BOUNDED SYMBOLS AND REPRODUCING KERNEL THESIS FOR TRUNCATED TOEPLITZ OPERATORS

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ABSTRACT. Compressions of Toeplitz operators to coinvariant subspaces of H^2 are called *truncated Toeplitz operators*. We study two questions related to these operators. The first, raised by Sarason, is whether boundedness of the operator implies the existence of a bounded symbol; the second is the Reproducing Kernel Thesis. We show that in general the answer to the first question is negative, and we exhibit some classes of spaces for which the answers to both questions are positive.

1. INTRODUCTION

Truncated Toeplitz operators on model spaces have been formally introduced by Sarason in [29], although special cases have long ago appeared in literature, most notably as model operators for contractions with defect numbers one and for their commutant. They are naturally related to the classical Toeplitz and Hankel operators on the Hardy space. This is a new area of study, and it is remarkable that many simple questions remain still unsolved. As a basic reference for their main properties, [29] is invaluable; further study can be found in [9, 10, 18] and in [30, Section 7].

The truncated Toeplitz operators live on the model spaces K_Θ. These are subspaces of H^2 (see Section 2 for precise definitions) that have attracted attention in the last decades; they are relevant in various subjects such as for instance spectral theory for general linear operators [26], control theory [25], and Nevanlinna domains connected to rational approximation [16]. Given a model space K_Θ and a function $\varphi \in L^2$, the truncated Toeplitz operator A_φ^Θ is defined on a dense subspace of K_Θ as the compression to K_Θ of multiplication by φ. The function φ is then called a symbol of the operator, and it is never uniquely defined.

In the particular case where $\varphi \in L^\infty$ the operator A_φ^Θ is bounded. In view of well-known facts about classical Toeplitz and Hankel operators, it is natural to ask whether the converse is true, that is, if a bounded truncated Toeplitz operator has necessarily a bounded symbol. This question has been posed in [29], where it is noticed that it is nontrivial even for rank one operators. In the present paper we will provide a class of inner functions Θ for which there exist rank one truncated Toeplitz operators on K_Θ without bounded symbols. On the other hand, we obtain positive results for some basic examples of model spaces. Therefore the situation is quite different from the classical Toeplitz and Hankel operators.

The other natural question that we address is the Reproducing Kernel Thesis for truncated Toeplitz operators. Recall that an operator on a reproducing kernel Hilbert space is said to satisfy the *Reproducing Kernel Thesis* (RKT) if its boundedness is determined by its behaviour on the reproducing kernels. This property has been studied for several classes of operators: Hankel and Toeplitz operators on the Hardy space of the unit disc [7, 21, 32], Toeplitz operators on the Paley–Wiener space [31], semicommutators of Toeplitz operators [26], Hankel operators on the Bergman space [5, 20], and Hankel operators on the Hardy space of the bidisk [17, 27]. It appears thus natural to ask the corresponding question for truncated Toeplitz operators. We will show that in this case it is more appropriate to assume the boundedness of the operator on the reproducing kernels as well as on a related "dual" family, and will discuss further its validity for certain model spaces.

The paper is organized as follows. The next two sections contain preliminary material concerning model spaces and truncated Toeplitz operators. Section 4 introduces the main two problems we are concerned with: existence of bounded symbols and the Reproducing Kernel Thesis. The counterexamples are presented in Section 5; in particular, Sarason's question on the general existence of bounded symbols is answered in the negative. Section 6 exhibits some classes of model spaces for which the answers to both questions are positive. Finally, in Section 7 we present another class of well behaved truncated Toeplitz operators, namely operators with positive symbols.

2. Preliminaries on model spaces

Basic references for the content of this section are [15, 19] for general facts about Hardy spaces and [26] for model spaces and operators.

2.1. **Hardy spaces.** The Hardy space H^p of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the space of analytic functions f on $\mathbb{D}$ satisfying $\|f\|_p < +\infty$, where

$$\|f\|_p = \sup_{0 \le r < 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}, \qquad 1 \le p < +\infty.$$

The algebra of bounded analytic functions on $\mathbb{D}$ is denoted by H^∞. We denote also $H_0^p = zH^p$. Alternatively, H^p can be identified (via radial limits) with the subspace of functions $f \in L^p = L^p(\mathbb{T})$ for which $\hat{f}(n) = 0$ for all $n < 0$. Here $\mathbb{T}$ denotes the unit circle with normalized Lebesgue measure m.

For any $\varphi \in L^\infty$, we denote by $M_\varphi f = \varphi f$ the *multiplication operator* on L^2; we have $\|M_\varphi\| = \|\varphi\|_\infty$. The *Toeplitz* and *Hankel* operators on H^2 are given by the formulas

$$T_\varphi = P_+ M_\varphi, \qquad T_\varphi : H^2 \to H^2;$$

$$H_\varphi = P_- M_\varphi, \qquad H_\varphi : H^2 \to H_-^2,$$

where P_+ is the Riesz projection from L^2 onto H^2 and $P_- = I - P_+$ is the orthogonal projection from L^2 onto $H_-^2 = L^2 \ominus H^2$. In case where φ is analytic, T_φ is just the restriction of M_φ to H^2. We have $T_\varphi^* = T_{\bar\varphi}$ and $H_\varphi^* = P_+ M_{\bar\varphi} P_-$; we also denote $S = T_z$ the usual shift operator on H^2.

Evaluations at points $\lambda \in \mathbb{D}$ are bounded functionals on H^2 and the corresponding reproducing kernel is $k_\lambda(z) = (1 - \bar\lambda z)^{-1}$; thus, $f(\lambda) = \langle f, k_\lambda \rangle$, for every function f in H^2. If $\varphi \in H^\infty$, then k_λ is an eigenvector for T_φ^*, and $T_\varphi^* k_\lambda = \overline{\varphi(\lambda)} k_\lambda$. By normalizing k_λ we obtain $h_\lambda = \frac{k_\lambda}{\|k_\lambda\|_2} = \sqrt{1 - |\lambda|^2} k_\lambda$.

2.2. **Model spaces.** Suppose now Θ is an inner function, that is, a function in H^∞ whose radial limits are of modulus one almost everywhere on $\mathbb{T}$. In what follows we consider only nonconstant inner functions. We define the corresponding *shift-coinvariant subspace* generated by Θ (also called *model space*) by the formula $K_\Theta^p = H^p \cap \Theta \overline{H_0^p}$, $1 \le p < +\infty$. We will be especially interested in the Hilbert case, that is, when $p = 2$. In this case we write $K_\Theta = K_\Theta^2$; it is easy to see that K_Θ is also given by

$$K_\Theta = H^2 \ominus \Theta H^2 = \left\{ f \in H^2 : \langle f, \Theta g \rangle = 0, \forall g \in H^2 \right\}.$$

The orthogonal projection of L^2 onto K_Θ is denoted by P_Θ; we have $P_\Theta = P_+ - \Theta P_+ \bar\Theta$. Since the Riesz projection P_+ acts boundedly on L^p, $1 < p < \infty$, this formula shows that P_Θ can also be regarded as a bounded operator from L^p onto K_Θ^p, $1 < p < \infty$.

The reproducing kernel in K_Θ for a point $\lambda \in \mathbb{D}$ is the function

$$(2.1) \qquad k_\lambda^\Theta(z) = (P_\Theta k_\lambda)(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \bar{\lambda}z};$$

we denote by h_λ^Θ the normalized reproducing kernel,

$$(2.2) \qquad h_\lambda^\Theta(z) = \sqrt{\frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2}}\, k_\lambda^\Theta(z).$$

Note that, according to (2.1), we have the orthogonal decomposition

$$(2.3) \qquad k_\lambda = k_\lambda^\Theta + \Theta\overline{\Theta(\lambda)}k_\lambda.$$

We will use the antilinear isometry $J : L^2 \to L^2$, given by $J(f)(\zeta) = \overline{\zeta f(\zeta)}$; it maps H^2 into $\overline{H_0^2} = L^2 \ominus H^2 = H_-^2$ and conversely. More often another antilinear isometry $\omega = \Theta J$ will appear, whose main properties are summarized below.

Lemma 2.1. *Define, for $f \in L^2$, $\omega(f)(\zeta) = \overline{\zeta f(\zeta)}\Theta(\zeta)$. Then:*
  (i) *ω is antilinear, isometric, onto;*
  (ii) *$\omega^2 = \mathrm{Id}$;*
  (iii) *$\omega P_\Theta = P_\Theta \omega$ (and therefore K_Θ reduces ω), $\omega(\Theta H^2) = H_-^2$ and $\omega(H_-^2) = \Theta H^2$.*

We define the *difference quotient* $\tilde{k}_\lambda^\Theta = \omega(k_\lambda^\Theta)$ and $\tilde{h}_\lambda^\Theta = \omega(h_\lambda^\Theta)$; thus

$$(2.4) \qquad \tilde{k}_\lambda^\Theta(z) = \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}, \qquad \tilde{h}_\lambda^\Theta(z) = \sqrt{\frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2}}\,\frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}.$$

In the sequel we will use the following simple lemmas.

Lemma 2.2. *Suppose Θ_1, Θ_2 are two inner functions, $f_1 \in K_{\Theta_1}$, $f_2 \in K_{\Theta_2} \cap H^\infty$. Then $f_1 f_2, z f_1 f_2 \in K_{\Theta_1 \Theta_2}$.*

Proof. Obviously $z f_1 f_2 \in H^2$. On the other side, $f_1 \in K_{\Theta_1}$ implies $f_1 = \Theta_1 \overline{z g_1}$, with $g_1 \in H^2$, and similarly $f_2 = \Theta_2 \overline{z g_2}$, $g_2 \in H^\infty$. Thus $z f_1 f_2 \in \Theta_1 \Theta_2 \overline{z H^2}$. Therefore $z f_1 f_2 \in H^2 \cap \Theta_1 \Theta_2 \overline{H_0^2} = K_{\Theta_1 \Theta_2}$. The claim about $f_1 f_2$ is an immediate consequence, since the model spaces are invariant under the backward shift operator S^*. $\qquad\square$

Recall that, given two inner functions θ_1, θ_2, we say that θ_2 divides θ_1 if there exists an inner function θ_3 such that $\theta_1 = \theta_2 \theta_3$.

Lemma 2.3. *Suppose that θ and Θ are two inner functions such that θ^3 divides $z\Theta$. Then:*

(a) $\theta K_\theta \subset K_{\theta^2} \subset K_\Theta$;

(b) *if $f \in H^\infty \cap \theta K_\theta$ and $\varphi \in K_\theta + \overline{K_\theta}$, then the functions φf and $\bar{\varphi} f$ belong to K_Θ.*

Proof. Since θ^3 divides $z\Theta$, there exists an inner function θ_1 such that $z\Theta = \theta^3 \theta_1$. In particular it follows from this factorization that $\theta(0)\theta_1(0) = 0$, which implies that $\theta\theta_1 H^2 \subset zH^2$.

Using $K_\theta = H^2 \cap \theta \overline{zH^2}$, we have

$$\theta K_\theta = \theta H^2 \cap \theta^2 \overline{zH^2} \subset H^2 \cap \theta^2 \overline{zH^2} = K_{\theta^2}.$$

Further,

$$K_{\theta^2} = H^2 \cap \theta^2 \overline{zH^2} = H^2 \cap \Theta \overline{z\Theta} \theta^2 \overline{H^2} = H^2 \cap \Theta \overline{\theta\theta_1 H^2} \subset H^2 \cap \Theta \overline{zH^2} = K_\Theta,$$

because $\theta\theta_1 H^2 \subset zH^2$; thus (a) is proved.

Let now $f = \theta f_1$ and $\varphi = \varphi_1 + \overline{\varphi_2}$, with $f_1 \in H^\infty \cap K_\theta$ and $\varphi_1, \varphi_2 \in K_\theta$. Since $\varphi_2 \in K_\theta$, using Lemma 2.1, we have $\varphi_2 = \theta \bar{z} \widetilde{\varphi_2}$, with $\tilde{\varphi}_2 \in K_\theta$, which implies that

$$\varphi f = \theta f_1 (\varphi_1 + \overline{\varphi_2}) = \theta f_1 \varphi_1 + z f_1 \tilde{\varphi}_2.$$

But it follows from Lemma 2.2 that $z f_1 \tilde{\varphi}_2 \in K_{\theta^2}$; by (a), we obtain $z f_1 \tilde{\varphi}_2 \in K_\Theta$. So it remains to prove that $\theta f_1 \varphi_1 \in K_\Theta$. Obviously $\theta f_1 \varphi_1 \in H^2$; moreover, for every function $h \in H^2$, we have

$$\langle \theta f_1 \varphi_1, \Theta h \rangle = \langle z\theta f_1 \varphi_1, z\Theta h \rangle = \langle z\theta f_1 \varphi_1, \theta^3 \theta_1 h \rangle = \langle z f_1 \varphi_1, \theta^2 \theta_1 h \rangle = 0,$$

because another application of Lemma 2.2 yields $z f_1 \varphi_1 \in K_{\theta^2}$. That proves that $\theta f_1 \varphi_1 \in K_\Theta$ and thus $\varphi f \in K_\Theta$. Since $K_\Theta + \overline{K_\Theta}$ is invariant under the conjugation, we obtain also the result for $\bar{\varphi} f$. $\square$

2.3. Angular derivatives and evaluation on the boundary.

The inner function Θ is said to have an *angular derivative in the sense of Carathéodory* at $\zeta \in \mathbb{T}$ if Θ and Θ' have a non-tangential limit at ζ and $|\Theta(\zeta)| = 1$. Then it is known [1] that evaluation at ζ is continuous on K_Θ, and the function k_ζ^Θ, defined by

$$k_\zeta^\Theta(z) := \frac{1 - \overline{\Theta(\zeta)}\Theta(z)}{1 - \bar{\zeta}z}, \qquad z \in \mathbb{D},$$

belongs to K_Θ and is the corresponding reproducing kernel. Replacing λ by ζ in the formula (2.4) gives a function $\tilde{k}_\zeta^\Theta$ which also belongs to K_Θ and $\omega(k_\zeta^\Theta) = \tilde{k}_\zeta^\Theta = \bar{\zeta}\Theta(\zeta)k_\zeta^\Theta$.

