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Global attractors of complete conformal foliations

N. I. Zhukova

Abstract. We prove that every complete conformal foliation $(M,\mathscr{F})$ of codimension $q\geqslant 3$ is either Riemannian or a $(\mathrm{Conf}(S^q),S^q)$-foliation. We further prove that if $(M,\mathscr{F})$ is not Riemannian, it has a global attractor which is either a nontrivial minimal set or a closed leaf or a union of two closed leaves. In this theorem we do not assume that the manifold M is compact. In particular, every proper conformal non-Riemannian foliation $(M,\mathscr{F})$ has a global attractor which is either a closed leaf or a union of two closed leaves, and the space of all nonclosed leaves is a connected q-dimensional orbifold. We show that every countable group of conformal transformations of the sphere S^q can be realized as the global holonomy group of a complete conformal foliation. Examples of complete conformal foliations with exceptional and exotic minimal sets as global attractors are constructed.

Bibliography: 20 titles.

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§ 1. Introduction. Main results

Let $(M,\mathscr{F})$ be a smooth foliation of codimension $q\geqslant 1$. Recall that a subset of a manifold M is called saturated if it is a union of some leaves of the foliation $(M,\mathscr{F})$.

A nonempty closed saturated subset $\mathscr{M}$ of a foliation $(M,\mathscr{F})$ admitting an open saturated neighbourhood $\mathscr{U}$ such that the closure of any leaf in $\mathscr{U}$ contains the set $\mathscr{M}$ is called an *attractor* of the foliation. The neighbourhood $\mathscr{U}$ is said to be a basin of the attractor $\mathscr{M}$ and is denoted by $\mathrm{Attr}(\mathscr{M})$. Moreover, if $\mathrm{Attr}(\mathscr{M})=M$, then the attractor $\mathscr{M}$ is *global*.

A minimal set of a foliation $(M,\mathscr{F})$ is a nonempty closed saturated subset of M that contains no proper subset with these properties. A nonempty closed saturated subset $\mathscr{M}$ of M is minimal if and only if every leaf of $\mathscr{M}$ is dense in $\mathscr{M}$.

A minimal set $\mathscr{M}$ is called *regular* if $\mathscr{M}$ is a submanifold of M. Let $\{U_i \mid i\in\mathbb{N}\}$ be an at most countable, locally finite open covering of the manifold M by foliated coordinate neighbourhoods, let T_i be a submanifold in U_i that intersects every leaf of the foliation $(U_i,\mathscr{F}_{U_i})$ transversally, and $T=\bigcup T_i$, $i\in\mathbb{N}$ be a total transversal.

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A minimal set $\mathscr{M}$ without interior points is called

- *exceptional* if the intersection $\mathscr{M} \cap T$ is the Cantor set;
- *exotic* if $\mathscr{M} \cap T$ is not a totally disconnected topological space.

Attractors and minimal sets for groups of homeomorphisms are defined similarly (§ 6).

A leaf L of a foliation $(M, \mathscr{F})$ is called proper if L is an embedded submanifold of M. A foliation $(M, \mathscr{F})$ is proper if all its leaves are proper. A leaf L is closed if L is a closed subset of M. It is known that every closed leaf is proper, but the converse does not hold in general.

At present the class of foliations admitting transversal geometric structures which is studied the most is the class of Riemannian foliations.

Molino developed a theory of Riemannian foliations on compact manifolds; see [1]. He proved that the closure of any leaf of such a foliation is a compact submanifold and a minimal set. Molino based this result on the construction of a foliated bundle over the original Riemannian foliation.

Similar results were obtained by Salem for Riemannian foliations on manifolds with a complete transversely projectable Riemannian metric; see [2]. Salem used the method of holonomy pseudogroups formed by local isometries of a (possibly, disconnected) transversal manifold.

These results of Molino and Salem were generalized by the author in [3] to complete Cartan foliations of arbitrary codimension (see the definitions in § 4.3) with transversal Cartan geometry of type $\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{h}$ is a compactly embedded subalgebra of a Lie algebra $\mathfrak{g}$. Here, in essence, we use the construction of a foliated bundle whose induced foliation is an e-foliation; the latter is called a lifted foliation.

Recall that a Cartan foliation $(M, \mathscr{F})$ is complete (see [3]) if there exists a transversal distribution $\mathfrak{M}$ on M with lifted e-foliation, which is $\mathfrak{N}$-complete in the sense of Conlon [4]; here $\mathfrak{N}$ is a distribution transversal to the lifted foliation and projectable to $\mathfrak{M}$. As we proved in [3], Proposition 3, the distribution $\mathfrak{M}$ for an $\mathfrak{M}$-complete Cartan foliation is an Ehresmann connection in the sense of Blumenthal and Hebda; see the definition in § 2.3. Note that the Riemannian foliations investigated by Molino and Salem are complete.

Conformal foliations $(M, \mathscr{F})$ were introduced by Vaisman (see [5]) as foliations of codimension $q \geqslant 3$ admitting a transversal conformal structure; see § 2.2 for a precise definition. Riemannian foliations form a subclass of conformal foliations. We consider conformal foliations to be Cartan foliations with a normal conformal connection. A conformal foliation is called complete if it is a complete Cartan foliation; see § 5.1 for the definition.

The aim of this work is to prove the existence of global attractors for complete non-Riemannian conformal foliations and to describe the structure of such foliations. Our first step is to prove Theorem 1 using the results of Salem [2]. It states that Riemannian foliations with Ehresmann connection have the properties of complete Riemannian foliations.

Theorem 1. *Let $(M, \mathscr{F})$ be a Riemannian foliation with Ehresmann connection. Then the closure $\overline{L}$ of any leaf L is a smooth embedded submanifold of M which is a minimal set of the foliation. Closures of leaves of the foliation $(M, \mathscr{F})$ form a Riemannian foliation with singularities.*

In particular, if $(M,\mathscr{F})$ is a proper foliation, then all its leaves are closed, and the space of leaves is a smooth q-dimensional orbifold.

In § 3 we recall the definitions of (G,N)-foliations, (G,N)-manifolds, and investigate (G,N)-foliations with Ehresmann connections. We prove the following theorem which plays the key role in our study of complete conformal foliations. (In what follows we always consider covering maps for spaces with base points, even if we do not say so explicitly.)

Theorem 2. *Let $(M,\mathscr{F})$ be a (G,N)-foliation admitting an Ehresmann connection. Then*

1) *there exists a regular covering map $f\colon \widehat{M}\to M$ such that the induced foliation $\widehat{\mathscr{F}}$ on $\widehat{M}$ is made up of fibres of the locally trivial bundle $r\colon \widehat{M}\to B$ over a simply connected (G,N)-manifold B;*

2) *there is an induced group of automorphisms Ψ on the (G,N)-manifold B and an epimorphism*

$$\chi\colon \pi_1(M,x)\to\Psi$$

of the fundamental group $\pi_1(M,x)$, $x\in M$, to Ψ. Further, the group of covering transformations of the covering $\widehat{M}$ is isomorphic to the group Ψ;

3) *for any points $y\in M$ and $z\in f^{-1}(y)$ the restriction $f|_{\widehat{L}}\colon \widehat{L}\to L$ to the leaf $\widehat{L}:=\widehat{L}(z)$ of the foliation $(\widehat{M},\widehat{\mathscr{F}})$ is a regular covering map to the leaf $L:=L(y)$ of the foliation $(M,\mathscr{F})$, and the group of covering transformations is isomorphic to the isotropy subgroup Ψ_b of the group Ψ at the point $b:=r(z)\in B$. Moreover, the subgroup Ψ_b is isomorphic to the holonomy group $\Gamma(L,y)$ of the leaf L.*

A group $\Psi=\Psi(M,\mathscr{F})$ satisfying the conditions of Theorem 2 is called a *global holonomy group* of a (G,N)-foliation $(M,\mathscr{F})$ admitting an Ehresmann connection.

It is known (see, for instance, [6]) that the Lie group $\mathrm{Conf}(S^q)$ of all conformal transformations of the q-sphere S^q is isomorphic to the Möbius group $\mathrm{Mob}(q)$. Denote q-dimensional Euclidean space by $\mathbb{E}^q$. The Lie group $\mathrm{Sim}(\mathbb{E}^q)$ of all similarities coincides with the group of all conformal transformations $\mathrm{Conf}(\mathbb{E}^q)$ of the space $\mathbb{E}^q$ and is a semidirect product $CO(q)\ltimes\mathbb{R}^q$ of the conformal group $CO(q)=\mathbb{R}^+\cdot O(q)$ and an Abelian normal subgroup $\mathbb{R}^q$. We show that the holonomy group of an arbitrary leaf of a conformal foliation is isomorphic to a subgroup of the Lie group $H=CO(q)\ltimes\mathbb{R}^q$, and this subgroup is defined up to conjugation (Proposition 5). This justifies the following definition. We call the holonomy group of a leaf of a conformal foliation *inessential*, if the corresponding subgroup of the Lie group H is relatively compact. Otherwise the holonomy group of a leaf is called *essential*.

We prove the following criterion for a conformal foliation to be Riemannian without assuming that the foliation is complete.

Theorem 3. *Let $(M,\mathscr{F})$ be a conformal foliation of codimension $q\geqslant 3$ that admits a model on conformal geometry $(N,[g])$. Then there exists a Riemannian metric $d\in[g]$ such that $(M,\mathscr{F})$ is a Riemannian foliation admitting a model on (N,d) if and only if all the holonomy groups of the foliation are inessential.*

Applying Theorems 2, 3 and some results due to Alexeevskiĭ (see [7]) and Ferrand (see [8]), we will prove that any complete conformal foliation that is not Riemannian is a $(\mathrm{Conf}(S^q), S^q)$-foliation. Moreover, the completeness of a non-Riemannian conformal foliation is equivalent to the existence of an Ehresmann connection (Theorem 4). Further we describe the structure of such foliations (Theorems 5 and 6). Note that we do not assume that the foliated manifold is compact.

Theorem 4. *For any non-Riemannian conformal foliation $(M,\mathscr{F})$ of codimension $q \geqslant 3$ the following two conditions are equivalent.*

(i) *There exists an Ehresmann connection for $(M,\mathscr{F})$.*

(ii) *The conformal foliation $(M,\mathscr{F})$ is complete.*

A foliation is called *transversally similar* if it is a $(\mathrm{Sim}(\mathbb{E}^q), \mathbb{E}^q)$-foliation. We have proved (see [3], Theorem 7) that every complete non-Riemannian transversally similar foliation of codimension $q \geqslant 1$ has a global attractor, and found a sufficient condition for this attractor to be regular (see [3], Theorem 9).

