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REGULAR SELECTIONS OF MULTIFUNCTIONS OF BOUNDED VARIATION

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1. The Problem on Regular Selections

Suppose that a multifunction $F : T \to \mathcal{P}(X)$ from a set T into the family $\mathcal{P}(X)$ of (nonempty) subsets of a set X is given. A selection F is a function $f : T \to X$ such that $f(t) \in F(t)$ for all $t \in T$. It is known [7] that under rather general assumptions about T and X, the lower-semicontinuous multifunction F taking closed convex values has a continuous selection. On the other hand, if the values of F are not convex, then, in general, continuous selections do not exist even for continuous multifunctions F defined on a closed interval T in $\mathbb{R}$ and having compact images in $\mathbb{R}^n$, $n \geq 2$ [5].

We give certain definitions that will be used in what follows. Let (X, d) be a metric space and $I = [a, b]$ be a closed interval in $\mathbb{R}$. A function $f : I \to X$ is said to be Lipschitzian if the following quantity is finite:

$$L(f) = \sup\{d(f(t), f(s))/|t - s|\};$$

here the supremum is taken over $t, s \in I$, $t \neq s$; this is written in the form $f \in \mathrm{Lip}(I; X)$. A function $f : I \to X$ is absolutely continuous if for $\varepsilon > 0$, one can find $\delta(\varepsilon) > 0$ such that for any finite tuple of nonintersecting intervals $\{(a_i, b_i)\}_{i=1}^m$ of I with the sum of lengths not exceeding $\delta(\varepsilon)$, we have

$$\sum_{i=1}^{m} d\big(f(b_i), f(a_i)\big) \leq \varepsilon.$$

For such f, we write $f \in AC(I; X)$. We say that a function $f : I \to X$ has bounded variation and write $f \in BV(I; X)$ if the following expression is finite:

$$V(f) = \sup\left\{\sum_{i=1}^{m} d\big(f(t_i), f(t_{i-1})\big)\right\};$$

here the supremum is taken over all partitions of the closed interval I, i.e., $m \in \mathbb{N}$ and $a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b$.

The Hausdorff metric on the family $\mathrm{c}(X)$ of compact subsets of X is defined in a standard way by the relation

$$D(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{x\in B}\inf_{y\in A} d(x, y)\right\}, \quad A, B \in \mathrm{c}(X).$$

Thus, one can highlight the classes of Lipschitzian and absolutely continuous multifunctions and multifunctions $F : I \to \mathrm{c}(X)$ of bounded variation.

In [6], it is shown that any multifunction $F \in \mathrm{Lip}(I; \mathrm{c}(\mathbb{R}^n))$ has selection f such that $L(f) \leq L(F)$; it is also shown there that if $F : I \to \mathrm{c}(\mathbb{R}^n)$ is continuous and has a bounded variation (in the metric D), then F has a continuous selection. In [10], it is proved that $F \in AC(I; \mathrm{c}(\mathbb{R}^n))$ has a selection $f \in AC(I; \mathbb{R}^n)$. In [8, D 1.8], the results of [6] are extended to the case of multifunctions $F : I \to \mathrm{c}(X)$ with compact graphs and range in an arbitrary Banach space X, and under the same conditions, it is proved in [1–3] that $F \in BV(I; \mathrm{c}(X))$ has a selection $f \in BV(I; X)$ that can be chosen as a continuous one if F is continuous, and also that a multifunction $F \in AC(I; \mathrm{c}(X))$ has a selection $f \in AC(I; X)$ preserving the function

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$\delta(\varepsilon)$ from the definition of absolute continuity. Thus, in the cases considered above, multifunctions have selections that inherit their characteristic properties. Such selections are called *regular*.

In the present paper (see Sec. 3), we will show that the constraints imposed on the graph in the results mentioned above can be removed, and multifunctions can be considered as only taking compact values in an arbitrary metric space. Our approach (see Sec. 2) is based on a generalization of the classical Helly selection principle [4] from the theory of functions of a real variable [9], which is more exact than that in [1, 2].

2. Generalization of Helly's Selection Principle

Throughout this paper, (X, d) is a metric space and $I = [a, b]$ is a closed interval in $\mathbb{R}$.

The following structural result allows us to extend the classical Helly selection principle to the case of functions with range in a metric space, when Jordan's idea on the decomposition of a function of bounded variation into the difference of two monotonic functions, which underlies the proof of this principle, does not work for obvious reasons.