Moreover we have $\|k_\zeta^\Theta\|_2 = |\Theta'(\zeta)|^{1/2}$. We denote by $E(\Theta)$ the set of points $\zeta \in \mathbb{T}$ where Θ has an angular derivative in the sense of Carathéodory.

In [1] and [13] precise conditions are given for the inclusion of k_ζ^Θ into L^p (for $1 < p < \infty$); namely, if (a_k) are the zeros of Θ in $\mathbb{D}$ and σ is the singular measure on $\mathbb{T}$ corresponding to the singular part of Θ, then $k_\zeta^\Theta \in L^p$ if and only if

$$(2.5) \qquad \sum_k \frac{1 - |a_k|^2}{|\zeta - a_k|^p} + \int_\mathbb{T} \frac{d\sigma(\tau)}{|\zeta - \tau|^p} < +\infty.$$

We will use in the sequel the following easy result.

Lemma 2.4. *Let $1 < p < +\infty$ and let Θ be an inner function. Then we have:*

(a) $E(\Theta^2) = E(\Theta)$;

(b) $\displaystyle\inf_{\lambda \in \mathbb{D} \cup E(\Theta)} \|k_\lambda^\Theta\|_2 > 0$;

(c) *for $\lambda \in \mathbb{D}$, we have*

$$(2.6) \qquad C\|k_\lambda^\Theta\|_p \leq \|k_\lambda^{\Theta^2}\|_p \leq 2\|k_\lambda^\Theta\|_p,$$

where $C = \|P_\Theta\|_{L^p \to L^p}^{-1}$ is a constant which depends only on Θ and p. Also, if $\zeta \in E(\Theta)$, then $k_\zeta^{\Theta^2} \in L^p$ if and only if $k_\zeta^\Theta \in L^p$, and (2.6) holds for $\lambda = \zeta$.

Proof. The proof of (a) is immediate using the definition. For the proof of (b) note that, for $\lambda \in \mathbb{D} \cup E(\Theta)$, we have

$$|1 - \overline{\Theta(0)}\Theta(\lambda)| = |k_0^\Theta(\lambda)| \leq \|k_0^\Theta\|_2 \|k_\lambda^\Theta\|_2 = (1 - |\Theta(0)|^2)^{1/2}\|k_\lambda^\Theta\|_2,$$

which implies $\|k_\lambda^\Theta\|_2 \geq \left(\frac{1 - |\Theta(0)|}{1 + |\Theta(0)|}\right)^{1/2}$.

It remains to prove (c). We have $k_\lambda^{\Theta^2} = (1 + \overline{\Theta(\lambda)}\Theta)k_\lambda^\Theta$, whence $P_\Theta k_\lambda^{\Theta^2} = k_\lambda^\Theta$. Thus the result follows from the fact that P_Θ is bounded on L^p and from the trivial estimate $|1 + \overline{\Theta(\lambda)}\Theta(z)| \leq 2$, $z \in \mathbb{T}$. $\qquad\square$

2.4. **The continuous case.** It is useful to remember the connection with the "continuous" case, for which we refer to [15, 22]. If $u(w) = \frac{w-i}{w+i}$, then u is a conformal homeomorphism of the Riemann sphere. It maps $-i$ to ∞, ∞ to 1, $\mathbb{R}$ onto $\mathbb{T}$ and $\mathbb{C}_+$ to $\mathbb{D}$ (here $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$).

The operator

$$(\mathcal{U}f)(t) = \frac{1}{\sqrt{\pi}(t+i)}f(u(t))$$

maps $L^2(\mathbb{T})$ unitarily onto $L^2(\mathbb{R})$ and H^2 unitarily onto $H^2(\mathbb{C}_+)$, the Hardy space of the upper half-plane. The corresponding transformation for functions in L^∞ is

$$\tilde{\mathcal{U}}(\varphi) = \varphi \circ u; \tag{2.7}$$

it maps $L^\infty(\mathbb{T})$ isometrically onto $L^\infty(\mathbb{R})$, H^∞ isometrically onto $H^\infty(\mathbb{C}_+)$ and inner functions in $\mathbb{D}$ into inner functions in $\mathbb{C}_+$. Now if Θ is an inner function in $\mathbb{D}$, we have $\mathcal{U}P_\Theta = \boldsymbol{P_\Theta}\mathcal{U}$ and then $\mathcal{U}K_\Theta = \boldsymbol{K_\Theta}$, where $\boldsymbol{\Theta} = \Theta \circ u$, $\boldsymbol{K_\Theta} = H^2(\mathbb{C}_+) \ominus \boldsymbol{\Theta}H^2(\mathbb{C}_+)$ and $\boldsymbol{P_\Theta}$ is the orthogonal projection onto $\boldsymbol{K_\Theta}$. Moreover

$$\mathcal{U}h_\lambda^\Theta = c_\mu \boldsymbol{h_\mu^\Theta} \quad \text{and} \quad \mathcal{U}\tilde{h}_\lambda^\Theta = \overline{c_\mu}\boldsymbol{\tilde{h}_\mu^\Theta}, \tag{2.8}$$

where $\mu = u^{-1}(\lambda) \in \mathbb{C}_+$, $c_\mu = \frac{\bar{\mu}-i}{|\mu+i|}$ is a constant of modulus one,

$$\boldsymbol{h_\mu^\Theta}(\omega) = \frac{i}{\sqrt{\pi}}\sqrt{\frac{\operatorname{Im}\mu}{1 - |\boldsymbol{\Theta}(\mu)|^2}}\frac{1 - \overline{\boldsymbol{\Theta}(\mu)}\boldsymbol{\Theta}(\omega)}{\omega - \bar{\mu}}, \qquad \omega \in \mathbb{C}_+,$$

is the normalized reproducing kernel for $\boldsymbol{K_\Theta}$, while

$$\boldsymbol{\tilde{h}_\mu^\Theta}(\omega) = \frac{1}{i\sqrt{\pi}}\sqrt{\frac{\operatorname{Im}\mu}{1 - |\boldsymbol{\Theta}(\mu)|^2}}\frac{\boldsymbol{\Theta}(\omega) - \boldsymbol{\Theta}(\mu)}{\omega - \mu}, \qquad \omega \in \mathbb{C}_+,$$

is the normalized difference quotient in $\boldsymbol{K_\Theta}$.

3. TRUNCATED TOEPLITZ OPERATORS

In [29], D. Sarason studied the class of *truncated Toeplitz operators* which are defined as the compression of Toeplitz operators to coinvariant subspaces of H^2.

Note first that we can extend the definitions of M_φ, T_φ, and H_φ in Section 2 to the case when the symbol is only in L^2 instead of L^∞, obtaining (possibly unbounded) densely defined operators. Then M_φ and T_φ are bounded if and only if $\varphi \in L^\infty$ (and $\|M_\varphi\| = \|T_\varphi\| = \|\varphi\|_\infty$), while H_φ is bounded if and only if $P_-\varphi \in BMO$ (and $\|H_\varphi\|$ is equivalent to $\|P_-\varphi\|_{BMO}$).

In [29], D. Sarason defines an analogous operator on K_Θ. Suppose Θ is an inner function and $\varphi \in L^2$; the *truncated Toeplitz operator* A_φ^Θ will in general be a densely defined, possibly unbounded, operator on K_Θ. Its domain is $K_\Theta \cap H^\infty$, on which it acts by the formula

$$A_\varphi^\Theta f = P_\Theta(\varphi f), \qquad f \in K_\Theta \cap H^\infty.$$

In particular, $K_\Theta \cap H^\infty$ contains all reproducing kernels k_λ^Θ, $\lambda \in \mathbb{D}$, and their linear combinations, and is therefore dense in K_Θ.

We will denote by $\mathcal{T}(K_\Theta)$ the space of all bounded truncated Toeplitz operators on K_Θ. It follows from [29, Theorem 4.2] that $\mathcal{T}(K_\Theta)$ is a Banach space in the operator norm.

Using Lemma 2.1 and the fact that $\omega M_\varphi \omega = M_{\bar\varphi}$, it is easy to check the useful formula

$$(3.1) \qquad\qquad \omega A_\varphi^\Theta \omega = A_{\bar\varphi}^\Theta = (A_\varphi^\Theta)^*.$$

We call φ a *symbol* of the operator A_φ^Θ. It is not unique; in [29], it is shown that $A_\varphi^\Theta = 0$ if and only if $\varphi \in \Theta H^2 + \overline{\Theta H^2}$. Let us denote $\mathfrak{S}_\Theta = L^2 \ominus (\Theta H^2 + \overline{\Theta H^2})$ and $P_{\mathfrak{S}_\Theta}$ the corresponding orthogonal projection. Two spaces that contain $\mathfrak{S}_\Theta$ up to a subspace of dimension at most 1 admit a direct description, and we will gather their properties in the next two lemmas.

Lemma 3.1. *Denote by Q_Θ the orthogonal projection onto $K_\Theta \oplus \bar{z}\overline{K_\Theta}$. Then:*

(a) $Q_\Theta(\bar\Theta) = \bar\Theta - \overline{\Theta(0)}^2 \Theta$;

(b) *we have*

$$K_\Theta \oplus \bar{z}\overline{K_\Theta} = \mathfrak{S}_\Theta \oplus \mathbb{C}q_\Theta,$$

where $q_\Theta = \|Q_\Theta(\bar\Theta)\|_2^{-1} Q_\Theta(\bar\Theta)$;

(c) *Q_Θ and $P_{\mathfrak{S}_\Theta}$ are bounded on L^p for $1 < p < \infty$.*

Proof. Since by Lemma 2.1 $\bar{z}\overline{K_\Theta} = \bar\Theta K_\Theta$, we have $K_\Theta \oplus \bar{z}\overline{K_\Theta} = K_\Theta \oplus \bar\Theta K_\Theta$, and therefore $Q_\Theta = P_\Theta + M_{\bar\Theta} P_\Theta M_\Theta$. Thus Q_Θ is bounded on L^p for all $p > 1$. Further, if we denote by $\mathbf{1}$ the constant function equal to 1, then

$$Q_\Theta(\bar\Theta) = P_\Theta(\bar\Theta) + M_{\bar\Theta} P_\Theta M_\Theta(\bar\Theta) = P_\Theta(\overline{\Theta(0)}\mathbf{1}) + M_{\bar\Theta} P_\Theta \mathbf{1}$$
$$= (\overline{\Theta(0)} + \bar\Theta)(1 - \overline{\Theta(0)}\Theta) = \bar\Theta - \overline{\Theta(0)}^2 \Theta.$$

Thus (a) is proved.

Since $L^2 = \Theta H^2 \oplus \overline{\Theta H_0^2} \oplus K_\Theta \oplus \bar{z}\overline{K_\Theta}$, it follows that $\mathfrak{S}_\Theta \subset K_\Theta \oplus \bar{z}\overline{K_\Theta}$ and thus

$$(3.2) \qquad K_\Theta \oplus \bar{z}\overline{K_\Theta} = Q_\Theta\left(\mathfrak{S}_\Theta + \Theta H^2 + \overline{\Theta H_0^2} + \mathbb{C}\bar\Theta\right) = \mathfrak{S}_\Theta \oplus \mathbb{C}Q_\Theta(\bar\Theta),$$

which proves (b). Note that according to (a), one easily see that $Q_\Theta(\bar\Theta) \not\equiv 0$.