Theorem 5. *Let $(M,\mathscr{F})$ be a complete conformal foliation of codimension $q \geqslant 3$. Then one of the following three conditions holds.*

1) *The foliation $(M,\mathscr{F})$ is Riemannian, complete, and the closure of every leaf is a minimal set.*

2) *The foliation $(M,\mathscr{F})$ is complete, transversally similar, and it has a global attractor $\mathscr{M}$, which is a minimal set containing all leaves with an essential holonomy group.*

3) *The foliation $(M,\mathscr{F})$ is covered by a bundle $r\colon \widehat{M} \to S^q$, where $f\colon \widehat{M} \to M$ is some regular covering, and the global holonomy group $\Psi = \Psi(M,\mathscr{F})$, which is a subgroup of the Lie group $\mathrm{Conf}(S^q)$, is well-defined. Moreover, in this case one of the following conditions holds.*

- (i) *The foliation $(M,\mathscr{F})$ has a global attractor $\mathscr{M}$, which is either a closed leaf or a union of two closed leaves.*
- (ii) *A global attractor $\mathscr{M}$ of the foliation $(M,\mathscr{F})$ exists that coincides with the closure of any leaf with essential holonomy group. It is a nontrivial minimal set and $\mathscr{M} = f(r^{-1}(\Lambda(\Psi)))$, where $\Lambda(\Psi)$ is a minimal set of the global holonomy group Ψ.*

In cases 2) and 3) the induced foliation $(M_0, \mathscr{F}_{M_0})$ on the complement $M_0 := M \setminus \mathscr{M}$ is a Riemannian foliation with Ehresmann connection. The closures of its leaves in M_0 form a Riemannian foliation with singularities, and the closure $\overline{L}$ in M of any leaf $L \in \mathscr{F}$ from M_0 equals $\mathfrak{L} \cup \mathscr{M}$, where $\mathfrak{L}$ is an embedded submanifold of M. The subset $\mathfrak{L}$ coincides with the closure of the leaf L in M_0.

Corollary 1. *For any complete conformal foliation of codimension $q \geqslant 3$ there exists a minimal set.*

Corollary 2. *Every complete conformal foliation of codimension $q \geqslant 3$ is either Riemannian or a $(\mathrm{Conf}(S^q), S^q)$-foliation.*

By Theorem 5, the transversal structure of global attractors of complete conformal foliations is defined by the structure of global attractors of global holonomy groups of these foliations. In particular, we obtain the following corollary.

Corollary 3. *A complete conformal foliation $(M,\mathscr{F})$ has a global attractor. It is an exceptional (exotic) minimal set if and only if its global holonomy group Ψ has a global attractor which is an exceptional (exotic) minimal set.*

As an application of Theorem 1 and Theorem 5 we obtain the following statement.

Theorem 6. *The structure of every complete proper conformal foliation $(M,\mathscr{F})$ of codimension $q\geqslant 3$ is of one of the following types.*

1) *A complete Riemannian foliation, where every leaf is closed and the space of leaves is a smooth q-dimensional orbifold.*

2) *A complete transversally similar non-Riemannian foliation with a unique closed leaf which is a global attractor.*

3) *A complete (Ψ, S^q)-foliation covered by a bundle $r\colon \widehat{M}\to S^q$, where Ψ is an elementary discrete subgroup of the conformal group* $\operatorname{Conf}(S^q)$. *There exists a global attractor which is either a closed leaf or a union of two closed leaves.*

Moreover, in cases 2) and 3) the union M_0 of nonclosed leaves is an open connected everywhere dense submanifold, and $(M_0,\mathscr{F}_{M_0})$ is a Riemannian foliation with an Ehresmann connection. The space of leaves is a smooth q-dimensional orbifold, and the closure $\overline{L}_\alpha$ in M of any leaf L_α from M_0 equals $L_\alpha\cup\mathscr{M}$, where $\mathscr{M}$ is a global attractor.

Corollary 4. *Every complete proper non-Riemannian conformal foliation of codimension $q\geqslant 3$ has either one or two closed leaves.*

We obtain a constructive proof of the following realization theorem.

Theorem 7. *Every countable subgroup Ψ of the group* $\operatorname{Conf}(S^q)$, *where* $q\geqslant 3$, *can be realized as a global holonomy group of some two-dimensional conformal foliation $(M,\mathscr{F})$ of codimension q. If the group Ψ is finitely generated, then such a foliation $(M,\mathscr{F})$ can be found on a closed manifold M of dimension $q+2$.*

In § 7 we construct examples of complete conformal foliations whose global attractors are exceptional or exotic minimal sets.

We will denote the module of vector fields on a manifold M by $\mathfrak{X}(M)$. If $\mathfrak{M}$ is a smooth distribution on M, then $\mathfrak{X}_{\mathfrak{M}}(M)$ denotes the set of vector fields in $\mathfrak{X}(M)$ tangent to $\mathfrak{M}$. Let $f\colon \widetilde{M}\to M$ be a submersion. Then the induced distribution $\widetilde{\mathfrak{M}}$ consisting of subspaces $\widetilde{\mathfrak{M}}_y:=\{Y\in T_y(\widetilde{M})\mid f_{*y}(Y)\in\mathfrak{M}_{f(y)}\}$ is denoted by $f^*\mathfrak{M}$. Similar notation is used for foliations.

Following [9], we denote the principal H-bundle with base M and total space P by $P(M,H)$.

In this paper all neighbourhoods are assumed to be open.

§ 2. Minimal sets of Riemannian foliations with Ehresmann connections

2.1. Representing a foliation by an N-cocycle. Suppose that we have:

1) an n-dimensional manifold M and, possibly, a disconnected q-dimensional manifold N with $q<n$;

2) an open covering $\{U_i\mid i\in J\}$ of the manifold M;

3) submersions with connected fibres $f_i\colon U_i \to V_i$ to $V_i \subset N$;
4) if $U_i \cap U_j \neq \varnothing$, then there exists a diffeomorphism

$$\gamma_{ij}\colon f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$$

of open subsets of the manifold N such that $f_i = \gamma_{ij} \circ f_j$ on $U_i \cap U_j$.

Condition 4) implies that if $U_i \cap U_j \cap U_k \neq \varnothing$, then the equality $\gamma_{ik} = \gamma_{ij} \circ \gamma_{jk}$ holds. Moreover, $\gamma_{ii} = \mathrm{id}|_{U_i}$.

A maximal N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ having the properties listed above defines a new topology $\tau_{\mathscr{F}}$ on M. It is called a *leaf* topology. Its base is the set of fibres of all submersions f_i. Arcwise connected components of the topological space $(M, \tau_{\mathscr{F}})$ form a decomposition of M which is denoted by $\mathscr{F} = \{L_\alpha \mid \alpha \in A\}$. It is called a *foliation represented by an N-cocycle* $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$, and L_α, $\alpha \in A$, are its leaves.

Since any N-cocycle is contained in a unique maximal N-cocycle, to represent a foliation $(M, \mathscr{F})$ it suffices to fix some N-cocycle with properties 1)–4).

2.2. Conformal and Riemannian foliations. Recall that a diffeomorphism $f\colon N_1 \to N_2$ of Riemannian manifolds (N_1, g_1) and (N_2, g_2) is *conformal* if there exists a smooth function λ on N_1 such that $f^*g_2 = e^\lambda g_1$.

A smooth foliation $(M, \mathscr{F})$ of codimension q is called *conformal* if it is represented by an N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ and the manifold N is equipped with a Riemannian metric g_N such that the $\{\gamma_{ij}\}$ are local conformal diffeomorphisms of the corresponding open subsets. If all the γ_{ij} are isometries, $(M, \mathscr{F})$ is a *Riemannian* foliation.

2.3. An Ehresmann connection for a foliation. The notion of an Ehresmann connection for a foliation was introduced by Blumenthal and Hebda [10].

Let $(M, \mathscr{F})$ be a foliation of arbitrary codimension q. A distribution $\mathfrak{M}$ on a manifold M is called *transversal* to a foliation $\mathscr{F}$ if for any $x \in M$ the equality $T_xM = T_x\mathscr{F} \oplus \mathfrak{M}_x$ holds, where $\oplus$ stands for a direct sum of vector spaces. Vectors from $\mathfrak{M}_x$, $x \in M$, are called horizontal. A piecewise smooth curve σ is horizontal if all its tangent vectors are horizontal. Equivalently, a piecewise smooth curve is horizontal (or $\mathfrak{M}$-horizontal) if each of its smooth segments is an integral curve of the distribution $\mathfrak{M}$. A distribution $T\mathscr{F}$ tangent to leaves of the foliation $\mathscr{F}$ is called vertical. One says that a curve h is vertical if h is contained in a leaf of the foliation $\mathscr{F}$.

A *vertical-horizontal homotopy* (*v.h.h.* for short) is a piecewise smooth map $H\colon I_1 \times I_2 \to M$, where $I_1 = I_2 = [0,1]$, such that for any $(s,t) \in I_1 \times I_2$ the curve $H|_{I_1\times\{t\}}$ is horizontal and the curve $H|_{\{s\}\times I_2}$ is vertical. A pair of curves $(H|_{I_1\times\{0\}}, H|_{\{0\}\times I_2})$ is called a *base of the v.h.h. H*. Two paths σ, h with common origin $\sigma(0) = h(0)$, where σ is a horizontal path and h is a vertical one, are called an *admissible pair of paths*.

A distribution $\mathfrak{M}$ transversal to a foliation $\mathscr{F}$ is called an *Ehresmann connection for $\mathscr{F}$* if for any admissible pair of paths σ, h there exists a v.h.h. with base (σ, h).

Let $\mathfrak{M}$ be an Ehresmann connection for a foliation $(M, \mathscr{F})$. Then for any admissible pair of paths σ, h there exists a unique v.h.h. H with base (σ, h). We say that

$\widetilde{\sigma} := H|_{I_1\times\{1\}}$ is the result of the parallel shift of the path σ along h with respect to the Ehresmann connection $\mathfrak{M}$. It is denoted by $\sigma \overset{h}{\twoheadrightarrow} \widetilde{\sigma}$.

2.4. $\mathfrak{M}$-adapted transversally projectable Riemannian metric. We recall that a submersion $f\colon U \to V$ of Riemannian manifolds (U, g) and (V, h) is said to be *Riemannian* if, for any point $x \in U$, the restriction of the differential $f_{*x}\colon \mathfrak{M}_x \to T_{f(x)}V$ to the normal vector subspace $\mathfrak{M}_x$ to the fibre of the submersion is an isomorphism of Euclidean vector spaces.

Proposition 1. *Let $(M, \mathscr{F})$ be a Riemannian foliation with Ehresmann connection $\mathfrak{M}$ represented by an N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$, and let $\mathfrak{M}$ be a distribution on M transversal to the foliation $(M, \mathscr{F})$. Then there exist Riemannian metrics g on M and g_N on N such that*

(i) *the distribution $\mathfrak{M}$ is orthogonal to the foliation $(M, \mathscr{F})$, and every submersion $f_i\colon U_i \to V_i = f_i(U_i)$, $i \in J$, is Riemannian*;

(ii) *any geodesic γ on the Riemannian manifold (M, g), which is tangent to $\mathfrak{M}$ at one point, is tangent to $\mathfrak{M}$ at every point*;

(iii) *for every admissible pair of paths of the form (σ, h), where σ is an $\mathfrak{M}$-horizontal geodesic, the result $\widetilde{\sigma}$ of the parallel shift $\sigma \overset{h}{\twoheadrightarrow} \widetilde{\sigma}$ is also an $\mathfrak{M}$-horizontal geodesic of the same length as σ, i.e., $l(\widetilde{\sigma}) = l(\sigma)$.*

Proof. By the definition of a Riemannian foliation, there exists a Riemannian metric g_N on N such that any element γ_{ij} is a local isometry of the Riemannian manifold (N, g_N). By assumption, $\mathfrak{M}$ is an Ehresmann connection for the foliation $(M, \mathscr{F})$. Consequently,

$$T_xM = T_x\mathscr{F} \oplus \mathfrak{M}_x \qquad \forall\, x \in M.$$

This shows that every vector field $X \in \mathfrak{X}(M)$ can be represented in the form $X = X_{\mathscr{F}} + X_{\mathfrak{M}}$, where $X_{\mathscr{F}} \in \mathfrak{X}_{\mathscr{F}}(M)$ and $X_{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M)$. Let g_0 be a Riemannian metric on M. Set

$$g(X, Y) := g_0(X_{\mathscr{F}}, Y_{\mathscr{F}}) + g_N(f_{i*}X, f_{i*}Y)$$

for any $X, Y \in \mathfrak{X}(M)$ on U_i, $i \in J$. If $U_i \cap U_j \neq \varnothing$, then $f_j = \gamma_{ji} \circ f_i$, where γ_{ji} is a local isometry of (N, g_N). Hence g is a well-defined Riemannian metric on M, which is transversally projectable with respect to the foliation $(M, \mathscr{F})$.

