the number of trials cannot be reduced, and it remains to be equal to $N_{\rm c}$ (the irreversibility of the number of trials), although the estimate for the expectation will be changed as the disaccumulation coefficient increases.

This gives a nonsmooth extension of the curve $T_r = const$ in the plane $E, Z = E/T_r$. Denote by $\widetilde{\widetilde{\mu}}$ the extension of the parameter $\widetilde{\mu}$ to this branch. Since $dN = 0$, it follows from (4) that, for this branch, we obtain the following formula for the entropy $S_c(T_r, \gamma)$ in the case of $\widetilde{\widetilde{\mu}} = 0$:

$$S_c(T_r, \gamma) = T_r^{\gamma+1}\zeta(\gamma + 2). \tag{17}$$

We can now find the value of γ for which the entropy on this branch is maximal. This is the point of reference for $\widetilde{\widetilde{\mu}}$ in formula (10) for this branch.

We can further carry out the dequantization procedure for the Wiener (tunnel) quantization. This means that we introduce the tunnel canonical operator (see Chap. 9, Sec. 3 in [17], and also [18]) on the Lagrangian manifold indicated above, and then consider the quantization parameter (the viscosity or the Planck constant multiplied by the imaginary unit) as an infinitesimal.

In this case, at the points of maximal accumulation $\widetilde{\mu} = 0$, we obtain the so-called critical indices, and obtain a phase transition of gas-liquid type for $T < T_c$ and $\widetilde{\mu} = \widetilde{\widetilde{\mu}} > 0$. A similar phase transition occurs also for the "jamming effect" and in genetics, at the passage from a mutation to stable transmission by heredity.

Seemingly, in economics, this passage corresponds to dropping/lifting an "iron curtain" (cf. [19]).

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REFERENCES

1. V. P. Maslov, "Number-Theoretic Internal Energy for a Gas Mixture", Russian J. Math. Phys. **18** (2), 163–175 (2011).
2. L. Hörmander, "Fourier integral operators, I", Acta Math., **127**, 79–183 (1971).
3. V. P. Maslov, *Perturbation Theory and Asymptotical Methods* (Izd. Moskov. Univ., Moscow, 1965; Dunod, Paris, 1972) [in Russian and French].
4. J. Heading, *An introduction to phase-integral methods* (Methuen & Co., Ltd., London; John Wiley & Sons, Inc., New York, 1962; Izdat. "Mir", Moscow, 1965).
5. V. P. Maslov and M. V. Fedoryuk, *Quasiclassical approximation for the equations of quantum mechanics*, (Izdat. "Nauka", Moscow, 1976) [in Russian].
6. V. P. Maslov, "A Remark to I. M. Gel'fand's Report "P_+^λ"", Uspekhi Mat. Nauk **23** (2), 205–206 (1968), in: "Meetings of the Moscow Mathematical Society", Uspekhi Mat. Nauk, Uspekhi Mat. Nauk **23** (2), (1968), 201–208 [in Russian].
7. V. P. Maslov, "A criterion for the discreteness of the spectrum of a Sturm–Liouville equation with operator coefficient (apropos the article of B. M. Levitan and G. A. Suvorčenkova)", Funktsional. Anal. i Prilozen., **2** (2), 63–67 (1968) [Funct. Anal. Appl. **2** (2), 153–157 (1968)].
8. V. P. Maslov, "The Intertwining of Two Lifelines (in Memoriam of V. I. Arnold)", Russ. J. Math. Phys. **17** (4), 395–398 (2010).
9. V. P. Maslov, "Mathematical Conception of "Phenomenological" Equilibrium Thermodynamics", Russ. J. Math. Phys. **18** (4), 363–370 (2011).
10. V. P. Maslov, "New Probability Theory Compatible with the New Conception of Modern Thermodynamics. Economics and Crisis of Debts", Russ. J. Math. Phys. **19** (1), 63–100 (2012).
11. E. M. Apfel'baum and V. S. Vorob'ev, "Correspondence between of the ideal Bose gas in a space of fractional dimension and a dense nonideal gas according to Maslov scheme", Russ. J. Math. Phys. **18** (1), 19–25 (2011).
12. L. D. Landau and E. M. Lifshits, *Theoretical Physics*, Vol. 5: *Statistical Physics* (Nauka, Moscow, 1964; Fizmatlit, Moscow, 2003) [in Russian].
13. V. P. Maslov and V. E. Nazaikinskii, "On the distribution of integer random variables related by a certain linear inequality: I," Mat. Zametki **83** (2), 232–263 (2008) [Math. Notes **83** (2), 211–237 (2008)].
14. V. P. Maslov and V. E. Nazaikinskii, "On the distribution of integer random variables related by a certain linear inequality: II," Mat. Zametki **83** (3), 381–401 (2008) [Math. Notes **83** (3), 345–363 (2008)].
15. V. P. Maslov and V. V. V'yugin."Concentration Theorems for Entropy and Free Energy',' Problemy Peredachi Informatsii **41** (2), 72–88 (2005) [Probl. Inf. Transm. **41** (2), 134–149 (2005)].
16. V. P. Maslov, "A Homogeneous Gas Mixture", Teoret. Mat. Fiz., **168** (2), 358–368 (2011) [Theoret. Math. Phys. **168** (2), 1165–1174 (2011)].
17. V. P. Maslov, Asymptotic Methods and Perturbation Theory (Nauka, Moscow, 1988) [in Russian].
18. V. P. Maslov, "Critical exponents as a consequence of the Wiener quantization of thermodynamics," Teoret. Mat. Fiz. **170** (3), 458–470 (2012) [in Russian].
19. V. P. Maslov, "Tropical Mathematics and the Financial Catastrophe of the 17th Century. Thermoeconomics of Russia in the Early 20th Century", Russ. J. Math. Phys, **17** (1), 126–140 (2010).