Now we have for $f \in L^\infty$

$$(3.3) \qquad\qquad P_{\mathfrak{S}_\Theta} f = Q_\Theta f - \langle f, q_\Theta \rangle q_\Theta.$$

and the second term is bounded in L^p, since q_Θ belongs to L^∞. This concludes the proof of (c). $\qquad\square$

Lemma 3.2. *We have $\mathfrak{S}_\Theta \subset K_\Theta + \overline{K_\Theta}$. Each truncated Toeplitz operator has a symbol φ of the form $\varphi = \varphi_+ + \overline{\varphi_-}$ with $\varphi_\pm \in K_\Theta$; any other such decomposition corresponds to $\varphi_+ + ck_0^\Theta$, $\varphi_- - \bar{c}k_0^\Theta$ for some $c \in \mathbb{C}$. In particular, $\varphi_\pm$ are uniquely determined if we fix (arbitrarily) the value of one of them in a point of $\mathbb{D}$.*

Proof. See [29, Section 3]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The formulas $\psi = \lim_{n \to \infty} \bar{z}^n T_\psi(z^n)$ and $P_-\psi = H_\psi(\mathbf{1})$ allow one to recapture simply the unique symbol of a Toeplitz operator as well as the unique symbol in H_-^2 of a Hankel operator. It is interesting to obtain a similar direct formula for the symbol of a truncated Toeplitz operator. Lemma 3.2 says that the symbol is unique if we assume, for instance, that $\varphi = \varphi_+ + \overline{\varphi_-}$, with $\varphi_\pm \in K_\Theta$ and $\varphi_-(0) = 0$. We can then recapture φ from the action of A_φ^Θ on k_λ^Θ and $\tilde{k}_\lambda^\Theta$. Indeed, one can check that

$$
(3.4) \qquad
\begin{aligned}
A_\varphi^\Theta k_0^\Theta &= \varphi_+ - \overline{\Theta(0)}\,\Theta\,\overline{\varphi_-}, \\
A_\varphi^\Theta \tilde{k}_0^\Theta &= \omega\left(\varphi_- + \overline{\varphi_+(0)} - \overline{\Theta(0)}\,\Theta\,\overline{\varphi_+}\right).
\end{aligned}
$$

From the first equation we obtain $\varphi_+(0) = \langle A_\varphi^\Theta k_0^\Theta, k_0^\Theta \rangle$. Then (3.4) imply, for any $\lambda \in \mathbb{D}$,

$$
\begin{aligned}
\varphi_+(\lambda) - \overline{\Theta(0)}\Theta(\lambda)\overline{\varphi_-(\lambda)} &= \langle A_\varphi^\Theta k_0^\Theta, k_\lambda^\Theta \rangle, \\
\overline{\varphi_-(\lambda)} - \Theta(0)\overline{\Theta(\lambda)}\varphi_+(\lambda) &= \langle A_\varphi^\Theta \tilde{k}_0^\Theta, \tilde{k}_\lambda^\Theta \rangle - \langle A_\varphi^\Theta k_0^\Theta, k_0^\Theta \rangle.
\end{aligned}
$$

This is a linear system in $\varphi_+(\lambda)$ and $\overline{\varphi_-(\lambda)}$, whose determinant is $1 - |\Theta(0)\Theta(\lambda)|^2 > 0$; therefore, $\varphi_\pm$ can be made explicit in terms of the products in the right hand side.

Note, however, that A_φ^Θ is completely determined by its action on reproducing kernels, so one should be able to recapture the values of the symbol only from $A_\varphi^\Theta k_\lambda^\Theta$. The next proposition shows how one can achieve this goal; moreover, one can also obtain an estimate of the L^2-norm of the symbol. Namely, for an inner function Θ and any (not necessarily bounded) linear operator T whose domain contains $K_\Theta \cap H^\infty$, define

$$
(3.5) \qquad\qquad\qquad \rho_r(T) := \sup_{\lambda \in \mathbb{D}} \|Th_\lambda^\Theta\|_2.
$$

We will have the occasion to come back to the quantity ρ_r in the next section.

To simplify the next statement, denote

$$
(3.6) \qquad F_{\lambda,\mu} = (I - \lambda S^*)\omega(A_\varphi^\Theta k_\lambda^\Theta) - (I - \mu S^*)\omega(A_\varphi^\Theta k_\mu^\Theta), \qquad \lambda, \mu \in \mathbb{D}.
$$

Proposition 3.3. *Let Θ be an inner function, A_φ^Θ a truncated Toeplitz operator, and $\mu \in \mathbb{D}$ such that $\Theta(\mu) \neq 0$. Suppose $\varphi = \varphi_+ + \overline{\varphi_-}$ is the unique decomposition of the symbol with $\varphi_\pm \in K_\Theta$, $\varphi_-(\mu) = 0$. Then*

$$\varphi_-(\lambda) = \frac{\langle (S - \mu)(I - \mu S^*)^{-1} F_{\lambda,\mu}, k_\mu^\Theta \rangle}{\Theta(\mu)(\overline{\Theta(0)}\Theta(\mu) - 1)}, \qquad \lambda \in \mathbb{D}, \tag{3.7}$$

and $\varphi_+ = \omega(\psi_+)$, where

$$\psi_+ = (I - \mu S^*)\omega(A_\varphi^\Theta k_\mu^\Theta) + \Theta(\mu)S^*\varphi_-. \tag{3.8}$$

Moreover, there exists a constant C depending only on Θ and μ such that

$$\max\{\|\varphi_-\|_2, \|\varphi_+\|_2\} \leq C\rho_r(A_\varphi^\Theta). \tag{3.9}$$

Proof. First note that for any $\lambda \in \mathbb{D}$, we have

$$(I - \lambda S^*)\omega(A_\varphi^\Theta k_\lambda^\Theta) = \psi_+ + \varphi_-(\lambda)S^*\Theta - \Theta(\lambda)S^*\varphi_-. \tag{3.10}$$

Indeed,

$$P_\Theta(\varphi_+ k_\lambda^\Theta) = P_\Theta\left(\varphi_+ \frac{1}{1 - \bar\lambda z}\right) = \varphi_+ + \bar\lambda P_\Theta\left(\frac{\Theta \bar z \overline{\psi_+}}{\bar z - \bar\lambda}\right)$$
$$= \varphi_+ + \bar\lambda \Theta \bar z \overline{\left(\frac{\psi_+ - \psi_+(\lambda)}{z - \lambda}\right)}.$$

Thus,

$$\omega(A_{\varphi_+}^\Theta k_\lambda^\Theta) = \psi_+ + \lambda \frac{\psi_+ - \psi_+(\lambda)}{z - \lambda} = \frac{z\psi_+ - \lambda\psi_+(\lambda)}{z - \lambda}.$$

One can easily check that

$$(I - \lambda S^*)^{-1} S^* f = \frac{f - f(\lambda)}{z - \lambda}, \tag{3.11}$$

for every function $f \in H^2$; then we obtain

$$(I - \lambda S^*)\omega(A_{\varphi_+}^\Theta k_\lambda^\Theta) = \psi_+. \tag{3.12}$$

On the other hand,

$$P_\Theta(\varphi_- k_\lambda^\Theta) = P_\Theta\left(\bar z \frac{\overline{\varphi_-} - \overline{\varphi_-(\lambda)}}{\bar z - \bar\lambda} + \frac{\overline{\varphi_-(\lambda)}}{1 - \bar\lambda z}\right.$$
$$\left. - \overline{\Theta(\lambda)}\bar z \Theta \frac{\overline{\varphi_-} - \overline{\varphi_-(\lambda)}}{\bar z - \bar\lambda} - \overline{\Theta(\lambda)}\Theta \frac{\overline{\varphi_-(\lambda)}}{1 - \bar\lambda z}\right)$$
$$= \overline{\varphi_-(\lambda)}k_\lambda^\Theta - \overline{\Theta(\lambda)}\bar z \Theta \overline{\left(\frac{\varphi_- - \varphi_-(\lambda)}{z - \lambda}\right)}.$$

Hence,

$$\omega(A_{\varphi_-}^\Theta k_\lambda^\Theta) = \varphi_-(\lambda)\frac{\Theta - \Theta(\lambda)}{z - \lambda} - \Theta(\lambda)\frac{\varphi_- - \varphi_-(\lambda)}{z - \lambda}$$

and

$$(3.13) \qquad (I - \lambda S^*)\omega(A_{\varphi_-}^\Theta k_\lambda^\Theta) = \varphi_-(\lambda)S^*\Theta - \Theta(\lambda)S^*\varphi_-.$$

Thus (3.10) follows immediately from (3.12) and (3.13). If we take $\lambda = \mu$ in (3.10), we get (remembering that $\varphi_-(\mu) = 0$)

$$(3.14) \qquad \psi_+ = (I - \mu S^*)\omega(A_\varphi^\Theta k_\mu^\Theta) + \Theta(\mu)S^*\varphi_-.$$

Now plugging (3.14) into (3.10) yields

$$\varphi_-(\lambda)S^*\Theta + (\Theta(\mu) - \Theta(\lambda))S^*\varphi_- = F_{\lambda,\mu}.$$

Therefore, applying $(S - \mu)(I - \mu S^*)^{-1}$ and using $\varphi_-(\mu) = 0$ and (3.11), we obtain

$$(3.15) \qquad \varphi_-(\lambda)(\Theta - \Theta(\mu)) + (\Theta(\mu) - \Theta(\lambda))\varphi_- = (S - \mu)(I - \mu S^*)^{-1}F_{\lambda,\mu}.$$

Finally, we take the scalar product of both sides with k_μ^Θ and use the fact that $\Theta \perp K_\Theta$, $P_\Theta \mathbf{1} = \mathbf{1} - \overline{\Theta(0)}\Theta$, and again $\varphi_-(\mu) = 0$. Therefore

$$-\varphi_-(\lambda)\Theta(\mu)(1 - \overline{\Theta(0)}\Theta(\mu)) = \langle(S - \mu)(I - \mu S^*)^{-1}F_{\lambda,\mu}, k_\mu^\Theta\rangle,$$

which immediately implies (3.7).

To obtain the boundedness of the L^2 norms, fix now $\lambda \in \mathbb{D}$ such that $\Theta(\lambda) \neq \Theta(\mu)$. Since

$$\|(I - \mu S^*)\omega(A_\varphi^\Theta k_\mu^\Theta)\|_2 \leq 2\|A_\varphi^\Theta k_\mu^\Theta\|_2 \leq 2\|k_\mu^\Theta\|_2\rho_r(A_\varphi^\Theta)$$

and a similar estimate holds for $\|(I - \lambda S^*)\omega(A_\varphi^\Theta k_\lambda^\Theta)\|_2$, we have $\|F_{\lambda,\mu}\|_2 \leq C_1\rho(A_\varphi^\Theta)$, where C_1, as well as the next constants appearing in this proof, depends only on Θ, λ, μ. By (3.15), it follows that

$$\|\varphi_-(\lambda)(\Theta - \Theta(\mu)) + (\Theta(\mu) - \Theta(\lambda))\varphi_-\|_2 \leq C_2\rho_r(A_\varphi^\Theta).$$

Projecting onto K_Θ decreases the norm; since $P_\Theta(\varphi_-(\lambda)\Theta) = 0$ and $P_\Theta(\mathbf{1}) = k_0^\Theta$, we obtain

$$\| - \Theta(\mu)\varphi_-(\lambda)k_0^\Theta + (\Theta(\mu) - \Theta(\lambda))\varphi_-\|_2 \leq C_2\rho_r(A_\varphi^\Theta).$$

Write now $\varphi_- = h + ck_0^\Theta$ with $h \perp k_0^\Theta$. Then $\|(\Theta(\mu) - \Theta(\lambda))h\|_2 \leq C_2\rho_r(A_\varphi^\Theta)$, whence $\|h\|_2 \leq C_3\rho_r(A_\varphi^\Theta)$. Since $\varphi_-(\mu) = 0$, we have $h(\mu) + ck_0^\Theta(\mu) = 0$, which implies that

$$|c| = |k_0^\Theta(\mu)|^{-1}|h(\mu)| \leq C_4\rho_r(A_\varphi^\Theta)$$

Therefore we have $\|\varphi_-\|_2 \leq C_5 \rho_r(A_\varphi^\Theta)$. Finally, (3.8) yields a similar estimate for ψ_+ and then for φ_+. $\qquad\square$

The following proposition yields a relation between truncated Toeplitz operators and usual Hankel operators.

Proposition 3.4. *With respect to the decompositions* $H_-^2 = \bar\Theta K_\Theta \oplus \bar\Theta H_-^2$, $H^2 = K_\Theta \oplus \Theta H^2$, *the operator* $H_{\bar\Theta}^* H_{\bar\Theta\varphi} H_{\bar\Theta}^* : H_-^2 \to H^2$ *has the matrix*

$$(3.16) \qquad \begin{pmatrix} A_\varphi^\Theta M_\Theta & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. If $f \in \bar\Theta H_-^2$, then $H_{\bar\Theta}^* f = 0$. If $f \in \bar\Theta K_\Theta$, then $H_{\bar\Theta}^* f = \Theta f \in K_\Theta$. Since $P_\Theta = P_+ M_\Theta P_- M_{\bar\Theta}$, it follows that, for $f \in K_\Theta$,

$$A_\varphi^\Theta f = P_\Theta M_\varphi f = P_+ M_\Theta P_- M_{\bar\Theta} M_\varphi f = H_{\bar\Theta}^* H_{\bar\Theta\varphi} f,$$

and therefore, if $f \in \bar\Theta K_\Theta$, then $A_\varphi^\Theta \Theta f = H_{\bar\Theta}^* H_{\bar\Theta\varphi} H_{\bar\Theta}^* f$ as required. $\qquad\square$

The non-zero entry in (3.16) consists in the isometry $M_\Theta : \bar\Theta K_\Theta \to K_\Theta$, followed by A_φ^Θ acting on K_Θ. There is therefore a close connection between properties of A_φ^Θ and properties of the corresponding product of three Hankel operators. Such products of Hankel operators have been studied for instance in [4, 8, 33].