Assertion (i) follows from the definition of g.

Well-known properties of Riemannian submersions (see [11]) and transversally projectable Riemannian metrics (see [1]) yield assertions (ii) and (iii).

Definition 1. The Riemannian metric introduced in the proof of Proposition 1 is called an $\mathfrak{M}$-*adapted transversally projectable metric* with respect to the Riemannian foliation $(M, \mathscr{F})$ with Ehresmann connection $\mathfrak{M}$.

We shall call $\mathfrak{M}$-horizontal geodesics $\mathfrak{M}$-geodesics. We assume that the geodesics under consideration are parametrized by arc length.

2.5. The holonomy pseudogroup of a foliation. We start with the basic definitions, see [2] and [12].

Definition 2. Let N be a smooth q-dimensional manifold, which may be disconnected. A *smooth pseudogroup* of local transformations $\mathscr{H}$ on N is a set of diffeomorphisms $h\colon D(h) \to R(h)$ between open subsets on the manifold N that satisfies the following axioms.

1. If $g, h \in \mathscr{H}$ and $R(h) \subset D(g)$ then $g \circ h \in \mathscr{H}$.
2. If $h \in \mathscr{H}$ then $h^{-1} \in \mathscr{H}$.
3. $\mathrm{id}_N \in \mathscr{H}$.
4. If $h \in \mathscr{H}$ and $W \subset D(h)$ is an open subset, then $h|_W \in \mathscr{H}$.
5. Let $h\colon D(h) \to R(h)$ be a diffeomorphism between open subsets in $\mathscr{N}$, and assume that for any $w \in D(h)$ there is a neighbourhood W in $D(h)$ such that $h|_W \in \mathscr{H}$. Then $h \in \mathscr{H}$.

Definition 3. Let A be a family of local diffeomorphisms of N containing id_N. The pseudogroup obtained by adding h^{-1} for every h from A together with the restrictions of local diffeomorphisms to open subsets, compositions and unions of elements from A, is called the *pseudogroup generated by* A.

Definition 4. Assume than an N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ represents a foliation $(M, \mathscr{F})$. The pseudogroup generated by local diffeomorphisms γ_{ij} of the manifold N is called the *holonomy pseudogroup* of this foliation and is denoted by $\mathscr{H} = \mathscr{H}(M, \mathscr{F})$.

We recall that a pseudogroup $\mathscr{H}$ of local diffeomorphisms of a manifold N is called *quasi-analytic* if the existence of an open connected subset V in N such that $g|_V = \mathrm{id}_V$ for an element $g \in \mathscr{H}$ implies that $g = \mathrm{id}_{D'(g)}$, where $D'(g)$ is the connected component of the domain of definition $D(g)$ of the element g that contains V.

For example, every pseudogroup of local isometries as well as every pseudogroup of local conformal transformations is quasi-analytic.

Definition 5. A quasi-analytic pseudogroup $\mathscr{H}$ of local diffeomorphisms of a manifold N is called *complete* if for every pair of points x and x' on N there exist open neighbourhoods U and U' such that:

if $y \in U$ and $y' = \varphi(y) \in U'$ for some $\varphi \in \mathscr{H}$, then there is an extension $\psi \in \mathscr{H}$ of the local diffeomorphism φ to the whole of the neighbourhood U.

2.6. A sufficient condition for the holonomy pseudogroup to be complete.

Proposition 2. *If a Riemannian foliation admits an Ehresmann connection, then its holonomy pseudogroup is complete.*

Proof. Let g be some $\mathfrak{M}$-adapted transversally projectable Riemannian metric on a manifold M with respect to a Riemannian foliation $(M, \mathscr{F})$ with Ehresmann connection $\mathfrak{M}$.

Let x and x' be two points on N. Consider submersions $f_i\colon U_i \to V$ and $f_j\colon U_j \to V'$ from an N-cocycle representing the foliation $(M, \mathscr{F})$, such that $x \in V$, $x' \in V'$. Take $a \in f_i^{-1}(x)$ and $a' \in f_j^{-1}(x')$. There exist a number $\varepsilon > 0$

and geodesically convex balls B_ε and B'_ε of radius ε, with centres at a and a', respectively, such that $B_\varepsilon \subset U_i$ and $B'_\varepsilon \subset U_j$. For $\delta = \varepsilon/3$ let $\mathscr{V}$ and $\mathscr{V}'$ be the geodesically convex balls of radius δ, with centres at x and x', respectively. Let S_δ and S'_δ be subsets of M consisting of points of $\mathfrak{M}$-geodesics of length $< \delta$ with origin at a and a', respectively. We will show that for any points $y \in \mathscr{V}$ and $y' = \varphi(y) \in \mathscr{V}'$, where $\varphi \in \mathscr{H}$, there exists an extension $\psi \in \mathscr{H}$ of a local isometry φ to the whole of $\mathscr{V}$.

For any point $z \in \mathscr{V}$ there is a unique geodesic γ in $\mathscr{V}$ connecting $y = \gamma(0)$ and $z = \gamma(s_0)$ of length $s_0 < 2\delta$. Take $b \in f_i^{-1}(y) \cap S_\delta$ and $b' \in f_j^{-1}(y') \cap S'_\delta$. First we prove the existence of an $\mathfrak{M}$-lift of the geodesic γ to the point b on U_i. By statement (i) of Proposition 1, f_i is a Riemannian submersion. Since an $\mathfrak{M}$-lift exists locally, there is a number $s_1 > 0$ such that there is an $\mathfrak{M}$-lift σ of the curve $\gamma|_{[0,s_1)}$ to the point b. Take $c \in f_i^{-1}(\gamma(s_1)) \in U_i$. For a point c there exist ε_0, $0 < \varepsilon_0 < s_1$, and a local lift τ of the geodesic $(\gamma|_{[s_1-\varepsilon_0,s_1]})^{-1}$ to the point c. Since the points $\tau(\varepsilon_0)$ and $\sigma(s_1 - \varepsilon_0)$ are in the same fibre $f_i^{-1}(\gamma(s_1 - \varepsilon_0))$ of the submersion f_i and the submersion has connected fibres, there is a smooth path h in this fibre connecting $h(0) = \tau(\varepsilon_0)$ with $h(1) = \sigma(s_1 - \varepsilon_0)$.

For an admissible pair of paths (τ^{-1}, h) there exists a shift $\tau^{-1} \overset{h}{\to}\!> \widetilde{\sigma}$ with respect to the Ehresmann connection $\mathfrak{M}$. By a property of Riemannian submersions, τ is $\mathfrak{M}$-geodesic. By Proposition 1, $\widetilde{\sigma} = \widetilde{\sigma}(s)$, $s \in [0, \varepsilon_0]$, is $\mathfrak{M}$-geodesic as well. Uniqueness implies that the $\mathfrak{M}$-lifts $\widetilde{\sigma}(s) = \sigma(s_1 - \varepsilon_0 + s)\ \forall\, s \in [0, \varepsilon_0)$ of the curve $\gamma|_{[s_1-\varepsilon_0,s_1)}$ to the point $\sigma(s_1-\varepsilon_0)$ coincide. Hence the point $\sigma(s_1) := \widetilde{\sigma}(\varepsilon_0)$ is well-defined. This shows that the set A of points t in the segment $[0, s_0]$, such that the $\mathfrak{M}$-lift $\sigma(s)$ is defined for all $s \in [0, t]$, is an open-and-closed subset of $[0, s_0]$. Since the segment $[0, s_0]$ is connected, we have $A = [0, s_0]$. The properties of the Riemannian submersion f_i yield $l(\sigma) = l(\gamma) = s_0 < 2\delta$.

Let f be a path in the leaf $L(b)$ connecting $b = f(0)$ with $b' = f(1)$, such that the germ of a holonomic diffeomorphism along f corresponds to the germ φ at the point y by means of f_i and f_j. Then the shift $\sigma \overset{f}{\to}\!> \sigma'$ with respect to the Ehresmann connection $\mathfrak{M}$ is defined. By Proposition 1, σ' is an $\mathfrak{M}$-geodesic of length s_0 with origin at the point $b' = \sigma'(0)$. Denote the distance function on the Riemannian manifold (M, g) by d. Then $d(a', \sigma'(s)) \leqslant d(a', b') + l(\sigma'(s)) < 3\delta = \varepsilon$ $\forall\, s \in [0, s_0]$. So we have $\sigma'(s) \in B'_\varepsilon \subset U_j\ \forall\, s \in [0, s_0]$. Consequently, there is a curve $\gamma' := f_j \circ \sigma'$, and γ' is a geodesic of length s_0 with origin at y'. It is easy to check that the equality $\widetilde{\varphi}(z) := \gamma'(s_0)$ defines a map $\widetilde{\varphi}\colon \mathscr{V} \to \mathscr{U}$ in $\mathscr{U} := f_j(B_\varepsilon)$, and $\widetilde{\varphi}$ is an extension of φ to $\mathscr{V}$. Since $\rho(y, z) = l(\gamma) = l(\gamma') = \rho(\widetilde{\varphi}(y), \widetilde{\varphi}(z))$, $\widetilde{\varphi}$ is an isometry. At each point of $\mathscr{V}$ the germ of the isometry $\widetilde{\varphi}$ coincides with the germ of an element of $\mathscr{H}$, and $\widetilde{\varphi} \in \mathscr{H}$.

2.7. Proof of Theorem 1. Let $(M, \mathscr{F})$ be a Riemannian foliation with Ehresmann connection $\mathfrak{M}$. By Proposition 2, the holonomy pseudogroup $\mathscr{H} \coloneqq \mathscr{H}(M, \mathscr{F})$ is a complete pseudogroup of local isometries. Consequently, we can apply the results of Salem [2].