APPENDIX

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REGULARIZATION OF THE CAUCHY PROBLEM FOR PSEUDODIFFERENTIAL EQUATIONS

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In this paper we discuss a modification of the construction of the regularizer (i.e., the selection of the nonsmooth part of the solution) of pseudohyperbolic systems (see [1]).[5]

First of all we note that we wish to investigate the nonsmooth part of the function $\phi(x) \in H_s$, $x \in R^n$. The smoothness of functions in H_s is defined in terms of their adherence to the domain of some power or other of the operator $\sqrt{-\Delta}$. For convenience we shall imbed the function $\phi(x)$ in a single-parameter family of functions by means of the formula $\psi(x,\tau) = \exp(i\sqrt{-\Delta}\tau)\phi(x)$. We shall regard $\psi(x,\tau)$ as a function $\psi(x)$ with values in a Banach space C of functions of the variable τ. The smoothness of such a function is defined in terms of its belonging to the domain of some power of the operator $A = i\partial/\partial\tau$, operating on only one variable. We proceed to the construction of a general theory of such functions.

Let H be a complex Hilbert space, and A be an unbounded selfadjoint operator with domain $D(A)$, and range $R(A)$, both dense in H. We consider a chain of Hilbert spaces ${H^A}_s$, $s = 1, -1, \dots$, with norms $\|g\|_s = \|(A+i)^s g\|_H$ (for $s > 0 \quad g \in D(A^s)$, and for $s < 0 \quad {H_s}^A$ is the completion of H in the norm $\|\cdot\|_s$).