Remark 3.5. Truncated Toeplitz operators can be defined also on model spaces of $H^2(\mathbb{C}_+)$, that is, $\boldsymbol{K_\Theta} = H^2(\mathbb{C}_+) \ominus \boldsymbol{\Theta} H^2(\mathbb{C}_+)$ for an inner function $\boldsymbol{\Theta}$ in the upper half-plane $\mathbb{C}_+$. We start then with a symbol $\boldsymbol{\varphi} \in (t+i)L^2(\mathbb{R})$ (which contains $L^\infty(\mathbb{R})$) and define (for $f \in \boldsymbol{K_\Theta} \cap (z+i)^{-1} H^\infty(\mathbb{C}_+)$, a dense subspace of $\boldsymbol{K_\Theta}$) the truncated Toeplitz operator $\boldsymbol{A_\varphi^\Theta} f = P_{\boldsymbol{\Theta}}(\boldsymbol{\varphi} f)$.

Let us briefly explain the relations between the truncated Toeplitz operators corresponding to model spaces on the upper half-plane and those corresponding to model spaces on the unit disk. If $\Theta = \boldsymbol{\Theta} \circ u^{-1}$ and $\psi = \varphi \circ u^{-1}$, using the fact that $\mathcal{U} P_\Theta \mathcal{U}^* = P_{\boldsymbol{\Theta}}$ and $\mathcal{U} M_\psi = M_\varphi \mathcal{U}$, we easily obtain

$$\boldsymbol{A_\varphi^\Theta} = \mathcal{U} A_\psi^\Theta \mathcal{U}^*.$$

In particular, if $\boldsymbol{A}$ is a linear operator on $\boldsymbol{K_\Theta}$, then $\boldsymbol{A}$ is a truncated Toeplitz operator on $\boldsymbol{K_\Theta}$ if and only if $A = \mathcal{U}^* \boldsymbol{A} \mathcal{U}$ is a truncated Toeplitz operator on K_Θ, and $\boldsymbol{\varphi}$ is a symbol for $\boldsymbol{A}$ if and only if $\psi := \varphi \circ u^{-1}$ is a symbol for A. It follows that $\boldsymbol{A}$ is bounded (or

has a bounded symbol) if and only if A is bounded (respectively, has a bounded symbol). Moreover we easily deduce from (2.8) that

$$\|\boldsymbol{A}_{\boldsymbol{\varphi}}^{\boldsymbol{\Theta}} \boldsymbol{h}_{\boldsymbol{\mu}}^{\boldsymbol{\Theta}}\|_2 = \|A_\psi^\Theta h_\lambda^\Theta\|_2 \quad \text{and} \quad \|\boldsymbol{A}_{\boldsymbol{\varphi}}^{\boldsymbol{\Theta}} \tilde{\boldsymbol{h}}_{\boldsymbol{\mu}}^{\boldsymbol{\Theta}}\|_2 = \|A_\psi^\Theta \tilde{h}_\lambda^\Theta\|_2,$$

for every $\mu \in \mathbb{C}_+$ and $\lambda = u(\mu)$. Finally, the truncated Toeplitz operator $\boldsymbol{A}_{\boldsymbol{\varphi}}^{\boldsymbol{\Theta}} = 0$ if and only if the symbol $\boldsymbol{\varphi} \in (t + i)\left(\boldsymbol{\Theta} H^2(\mathbb{C}_+) \oplus \overline{\boldsymbol{\Theta} H^2(\mathbb{C}_+)}\right)$ (note that the sum is in this case orthogonal, since $H^2(\mathbb{C}_+) \perp \overline{H^2(\mathbb{C}_+)}$).

4. Existence of bounded symbols and the Reproducing Kernel Thesis

As noted in Section 3, a Toeplitz operator T_φ has a unique symbol, T_φ is bounded if and only if this symbol is in L^∞, and the map $\varphi \mapsto T_\varphi$ is isometric from L^∞ onto the space of bounded Toeplitz operators on H^2. The situation is more complicated for Hankel operators: there is no uniqueness of the symbol, while the map $\varphi \mapsto H_\varphi$ is contractive and onto from L^∞ to the space of bounded Hankel operators (the boundedness condition $P_-\varphi \in BMO$ is equivalent to the fact that any bounded Hankel operator has a symbol in L^∞).

In the case of truncated Toeplitz operators, the map $\varphi \mapsto A_\varphi^\Theta$ is again contractive from L^∞ to $\mathcal{T}(K_\Theta)$. It is then natural to ask whether it is onto, that is, whether any bounded truncated Toeplitz operator is a compression of a bounded Toeplitz operator in H^2. This question has been asked by Sarason in [29].

Question 1. *Does every bounded truncated Toeplitz operator on K_Θ possess an L^∞ symbol?*

One may expect the answer to depend on the function Θ, and indeed we show below that it is the case. Assume that for some inner function Θ, any operator in $\mathcal{T}(K_\Theta)$ has a bounded symbol. Then if follows from the open mapping theorem that there exists a constant C such that for any $A \in \mathcal{T}(K_\Theta)$ one can find $\varphi \in L^\infty$ with $A = A_\varphi^\Theta$ and $\|\varphi\|_\infty \leq C\|A\|$.

A second natural question that may be asked about truncated Toeplitz operators is the Reproducing Kernel Thesis (RKT). This is related to the quantity ρ_r defined in (3.5). The functions h_λ^Θ have all norm 1, so if A_φ^Θ is bounded then obviously $\rho_r(A_\varphi^\Theta) \leq \|A_\varphi^\Theta\|_2$. The following question is then natural:

Question 2. *(RKT for truncated Toeplitz operators): let Θ be an inner function and $\varphi \in L^2$. Assume that $\rho_r(A_\varphi^\Theta) < +\infty$. Is A_φ^Θ bounded on K_Θ?*

As we have seen in the introduction, the RKT is true for various classes of operators related to the truncated Toeplitz operators, and it seems natural to investigate it for this class. We will see in Section 5 that the answer to this question is in general negative.

As we will show below, it is more natural to restate the RKT by including in the hypothesis also the functions $\tilde{h}_\lambda^\Theta$. Thus, for any linear operator T whose domain contains $K_\Theta \cap H^\infty$, define

$$\rho_d(T) = \sup_{\lambda \in \mathbb{D}} \|T\tilde{h}_\lambda^\Theta\|_2,$$

and $\rho(T) = \max\{\rho_r(T), \rho_d(T)\}$. The indices r and d in notation ρ_r and ρ_d stand for "reproducing kernels" and "difference quotients".

Note that if A_φ^Θ is a truncated Toeplitz operator, then by (3.1), we have $\rho_d(A_\varphi^\Theta) = \rho_r((A_\varphi^\Theta)^*)$, and then

$$\rho(A_\varphi^\Theta) = \max\{\rho_r(A_\varphi^\Theta), \rho_r((A_\varphi^\Theta)^*)\}.$$

Question 3. *Let Θ be an inner function and $\varphi \in L^2$. Assume that $\rho(A_\varphi^\Theta) < \infty$. Is A_φ^Θ bounded on K_Θ?*

In Section 5, we will show that the answer to Questions 1 and 2 may be negative. Question 3 remains in general open. In Section 6, we will give some examples of spaces K_Θ on which the answers to Questions 1 and 3 are positive.

In the rest of this section we will discuss the existence of bounded symbols and the RKT for some simple cases.

First, it is easy to deal with analytic or antianalytic symbols. The next proposition is a straightforward consequence of Bonsall's theorem [7] and the commutant lifting theorem. The equivalence between (i) and (ii) has already been noticed in [29].

Proposition 4.1. *Let $\varphi \in H^2$ and let A_φ^Θ be a truncated Toeplitz operator. Then the following assertions are equivalent:*

   (i) *A_φ^Θ has a bounded symbol;*
   (ii) *A_φ^Θ is bounded;*
   (iii) *$\rho_r(A_\varphi^\Theta) < +\infty$.*

More precisely there exists a universal constant $C > 0$ such that any truncated Toeplitz operator A_φ^Θ has a symbol φ_0 with $\|\varphi_0\|_\infty \leq C\rho_r(A_\varphi^\Theta)$.

Proof. It is immediate that (i) $\implies$ (ii) $\implies$ (iii). The implication (ii) $\implies$ (i) has already been noted in [29]; indeed if $\varphi \in H^2$ and A_φ^Θ is bounded, then A_φ^Θ commutes with $S_\Theta := A_z^\Theta$

and then, by a corollary of the commutant lifting theorem, A_φ^Θ has an H^∞ symbol with norm equal to the norm of A_φ^Θ.

So it remains to prove that there exists a constant $C > 0$ such that $\|A_\varphi^\Theta\| \leq C\rho_r(A_\varphi^\Theta)$. If $f \in K_\Theta \cap H^\infty$, then $\varphi f \in H^2$. Therefore $P_\Theta(\varphi f) = \Theta P_-(\bar\Theta \varphi f)$, or, in other words, $A_\varphi^\Theta(f) = \Theta H_{\bar\Theta\varphi} f$.

On the other hand, $\Theta H^2 \subset \ker H_{\bar\Theta\varphi}$, and therefore, with respect to the decompositions $H^2 = K_\Theta \oplus \Theta H^2$, $H_-^2 = \bar\Theta K_\Theta \oplus \bar\Theta H_-^2$, one can write

$$(4.1) \qquad H_{\bar\Theta\varphi} = \begin{pmatrix} \bar\Theta A_\varphi^\Theta & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that A_φ^Θ is bounded if and only if $H_{\bar\Theta\varphi}$ is. By Bonsall's Theorem [7], there exists a universal constant C (independent of φ and Θ) such that the boundedness of $H_{\bar\Theta\varphi}$ is equivalent to $\sup_{\lambda\in\mathbb{D}} \|H_{\bar\Theta\varphi} h_\lambda\|_2 < \infty$, and

$$\|H_{\bar\Theta\varphi}\| \leq C \sup_{\lambda\in\mathbb{D}} \|H_{\bar\Theta\varphi} h_\lambda\|_2.$$

But, again by (4.1) and using (2.1) and (2.2), we have

$$H_{\bar\Theta\varphi} h_\lambda = \bar\Theta A_\varphi^\Theta P_\Theta h_\lambda = \bar\Theta (1 - |\Theta(\lambda)|^2)^{1/2} A_\varphi^\Theta h_\lambda^\Theta,$$

and thus $\sup_{\lambda\in\mathbb{D}} \|H_{\bar\Theta\varphi} h_\lambda\|_2 \leq \sup_{\lambda\in\mathbb{D}} \|A_\varphi^\Theta h_\lambda^\Theta\|_2 = \rho_r(A_\varphi^\Theta)$. The proposition is proved. $\square$

A similar result is valid for antianalytic symbols.

Proposition 4.2. *Let $\varphi \in \overline{H^2}$ and let A_φ^Θ be a truncated Toeplitz operator. Then the following assertions are equivalent:*

(i) *A_φ^Θ has a bounded symbol;*
(ii) *A_φ^Θ is bounded;*
(iii) *$\rho_d(A_\varphi^\Theta) < +\infty$.*

More precisely there exists a universal constant $C > 0$ such that any truncated Toeplitz operator A_φ^Θ has a symbol φ_0 with $\|\varphi_0\|_\infty \leq C\rho_d(A_\varphi^\Theta)$.