By [2], Proposition 2.3, for a complete pseudogroup of local isometries $\mathscr{H}$ of a Riemannian manifold (N, g_N) there exists a complete pseudogroup $\overline{\mathscr{H}}$ of local isometries of (N, g_N) such that the orbits of $\overline{\mathscr{H}}$ are the closures of orbits of the pseudogroup $\mathscr{H}$. Moreover, by [2], Corollary 3.2, the orbits of the pseudogroup $\overline{\mathscr{H}}$ are closed embedded submanifolds of the manifold N, and an $\mathscr{H}$-orbit of a point from $\overline{\mathscr{H}}\cdot x$, $x \in N$, is everywhere dense in $\overline{\mathscr{H}}\cdot x$. In other words, for any $x \in N$ we have $\overline{\mathscr{H}\cdot x} = \overline{\mathscr{H}}\cdot x$ and for any $y \in \overline{\mathscr{H}\cdot x}$ the conditions $\overline{\mathscr{H}}\cdot y = \overline{\mathscr{H}\cdot y} = \overline{\mathscr{H}\cdot x}$ hold. Let L be a leaf of the foliation $(M,\mathscr{F})$ and let $a \in U_i \cap L$ for some neighbourhood U_i from the N-cocycle representing this foliation. Recall that the closure $\overline{L}$ of the leaf L corresponds to the closure of the orbit $\mathscr{H}\cdot x$ of point $x = f_i(a)$ with respect to the holonomy pseudogroup $\mathscr{H}$. Hence the closure $\overline{L}$ of L is a closed embedded submanifold in M, and every leaf $L' \subset \overline{L}$ is everywhere dense in $\overline{L}$. This means that $\overline{L}$ is a minimal set of the foliation $(M,\mathscr{F})$. The foliation formed by closures of leaves of the foliation $(M,\mathscr{F})$ is a Riemannian foliation with singularities.

If $(M,\mathscr{F})$ is a proper foliation, then all orbits of the holonomy pseudogroup are discrete subsets in N. It is known (see [2]) that the orbit space of a complete pseudogroup of local isometries, all of whose orbits are discrete, is a q-dimensional orbifold. Since the space of leaves $M/\mathscr{F}$ of the foliation $(M,\mathscr{F})$ is homeomorphic to the orbit space of the complete pseudogroup of local isometries $\mathscr{H}(M,\mathscr{F})$, the space $M/\mathscr{F}$ is also a q-dimensional orbifold.

Example 1. Let $(T^2,\mathscr{F}_\alpha)$ be a Kronecker foliation on the two-dimensional locally Euclidean torus T^2 corresponding to an irrational number $\alpha > 0$. Take two distinct points a and b on one leaf $L \in \mathscr{F}_\alpha$. Let $M := T^2 \setminus \{a,b\}$ and $\mathscr{F} := \mathscr{F}_\alpha|_M$. Then $(M,\mathscr{F})$ is a Riemannian foliation. The leaf L contains leaves L', L'' and L''' of the induced foliation $(M,\mathscr{F})$. One of these three leaves is closed in M. Assume that this is the leaf L'. The Riemannian foliation $(M,\mathscr{F})$ has a unique minimal set, namely the leaf L'. By Theorem 1, the foliation $(M,\mathscr{F})$ does not admit an Ehresmann connection. The holonomy pseudogroup of this foliation is not complete.

Proposition 3. *Let $(M,\mathscr{F})$ be a proper Riemannian foliation of codimension q admitting an Ehresmann connection, and suppose that the holonomy group of every leaf of this foliation is trivial. Then the space of leaves $B := M/\mathscr{F}$ is a q-dimensional manifold, and the foliation $(M,\mathscr{F})$ is formed by fibres of the locally trivial bundle $r\colon M \to B$. If M is simply connected, so is the base B.*

Proof. By Theorem 1, the space of leaves of the foliation $(M,\mathscr{F})$ is a q-dimensional orbifold. By assumption, all holonomy groups of the foliation $(M,\mathscr{F})$ are trivial, so all isotropy holonomy pseudogroups are trivial too. Consequently, this orbifold is a smooth q-dimensional manifold, and the foliation $(M,\mathscr{F})$ is formed by fibres of the submersion $f\colon M \to B = M/\mathscr{F}$. Here an Ehresmann connection for the foliation $(M,\mathscr{F})$ is an Ehresmann connection for the submersion f. It is known that a submersion with an Ehresmann connection is a locally trivial bundle. Therefore the foliation $(M,\mathscr{F})$ is formed by fibres of the locally trivial bundle $f\colon M \to B$. In the case when M is simply connected, the exact homotopy sequence of this bundle and the fact that the fibres of the bundle are connected yield that the base B is simply connected too.

§ 3. (G,N)-foliations with Ehresmann connections

3.1. (G,N)-foliations and (G,N)-manifolds. Let N be a connected manifold and G a group of diffeomorphisms of N. The group G is said to act quasi-analytically on N if, for any open subset U in N and an element $g \in G$, the condition $g|_U = \mathrm{id}_U$ implies $g = e$, where e is the identity transformation of N.

In this section we assume that the group G of diffeomorphisms of a manifold N acts on N quasi-analytically.

Definition 6. A foliation $(M, \mathscr{F})$ represented by an N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ is called a (G,N)-*foliation* if for any $U_i \cap U_j \neq \varnothing$, $i, j \in J$, there exists an element $g \in G$ such that $\gamma_{ij} = g|_{f_j(U_i\cap U_j)}$.

Definition 7. A manifold M is called a (G,N)-*manifold* if it has an atlas $\{(U_i, \varphi_i)\}_{i,j\in J}$ with the following properties:

1) $\varphi_i(U_i) \subset N$;

2) if $U_i \cap U_j \neq \varnothing$, then the diffeomorphism $\gamma_{ij} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is the restriction of some transformation $g \in G$.

Equivalently, M is called a (G,N)-manifold if its zero-dimensional foliation is a (G,N)-foliation.

3.2. Proof of Theorem 2. Fix a point $x_0 \in M$. Let x be a point on M and $h(t)$, $t \in I = [0,1]$, be a piecewise smooth path connecting $x_0 = h(0)$ to $x = h(1)$. Cover $h(I)$ by a finite chain of neighbourhoods $U_1, \dots, U_s$ from an N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ representing the foliation $(M, \mathscr{F})$. Fix neighbourhoods $U_0 \ni x_0$ and $U \ni x_1$ from the N-cocycle and consider chains of the form $U_0, U_1, \dots, U_s, U_{s+1} := U$. Since $U_k \cap U_{k+1} \neq \varnothing$, $k = 0, \dots, s$, there is a local diffeomorphism $\gamma_{(k+1)k} \colon f_k(U_k \cap U_{k+1}) \to f_{k+1}(U_k \cap U_{k+1})$ such that $f_{k+1} = \gamma_{(k+1)k} \circ f_k$, where $f_{s+1} = f \colon U \to V$. By the definitions of (G,N)-foliations and quasi-analytic actions of a Lie group, there exists a unique element $g_{(k+1)k} \in G$ such that $g_{(k+1)k}|_{f_k(U_k\cap U_{k+1})} = \gamma_{(k+1)k}$. Consider the element

$$g := g_{(s+1)s} \circ g_{s(s-1)} \circ \cdots \circ g_{21} \circ g_{10} \tag{$*$}$$

of the group G. It follows in the standard way that g is independent of both the choice of the chain of neighbourhoods $U_1, \dots, U_s$ covering $h(I)$ and the choice of the path from the homotopy class $[h]$ in M with fixed end-points x_0 and x. It is known that the set of homotopy classes of paths starting at $x_0 \in M$ forms a universal covering space $\widetilde{M}$ of M, and the projection $p \colon \widetilde{M} \to M \colon [h] \mapsto h(1)$ is the universal covering map.

We define a map $D \colon \widetilde{M} \to N$ by setting $D([h]) = g^{-1}(f_{s+1}(h(1)))$, where the element $g \in G$ is given by equality $(*)$. Note that $D = D(U_0)$ depends only on the choice of the neighbourhood U_0. Since f_k and g are submersions, $D \colon \widetilde{M} \to N$ is too. But any submersion is an open map. So $D(\widetilde{M})$ is an open subset of N. The submersion $D \colon \widetilde{M} \to N$ is called a decomposing map.

Let $\widetilde{\mathscr{F}} := p^*\mathscr{F}$ be the induced foliation on $\widetilde{M}$. We remark that every leaf $\widetilde{L}$ of the foliation $(\widetilde{M}, \widetilde{\mathscr{F}})$ is contained in some fibre $D^{-1}(b)$, $b \in D(\widetilde{M})$, of the submersion $D \colon \widetilde{M} \to N$. Since the fibres of D have dimension $n - p = \dim(\widetilde{\mathscr{F}})$, we conclude that $\widetilde{L}$ is an open subset of $D^{-1}(b)$. The set $D^{-1}(b) \setminus \widetilde{L}$ is a union of some leaves of the foliation $(\widetilde{M}, \widetilde{\mathscr{F}})$, so $D^{-1}(b) \setminus \widetilde{L}$ is also an open subset of $D^{-1}(b)$, and $\widetilde{L}$ is open and closed in $D^{-1}(b)$. Consequently, the foliation $(\widetilde{M}, \widetilde{\mathscr{F}})$ is formed by connected components of fibres of the submersion D. The leaf $\widetilde{L}$ is closed in the closed subspace $D^{-1}(b)$ of the space $\widetilde{M}$, so $\widetilde{L}$ is closed in $\widetilde{M}$. This shows that all leaves of the foliation $(\widetilde{M},\widetilde{\mathscr{F}})$ are closed subsets of $\widetilde{M}$. By assumption, the foliation $(M,\mathscr{F})$ admits an Ehresmann connection $\mathfrak{M}$. One can easily check that $\widetilde{\mathfrak{M}} := p^*\mathfrak{M}$ is an Ehresmann connection for the foliation $(\widetilde{M},\widetilde{\mathscr{F}})$. Let k be a Riemannian metric on the open submanifold $B = D(\widetilde{M})$ of N. Then

$$\widetilde{g}|_{\widetilde{\mathfrak{M}}}(X,Y) := k(D_*(X),D_*(Y)) \qquad \forall\, X,Y \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\widetilde{M})$$

is a Riemannian metric on the transversal bundle $T\widetilde{M}/T\widetilde{\mathscr{F}}$ that is identified with $\widetilde{\mathfrak{M}}$, and it is projectable with respect to the foliation $(\widetilde{M},\widetilde{\mathscr{F}})$.

So $(\widetilde{M},\widetilde{\mathscr{F}})$ is a Riemannian foliation with Ehresmann connection such that all leaves are closed subsets of $\widetilde{M}$, and all the holonomy groups are trivial. It is known that every closed leaf of a foliation is proper, thus the foliation $(\widetilde{M},\widetilde{\mathscr{F}})$ is proper. By Proposition 3, leaves of this foliation are formed by fibres of a locally trivial bundle $\widetilde{r}\colon \widetilde{M} \to B$ over a simply connected manifold B. Since $(M,\mathscr{F})$ is a (G,N)-foliation, we obtain that $(\widetilde{M},\widetilde{\mathscr{F}})$ is a (G,N)-foliation and B is a (G,N)-manifold.