Let M^n be an $n-$dimensional Lagrange manifold in $2n-$dimensional (Euclidean) phase space p, q [1, 2] with the infinitely smooth measure d_σ. We shall consider functions on M^n with values in H. On M^n we have the following "weighting" partition of unity:

$$l = \sum_{i_1=i_l}^{w} \rho_{i_1,\dots,i_k}(y_{i_1,\dots,i_k}); \quad y_{i_1,\dots,i_k} = q_{i_1}, q_{i_2}, \dots, q_{i_k}, p_{i_{k+1}}, \dots, p_{i_n} \quad (i_{h-\nu} \neq i_{k+\mu}),$$

$\rho_{i_1,\dots,i_k} \in C^\infty, \quad \rho_{i_1,\dots,i_k} = 0$ in a neighborhood of points for which $J_{i_1,\dots,i_k} = dy_{i_1,\dots,i_k}/d\sigma = 0$. We shall set

$$\Delta_{i_1,\dots,i_k} = \partial^2/\partial q_{i_1} + \dots + \partial^2/\partial q_{i_k} + \partial^2/\partial p_{i_{k+1}} + \dots + \partial^2/\partial p_{i_n}.$$

The sequence of norms

$$\int \sum_{i_j \neq i_l} \| \, (\sqrt{-\Delta_{i_j,\dots,i_k}} + i)^r \varphi(y_{i_1,\dots,i_k}) \, \|_s^2 \, {J^{-1}}_{i_1,\dots,i_k} \rho_{i_1,\dots,i_k} dy_{i_1,\dots,i_k}$$

for $r = 1, 2, \dots$ defines a countably-normed space v_s, and for $r = 1, 2, \dots$, $s = 1, 2, \dots$, defines the countably-normed space v_∞. The completion of the algebraic factor-space v_s/v_∞ in the weak topology

$$g_N - g \in v_N \Rightarrow \lim_{N\to\infty} g_N = g$$

we shall denote by V_s.

We now will formulate a rule for transforming functions $\phi \in V_s$ under a canonical change of variables. 1) Under a canonical change which transforms the coordinates $q_{i_1}, q_{i_2}, \dots, q_{i_k}$ into themselves, $\phi(y_{i_1,\dots,i_k})$ changes like a scalar. 2) In the case of a change $q_{i_1}, q_{i_2}, \dots, q_{i_{m-1}} \longrightarrow p_{i_1}, p_{i_2}, \dots, p_{i_{m-1}}$, $m - 1 \leqslant k$, we introduce the operator $V^{i_m,\dots,i_k}_{i_1,\dots,i_k} \rho_{i_1,\dots,i_k} \phi(y_{i_1,\dots,i_k}) = \psi(y_{i_m,\dots,i_k}, \pi/2)$, where

[5] Presented at the International Congress of Mathematicians, Moscow, 1966

$\psi(y_{i_m,\dots,i_k},t)$ is obtained by the method of successive approximations in the powers of $(A+i)^{-1}$ from the equations

$$\psi(y_{i_m,\dots,i_k},t)=\frac{A}{A+i}\rho_{i_1,\dots,i_k}(y_{i_m,\dots,i_k})\phi(y_{i_m,\dots,i_k})$$

$$+\frac{1}{A+i}\int_0^t\sqrt{J}\sum_{\mu=1}^{m}(\frac{1}{\partial x_{i_\mu}/\partial p_{i_\nu}}\frac{\partial}{\partial p_{i_\nu}})^2\frac{1}{\sqrt{J}}\phi(y_{i_m,\dots,i_k},t')dt'$$

$$+\frac{i}{A+i}\phi(y_{i_m,\dots,i_k},t),$$

where: $x=q\cos(t)+p\sin(t)$, $p=p_{i_1,\dots,i_m}$, $q=q(y_{i_m,\dots,i_k})=q_{i_1,\dots,i_m}$, $J=J_{i_m,\dots,i_k}Dx/Dp$, and all integrals are understood in the sense of circling the actual poles in the lower halfplane.

Let k be a canonical atlas, the maps of which are presented in the form u_j, $y_{i_1,\dots,i_k}$. Let the support of $\phi_{u_j}\in V_s$ lie in u_j. We shall denote by

$$U_{u_j}\Phi_{u_j}=\sum_{j_\nu}V^{i_1,\dots,i_k}_{j_1,\dots,j_m}\rho_{j_1,\dots,j_m}\phi_{u_j}(y_{j_1,\dots,j_m})$$

the operator of transitions of the coordinates $y_{i_1,\dots,i_k}$.