Proof. Suppose $\varphi \in \overline{H^2}$. Since $\|A_\varphi^\Theta\| = \|(A_\varphi^\Theta)^*\| = \|A_{\bar\varphi}^\Theta\|$, and $\bar\varphi \in H^2$, we may apply Proposition 4.1 to $A_{\bar\varphi}^\Theta$ because by (3.1), we have

$$\rho_r(A_{\bar\varphi}^\Theta) = \sup_{\lambda\in\mathbb{D}} \|A_{\bar\varphi}^\Theta h_\lambda^\Theta\|_2 = \sup_{\lambda\in\mathbb{D}} \|A_\varphi^\Theta \omega h_\lambda^\Theta\|_2 = \sup_{\lambda\in\mathbb{D}} \|A_\varphi^\Theta \tilde{h}_\lambda^\Theta\|_2 = \rho_d(A_\varphi^\Theta). \qquad \square$$

As we have seen, if φ is bounded, then obviously the truncated Toeplitz operator A_φ^Θ is bounded. We will see now that one can get a slightly more general result. It involves the so-called *Carleson curves* associated with an inner function (see for instance [19]). Recall that if Θ is an inner function and $\alpha \in (0,1)$, then the system of Carleson curves Γ_α associated to Θ and α is the countable union of closed simple and rectifiable curves in $\operatorname{clos} \mathbb{D}$ such that:

- The interior of curves in Γ_α are pairwise disjoint.
- There is a constant $\eta(\alpha) > 0$ such that for every $z \in \Gamma_\alpha \cap \mathbb{D}$ we have

$$(4.2) \qquad\qquad \eta(\alpha) \leq |\Theta(z)| \leq \alpha.$$

- Arclength $|dz|$ on Γ_α is a Carleson measure, which means that there is a constant $C > 0$ such that

$$\int_{\Gamma_\alpha} |f(z)|^2 \, |dz| \leq C\|f\|_2^2,$$

   for every function $f \in H^2$.
- For every function $\varphi \in H^1$, we have

$$(4.3) \qquad\qquad \int_{\mathbb{T}} \frac{\varphi(z)}{\Theta(z)} dz = \int_{\Gamma_\alpha} \frac{\varphi(z)}{\Theta(z)} dz.$$

Proposition 4.3. *Let $\varphi \in H^2$ and assume that $|\varphi||dz|$ is a Carleson measure on Γ_α. Then A_φ^Θ is a bounded truncated Toeplitz operator on K_Θ and it has a bounded symbol.*

Proof. Let $f, g \in K_\Theta$ and assume further that $f \in H^\infty$. Then we have

$$\langle A_\varphi^\Theta f, g \rangle = \langle \varphi f, g \rangle = \int_{\mathbb{T}} \varphi(z) f(z) \overline{g(z)} dz.$$

Since $g \in K_\Theta$, we can write (on $\mathbb{T}$), $g(z) = \bar{z} \overline{h(z)} \Theta(z)$, with $h \in K_\Theta$. Therefore

$$\langle A_\varphi^\Theta f, g \rangle = \int_{\mathbb{T}} \frac{z \varphi(z) f(z) h(z)}{\Theta(z)} dz.$$

But $z f(z) \varphi(z) h(z) \in H^1$ and using (4.3), we can write

$$\langle A_\varphi^\Theta f, g \rangle = \int_{\Gamma_\alpha} \frac{z \varphi(z) f(z) h(z)}{\Theta(z)} dz.$$

Therefore, according to (4.2), we have

$$|\langle A_\varphi^\Theta f, g \rangle| \leq \int_{\Gamma_\alpha} \frac{|z \varphi(z) f(z) h(z)|}{|\Theta(z)|} |dz| \leq \frac{1}{\eta(\alpha)} \int_{\Gamma_\alpha} |f(z)||h(z)||\varphi(z)||dz|.$$

Hence, by the Cauchy-Schwarz inequality and using the fact that $|\varphi||dz|$ is a Carleson measure on Γ_α, we have

$$|\langle A_\varphi^\Theta f, g\rangle| \le C\frac{1}{\eta(\alpha)}\|f\|_2\|g\|_2.$$

Finally, we get that A_φ^Θ is bounded. Since φ is analytic it follows from Proposition 4.1 that A_φ^Θ has a bounded symbol. $\qquad\square$

Corollary 4.4. *Let $\varphi = \varphi_1 + \overline{\varphi_2}$, with $\varphi_i \in H^2$, $i = 1, 2$. Assume that $|\varphi_i||dz|$ are Carleson measures on Γ_α for $i = 1, 2$. Then A_φ^Θ is bounded and has a bounded symbol.*

Proof. Using Proposition 4.3, we get immediately that $A_{\varphi_i}^\Theta$ is bounded and has a bounded symbol $\widetilde{\varphi_i}$, for $i = 1, 2$. Therefore, $A_{\overline{\varphi_2}}^\Theta = (A_{\varphi_2}^\Theta)^*$ is also bounded and has a bounded symbol $\overline{\widetilde{\varphi_2}}$. Hence we get that $A_\varphi^\Theta = A_{\varphi_1}^\Theta + A_{\overline{\varphi_2}}^\Theta$ is bounded and it has a bounded symbol, say $\widetilde{\varphi_1} + \overline{\widetilde{\varphi_2}}$. $\qquad\square$

Remark 4.5. By the construction of the Carleson curves Γ_α associated to an inner function Θ, we know that $|dz|$ is a Carleson measure on Γ_α. Therefore, Proposition 4.3 can be applied if φ is bounded on Γ_α and Corollary 4.4 can be applied if φ_1, φ_2 are bounded on Γ_α.

5. COUNTEREXAMPLES

We will show that under certain conditions on the inner function Θ there exist rank one bounded truncated Toeplitz operators that have no bounded symbol. It is proven in [29, Theorem 5.1] that any rank one truncated Toeplitz operator is either of the form $k_\lambda^\Theta \otimes \tilde{k}_\lambda^\Theta$ or $\tilde{k}_\lambda^\Theta \otimes k_\lambda^\Theta$ for $\lambda \in \mathbb{D}$, or of the form $k_\zeta^\Theta \otimes k_\zeta^\Theta$ where $\zeta \in \mathbb{T}$ and Θ has an angular derivative at ζ. In what follows we will use a representation of the symbol of a rank one operator which differs slightly from the one given in [29].

Lemma 5.1. *If $\lambda \in \mathbb{D} \cup E(\Theta)$, then $\varphi_\lambda = \Theta\bar{z}\overline{k_\lambda^{\Theta^2}} \in K_\Theta \oplus \bar{z}\overline{K_\Theta}$ is a symbol for $\tilde{k}_\lambda^\Theta \otimes k_\lambda^\Theta$. In particular, if $\zeta \in E(\Theta)$, then $\varphi_\zeta = \Theta\bar{z}\overline{k_\zeta^{\Theta^2}}$ is a symbol for $\Theta(\zeta)\bar{\zeta}\, k_\zeta^\Theta \otimes k_\zeta^\Theta$.*

Proof. If $\zeta \in E(\Theta)$, then by Lemma 2.4, Θ^2 has an angular derivative at ζ, and so $k_\zeta^{\Theta^2} \in K_{\Theta^2} = K_\Theta \oplus \Theta K_\Theta$. It follows from Lemma 2.1 that $\Theta\bar{z}\overline{k_\lambda^{\Theta^2}} \in K_\Theta \oplus \bar{z}\overline{K_\Theta}$ for $\lambda \in \mathbb{D} \cup E(\Theta)$.

Take $g, h \in K_\Theta$, and, moreover, let $g \in L^\infty$. Then

$$\langle A_{\varphi_\lambda}^\Theta g, h\rangle = \langle \varphi_\lambda g, h\rangle = \int_\mathbb{T} \Theta\bar{z}\overline{k_\lambda^{\Theta^2}}g\bar{h}\, dm.$$

But $\Theta\bar{z}\bar{h} = \omega(h) \in K_\Theta$, $g \in K_\Theta \cap L^\infty$, and so by Lemma 2.2 $g\Theta\bar{z}\bar{h} \in K_{\Theta^2}$. Therefore

$$\int_{\mathbb{T}} \Theta\bar{z}\overline{k_\lambda^{\Theta^2}} \, g\bar{h} \, dm = \langle g\omega(h), k_\lambda^{\Theta^2}\rangle = g(\lambda)(\omega(h))(\lambda) = \langle g, k_\lambda^\Theta\rangle\langle\omega(h), k_\lambda^\Theta\rangle$$

$$= \langle g, k_\lambda^\Theta\rangle\overline{\langle h, \omega(k_\lambda^\Theta)\rangle} = \langle g, k_\lambda^\Theta\rangle\overline{\langle h, \tilde{k}_\lambda^\Theta\rangle} = \langle(\tilde{k}_\lambda^\Theta \otimes k_\lambda^\Theta)g, h\rangle.$$

Therefore $A_{\varphi_\lambda}^\Theta = \tilde{k}_\lambda^\Theta \otimes k_\lambda^\Theta$ as claimed. Finally, recall that, for $\zeta \in E(\Theta)$, we have $\tilde{k}_\zeta^\Theta = \omega(k_\zeta^\Theta) = \Theta(\zeta)\bar{\zeta} \, k_\zeta^\Theta$. $\qquad\square$

The construction of bounded truncated Toeplitz operators that have no bounded symbol is based on the next lemma.

Lemma 5.2. *Let Θ be an inner function and $1 < p < \infty$. There exists a constant C depending only on Θ and p such that, if $\varphi, \psi \in L^2$ are two symbols for the same truncated Toeplitz operator, with $\varphi \in K_\Theta \oplus \bar{z}\overline{K_\Theta}$, then*

$$\|\varphi\|_p \le C(\|\psi\|_p + \|\varphi\|_2).$$

In particular, if $\psi \in L^p$, then $\varphi \in L^p$.

Proof. By hypothesis $P_{\mathfrak{S}_\Theta}\varphi = P_{\mathfrak{S}_\Theta}\psi$; therefore, using (3.3),

$$\varphi = Q_\Theta\varphi = P_{\mathfrak{S}_\Theta}\varphi + \langle\varphi, q_\Theta\rangle q_\Theta = P_{\mathfrak{S}_\Theta}\psi + \langle\varphi, q_\Theta\rangle q_\Theta.$$

By Lemma 3.1 we have $\|P_{\mathfrak{S}_\Theta}\psi\|_p \le C_1\|\psi\|_p$, while

$$\|\langle\varphi, q_\Theta\rangle q_\Theta\|_p \le \|\varphi\|_2 \cdot \|q_\Theta\|_p,$$

whence the lemma follows. $\qquad\square$

If Θ is an inner function and $\zeta \in E(\Theta)$, then, as noted above, $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is a rank one operator in $\mathcal{T}(K_\Theta)$. In [29] Sarason has asked specifically whether this operator has a bounded symbol. We can now show that in general this question has a negative answer.

Theorem 5.3. *Suppose that Θ is an inner function which has an angular derivative at $\zeta \in \mathbb{T}$. Let $p \in (2, +\infty)$. Then the following are equivalent:*

(1) *the bounded truncated Toeplitz operator $k_\zeta^\Theta \otimes k_\zeta^\Theta$ has a symbol $\psi \in L^p$;*
(2) *$k_\zeta^\Theta \in L^p$.*

In particular, if $k_\zeta^\Theta \notin L^p$ for some $p \in (2, \infty)$, then $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is a bounded truncated Toeplitz operator with no bounded symbol.

Proof. A symbol for the operator $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is, by Lemma 5.1, $\varphi = \overline{\Theta(\zeta)}\zeta\Theta\bar{z}\overline{k_\zeta^{\Theta^2}}$. Since by Lemma 2.4 $\varphi \in L^p$ if and only if $k_\zeta^\Theta \in L^p$, we obtain that (2) implies (1). Conversely, assume that $\psi \in L^p$ is a symbol for $k_\zeta^\Theta \otimes k_\zeta^\Theta$. We may then apply Lemma 5.2 and obtain that $\varphi \in L^p$. Once again according to Lemma 2.4, we get that $k_\zeta^\Theta \in L^p$, which proves that (1) implies (2). $\qquad\square$

To obtain a bounded truncated Toeplitz operator with no bounded symbol, it is sufficient to have a point $\zeta \in \mathbb{T}$ such that (2.5) is true for $p = 2$ but not for some strictly larger value of p. It is now easy to give concrete examples, as, for instance:

(1) a Blaschke product with zeros a_k accumulating to the point 1, and such that

$$\sum_k \frac{1 - |a_k|^2}{|1 - a_k|^2} < +\infty, \quad \sum_k \frac{1 - |a_k|^2}{|1 - a_k|^p} = +\infty \quad \text{for some } p > 2;$$

(2) a singular function $\sigma = \sum_k c_k \delta_{\zeta_k}$ with $\sum_k c_k < +\infty$, $\zeta_k \to 1$, and

$$\sum_k \frac{c_k}{|1 - \zeta_k|^2} < +\infty, \quad \sum_k \frac{c_k}{|1 - \zeta_k|^p} = +\infty \quad \text{for some } p > 2.$$

Remark 5.4. A related question raised in [29] remains open. Let μ be a positive measure on $\mathbb{T}$ such that the support of the singular part of μ (with respect to the Lebesgue measure) is contained in $\mathbb{T} \setminus \sigma(\Theta)$, where $\sigma(\Theta)$ is the spectrum of the inner function Θ. Then we say that μ is a *Carleson measure* for K_Θ if there is a constant $c > 0$ such that

$$(5.1) \qquad \int_{\mathbb{T}} |f|^2 \, d\mu \le c\|f\|_2^2, \qquad f \in K_\Theta.$$

It is easy to see (and had already been noticed in [12]) that (5.1) is equivalent to the boundedness of the operator A_μ^Θ defined by the formula

$$(5.2) \qquad \langle A_\mu^\Theta f, g \rangle = \int_{\mathbb{T}} f\bar{g} \, d\mu, \qquad f, g \in K_\Theta;$$

it is shown in [29] that A_μ^Θ is a truncated Toeplitz operator. More generally, a complex measure ν on $\mathbb{T}$ is called a Carleson measure for K_Θ if its total variation $|\nu|$ is a Carleson measure for K_Θ. In this case there is a corresponding operator A_ν^Θ, defined also by formula (5.2), which belongs to $\mathcal{T}(K_\Theta)$. Now if a truncated Toeplitz operator A_φ^Θ has a bounded symbol $\psi \in L^\infty$ then the measure $d\mu = \psi \, dm$ is a Carleson measure for K_Θ and $A_\varphi^\Theta = A_\mu^\Theta$. The natural question whether every operator in $\mathcal{T}(K_\Theta)$ is of the form A_μ^Θ (for some Carleson measure μ for K_Θ) is not answered by our counterexample; indeed (as already noticed

in [29]) if Θ has an angular derivative in the sense of Carathéodory at $\zeta \in \mathbb{T}$, then δ_ζ is a Carleson measure for K_Θ and $k_\zeta^\Theta \otimes k_\zeta^\Theta = A_{\delta_\zeta}^\Theta$.