Let Φ be the group of covering transformations of the universal covering $p\colon \widetilde{M} \to M$. Then there is a group isomorphism $\beta\colon \pi_1(M,x) \to \Phi$. Since any transformation $\phi \in \Phi$ is an automorphism of the bundle $\widetilde{r}\colon \widetilde{M} \to B$, ϕ induces a transformation $\psi = \psi(\phi)$ of the manifold B, which is an automorphism of the (G,N)-manifold B and satisfies $\widetilde{r}\circ\phi = \psi(\phi)\circ\widetilde{r}$. The set of all such transformations is a group $\Psi = \{\psi(\phi) \mid \phi \in \Phi\}$, and the map $\mu\colon \Phi \to \Psi\colon \phi \mapsto \psi(\phi)$ is a group epimorphism. Since $\widetilde{\Phi} := \operatorname{Ker}\mu$ is a normal subgroup of the group Φ whose action on $\widetilde{M}$ is free and properly disconnected, we have a quotient manifold $\widehat{M} := \widetilde{M}/\widetilde{\Phi}$ and a free properly disconnected action of the factor group $\widehat{\Phi} := \Phi/\widetilde{\Phi}$ on $\widehat{M}$. Moreover, $M \cong \widehat{M}/\widehat{\Phi}$, and the quotient map $\widetilde{p}\colon \widetilde{M} \to \widehat{M} = \widetilde{M}/\widetilde{\Phi}$ satisfies $p = \widehat{p}\circ\widetilde{p}$, where $\widehat{p}\colon \widehat{M} \to M = \widehat{M}/\widehat{\Phi}$ is the projection on the orbit space of the group $\widehat{\Phi}$. Note that $\widehat{p}\colon \widehat{M} \to M$ is a regular covering map with $\widehat{\Phi}$ as the group of covering transformations. The definition of $\widehat{M}$ implies that the foliation $\widehat{F} := \widehat{p}^*F$ is formed by fibres of a locally trivial bundle $r\colon \widehat{M} \to B$, and $r\circ\widetilde{p} = \widetilde{r}$. The natural group epimorphism $\widehat{\mu}\colon \widehat{\Phi} \to \Psi$ is an isomorphism. So assertions 1) and 2) of Theorem 2 are proved.

Assertion 3) follows from the fact that the action of the group Ψ is quasi-analytic.

§ 4. A criterion for a conformal foliation to be Riemannian

4.1. Foliated bundles and foliated reductions. Let $(M,\mathscr{F})$ be any smooth foliation of codimension q. A locally trivial bundle $\pi\colon \mathscr{R} \to M$ with foliation $(\mathscr{R},F)$ is called a *foliated bundle* if the following conditions hold.

1) If $\mathfrak{N}$ is a distribution, tangent to fibres of the submersion $\pi\colon \mathscr{R} \to M$, then $\mathfrak{N}_u \cap T_uF = \{0\}\ \forall\, u \in \mathscr{R}$.

2) The projection $\pi\colon (\mathscr{R},F) \to (M,\mathscr{F})$ is a morphism of foliations.

We denote a foliated bundle by $(\mathscr{R},\pi,M,F)$. If $\pi\colon \mathscr{R} \to M$ is a principal H-bundle, and the foliation $(\mathscr{R},F)$ is H-invariant, then we say that this is a *foliated H-bundle* over $(M,\mathscr{F})$. We denote a foliated H-bundle as $(\mathscr{R}(M,H),F)$.

Let $(\mathscr{R}'(M',H'),F')$ and $(\mathscr{R}(M,H),F)$ be two principal foliated bundles with projections $\pi'\colon \mathscr{R}' \to M'$ and $\pi\colon \mathscr{R} \to M$, respectively. Let $f\colon \mathscr{R}' \to \mathscr{R}$ be a morphism from the foliation $(\mathscr{R}',F')$ to the foliation $(\mathscr{R},F)$ and $\widehat{f}\colon H' \to H$ be a Lie group homomorphism. Then the pair $(f,\widehat{f})$ is called a *morphism in the category of principal foliated bundles* if $f(u'a') = f(u')\widehat{f}(a')$ for all $u' \in \mathscr{R}'$ and $a' \in H'$. The morphism $(f,\widehat{f})\colon (\mathscr{R}'(M',H'),F') \to (\mathscr{R}(M,H),F)$ defines a projection $f'\colon M' \to M$ satisfying the condition $\pi \circ f = f' \circ \pi'$.

Let $(f,\widehat{f})\colon (\mathscr{R}'(M,H'),F') \to (\mathscr{R}(M,H),F)$ be a morphism of principal foliated bundles, where $f\colon \mathscr{R}' \to \mathscr{R}$ is an embedding with projection $f' = \mathrm{id}_M$, the image $f(\mathscr{R}')$ is an F-saturated submanifold in $\mathscr{R}$, and $\widehat{f}\colon H' \to H$ is an immersion of Lie groups. We identify $\mathscr{R}'$ with $f(\mathscr{R}')$, the group H' with the Lie subgroup $\widehat{f}(H')$ of the group H, and the foliation F' with the foliation $F|_{f(\mathscr{R}')}$. In this case the foliated bundle $(\mathscr{R}'(M,H'),F|_{f(\mathscr{R}')})$ is called *a reduced foliated bundle* and is denoted by $(\mathscr{R}'(M,H'),F')$. The map $\mathscr{R}' \to \mathscr{R}$ is called *a reduction of the principal foliated bundle $(\mathscr{R}(M,H),F)$ to a subgroup H'.*

4.2. Foliated sections. Let $(M,\mathscr{F})$ be a foliation and U an open subset of the manifold M. Let F_U denote the induced foliation on U whose leaves are connected components of the intersections $L_\alpha \cap U$ of the leaves L_α of the foliation $\mathscr{F}$ with U. Let (E,π,M,F) be some foliated bundle with standard fibre Y over $(M,\mathscr{F})$.

Definition 8. Let U be an open subset of M and let A be any subset of U. A section $\sigma\colon A \to E$ of the bundle $\pi\colon E \to M$ is called an *$\mathscr{F}_U$-section*, if for any two points x,y in $A \cap L$, where $L \in \mathscr{F}_U$, there exists a leaf $\mathcal{L} \in F_{\pi^{-1}(U)}$ such that $\sigma(x),\sigma(y) \in \sigma(A) \cap \mathcal{L}$.

Definition 9. An $\mathscr{F}_U$-section $\widetilde{\sigma}\colon B \to E$ is called an *$\mathscr{F}_U$-extension* of the $\mathscr{F}_U$-section $\sigma\colon A \to E$, if $A \subset B \subset U$ and $\widetilde{\sigma}|_A = \sigma$. If $U = M$, an $\mathscr{F}_U$-section is called an *$\mathscr{F}$-section*, and an $\mathscr{F}_U$-extension is called an *$\mathscr{F}$-extension*.

Lemma 1. *Let (E,π,M,F) be a foliated bundle with standard fibre $\mathbb{R}^m$ over a foliation $(M,\mathscr{F})$. Let a be a point of M and let (U,φ) be a foliated chart with centre at the point a. Then for any compact subset A of U every $\mathscr{F}_U$-section $\sigma\colon A \to E$ admits an $\mathscr{F}_U$-extension to U.*

Proof. Let $r\colon U \to U/\mathscr{F}_U = \mathbb{R}^q$ be the projection to the space of leaves of the foliation $(U,\mathscr{F}_U)$ identified with $\mathbb{R}^q$, where $q = \operatorname{codim}\mathscr{F}$. Since the neighbourhood U is contractible, the total space of the R^m-bundle $\pi^{-1}(U)$ can be identified with $U \times \mathbb{R}^m$. Here the foliation $F_{\pi^{-1}(U)}$ is identified with the trivial foliation $F_{st} := \{L \times r(L) \times v \mid L \in \mathscr{F}_U,\ v \in \mathbb{R}^m\}$, and the $\mathscr{F}_U$-section σ can be written in the form $\sigma(x) = (x, \widetilde{\sigma} \circ r(x)) \in U \times \mathbb{R}^m$, $x \in A$, where $\widetilde{\sigma}\colon \widetilde{A} := r(A) \to \mathbb{R}^m$ is the induced smooth map satisfying $\widetilde{\sigma} \circ r = \mathrm{pr}_2 \circ \sigma$, and $\mathrm{pr}_2\colon U \times \mathbb{R}^m \to \mathbb{R}^m$ is the projection to the second factor.

The map $\widetilde{\sigma}\colon \widetilde{A} \to \mathbb{R}^m$ can be identified with a tuple of m smooth functions $h_1,\dots,h_m\colon \widetilde{A} \to \mathbb{R}$. By assumption, the set A is compact, so $\widetilde{A}$ is compact too. Consequently, $\widetilde{A}$ is a closed subset of $\mathbb{R}^q$. It is known that any smooth function on a closed subset of $\mathbb{R}^q$ admits a smooth extension to the whole of $\mathbb{R}^q$. This implies that there exists a smooth extension $\widetilde{\delta}\colon \mathbb{R}^q \to \mathbb{R}^m$ of the map $\widetilde{\sigma}\colon \widetilde{A} \to \mathbb{R}^m$. Hence the equality $\delta(x) := (x, \widetilde{\delta} \circ r(x))\ \forall\, x \in U$ defines a smooth $\mathscr{F}_U$-section δ, which is an extension of the section σ.

Remark 1. Lemma 1 plays an important role in this section. In general, it does not hold for arbitrary closed subsets A, because the projection to the space of leaves of a foliation is not a closed map.

Proposition 4. *Let (E, π, M, F) be a foliated bundle with standard fibre $\mathbb{R}^m$ over a foliation $(M, \mathscr{F})$, and let $\pi|_{\mathcal{L}}\colon \mathcal{L} \to L\ \forall\, \mathcal{L} \in F$ be a diffeomorphism. Then for any closed subset A of M every $\mathscr{F}$-section $\sigma\colon A \to E$ defined on A admits an extension to an $\mathscr{F}$-section on M. In particular, for any points $x \in M$, $u \in \pi^{-1}(x) \subset E$, there exists an $\mathscr{F}$-section $\sigma\colon M \to E$ such that $\sigma(x) = u$.*

Proof. Since M is paracompact, there is a covering $\{U_i \mid i \in \mathbb{N}\}$ of the space M by foliated neighbourhoods and a locally finite open covering $\{V_j \mid j \in J\}$ of the manifold M with the following property:

for any V_j there exists U_i such that the closure $\overline{V}_j$ is compact and $\overline{V}_j \subset U_i$

(see [12], for instance). If $Y \subset J$ is an arbitrary subset of indices, we set $B_Y := \bigcup_{j \in Y} \overline{V}_j$.

We will show that B_Y is closed in M. If $x \in \overline{B}_Y$, then there exists a sequence $\{x_n\} \subset B_Y$ that converges to x. As the covering $\{V_j \mid j \in J\}$ is locally finite, there is a neighbourhood of x that intersects only finitely many elements of this covering. Therefore there is a subsequence $\{x_{n_k}\}$ that is contained in one subset V_{j_0}, $j_0 \in Y$. Consequently, $x \in \overline{V}_{j_0} \subset B_Y$ and B_Y is closed in M.

We let $\mathcal{P}$ denote the set of triples (δ, Y, U), where $Y \subset J$, U is an open neighbourhood of B_Y, and $\delta\colon B_Y \to E$ is an $\mathscr{F}_U$-section, which coincides with the restriction $\sigma|_{A \cap B_Y}$ of σ to $A \cap B_Y$.