We shall now define two zero-dimensional cochains. Suppose that on M^n the characteristic class is trivial [1, 2], and $\oint pdq=0$.

Let a^0 be some fixed point in M^n. 1) we set

$$S(u_j,\alpha)\overset{\text{def}}{=}\int_{\alpha^0}^{\alpha}p\,\mathrm{d}q-\sum_{\nu=1}^{n-k}p_{i_{k+\nu}}q_{i_{k+\nu}}(\alpha),\quad \alpha=\alpha(y_{i_1,\dots,i_k})\in u_j.$$

This zero-dimensional cochain with values in a bundle of functions shall be called an action.

2) We set $\gamma_{u_j}=ind[\alpha^0,\alpha]$ - the index of inertia $\|\partial q_{i_{k+\nu}}/\partial p_{i_{k+\mu}}\|_{\nu,\mu=1,\dots,n-k}$ where $ind[\alpha^0,\alpha]$ is the index of the intersection of any path from α^0 to α with the singularities of the projection of M^n on the q-plane (see [1, 2]). It turns out that γ_{uj} does not depend on α, and is the integer zero-dimensional cochain.

We now proceed to the definition of the canonical operator (c.o.) K^{α^0} which maps the space V_s into the completion W_s in the weak topology ($g_N-g\in K_N\Rightarrow\lim_{N\to\infty}g_N=g$) of the algebraic factor-space $K_s/\cap_N K_N$, where K_s is the Hilbert space of function from q with values in H and norms

$$\|\,f\,\|^2{}_{K_s}=\int\|\,(\sqrt{-\Delta+A^2q^2}+i)^3f(q)\,\|^2{}_H dq:$$

$$K^{\alpha^0}\varphi_{u_j}(\alpha)=\Phi_A^{p_{i_{k+1}},\dots,p_{i_n}}\frac{I^{\gamma u_j}}{\sqrt{|J_{i_1,\dots i_k}|}}\exp[iAS(u_j,\alpha)-\frac{i\pi}{2}\Upsilon_{u_j}]U_{u_j}\varphi_{u_j}(\alpha),$$

where $\Phi_A^{p_{i_{k+1}},\dots,p_{i_n}}$ is the A-Fourier transform in the variables $p_{i_{k+1}},\dots,p_{i_n}$ [1].[6]

We have defined zero-dimensional operator cochain. It turns out that the c.o. does not depend on the canonical atlas. In fact, we have:

[6]Let us recall the definition of the A-Fourier transform. Let $H^+(H^-)$ be a subspace of H, invariant with respect to A, such that A is positive (negative) definite on $H^+(H^-)$. Clearly $H=H^+\oplus H^-$. We define by $I(I^{1/2})$ the linear operator such that $Ig=g(I^{1/2}g=g)$ for $g\in H^+$ and $Ig=-g(I^{1/2}g=ig)$ for $g\in H^-$. The operator $|A|=AI=IA$ is positive definite. The operator

$$\Phi_A^{v_1,\dots,p_n}\phi\overset{def}{=}(2\pi)^{-n/2}|A|^{n/2}J^{n/2}\int\exp[-ipqA]\phi(p)\mathrm{d}p$$

is unitary in K_s.

Theorem 1. *The c.o. is a zero-dimensional cocycle with coefficients in an operator bundle.*

Note. Consider $D^n \in M^n$. Let $dD^n = M^{n-1}$ so that dimensionalities of the projection operators of M^n and M^{n-1} onto the q-plane coincide at points $\alpha \in M^{n-1}$. Then a c.o. on M^n might be contracting on D^n.