Remark 5.5. We arrive at the same class of counterexamples as in Theorem 5.3 if we follow an idea due to Sarason [29] (we would like to emphasize that our first counterexample was obtained in this way). It is shown in [29, Section 5] that, for an inner function Θ which has an angular derivative at the point $\zeta \in \mathbb{T}$, the rank one operator $k_\zeta^\Theta \otimes k_\zeta^\Theta$ has a bounded symbol if and only if there exists a function $h \in H^2$ such that

$$(5.3) \qquad \mathrm{Re}\left(\frac{\overline{\Theta(\zeta)}\,\Theta}{1 - \bar{\zeta}z} + \Theta h \right) \in L^\infty.$$

Since $\mathrm{Re}(1 - \bar{\zeta}z)^{-1} = 1/2$ a.e. on $\mathbb{T}$, condition (5.3) is, obviously, equivalent to

$$\mathrm{Re}\left(k_\zeta^\Theta + \Theta h \right) \in L^\infty.$$

Then, by the M. Riesz theorem, $k_\zeta^\Theta + \Theta h \in L^p$ for any $p \in (2, \infty)$ and the boundedness of the projection P_Θ in L^p implies that $k_\zeta^\Theta \in L^p$.

The next theorem provides a wider class of examples.

Theorem 5.6. *Suppose that Θ is an inner function with the property that each bounded operator in $\mathcal{T}(K_\Theta)$ has a bounded symbol. Then for each $p > 2$ we have*

$$(5.4) \qquad \sup_{\lambda \in \mathbb{D}} \frac{\|k_\lambda^\Theta\|_p}{\|k_\lambda^\Theta\|_2^2} < \infty.$$

Proof. As mentioned in the previous section, it follows from the open mapping theorem that there exists a constant $C > 0$ such that for any operator $A \in \mathcal{T}(K_\Theta)$ one can always find a symbol $\psi \in L^\infty$ with $\|\psi\|_\infty \leq C\|A\|$.

Fix $\lambda \in \mathbb{D}$, and consider the rank one operator $\tilde{k}_\lambda^\Theta \otimes k_\lambda^\Theta$, which has operator norm $\|k_\lambda^\Theta\|_2^2$. Therefore there exists $\psi_\lambda \in L^\infty$ with $A_{\psi_\lambda}^\Theta = \tilde{k}_\lambda^\Theta \otimes k_\lambda^\Theta$ and

$$(5.5) \qquad \|\psi_\lambda\|_p \leq \|\psi_\lambda\|_\infty \leq C\|k_\lambda^\Theta\|_2^2.$$

On the other hand, $\varphi_\lambda = \Theta \bar{z} \overline{k_\lambda^{\Theta^2}} \in K_\Theta \oplus \bar{z}\overline{K_\Theta}$ is also a symbol for $\tilde{k}_\lambda^\Theta \otimes k_\lambda^\Theta$ by Lemma 5.1. Applying Lemma 5.2, it follows that there exists a constant $C_1 > 0$ such that

$$\|\varphi_\lambda\|_p \leq C_1(\|\psi_\lambda\|_p + \|\varphi_\lambda\|_2).$$

By (2.6) and Lemma 2.4 (b), we have

$$(5.6) \qquad \|\varphi_\lambda\|_2 = \|k_\lambda^{\Theta^2}\|_2 \leq 2\|k_\lambda^\Theta\|_2 \leq C_2\|k_\lambda^\Theta\|_2^2.$$

Therefore (5.5) and (5.6) yield

$$\|\varphi_\lambda\|_p \le C_1(C + C_2)\|k_\lambda^\Theta\|_2^2.$$

Since $\|\varphi_\lambda\|_p = \|k_\lambda^{\Theta^2}\|_p$, using once more (2.6) concludes the proof. $\quad\square$

It is easy to see that if there exists $\zeta \in E(\Theta)$ such that $k_\zeta^\Theta \notin L^p$, then

$$\sup_{r<1} \frac{\|k_{r\zeta}^\Theta\|_p}{\|k_{r\zeta}^\Theta\|_2^2} = \infty.$$

Therefore the existence of operators in $\mathcal{T}(K_\Theta)$ without bounded symbol, under the hypothesis of Theorem 5.3, is also a consequence of Theorem 5.6. Note however that Theorem 5.6 does not show that the particular operator $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is a bounded truncated Toeplitz operator without bounded symbol. A larger class of examples is described below.

Example 5.7. Let Θ be a Blaschke product such that for some sequence of its zeros z_n and some points $\zeta_n \in \mathbb{T}$ (which are "close to z_n"), we have, for some $p \in (2, \infty)$,

$$(5.7) \qquad |\Theta'(\zeta_n)| = \|k_{\zeta_n}^\Theta\|_2^2 \asymp \frac{1 - |z_n|}{|\zeta_n - z_n|^2}, \qquad \|k_{\zeta_n}^\Theta\|_p^p \asymp \frac{1 - |z_n|}{|\zeta_n - z_n|^p}$$

(notation $X \asymp Y$ means that the fraction X/Y is bounded above and below by some positive constants), and

$$(5.8) \qquad \lim_{n \to +\infty} \frac{(1 - |z_n|)^{1-\frac{1}{p}}}{|\zeta_n - z_n|} = 0.$$

Condition (5.7) means that the main contribution to the norms of $k_{\zeta_n}^\Theta$ is due to the closest zero z_n. Then, by Theorem 5.6, there exists a bounded truncated Toeplitz operator without bounded symbol.

Such examples may be easily constructed. Take a sequence $w_k \in \mathbb{D}$ such that $w_k \to \zeta$ and

$$\lim_{k \to +\infty} \frac{(1 - |w_k|)^\gamma}{|w_k - \zeta|} = 0$$

for some $\zeta \in \mathbb{T}$ and $\gamma \in (0, 1)$. Then it is not difficult to see that for any $p > \max(2, (1 - \gamma)^{-1})$ one can construct recurrently a subsequence $z_n = w_{k_n}$ of w_k and a sequence $\zeta_n \in \mathbb{T}$ with the properties (5.7) and (5.8).

Although related to the examples of Theorem 5.3, this class of examples may be different. Indeed, it is possible that Θ has no angular derivative at ζ, e.g., if $1 - |z_n| = |\zeta - z_n|^2$. Also,

if the zeros tend to ζ "very tangentially", it is possible that k_ζ^Θ is in L^p for any $p \in (2, \infty)$, but there exists a bounded truncated Toeplitz operator without a bounded symbol.

We pass now to the Reproducing Kernel Thesis. The next example shows that in general Question 2 has a negative answer.

Example 5.8. Suppose Θ is a singular inner function and $s \in [0, 1)$. Then

$$
\begin{aligned}
A_{\bar{\Theta}^s}^\Theta k_\lambda^\Theta &= P_\Theta \left(\frac{\bar{\Theta}^s - \overline{\Theta(\lambda)}\Theta^{1-s}}{1 - \bar{\lambda}z} \right) \\
&= P_\Theta \left(\frac{\bar{\Theta}^s - \overline{\Theta(\lambda)^s} + \overline{\Theta(\lambda)^s}(1 - \overline{\Theta(\lambda)^{1-s}}\Theta^{1-s})}{1 - \bar{\lambda}z} \right) \\
&= P_\Theta \left(\bar{z}\frac{\bar{\Theta}^s - \overline{\Theta^s(\lambda)}}{\bar{z} - \bar{\lambda}} \right) + \overline{\Theta(\lambda)^s}P_\Theta \left(\frac{1 - \overline{\Theta(\lambda)^{1-s}}\Theta^{1-s}}{1 - \bar{\lambda}z} \right) \\
&= P_\Theta \left(\bar{z}\overline{\widetilde{k}_\lambda^{\Theta^s}} \right) + \overline{\Theta(\lambda)^s}P_\Theta \left(k_\lambda^{\Theta^{1-s}} \right).
\end{aligned}
$$

The first term $\bar{z}\overline{\widetilde{k}_\lambda^{\Theta^s}}$ is in $\bar{z}\overline{H^2}$, which is orthogonal to K_Θ, while the second $k_\lambda^{\Theta^{1-s}}$ is contained in $K_{\Theta^{1-s}} \subset K_\Theta$. Therefore we have

$$
A_{\bar{\Theta}^s}^\Theta k_\lambda^\Theta = \overline{\Theta(\lambda)^s}k_\lambda^{\Theta^{1-s}},
$$

and

$$
\|A_{\bar{\Theta}^s}^\Theta k_\lambda^\Theta\|_2^2 = |\Theta(\lambda)|^{2s}\frac{1 - |\Theta(\lambda)|^{2-2s}}{1 - |\lambda|^2}, \qquad \|A_{\bar{\Theta}^s}^\Theta h_\lambda^\Theta\|_2^2 = \frac{|\Theta(\lambda)|^{2s}(1 - |\Theta(\lambda)|^{2-2s})}{1 - |\Theta(\lambda)|^2}.
$$

It is easy to see that $\sup_{y \in [0,1)} \frac{y^s - y}{1 - y} \leq 1 - s \to 0$ when $s \to 1$, and therefore

$$
\rho_r(A_{\bar{\Theta}^s}^\Theta) = \sup_{\lambda \in \mathbb{D}} \|A_{\bar{\Theta}^s}^\Theta h_\lambda^\Theta\|_2^2 \to 0 \quad \text{for} \ \ s \to 1.
$$

On the other hand, $\Theta^s K_{\Theta^{1-s}} \subset K_\Theta$ and $\bar{\Theta}^s(\Theta^s K_{\Theta^{1-s}}) = K_{\Theta^{1-s}} \subset K_\Theta$; therefore $A_{\bar{\Theta}^s}^\Theta$ acts isometrically on $\Theta^s K_{\Theta^{1-s}}$, so it has norm 1. Thus there is no constant M such that

$$
\|A_\varphi^\Theta\| \leq M \sup_{\lambda \in \mathbb{D}} \rho_r(A_\varphi^\Theta)
$$

for all φ.

It seems natural to deduce that in the previous example we may actually have a truncated Toeplitz operator which is uniformly bounded on reproducing kernels but not bounded. This is indeed true, by an abstract argument based on Proposition 3.3. Note that the

quantity ρ_r introduced in (3.5) is a norm, and $\rho_r(T) \leq \|T\|$, for every linear operator T whose domain contains $H^\infty \cap K_\Theta$.

Proposition 5.9. *Assume that for any (not necessarily bounded) truncated Toeplitz operator A on K_Θ the inequality $\rho_r(A) < \infty$ implies that A is bounded. Then $\mathcal{T}(K_\Theta)$ is complete with respect to ρ_r, and ρ_r is equivalent to the operator norm on $\mathcal{T}(K_\Theta)$.*

Proof. Fix $\mu \in \mathbb{D}$ such that $\Theta(\mu) \neq 0$. Let $A_{\varphi_n}^\Theta$ be a ρ_r-Cauchy sequence in $\mathcal{T}(K_\Theta)$. Suppose all φ_n are written as $\varphi_n = \varphi_{n,+} + \overline{\varphi_{n,-}}$, with $\varphi_{n,+}, \varphi_{n,-} \in K_\Theta$, and $\varphi_{n,-}(\mu) = 0$. According to (3.9), the sequences $\varphi_{n,\pm}$ are Cauchy sequences in K_Θ and thus converge to functions $\varphi_\pm \in K_\Theta$; moreover we also have $\varphi_-(\mu) = 0$ (because norm convergence in H^2 implies pointwise convergence). Define then $\varphi = \varphi_+ + \overline{\varphi_-} \in L^2$. By (3.10), we have

$$A_{\varphi_n}^\Theta k_\lambda^\Theta = \omega \left[(I - \lambda S^*)^{-1} \left(\omega(\varphi_{n,+}) + \varphi_{n,-}(\lambda) S^* \Theta - \Theta(\lambda) S^* \varphi_{n,-} \right) \right],$$

so the sequence $A_{\varphi_n}^\Theta k_\lambda^\Theta$ tends (in K_Θ) to $A_\varphi^\Theta k_\lambda^\Theta$, for all $\lambda \in \mathbb{D}$. In particular, we have $\rho_r(A_\varphi^\Theta) < +\infty$, whence $A_\varphi^\Theta \in \mathcal{T}(K_\Theta)$. Now it is easy to see that $A_{\varphi_n}^\Theta \to A_\varphi^\Theta$ in the ρ_r-norm.