Lemma 1 ([2, Sec. 3]). *A function $f : I \to X$ belongs to the set $BV(I; X)$ if and only if there exist a nondecreasing bounded function $\varphi : I \to \mathbb{R}$ and a function $g \in \mathrm{Lip}(J; X)$, where J is the image of φ, such that $L(g) \le 1$ and $f(t) = g(\varphi(t))$ for all $t \in I$.*

We say that the family of functions $\mathcal{F} \subset X^I$: *has uniformly bounded variation* if there exists a constant $C \ge 0$ such that $V(f) \le C$ for all $f \in \mathcal{F}$; it is said to be *pointwise precompact* if for any $t \in I$ the closure of the set $\mathcal{F}(t) = \{f(t) \mid f \in \mathcal{F}\}$ is compact in X. The following theorem is an exact generalization of Helly's selection principle.

Theorem 1. *Any pointwise precompact family $\mathcal{F} \subset X^I$ which has uniformly bounded variation contains a subsequence that converges pointwise on I to a certain function $f \in BV(I; X)$.*

The condition of pointwise precompactness in Theorem 1 cannot be replaced by the closedness and boundedness of sets $\mathcal{F}(t)$ at each point t. Consider a Banach space ℓ^1 of absolutely converging sequences $x = \{x_i\}_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ with the norm $\|x\| = \sum_{i=1}^{\infty} |x_i|$ and with standard basis $e_n = \{\delta_{in}\}_{i=1}^{\infty}$ ($n \in \mathbb{N}$), where $\delta_{in} = 0$ for $i \ne n$ and $\delta_{nn} = 1$. We set $\mathcal{F} = \{f_n\}_{n=1}^{\infty}$, where $f_n : [0, 1] \to \ell^1$ is defined by the following rule: $f_n(t) = te_n$, $t \in [0, 1]$. Then all the sets $\mathcal{F}(t)$ are bounded and closed in ℓ^1, and the family $\mathcal{F}$ has a uniformly bounded variation: $V(f) = 1$ for all $f \in \mathcal{F}$. The sequence $\mathcal{F}$ does not contain a subsequence converging in ℓ^1 at least at a single point of the half-interval $(0, 1]$. This example also demonstrates that in contrast to the classical case of $X = \mathbb{R}$, it is not sufficient to require that $\mathcal{F}(t)$ be precompact only at one point of the closed interval I.

Applications of Theorem 1 are discussed in the next section.

3. Selections of Bounded Variation

Based on Lemma 1 and Theorem 1, we prove the following theorem on the existence of regular selections for multifunctions.

Theorem 2. *Suppose that $F : I \to \mathrm{c}(X)$, $t_0 \in I = [a, b]$, and $x_0 \in F(t_0)$. In this case,*

(a) *if $F \in BV(I; \mathrm{c}(X))$, then F has a selection $f \in BV(I; X)$; if, in addition, F is continuous on I, then the selection f can be chosen as a continuous one as well*;
(b) *if $F \in \mathrm{Lip}(I; \mathrm{c}(X))$, then F has a selection $f \in \mathrm{Lip}(I; X)$ such that $L(f) \le L(F)$*;
(c) *if $F \in AC(I; \mathrm{c}(X))$, then F has a selection $f \in AC(I; X)$.*

In addition, in all the cases listed above, the regular selection f can be chosen in such a way that $f(t_0) = x_0$ and $V(f) \le V(F)$.

The condition of this theorem stating that the images of $F(t)$ are compact cannot be replaced by the condition of closedness and boundedness of these images. As above, we consider the space ℓ^1, set $A = \{(1 + 1/n)e_n\}_{n=2}^{\infty}$ and $B = \{e_1\} \cup A$, and then consider the mapping $F : [0, 1] \to \mathcal{P}(\ell^1)$ such that $F(t) = A$ if $0 \leq t < 1$ and $F(1) = B$. Then, for any selection $f : [0, 1] \to \ell^1$ of the multifunction F for which $f(1) = e_1$ we have $V(f) > 2 = D(A, B) = V(F)$.

Now, consider regular selections f of a multivalued mapping F of bounded variation that pass through the two given points, i.e., $f(a) = x_1 \in F(a)$ and $f(b) = x_2 \in F(b)$. Using Theorems 1 and 2, one can prove the following theorem.

Theorem 3. *Suppose that $F \in BV([a, b]; \mathrm{c}(X))$ and $x_1 \in F(a)$ and $x_2 \in F(b)$. Then among all regular selections $f \in BV([a, b]; X)$ of the multifunction F such that $f(a) = x_1$ and $f(b) = x_2$, there exists a selection of minimal variation.*

As a remark to all the theorems presented above, we note that under the hypotheses of these theorems, the closed interval $I = [a, b]$ can be replaced by an open, semiopen, finite, or infinite interval from $\mathbb{R}$.

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