Example. Let H be the space of m-component vector-functions in τ with square-integrable modulus, and $A = i\partial/\partial\tau$. It is not hard to see that for sufficiently small τ, $\exp(i\sqrt{-\Delta}\tau)\delta(q-\xi)E = K^{\alpha^0}\phi(\alpha)g(\tau)$; $\quad g(\tau) = (n-1)!(\tau - i_0)^{-n/2}/(n/2)!(2\pi)^{n/2}$ where E is the unit $m \times m$ matrix; M^n is a cylinder; $|p| = 1$; $q = \xi + p\,t'$ (ξ fixed, t' is a parameter); $\phi(\alpha) = \phi(t')$ is finite and is equal to unity in a neighborhood of $t' = 0$; $\mathrm{d}\sigma = \mathrm{d}t'\mathrm{d}\omega$, where $\mathrm{d}w$ is the global measure on the sphere $|p| = 1$. On this manifold we shall denote by M^n_ξ the support of $\phi(\alpha)$.

We shall now proceed to the pseudohyperbolic equation.

Let $L(p,q,t)$ be a $(2n+1)$ -parameter $m \times m$ matrix, having the following properties: 1) the coefficients of the matrices $L(p,q,t)$,

$$L^0(p,q,t) \stackrel{def}{=} \lim\nolimits_{h\to 0} hL(h^{-1}p,q,t)$$
$$L^1(p,q,t) \stackrel{def}{=} \lim\nolimits_{h\to 0} h(hL(h^{-1}p,q,t) - L^0(p,q,t))$$

belong to C^∞. 2) The eigenvalues $\lambda^1(p,q,t),\dots,\lambda^m(p,q,t)$ of L^0 are real, their multiplicity is constant, and they have no root-subspaces. 3) A solution of the system $\dot p = -\lambda^i_q, \dot q = \lambda^i_p, \quad i = 1,\dots,m$ exists in the large.[7]

We shall consider the problem[8]

$$i\partial u/\partial t + \hat L u = f(q,t), \qquad f(q,t) \in H_s, \tag{A1}$$
$$u|_{t=0} = 0, \tag{A2}$$

where $\hat L$ is a pseudodifferential operator with symbol $L(p,q,t)$. Notation: M^{n+1}_i is a Lagrange manifold with boundary formed by the trajectories of the system $\dot p = -\lambda^i_q, \dot q = \lambda^i_p, \quad 0 \leqslant t \leqslant t_1$, with boundary conditions on $M^n_\xi : M^{n+1}_i|_{t=0} = M^n_\xi$; $K^{\alpha^0}_i$ is a c.o. corresponding to the manifold M^{n+1}_i, the measure $\mathrm{d}\sigma = \mathrm{d}t\,\mathrm{d}t'\,\mathrm{d}\omega$, and some fixed weighting partition of unity; L^i is a matrix adjoint to the matrix consisting of the complementary minors of $L^0 - \lambda^i$, multiplied by $\prod_{j\neq i}(\lambda^i - \lambda^j)^{-1}$, $L^i_0 \equiv {}^{def} L^i|_{t=0}$; R^i is the matrix inverse to $L^0 - \lambda^i$ on the region $R(L^0 - \lambda^i)$ of its values;

$$P^i = \left[\sum_{j=0} L^i_{p_j} L^0_{q_j} + L^1 L^i\right]_{p,q\in M^{n+1}_i, p_0=\lambda^i_{q_0=t}}$$

is the polarization matrix;[9] $T^j = id/dt + P^j$ is an operator on M^{n+1}_i, defined on functions which are equal to zero when $t = 0$ (on M^n_ξ); $\hat L_1, \hat L_2$ are some infinitely differentiable operators in $W_{-[n/2]-1}$, the analytic expressions of which we will not introduce here. The sign $\doteq$ indicates equality in $W_{-[n/2]-1}$, i.e., equality up to a function in $\bigcap_N K_N$.

Theorem 2. *Under conditions 1)–3), a solution of problem (A1), (A2) exists and is unique in the space H_s, and can be represented, up to an arbitrarily smooth function, in the form $u = Rf(q,t)$, where the operator R is written below.*

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[7]It is sufficient that the derivative of λ^i in p in the direction q grow faster than $|q|\log|q|\log\log|q|\dots\underbrace{\log\log\dots\log|q|}_{k}$ for one k.