Thus $\mathcal{T}(K_\Theta)$ is indeed complete with respect to the ρ_r-norm. The equivalence of the norms is then a consequence of the open mapping theorem. $\square$

Proposition 5.9 and Example 5.8 imply that, if Θ is a singular inner function, then there exist truncated Toeplitz operators A_φ^Θ with $\rho_r(A_\varphi^\Theta)$ finite, but A_φ^Θ unbounded. Therefore Question 2 has a negative answer for a rather large class of inner functions Θ. If we consider such a truncated Toeplitz operator, then its adjoint, $A_{\overline{\varphi}}^\Theta$, is an unbounded truncated Toeplitz operator with $\rho_d(A_{\overline{\varphi}}^\Theta) = \rho_r(A_\varphi^\Theta) < +\infty$.

It is easy to see, however, that in Example 5.8 $\rho_d(A_{\overline{\Theta}^s}^\Theta) = 1$ for all $s < 1$. This suggests that we should rather consider boundedness of the action of the operator on both the reproducing kernels and the difference quotients, and that the quantity ρ might be a better estimate for the norm of a truncated Toeplitz operator than either ρ_r or ρ_d. We have been thus lead to formulate Question 3 as a more relevant variant of the RKT; further arguments will appear in the next section.

6. Positive results

There are essentially two cases in which one can give positive answers to Questions 1 and 3. There are similarities between them: in both one obtains a convenient decomposition

of the symbol in three parts: one analytic, one coanalytic, and one that is neither analytic nor coanalytic, but well controlled.

6.1. **A general result.** As we have seen in Proposition 4.1 and 4.2, the answers to Questions 1 and 3 are positive for classes of truncated Toeplitz operators corresponding to analytic and coanalytic symbols. We complete these propositions with a different boundedness result, which covers certain cases when the symbol is neither analytic nor coanalytic. The proof is based on an idea of Cohn [14].

Theorem 6.1. *Suppose θ and Θ are two inner functions such that θ^3 divides $z\Theta$ and Θ divides θ^4. If $\varphi \in K_\theta + \overline{K_\theta}$ then $\|\varphi\|_\infty \leq 2\rho_r(A_\varphi^\Theta)$.*

Proof. Using Lemma 2.3, if $f \in L^\infty \cap \theta K_\theta$, then $f \in K_\Theta$ and $\bar{\varphi}f \in K_\Theta$; thus $A_{\bar\varphi}^\Theta f = \bar{\varphi}f$. If we write $f = \theta f_1$, $\varphi_1 = \theta\bar{\varphi}$, then $\varphi_1 \in H^2$, $f_1 \in K_\theta$, and $\varphi_1 f_1 = \bar{\varphi}f = A_{\bar\varphi}^\Theta f \in K_\Theta$. Therefore, for $\lambda \in \mathbb{D}$,

$$|\varphi_1(\lambda)f_1(\lambda)| = |\langle \varphi_1 f_1, k_\lambda^\Theta \rangle| = |\langle \theta f_1, \varphi k_\lambda^\Theta \rangle| = |\langle \theta f_1, A_\varphi^\Theta k_\lambda^\Theta \rangle|$$
$$\leq \|f_1\| \|A_\varphi^\Theta k_\lambda^\Theta\|_2 \leq \|f_1\| \|k_\lambda^\Theta\|_2 \rho_r(A_\varphi^\Theta),$$

where we used the fact that $\theta f_1 \in K_\Theta$.

For a fixed $\lambda \in \mathbb{D}$,

$$\sup_{\substack{f_1 \in K_\theta \cap L^\infty \\ \|f_1\|_2 \leq 1}} |f_1(\lambda)| = \sup_{\substack{f_1 \in K_\theta \cap L^\infty \\ \|f_1\|_2 \leq 1}} |\langle f_1, k_\lambda^\theta \rangle| = \|k_\lambda^\theta\|_2,$$

and thus

$$|\varphi_1(\lambda)| \leq \rho_r(A_\varphi^\Theta) \frac{\|k_\lambda^\Theta\|_2}{\|k_\lambda^\theta\|_2} = \rho_r(A_\varphi^\Theta) \frac{(1 - |\Theta(\lambda)|^2)^{1/2}}{(1 - |\theta(\lambda)|^2)^{1/2}}.$$

If Θ divides θ^4, then $|\Theta(\lambda)| \geq |\theta(\lambda)|^4$, and therefore

$$1 - |\Theta(\lambda)|^2 \leq 1 - |\theta(\lambda)|^8 \leq 4(1 - |\theta(\lambda)|^2).$$

It follows that $|\varphi_1(\lambda)| \leq 2\rho_r(A_\varphi^\Theta)$ for all $\lambda \in \mathbb{D}$, and thus $\|\varphi_1\|_\infty \leq 2\rho_r(A_\varphi^\Theta)$. The proof is finished by noting that $\|\varphi\|_\infty = \|\varphi_1\|_\infty$. $\qquad\square$

As a consequence, we obtain a general result for the existence of bounded symbols and Reproducing Kernel Thesis.

Corollary 6.2. *Let Θ be an inner function and assume that there is another inner function θ such that θ^3 divides $z\Theta$ and Θ divides θ^4. Suppose also there are constants $C_i > 0$, $i = 1, 2, 3$ such that any $\varphi \in L^2$ can be written as $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, with:*

(a) $\varphi_1 \in K_\theta + \overline{K_\theta}$, $\varphi_2 \in H^2$, and $\varphi_3 \in \overline{H^2}$;

(b) $\rho(A_{\varphi_i}^\Theta) \leq C_i \rho(A_\varphi^\Theta)$ for $i = 1, 2, 3$.

Then the following are equivalent:

(i) A_φ^Θ has a bounded symbol;

(ii) A_φ^Θ is bounded;

(iii) $\rho(A_\varphi^\Theta) < +\infty$.

More precisely, there exists a constant $C > 0$ such that any truncated Toeplitz operator A_φ^Θ has a symbol φ_0 with $\|\varphi_0\|_\infty \leq C\rho(A_\varphi^\Theta)$.

There are of course many decompositions of φ as in (a); the difficulty consists in finding one that satisfies (b).

Proof. It is immediate that (i) $\implies$ (ii) $\implies$ (iii), so it remains to prove (iii) $\implies$ (i). Since $\rho(A_{\varphi_i}^\Theta) < +\infty$, $i = 2, 3$, Proposition 4.1 and 4.2 imply that $A_{\varphi_i}^\Theta$ have bounded symbols $\tilde{\varphi}_i$ with $\|\tilde{\varphi}_i\|_\infty \leq \tilde{C}\rho(A_{\varphi_i}^\Theta) \leq \tilde{C}C_i\rho(A_\varphi^\Theta)$. As for φ_1, we can apply Theorem 6.1 which gives that φ_1 is bounded with $\|\varphi_1\|_\infty \leq 2\rho_r(A_{\varphi_1}^\Theta) \leq 2C_1\rho(A_\varphi^\Theta)$. Finally A_φ^Θ has the bounded symbol $\varphi_0 = \varphi_1 + \tilde{\varphi}_2 + \tilde{\varphi}_3$ whose norm is at most $(2C_1 + \tilde{C}(C_2 + C_3))\rho(A_\varphi^\Theta)$. □

6.2. Classical Toeplitz matrices.

Suppose $\Theta(z) = z^N$; the space K_Θ is then an N-dimensional space with orthonormal basis formed by monomials, and truncated Toeplitz operators have a (usual) Toeplitz matrix with respect of this basis. Of course every truncated Toeplitz operator has a bounded symbol; it is however interesting that there exists a universal estimate of this bound. The question had been raised in [29, Section 7]; the positive answer had actually been already independently obtained in [6] and [24]. The following result is stronger, giving a universal estimate for the symbols in terms of the action on the reproducing kernels.

Theorem 6.3. *Suppose $\Theta(z) = z^N$. There exists a constant $C > 0$, independent of N, such that any truncated Toeplitz operator A_φ^Θ has a symbol $\varphi_0 \in L^\infty$ such that $\|\varphi_0\|_\infty \leq C\rho(A_\varphi^\Theta)$.*

Proof. Consider a smooth function η_k on $\mathbb{T}$, and the convolution (on $\mathbb{T}$) $\varphi_k = \eta_k * \varphi$, that is,

$$\varphi_k(e^{is}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \eta_k(e^{it}) \varphi(e^{i(s-t)}) \, dt.$$

We have then $\hat{\varphi}_k(n) = \hat{\eta}_k(n)\hat{\varphi}(n)$, $n \in \mathbb{Z}$.

The map τ_t defined by $\tau_t : f(z) \mapsto f(e^{it}z)$ is a unitary on K_Θ and straightforward computations show that

(6.1) $$\tau_t h_\lambda^\Theta = h_{e^{-it}\lambda}^\Theta \quad \text{and} \quad \tau_t \tilde{h}_\lambda^\Theta = e^{i(N-1)t} \tilde{h}_{e^{-it}\lambda}^\Theta,$$

for every $\lambda \in \mathbb{D}$. By Fubini's Theorem and a change of variables we have

$$\langle A_{\varphi_k}^\Theta f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \eta_k(e^{it}) \langle A_\varphi^\Theta \tau_t(f), \tau_t(g) \rangle \, dt,$$

for every $f, g \in K_\Theta$. That implies that

$$\|A_{\varphi_k}^\Theta h_\lambda^\Theta\| = \sup_{\substack{g \in K_\Theta \\ \|g\|_2 \leq 1}} \left| \langle A_{\varphi_k}^\Theta h_\lambda^\Theta, g \rangle \right| \leq \sup_{\substack{g \in K_\Theta \\ \|g\|_2 \leq 1}} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\eta_k(e^{it})| |\langle A_\varphi^\Theta \tau_t(h_\lambda^\Theta), \tau_t(g) \rangle| \, dt,$$

and using (6.1), we obtain

$$\|A_{\varphi_k}^\Theta h_\lambda^\Theta\| \leq \|\eta_k\|_1 \rho_r(A_\varphi^\Theta) \leq \|\eta_k\|_1 \rho(A_\varphi^\Theta).$$

A similar argument shows that

$$\|A_{\varphi_k}^\Theta \tilde{h}_\lambda^\Theta\| \leq \|\eta_k\|_1 \rho(A_\varphi^\Theta)$$

and thus

(6.2) $$\rho(A_{\varphi_k}^\Theta) \leq \|\eta_k\|_1 \rho(A_\varphi^\Theta).$$

Now consider the Fejér kernel F_m, defined by the formula $\hat{F}_m(n) = 1 - \frac{|n|}{m}$ for $|n| \leq m$ and $\hat{F}_m(n) = 0$ otherwise. It is well known that $\|F_m\|_1 = 1$ for all $m \in \mathbb{N}$. If we take $M = \left[\frac{N+1}{3}\right]$ and define η_i ($i = 1, 2, 3$) by

$$\eta_1 = F_M, \qquad \eta_2 = 2e^{2iMt} F_{2M} - e^{2iMt} F_M, \qquad \eta_3 = \bar{\eta}_2,$$

then $\hat{\eta}_2(n) = 0$ for $n < 0$, $\hat{\eta}_3(n) = 0$ for $n > 0$, $\hat{\eta}_1(n) + \hat{\eta}_2(n) + \hat{\eta}_3(n) = 1$ for $|n| \leq N$, and $\|\eta_1\|_1 = 1$, $\|\eta_i\|_1 \leq 3$ for $i = 2, 3$. If we denote $\varphi_i = \eta_i * \varphi$, then $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, $\varphi_1 \in K_{z^M} + \overline{K_{z^M}}$, φ_2 is analytic and φ_3 is coanalytic. Moreover z^{3M} divides z^{N+1} and z^N divides z^{4M}. According to (6.2), we can apply Corollary 6.2 to obtain that there exists a universal constant $C > 0$ such that A_φ^Θ has a bounded symbol φ_0 with $\|\varphi_0\|_\infty \leq C\rho(A_\varphi^\Theta)$.

$\square$

In particular, it follows from Theorem 6.3 that any (classical) Toeplitz matrix $A_\varphi^{z^N}$ has a symbol φ_0 such that $\|\varphi_0\|_\infty \leq C\|A_\varphi^{z^N}\|$. The similar statement is proved with explicit estimates $\|\varphi_0\|_\infty \leq 4\|A_\varphi^{z^N}\|$ in [6] and $\|\varphi_0\|_\infty \leq 3\|A_\varphi^{z^N}\|$ in [24].