Assume that $A \neq \varnothing$. Then we have $V_j \cap A \neq \varnothing$ for some $j \in J$. The set $A_j = \overline{V}_j \cap A$ is compact as a closed subset of the compact set $\overline{V}_j$. Let $\overline{V}_j \subset U_i$. We will show that $\sigma_j := \sigma|_{A_j}\colon A_j \to E$ is an $\mathscr{F}_{U_i}$-section. Take any two points x, y on a leaf $L_0 \in \mathscr{F}_{U_i}$ contained in A_j. Then x, y are in the same leaf L of the foliation $\mathscr{F}$ that contains L_0. Since by assumption $\sigma\colon A \to E$ is an $\mathscr{F}$-section, the points $\sigma(x)$ and $\sigma(y)$ are in the same leaf $\mathcal{L}$ of the lifted foliation (E, F), and $\pi(\mathcal{L}) = L$. Moreover, by hypothesis the restriction $\pi|_{\mathcal{L}}\colon \mathcal{L} \to L$ is a diffeomorphism, so the intersection of $\pi^{-1}(L_0)$ with $\mathcal{L}$ is arcwise connected, and the points $\sigma(x)$, $\sigma(y)$ are in the same leaf of the foliation $F_{\pi^{-1}(U_i)}$. This means that σ_j is an $\mathscr{F}_{U_i}$-section. By Lemma 1, there exists an $\mathscr{F}_{U_i}$-extension δ of the section σ_j to U_i. Thus $\overline{\delta}_j := \delta|_{\overline{V}_j}\colon \overline{V}_j \to E$ is an F_{U_i}-section, and $\overline{\delta}_j|_{\overline{V}_j \cap A} = \sigma|_{\overline{V}_j \cap A}$, so $(\overline{\delta}_j, j, U_i) \in \mathcal{P}$ and $\mathcal{P} \neq \varnothing$.

We introduce a partial order on $\mathcal{P}$ by $(\delta_1, Y_1, U_1) \preceq (\delta_2, Y_2, U_2)$ if $Y_1 \subset Y_2$, $U_1 \subset U_2$ and $\delta_2|_{B_{Y_1}} = \delta_1$. The same arguments as in the proof of Theorem 3.4.1 in [13] show that every linearly ordered subset of $\mathcal{P}$ is bounded above. By the Zorn Lemma, the set $(\mathcal{P}, \preceq)$ has a maximal element (δ_0, Y_0, U_0).

Assume that $Y_0 \neq J$ and $s \in J \setminus Y_0$. Set $C = (A \cup B_{Y_0}) \cap \overline{V}_s \subset \overline{V}_s$. There exists U_k, $k \in \mathbb{N}$, that contains $\overline{V}_s$. Then we can find a section $\varepsilon\colon C \to E$ which coincides with σ on $A \cap \overline{V}_s$ and with δ_0 on $B_{Y_0} \cap \overline{V}_s$. As above, we can check that ε is an $\mathscr{F}_{U_k}$-section. Moreover, C is compact as the trace of the compact set

$A \cup B_{Y_0}$ on the compact set $\overline{V}_s$. Thus the conditions of Lemma 1 are fulfilled. This implies there exists an F_{U_k}-extension $\overline{\varepsilon}\colon \overline{V}_s \to E$ of the section ε that coincides with σ on $A \cap \overline{V}_s$. Therefore $(\overline{\varepsilon}, s, U_k) \in \mathcal{P}$. Let $U' := U_0 \cup U_k$ and $Y_1 := Y_0 \cup \{s\}$. Then $B_{Y_1} = B_{Y_0} \cup \overline{V}_s$. We define a section $\tau\colon B_{Y_1} \to E$, setting $\tau|_{B_{Y_0}} := \delta_0$, $\tau|_{\overline{V}_s} = \overline{\varepsilon}$. The equalities $\overline{\varepsilon}|_{B_{Y_0} \cap \overline{V}_s} = \delta_0$ and $\overline{\varepsilon}|_{A \cap \overline{V}_s} = \delta_0|_{A \cap \overline{V}_s} = \sigma|_{A \cap \overline{V}_s}$ yield that the section τ is well-defined and τ is an $\mathscr{F}_{U'}$-section with $\tau|_{B_{Y_0}} = \delta_0$. Consequently, $(\tau, Y_1, U') \in \mathcal{P}$ and $(\delta_0, Y_0, U_0) \preceq (\tau, Y_1, U')$. This contradicts the maximality of the element (δ_0, Y_0, U_0) in $\mathcal{P}$. Thus $Y_0 = J$ and δ_0 is an $\mathscr{F}$-extensions of the section σ to the whole manifold M.

Now we fix points $x \in M$, $u \in \pi^{-1}(x) \subset E$. Let $A = \{x\}$ be a one-point set and $\sigma_0\colon A \to E\colon x \to u$. Then σ_0 is an $\mathscr{F}$-section. By the above there exists an extension of σ_0 to an $\mathscr{F}$-section $\sigma\colon M \to E$.

Remark 2. If $\mathscr{F}$ is a zero-dimensional foliation of a manifold M, then Theorem 5.7 from [9], Ch. I follows from Proposition 4.

4.3. Cartan foliations and their holonomy groups. A Cartan foliation $(M, \mathscr{F})$ of type (G, H) (or $\mathfrak{g}/\mathfrak{h}$) is represented by the following objects:

1) a Lie group G and its closed Lie subgroup H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively;

2) a principal H-bundle $\pi\colon \mathscr{R} \to M$;

3) an H-invariant foliation $(\mathscr{R}, F)$ such that π is a morphism of the foliation $(\mathscr{R}, F)$ to $(M, \mathscr{F})$;

4) a $\mathfrak{g}$-valued H-equivariant 1-form ω on $\mathscr{R}$ with the following properties:

(i) $\omega(A^*) = A$ for any $A \in \mathfrak{h}$, where A^* is the fundamental vector field corresponding to A;

(ii) for any point $u \in \mathscr{R}$ the map $\omega_u\colon T_u\mathscr{R} \to \mathfrak{g}$ is surjective, and $\operatorname{Ker}\omega = TF$ is a distribution tangent to the foliation $(\mathscr{R}, F)$;

(iii) the Lie derivative $L_X\omega$ vanishes for any vector field X from $\mathfrak{X}_F(\mathscr{R})$.

The H-bundle $\pi\colon \mathscr{R} \to M$ under consideration is called a *foliated bundle* for $(M, \mathscr{F})$, and $(\mathscr{R}, F)$ is called a *lifted foliation*. It is known that $(\mathscr{R}, F)$ is a transversally parallelizable foliation, or an e-foliation.

In what follows we assume that the action of the group G on the space G/H by left translations is effective. Then definitions of the Cartan foliation given by Blumenthal (see [14]) and by the author (see [3]) are equivalent.

The following proposition can be proved in the same way as Theorem 4 from our paper [15]. It does not use the completeness of the foliation.

Proposition 5. *Let $(M, \mathscr{F})$ be a Cartan foliation represented by an N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}$ and let $(\mathscr{R}(M, H), F)$ be a foliated bundle over $(M, \mathscr{F})$ with projection $\pi\colon \mathscr{R} \to M$. Let $L = L(x)$, $x \in M$, be a leaf of the foliation $(M, \mathscr{F})$, and let $\mathcal{L} = \mathcal{L}(u)$, $u \in \pi^{-1}(x)$, be a leaf of the lifted foliation $(\mathscr{R}, F)$. Then the germ holonomy group $\Gamma(L, x)$ of the leaf L is isomorphic to each of the following groups*:

1) *the subgroup $H(\mathcal{L}) := \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$ of the group H;*

2) *the group of covering transformations of a regular covering map $\pi|_{\mathcal{L}}\colon \mathcal{L} \to L$;*

3) *the group $\mathscr{H}_v$, where $v = f_i(x)$, $x \in U_i$, consisting of germs of local diffeomorphisms from the holonomy pseudogroup $\mathscr{H} = \mathscr{H}(M, \mathscr{F})$ fixing the point v.*

Now we turn to another point $u' \in \pi^{-1}(x)$. Let $\mathcal{L}' = \mathcal{L}(u')$. Then the subgroup $H(\mathcal{L})$ is replaced by a conjugate subgroup $H(\mathcal{L}')$ in the Lie group H. Therefore the following definition is well-posed.

Definition 10. The holonomy group of a leaf L is called *inessential* if the subgroup $H(\mathcal{L})$ of the Lie group H isomorphic to it (by Proposition 5) has a compact closure in H. Otherwise the holonomy group of the leaf L is called *essential.*

4.4. Conformal foliations as Cartan foliations. Let $G = \mathrm{Conf}(S^q)$ be the Lie group of all conformal transformations of a q-dimensional sphere S^q, where $q \geqslant 3$. Let H denote the isotropy subgroup of some point $x_0 \in S^q$ in the group G. Then H can be considered as the Lie group of all conformal transformations of the q-dimensional Euclidean space $\mathbb{E}^q$. Consequently, $H = \mathrm{Sim}(\mathbb{E}^q) = CO(q) \ltimes \mathbb{R}^q$ is the Lie group of all similarities of the space $\mathbb{E}^q$.

Now we will consider an arbitrary conformal foliation $(M, \mathscr{F})$ of codimension $q \geqslant 3$ as a Cartan foliation modelled on an effective transversal Cartan geometry $\eta = (P(H,N), \alpha)$ of type (G,H), where G and H are the Lie groups indicated above, N is a conformal manifold (possibly disconnected), and $P(H,N)$ is a principal H-bundle over N with normal conformal connection α in the sense of [16]. Let $(\mathscr{R}, F)$ be the lifted foliation for the conformal foliation $(M, \mathscr{F})$, and let $\pi\colon \mathscr{R} \to M$ be the projection of the foliated bundle to M.

4.5. Conformal transformations of Euclidean space. We denote an element of the group $H = \mathrm{Sim}(\mathbb{E}^q) = \mathbb{R}^+ \cdot O(q) \ltimes \mathbb{R}^q$ by the pair $\langle \lambda \cdot A, a\rangle$, where $\lambda \in \mathbb{R}^+$, $A \in O(q)$, $a \in \mathbb{R}^q$. The composition of maps has the form

$$\langle \lambda \cdot A, a\rangle \langle \mu \cdot B, b\rangle = \langle \lambda\mu \cdot AB, \lambda \cdot Ab + a\rangle,$$

where $\langle \lambda \cdot A, a\rangle$, $\langle \mu \cdot B, b\rangle \in H$. We set $H^0 := \mathbb{R}^+ \cdot E \ltimes \mathbb{R}^q$.

Lemma 2. *The group H^0 is a normal subgroup of H.*

Proof. For any elements $\langle C, c\rangle \in H$ and $\langle \lambda \cdot E, a\rangle \in H^0$ there exists an element $\langle \lambda \cdot E, d\rangle = \langle \lambda \cdot E, Ca + (1-\lambda)c\rangle \in H^0$ which satisfies $\langle C, c\rangle\langle \lambda \cdot E, a\rangle = \langle \lambda \cdot E, d\rangle\langle C, c\rangle$.

Lemma 3. *For every maximal compact subgroup K of H the intersection $K \cap H^0$ is the unit subgroup.*

Proof. Since $O(q)$ is a maximal compact subgroup of H, any other maximal subgroup K in H is conjugate to $O(q)$. Note that $H^0 = \{\langle \lambda \cdot E, a\rangle \mid \lambda \in \mathbb{R}^+,\ a \in \mathbb{R}^q\}$. If $K = \langle \mu \cdot B, b\rangle O(q) \langle \mu \cdot B, b\rangle^{-1}$, where $\langle \mu \cdot B, b\rangle \in H$, $\mu \in \mathbb{R}^+$, $B \in O(q)$ and $b \in \mathbb{R}^q$, then

$$K = \bigl\{\langle A, (E-A)b\rangle \mid A \in O(q),\ b \in \mathbb{R}^q\bigr\}.$$

Consequently, $K \cap H^0 = \langle E, 0\rangle$, where $\langle E, 0\rangle$ is the unit of H.