[8]We note that $\widehat{L}\psi(q) = \Phi^{*\,q_1\cdots q_n}_A L(Ap,q,t)\Phi_A{}^{p_1\cdots p_n}$

[9]This matrix is significant in physics. It indicates the divergence of spin and dissipative character in the jth component of the regularizer R from the classical selfadjoint Schroedinger equation $i\partial\psi/\partial t = \lambda^j(\hat p,q,t)\psi, \hat p = i\partial/\partial q$, where $\lambda^j(\hat p,q,t)$ is an operator pseudodifferential in the sense of Weyl with symbol $\lambda^j(p,q,t)$.

We outline the scheme of the proof.

Having made the change $u \doteq K_i^{\alpha^0} G^i(a,t)g(\tau)$, we obtain.

$$\left(i\frac{\partial}{\partial t} + \hat{L}\right) K_i^{\alpha^0} G^i(a,t)g(\tau) = K_i^{\alpha^0}\left[(L^0 - \lambda^i) + \tfrac{1}{A+i}\hat{L}_1 + \tfrac{1}{(A+i)^2}\hat{L}_2\right] \times G^i(\alpha,t)g(\tau)$$

Developing $G^i(a,t)$ in power of $(A+i)^{-1}$ under the condition $G^i|_{t=0} = L_0^i\phi(t')$, we obtain [1]

$$G^j \doteq \sum_{k=0}^{\infty}(A+i)^{-k} f_k^j,$$

$$f_k^j = (1 - B_j^{-1}L^j\hat{L}_1)R^j(\hat{L}_1 f_{k-1}^j + \hat{L}_2 f_{k-2}^j) - B_j^{-1}L^j\hat{L}_2 f_{k-1}^j;\, B_j = T_j + \frac{1}{2}\sum_k \lambda_{p_k q_k}$$

$$f_0^j = \exp\left[1/2\int_0^t \sum_k \lambda_{p_k q_k}^j\,\mathrm{d}t\right] T_j^{-1}P^jL_0^j\varphi(t')$$

The sum $\sum_{i=1}^m K_i^{\alpha^0} G^i(a,t)g(\tau)$ gives in the form of a series in power of $(A+i)^{-1}$ the solution of equation (A1) for $f(q,t) = 0$ and under the condition $u|_{t=0} = \exp[-i\sqrt{\Delta}\tau] \times\delta(q-\xi)E$, as an operator from $W_{-[n/2]-1}$ to $W_{-[n/2]-1}$.

Let us take a sufficiently large segment of the series we obtain and denote the resulting $m\times m$ matrix by $G(q,\xi,t,\tau)$. The operator

$$RF(q,t) = \int_0^t \mathrm{d}t^* \int G(q,\xi,t-t^*,\tau)\exp[-i\sqrt{-\Delta}\tau]F(\xi,t^*)\mathrm{d}\xi$$

is the regularizer of the solution of equation (A1): it is clear that $(i\partial/\partial t + \hat{L})RF(q,t) = F + vF$, where v is a Volterra operator which arbitrarily smooth kernel.

From this it follows that for any $f(q,t) \in H_s$ there exists a solution of problem (A1)–(A2) (precisely, $u = R(1+v)^{-1}f(q,t)$) which up to a sufficiently smooth function is representable in the form $RF(q,t)$. An analogous assertion for the adjoint operator leads to the uniqueness of the solution of problem (A1)–(A2). Since the solution found here has an integral representation, its smoothness may be investigated just as is done for equations with constant coefficients. Analogous theorems of existence and uniqueness can be obtained in significantly more general situations.

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REFERENCES

[A1] V. P. Maslov, *Theory of Perturbations and Asymptotic Methods* (Izdat. "Nauka", Moscow, 1965) [in Russian].

[A2] V. I. Arnol'd, "On a characteristic class entering into conditions of quantization", Funkcional. Anal. Prilozhen. 1 (1), 1–14 (1967) [in Russian]..

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