We can obtain a slightly more general result (in the choice of the function Θ).

Corollary 6.4. *Suppose $\Theta = b_\alpha^N$, with $b_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ a Blaschke factor. There exists a universal constant $C > 0$ such that any truncated Toeplitz operator A_φ^Θ has a symbol $\varphi_0 \in L^\infty$ such that $\|\varphi_0\|_\infty \leq C\rho(A_\varphi^\Theta)$.*

Proof. The mapping U defined by

$$(U(f))(z) := \frac{(1 - |\alpha|^2)^{1/2}}{1 - \bar{\alpha}z} f(b_\alpha(z)), \qquad z \in \mathbb{D}, \, f \in H^2,$$

is unitary on H^2 and one easily checks that $UP_{z^N} = P_\Theta U$. In particular, it implies that $U(K_{z^N}) = K_\Theta$; straightforward computations show that

$$(6.3) \qquad Uh_\lambda^{z^N} = c_\lambda h_{b_\alpha(\lambda)}^\Theta \quad \text{and} \quad U\tilde{h}_\lambda^{z^N} = -\bar{c}_\lambda \tilde{h}_{b_\alpha(\lambda)}^\Theta,$$

for every $\lambda \in \mathbb{D}$, where $c_\lambda := |1 - \bar{\lambda}\alpha|(1 - \bar{\lambda}\alpha)^{-1}$ is a constant of modulus one.

Suppose A_φ^Θ is a (bounded) truncated Toeplitz operator; if $\Phi = \varphi \circ b_\alpha$, then the relation $UP_{z^N} = P_\Theta U$ yields $A_\Phi^{z^N} = U^* A_\varphi^\Theta U$. Thus, using (6.3), we obtain

$$\|A_\Phi^{z^N} h_\lambda^{z^N}\|_2 = \|U^* A_\varphi^\Theta U h_\lambda^{z^N}\|_2 = \|A_\varphi^\Theta h_{b_\alpha(\lambda)}^\Theta\|_2$$

and

$$\|A_\Phi^{z^N} \tilde{h}_\lambda^{z^N}\|_2 = \|U^* A_\varphi^\Theta U \tilde{h}_\lambda^{z^N}\|_2 = \|A_\varphi^\Theta \tilde{h}_{b_\alpha(\lambda)}^\Theta\|_2,$$

which implies that

$$(6.4) \qquad \rho(A_\Phi^{z^N}) = \rho(A_\varphi^\Theta).$$

Now it remains to apply Theorem 6.3 to complete the proof. $\square$

6.3. **Elementary singular inner functions.** Let us now take $\Theta(z) = \exp(\frac{z+1}{z-1})$. A positive answer to Questions 1 and 3 is a consequence of results obtained by Rochberg [28] and Smith [31] on the Paley–Wiener space. We sketch the proof for completeness, without entering into details.

Theorem 6.5. *If* $\Theta(z) = \exp(\frac{z+1}{z-1})$ *and* A_φ^Θ *is a truncated Toeplitz operator, then the following are equivalent:*

(i) A_φ^Θ *has a bounded symbol;*

(ii) A_φ^Θ *is bounded;*

(iii) $\rho(A_\varphi^\Theta) < \infty.$

More precisely, there exists a constant $C > 0$ *such that any truncated Toeplitz operator* A_φ^Θ *has a symbol* φ_0 *with* $\|\varphi_0\|_\infty \leq C\rho(A_\varphi^\Theta).$

Proof. By Remark 3.5 it is enough to prove the corresponding result for the space $\boldsymbol{K_\Theta}$, where $\boldsymbol{\Theta}(w) = e^{iw}$, and $\boldsymbol{\rho}$ is the analogue of ρ for operators on $\boldsymbol{K_\Theta}$. If $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}$, then $\boldsymbol{K_\Theta} = \mathcal{F}^{-1}(L^2([0,1]))$, and we may suppose that the symbol $\boldsymbol{\varphi} \in (t+i)\mathcal{F}^{-1}(L^2([-1,1])).$

For a rapidly decreasing function η on $\mathbb{R}$, define

(6.5) $$\Psi(s) = \int_{\mathbb{R}} \eta(t)\boldsymbol{\varphi}(s-t)\,dt.$$

We have then $\hat{\Psi} = \hat{\eta}\hat{\boldsymbol{\varphi}}$ and $\boldsymbol{\rho}(\boldsymbol{A_\psi^\Theta}) \leq \|\eta\|_1 \cdot \boldsymbol{\rho}(\boldsymbol{A_\varphi^\Theta}).$

Take now ψ_i, $i = 1,2,3$, such that $\operatorname{supp}\hat{\psi}_1 \subset [-1/3, 1/3]$, $\operatorname{supp}\hat{\psi}_2 \subset [0,2]$, $\operatorname{supp}\hat{\psi}_3 \subset [-2,0]$, and $\hat{\psi}_1 + \hat{\psi}_2 + \hat{\psi}_3 = 1$ on $[-1,1]$. If we define $\boldsymbol{\varphi}_i$ by replacing η with ψ_i in (6.5), then there is a constant $C_1 > 0$ such that $\boldsymbol{\rho}(\boldsymbol{A_{\varphi_i}^\Theta}) \leq C_1 \boldsymbol{\rho}(\boldsymbol{A_\varphi^\Theta})$ for $i = 1,2,3$.

On the other hand, $\boldsymbol{\varphi} = \boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2 + \boldsymbol{\varphi}_3$, $\boldsymbol{\varphi}_1 \in \boldsymbol{K_{\Theta^{1/3}}} + \overline{\boldsymbol{K_{\Theta^{1/3}}}}$, $\boldsymbol{\varphi}_2$ is analytic, $\boldsymbol{\varphi}_3$ is antianalytic. We may then apply the analogue of Corollary 6.2 for the upper half-plane which completes the proof.

$\square$

One can see easily that a similar result is valid for any elementary singular function $\Theta(z) = \exp\left(a\frac{z+\zeta}{z-\zeta}\right)$, for $\zeta \in \mathbb{T}$, $a > 0$.

Remark 6.6. Truncated Toeplitz operators on the model space $\boldsymbol{K_\Theta}$ with $\boldsymbol{\Theta}(w) = e^{iaw}$ are closely connected with the so-called truncated Wiener–Hopf operators. Let $\boldsymbol{\varphi} \in L^1(\mathbb{R})$ and let

$$(W_{\boldsymbol{\varphi}}f)(x) = \int_0^a f(t)\boldsymbol{\varphi}(x-t)dt, \quad x \in (0,a),$$

for $f \in L^2(0,a) \cap L^\infty(0,a)$. If W extends to a bounded operator on $L^2(0,a)$, then it is called a *truncated Wiener–Hopf operator*. If $\boldsymbol{\varphi} = \hat{\boldsymbol{\psi}}$ with $\boldsymbol{\psi} \in (t+i)L^2(\mathbb{R})$ (the Fourier

transform may be understood in the distributional sense), then

$$W_{\boldsymbol{\varphi}} f = \mathcal{F} P_{\boldsymbol{\Theta}}(\boldsymbol{\psi} g)$$

for $g = \check{f} \in \boldsymbol{K_{\Theta}}$. Thus, the Wiener–Hopf operator $W_{\boldsymbol{\varphi}}$ is unitarily equivalent to $\boldsymbol{A_{\psi}^{\Theta}}$.

7. Truncated Toeplitz operators with positive symbols

As noted in Remark 5.4, if $\varphi \in L^2$ is a positive function, then A_{φ}^{Θ} is bounded if and only if $\varphi\, dm$ is a Carleson measure for K_{Θ}. As a consequence mainly of results of Cohn [11, 12], one can say more for positive symbols φ for a special class of model spaces. Recall that Θ is said to satisfy the *connected level set condition* (and we write $\Theta \in (CLS)$) if there is $\varepsilon \in (0, 1)$ such that the level set

$$\Omega(\Theta, \varepsilon) := \{z \in \mathbb{D} : |\Theta(z)| < \varepsilon\}$$

is connected. Such inner functions are also referred to as *one-component* inner functions.

Theorem 7.1. *Let Θ be an inner function such that $\Theta \in (CLS)$. If φ is a positive function in L^2, then the following conditions are equivalent:*

(1) A_{φ}^{Θ} *is a bounded operator on K_{Θ}^2;*
(2) $\sup_{\lambda \in \mathbb{D}} \|A_{\varphi}^{\Theta} h_{\lambda}^{\Theta}\|_2 < +\infty$;
(3) $\sup_{\lambda \in \mathbb{D}} |\langle A_{\varphi}^{\Theta} h_{\lambda}^{\Theta}, h_{\lambda}^{\Theta}\rangle| < +\infty$;
(4) A_{φ}^{Θ} *has a bounded symbol.*

Proof. The implications $(4) \Longrightarrow (1) \Longrightarrow (2) \Longrightarrow (3)$ are obvious.

We have

$$(7.1) \qquad \int_{\mathbb{T}} \varphi |h_{\lambda}^{\Theta}|^2 \, dm = \langle \varphi h_{\lambda}^{\Theta}, h_{\lambda}^{\Theta}\rangle = \langle P_{\Theta} \varphi h_{\lambda}^{\Theta}, h_{\lambda}^{\Theta}\rangle = \langle A_{\varphi}^{\Theta} h_{\lambda}^{\Theta}, h_{\lambda}^{\Theta}\rangle.$$

It is shown in [11] that, for $\Theta \in (CLS)$, a positive μ satisfies $\sup_{\lambda \in \mathbb{D}} \|h_{\lambda}^{\Theta}\|_{L^2(\mu)} < \infty$ if and only if it is a Carleson measure for K_{Θ}. Thus (3) implies that $\varphi\, dm$ is a Carleson measure for K_{Θ}, which has been noted above to be equivalent to A_{φ}^{Θ} bounded; so $(1) \Longleftrightarrow (3)$.

On the other hand, it is proved in [12] that if A_{φ}^{Θ} is bounded, then there are functions $v \in L^{\infty}(\mathbb{T})$ and $h \in H^2$ such that $\varphi = \mathrm{Re}(v + \Theta h)$. Write then

$$\varphi = \mathrm{Re}\, v + \frac{1}{2}(\Theta h + \bar{\Theta}\bar{h}),$$

which implies that $\varphi - \mathrm{Re}\, v \in \Theta H^2 + \overline{\Theta H^2}$. Therefore $A_{\varphi}^{\Theta} = A_{\mathrm{Re}\, v}^{\Theta}$ and $\mathrm{Re}\, v \in L^{\infty}(\mathbb{T})$. Thus the last remaining implication $(1) \Longrightarrow (4)$ is proved. $\square$

Remark 7.2. In [11], Cohn asked the following question: let Θ be an inner function and let μ be a positive measure on $\mathbb{T}$ such that the singular part of μ is supported on a subset of $\mathbb{T} \setminus \sigma(\Theta)$; is it sufficient to have

$$\sup_{\lambda \in \mathbb{D}} \int_{\mathbb{T}} |h_\lambda^\Theta|^2 \, d\mu < +\infty,$$

to deduce that μ is a Carleson measure for K_Θ? In [23] Nazarov and Volberg construct a counterexample to this question with a measure μ of the form $d\mu = \varphi \, dm$ where φ is some positive function in L^2. In our context, this means that they provide an inner function Θ and a positive function $\varphi \in L^2$ such that

$$(7.2) \qquad \sup_{\lambda \in \mathbb{D}} |\langle A_\varphi^\Theta h_\lambda^\Theta, h_\lambda^\Theta \rangle| < +\infty,$$

while A_φ^Θ is not bounded. But the condition (7.2) is obviously weaker than $\rho_r(A_\varphi^\Theta) < +\infty$ (note that since φ is positive, the truncated Toeplitz operator is positive and $\rho_r(A_\varphi^\Theta) = \rho(A_\varphi^\Theta)$). Thus an answer to Question 3 does not follow from the Nazarov–Volberg result.

Remark 7.3. It is shown by Aleksandrov [3, Theorem 1.2] that the condition

$$\sup_{\lambda \in \mathbb{D}} \frac{\|k_\lambda^\Theta\|_\infty}{\|k_\lambda^\Theta\|_2^2} < +\infty$$

is equivalent to $\Theta \in (CLS)$. On the other hand, as we have seen in Theorem 5.6, the condition

$$\sup_{\lambda \in \mathbb{D}} \frac{\|k_\lambda^\Theta\|_p}{\|k_\lambda^\Theta\|_2^2} = +\infty$$

for some $p \in (2, \infty)$ implies that there exists a bounded operator in $\mathcal{T}(K_\Theta)$ without a bounded symbol. Therefore, based on Theorem 7.1 and Theorem 5.6, it seems reasonable to state the following conjecture.

Conjecture. *Let Θ be an inner function. Then any bounded truncated Toeplitz operator has a bounded symbol if and only if $\Theta \in (CLS)$.*

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