4.6. Proof of Theorem 3. Let $(M, \mathscr{F})$ be a conformal foliation of codimension $q \geqslant 3$ considered as a Cartan foliation modelled on a Cartan geometry $(P(N,H), \omega)$, where $G = \mathrm{Conf}(S^q)$, $H = CO(q) \ltimes \mathbb{R}^q$, and ω is a normal conformal connection. We let $(\mathscr{R}(M,H), F)$ denote a foliated H-bundle over $(M, \mathscr{F})$.

Assume that all the holonomy groups $H(u)$, $u \in \mathscr{R}$, are relatively compact.

Since we have a free proper action of the Lie group H on $\mathscr{R}$, the action of the normal subgroup $H^0 := \mathbb{R}^+ \cdot E \ltimes \mathbb{R}^q$ of the group H on $\mathscr{R}$ is free and proper as well. Therefore the orbit space $\widehat{\mathscr{R}} := \mathscr{R}/H^0$ is a smooth manifold, and the natural projection $r\colon \mathscr{R} \to \widehat{\mathscr{R}}$ onto the orbit space is a principal H^0-bundle. In this case the action of the factor group $H/H^0 = O(q)$, defined on the orbit space $\widehat{\mathscr{R}} := \mathscr{R}/H^0$ by the formula $(uH^0)a = uaH^0$ for any $uH^0 \in \widehat{\mathscr{R}}$ and $a \in O(q)$, is free. Moreover, the orbit space $\widehat{\mathscr{R}}/O(q)$ coincides with M, and the projection $\widehat{\pi}\colon \widehat{\mathscr{R}} \to M$ satisfies $\widehat{\pi} \circ r = \pi$. The foliation $(\mathscr{R}, F)$ is H-invariant, so it is H^0-invariant. This shows that there is an induced foliation $\widehat{F}$ on $\widehat{\mathscr{R}}$, whose leaves are the images of the leaves of the foliation F under the map r. So $(\mathscr{R}(\widehat{\mathscr{R}}, H^0), F)$ is a foliated principal H^0-bundle with standard fibre $\mathbb{R}^{q+1}$, diffeomorphic to H^0, with the projection $r\colon \mathscr{R} \to \widehat{\mathscr{R}}$ over the foliation $(\widehat{\mathscr{R}}, \widehat{F})$.

For any point $u \in \mathscr{R}$ and a leaf $\mathscr{L} = \mathscr{L}(u)$ the restriction $r|_{\mathscr{L}}\colon \mathscr{L} \to \widehat{\mathscr{L}}$ is a regular covering with the group of covering transformations isomorphic to the group $H^0(u) := \{b \in H^0 \mid R_b(\mathscr{L}) = \mathscr{L}\}$, to the leaf $\widehat{\mathscr{L}}$ of the foliation $(\widehat{\mathscr{R}}, \widehat{F})$. Note that $H^0(u) = H(u) \cap H^0$, where $H(u) = \{a \in H \mid R_a(\mathscr{L}) = \mathscr{L}\}$. By assumption, the group $H(u)$ is relatively compact. In this case the closure of the group $H(u)$, and hence also $H(u)$ itself, is contained in some maximal compact subgroup K of H, which is conjugate to $O(q)$. Therefore $H^0(u) \subseteq K \cap H^0$. By Lemma 3, $K \cap H^0 = \{e\}$, where e is the unit of H. This shows that $H^0(u) = \{e\}$ and $r|_{\mathscr{L}}\colon \mathscr{L} \to \widehat{\mathscr{L}}$ is a diffeomorphism. Consequently, the conditions of Proposition 4 are satisfied, and there exists a foliated section $\sigma\colon \widehat{\mathscr{R}} \to \mathscr{R}$ of the foliated principle H^0-bundle $(\mathscr{R}(\widehat{\mathscr{R}}, H^0), F)$. The section σ defines a trivialization of this bundle by the formula

$$f\colon \mathscr{R} \to \widehat{\mathscr{R}} \times H^0\colon \sigma(u)\cdot h \mapsto (u,h) \qquad \forall\, u \in \widehat{\mathscr{R}}, \quad \forall\, h \in H^0.$$

Here $f(\sigma(u)) = (u, e)$ and $\sigma(ua) = \sigma(u)\cdot a$ for each $a \in O(q)$. The equalities $\widehat{\pi} \circ r = \pi$ and $r \circ \sigma = \mathrm{id}_{\widehat{R}}$ yield $\widehat{\pi} = \pi \circ \sigma$.

Identifying $\widehat{\mathscr{R}}$ with $\sigma(\widehat{\mathscr{R}})$, we obtain a foliated reduction of the bundle $(\mathscr{R}(M, H), F)$ to the orthogonal subgroup $O(q)$. Similarly to [17], § 2.2.2 we can check that $(M, \mathscr{F})$ is a Cartan foliation of type $(\widehat{G}, O(q))$, where $\widehat{G} = O(q) \ltimes \mathbb{R}^q$ is the Lie group of all isometries of the Euclidean space $\mathbb{E}^q$, and $(\widehat{R}(M, O(q)), \widehat{F})$ is its foliated bundle. This means that the foliation $(M, \mathscr{F})$ is Riemannian. Moreover, if the Cartan geometry $\xi = (P(N, H), \omega)$, on which the foliation $(M, \mathscr{F})$ is modelled, corresponds to the conformal geometry $(N, [g])$, then the reduction of the foliated bundle $(\mathscr{R}(M, H), F)$ to the compact subgroup $O(q)$ yields a reduction of the H-bundle $P(N, H)$ to the subgroup $O(q)$. This implies a Riemannian metric $d \in [g]$ exists on N such that $(M, \mathscr{F})$ is a Riemannian foliation modelled on the Riemannian manifold (N, d).

Conversely, assume that there exists a Riemannian metric $d \in [g]$ on N such that a conformal foliation $(M, \mathscr{F})$ is a Riemannian foliation modelled on a Riemannian manifold (N, d). Then there exists a reduction of the H-bundle $P(N, H)$ to a closed subgroup of $O(q)$. This reduction induces the corresponding foliated reduction of the foliated bundle $(\mathscr{R}(M, H), F)$ to the subgroup $O(q)$. Since $O(q)$ is compact, using Proposition 5 all the holonomy groups of the original foliation $(M, \mathscr{F})$ are relatively compact.

§ 5. A complete non-Riemannian conformal foliation is transversally flat

5.1. Completeness of conformal foliations. Let $(\mathscr{R}, F)$ be the lifted foliation for a conformal foliation $(M, \mathscr{F})$, and let $\pi\colon \mathscr{R} \to M$ be the projection of the foliated bundle to M. It is known that $(\mathscr{R}, F)$ is a transversally parallelizable foliation, or an e-foliation.

We recall (see [4]) that an e-foliation $(\mathscr{R}, F)$ is called *complete* (or $\mathfrak{N}$-*complete*) if there exists a distribution $\mathfrak{N}$ on $\mathscr{R}$ transversal to $(\mathscr{R}, F)$ such that any vector field on $X \in \mathfrak{X}_{\mathfrak{N}}(\mathscr{R})$ with $\omega(X) = \mathrm{const}$ is complete. A conformal foliation $(M, \mathscr{F})$ is *complete* (or $\mathfrak{M}$-*complete*) if there exists a distribution $\mathfrak{M}$, transversal to $(M, \mathscr{F})$, such that the lifted e-foliation $(\mathscr{R}, F)$ is $\mathfrak{N}$-complete, where $\mathfrak{N} = \pi^*\mathfrak{M}$. In other words, a conformal foliation is called *complete* if it is a complete Cartan foliation in the sense of [3].

Lemma 4. *Let $(M, \mathscr{F})$ be a conformal foliation of codimension $q \geqslant 3$ with Ehresmann connection $\mathfrak{M}$. Assume that it is covered by a bundle $r\colon \widetilde{M} \to S^q$, where $f\colon \widetilde{M} \to M$ is some covering map for M. Then $(M, \mathscr{F})$ is an $\mathfrak{M}$-complete conformal foliation.*

Proof. We note that $(M, \mathscr{F})$ is a $(\mathrm{Conf}(S^q), S^q)$-foliation modelled on a Cartan geometry $\eta_0 = (G(S^q, H), \omega_G)$, where $G = \mathrm{Conf}(S^q)$, $H = CO(q) \ltimes \mathbb{R}^q$, and ω_G is the Cartan-Maurer form on the group G. We denote the foliated H-bundle for $(M, \mathscr{F})$ by $\pi\colon \mathscr{R} \to M$. Consider

$$\widetilde{\mathscr{R}} = f^*\mathscr{R} = \bigl\{(u, x) \in \mathscr{R} \times \widetilde{M} \mid \pi(u) = f(x)\bigr\},$$

the preimage of the H-bundle $\pi\colon \mathscr{R} \to M$ under the map $f\colon \widetilde{M} \to M$. Then $\widetilde{\pi}\colon \widetilde{\mathscr{R}} \to \widetilde{M}\colon (u, x) \mapsto x$ is a principal H-bundle, and $\widetilde{f}\colon \widetilde{\mathscr{R}} \to \mathscr{R}\colon (u, x) \mapsto u$ is a covering map with $f \circ \widetilde{\pi} = \pi \circ \widetilde{f}$. Since $r\colon \widetilde{M} \to S^q$ is a submersion with Ehresmann connection $\widetilde{\mathfrak{M}} = f^*\mathfrak{M}$, there is a surjective submersion $\widetilde{r}\colon \widetilde{\mathscr{R}} \to G$ with Ehresmann connection $\widetilde{\mathfrak{N}} = \widetilde{\pi}^*\widetilde{\mathfrak{M}}$. We have an induced $\mathfrak{g}$-valued 1-form $\widetilde{\omega}$ on $\widetilde{\mathscr{R}}$ such that $\widetilde{\omega} = \omega \circ \widetilde{f}_*$. This form has properties (i)–(iii) (see 4.3) with respect to the foliation $(\widetilde{M}, \widetilde{F})$, where $\widetilde{F} = f^*F$. Since any vector field $Y \in \mathfrak{X}_{\widetilde{\mathfrak{N}}}(\widetilde{\mathscr{R}})$, such that $\widetilde{\omega}(Y) = \mathrm{const} \in \mathfrak{g}$, projects to a left-invariant vector field $Z := \widetilde{r}_*Y$ on the Lie group G, and $\omega_G(Z) = \widetilde{\omega}(Y) = \mathrm{const} \in \mathfrak{g}$, the vector field Y is complete. This means that $(\widetilde{M}, \widetilde{\mathscr{F}})$ is an $\widetilde{\mathfrak{M}}$-complete conformal foliation. Finally, we conclude that $(M, \mathscr{F})$ is an $\mathfrak{M}$-complete conformal foliation.

5.2. A neighbourhood of a leaf with essential holonomy group. The following lemma is proved without assuming that the conformal foliation is complete.

Lemma 5. *For every leaf L with essential holonomy group of a conformal foliation $(M, \mathscr{F})$ there exists an open saturated neighbourhood $\mathscr{U}$ such that $(\mathscr{U}, \mathscr{F}_{\mathscr{U}})$ is a $(\mathrm{Conf}(S^q), S^q)$-foliation, and the closure $\overline{L}_\alpha$ of any leaf L_α from $\mathscr{U}$ contains L.*

Proof. Let $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ be an N-cocycle representing the conformal foliation $(M, \mathscr{F})$ of codimension $q \geqslant 3$. Since this foliation is non-Riemannian, Theorem 3 implies that there exists a leaf L with essential holonomy group. By Proposition 5, the group $H(\mathscr{L})$ contains an element of the form $\langle\lambda\cdot A, a\rangle \in H$, where $\lambda\cdot A \in CO(q)$ and $a \in \mathbb{R}^q$, satisfying at least one of the conditions: 1) $\lambda \neq 1$; 2) $a \neq 0$. As in the proof of Theorem 7 from [7] we show that a local conformal diffeomorphism γ of a Riemannian manifold (N, g_N) from the holonomy pseudogroup $\mathscr{H}$, that corresponds to the element $\langle\lambda\cdot A, a\rangle$, satisfies the following conditions at the point $v = f_i(x)$, $x \in L$: $\gamma(v) = v$, $\gamma_{*v} = \lambda\cdot A$ and $a = db|_v$, where $\gamma^* g_N = e^b g_N$. Therefore γ is an essential conformal transformation at the point v in the terminology of [8]. If $\lambda \neq 1$ then we can assume that $\lambda \in (0, 1)$. Thus there exists a neighbourhood $\mathscr{V}$ of the point v such that $\gamma^n(z) \to v$ as $n \to +\infty$ for any $z \in \mathscr{V}$. If $\lambda = 1$ then it is shown in [8] that there exists an open neighbourhood $\mathscr{V} \subset V_i = f_i(U_i)$ of v in N such that at any point y of $\mathscr{V}$ at least one of subsequences $\gamma^k(y)$, $\gamma^{-k}(y)$, where k is any positive integer, is defined, and this sequence tends to v as $k \to +\infty$.

Denote by U the preimage of $\mathscr{V}$ under the map f_i, that is, $U = f_i^{-1}(\mathscr{V})$. Then $\mathscr{U} := \bigcup_\alpha L_\alpha$, where $L_\alpha \cap U \neq \varnothing$ and $L_\alpha \in \mathscr{F}$ is a connected open saturated subset of M. Using the properties of the neighbourhood $\mathscr{V}$, for every leaf L_α in $\mathscr{U}$ the inclusion $L \subset \overline{L}_\alpha$ holds.

The Weyl tensor W of the conformal curvature of the Riemannian manifold $(\mathscr{V}, g_N)$ is invariant with respect to conformal transformations. This shows that W does not depend on a choice of the point $y \in \mathscr{V}$, and $\|W\| = \mathrm{const}$. We can check directly that if $\|W\| \neq 0$ then any local conformal transformation of the Riemannian manifold $(\mathscr{V}, g_N)$ is a local isometry of $(\mathscr{V}, h)$, where $h = \|W\| g_N$. Hence $(\mathscr{U}, \mathscr{F}_{\mathscr{U}})$ is a Riemannian foliation. This contradiction to Theorem 3 shows that $W \equiv 0$ on $\mathscr{V}$. If $q = \dim \mathscr{V} = 3$, then $W \equiv 0$, and instead of the Weyl tensor we should take other conformal invariant, the Schouten tensor V. In the same way we check that $V \equiv 0$ on $\mathscr{V}$. By the Schouten-Weyl Theorem (see [18]), the identities $W \equiv 0$ with $q \geqslant 4$ and $V \equiv 0$ with $q = 3$ mean that the q-dimensional Riemannian manifold $(\mathscr{V}, g_N)$ is locally conformally flat for any $q \geqslant 3$. This yields that $(\mathscr{U}, \mathscr{F}_{\mathscr{U}})$ is a $(\mathrm{Conf}(S^q), S^q)$-foliation.

5.3. Non-Riemannian conformal foliations. We recall that a neighbourhood U_k from an N-cocycle η representing a foliation $(M, \mathscr{F})$ is called *distinguished* if there exists another connected neighbourhood U_m from η such that the closure $\overline{U}_k$ is contained in U_m and $f_k = f_m|_{U_k}$. In this case we say that U_k is subordinated to the neighbourhood U_m. Note that for any point of M there exists a distinguished neighbourhood.

Lemma 6. *Let $(M, \mathscr{F})$ be a non-Riemannian conformal foliation of codimension $q \geqslant 3$ admitting an Ehresmann connection $\mathfrak{M}$. Then $(M, \mathscr{F})$ is an $\mathfrak{M}$-complete transversally flat conformal foliation covered by a bundle $r\colon \widehat{M} \to B$, where $f\colon \widehat{M} \to M$ is a regular covering map satisfying the conditions of Theorem 2, and B is either equal to $\mathbb{E}^q$ or S^q. Moreover, $B = \mathbb{E}^q$ if and only if $(M, \mathscr{F})$ is a transversally similar foliation.*

Proof. By assumption, $(M, \mathscr{F})$ is a non-Riemannian conformal foliation of codimension $q \geqslant 3$. We apply Theorem 3 and conclude that there exists a leaf L with essential holonomy group. By Lemma 5, there exists an open saturated neighbourhood $\mathscr{U}$ of the leaf L, such that $(\mathscr{U}, \mathscr{F}_{\mathscr{U}})$ is a $(\mathrm{Conf}(S^q), S^q)$-foliation, which is not Riemannian. Since $\mathfrak{M}_{\mathscr{U}}$ is an Ehresmann connection for $(\mathscr{U}, \mathscr{F}_{\mathscr{U}})$, Theorem 2 shows that there exists a regular covering map $f\colon \widehat{\mathscr{U}} \to \mathscr{U}$. For this map the foliation $(\widehat{\mathscr{U}}, \widehat{\mathscr{F}})$, where $\widehat{\mathscr{F}} := f^*\mathscr{F}_{\mathscr{U}}$, is formed by fibres of a locally trivial bundle $r\colon \widehat{\mathscr{U}} \to B$, where B is a $(\mathrm{Conf}(S^q), S^q)$-manifold. Therefore B admits a Riemannian metric g_B such that the global holonomy group Ψ of the foliation $(\mathscr{U}, \mathscr{F}_{\mathscr{U}})$ is a group of conformal transformations of the Riemannian manifold (B, g_B). The holonomy group of the leaf L is isomorphic to the isotropy subgroup Ψ_z, where $z \in r(f^{-1}(L))$. Thus Ψ_z is an essential group of conformal transformations of (B, g_B). Therefore there is an essential transformation $\psi \in \Psi_z$. Lemma 5 from [7] implies that there exists an open neighbourhood U of z such that the subset $P = \bigcup_{n\in\mathbb{Z}} \psi^n(U)$ of the Riemannian manifold (B, g_B) is conformally equivalent to either the standard sphere S^q or the Euclidean space $\mathbb{E}^q$.

Case 1. Let $B = S^q$. By Lemma 4, $(\mathscr{U}, \mathscr{F}_{\mathscr{U}})$ is an $\mathfrak{M}$-complete conformal foliation. This implies that $\mathscr{U}$ is a closed subset of M. Therefore as M is connected, $\mathscr{U} = M$. This shows that $(M, \mathscr{F})$ is an $\mathfrak{M}$-complete foliation covered by the bundle $r\colon \widehat{M} \to S^q$.

Case 2. Let $B = \mathbb{E}^q$ and $\mathscr{U} = M$. Again by analogy with Lemma 4 we obtain that $(M, \mathscr{F})$ is an $\mathfrak{M}$-complete $(\mathrm{Sim}\,\mathbb{E}^q, \mathbb{E}^q)$-foliation, that is, $(M, \mathscr{F})$ is a complete transversally similar foliation.

Case 3. Suppose that $B = \mathbb{E}^q$ and $\mathscr{U} \neq M$. Let $x \in U_i \cap L$, where U_i is a neighbourhood from the N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ representing the foliation, and $v = f_i(x)$. We can assume that $U_i \subset \mathscr{U}$. Since $(\mathscr{U}, \mathscr{F}_{\mathscr{U}})$ is a $(\mathrm{Conf}(S^q), S^q)$-foliation, its holonomy pseudogroup $\mathscr{H}$ is complete. Therefore for an essential local conformal transformation $\gamma \in \mathscr{H}$ at the point v there is an open neighbourhood $V \subset f_i(U_i)$ such that γ^n is defined on V for any integer n. By the same arguments as in the proof of Lemma 5 from [7], the subset $P = \bigcup_{n\in\mathbb{Z}} \gamma^n(V)$ of a Riemannian manifold N is conformally equivalent to either the standard sphere S^q or the Euclidean space $\mathbb{E}^q$. In the first case we have $B = S^q$, a contradiction.

So $P \cong \mathbb{E}^q \cong S^q \setminus \{a\}$. Since the conformal foliation $(M, \mathscr{F})$ is $\mathfrak{M}$-complete, every two leaves of this foliation can be connected by an $\mathfrak{M}$-curve. We have $M \setminus \mathscr{U} \neq \varnothing$, so there is an $\mathfrak{M}$-curve $\sigma\colon [0,1] \to M$ such that

$$\sigma([0,\epsilon)) \subset U_i \subset \mathscr{U}, \qquad x = \sigma(0), \qquad y = \sigma(\epsilon) \in M \setminus \mathscr{U}.$$

We can assume that U_i is a distinguished neighbourhood and $\sigma(0,\epsilon) \subset U_i \cap U_j$, where $y \in U_j$.

Since $U_i \cap U_j \neq \varnothing$, there is a local conformal diffeomorphism

$$\gamma_{ij}\colon V' \to V'', \qquad V' = f_j(U_i \cap U_j), \quad V'' = f_i(U_i \cap U_j),$$

such that $f_i = \gamma_{ij} \circ f_j$ on $U_i \cap U_j$. Let $S^q_{(1)}$ and $S^q_{(2)}$ be two copies of the standard sphere. Since $(\mathscr{U}, \mathscr{F}_{\mathscr{U}})$ is a $(\mathrm{Conf}(S^q), S^q)$-foliation, there exist conformal embeddings $\delta_{(1)}\colon V' \to S^q_{(1)}$, $\delta_{(2)}\colon V'' \to S^q_{(2)}$ and a conformal diffeomorphism $\Gamma\colon S^q_{(1)} \to S^q_{(2)}$ such that

$$\delta_{(2)} \circ \gamma_{ij} = \Gamma \circ \delta_{(1)}.$$

We let N_v denote the connected component of a manifold N that contains v. Since U_i is a distinguished neighbourhood, there exists a connected neighbourhood U_m such that U_i is subordinated to U_m, that is, $\overline{U}_i \subset U_m$. Hence $f_m(y) \in N_v$. The definition of $\mathscr{U}$ and the inclusion $U_i \subset \mathscr{U}$ imply that $f_m(y) = a$ and $\gamma_{mj}(a) = v'$, where $v' = f_j(y)$. This shows that $P \cup a \cong S^q$ is a compact q-dimensional submanifold of N, thus $P \cup a \cong S^q$ is a closed submanifold of N. As the dimensions are equal, the union $P \cup a$ is open in N. So we obtain $N_v = S^q$.

Here $\mathscr{U}' = \mathscr{U} \cup L(y)$ is an open saturated subset of M, and $(\mathscr{U}', \mathscr{F}_{\mathscr{U}'})$ is a conformal foliation with Ehresmann connection covered by a locally trivial bundle $r\colon \widehat{\mathscr{U}'} \to S^q$. This shows that the foliation $(\mathscr{U}', \mathscr{F}_{\mathscr{U}'})$ corresponds to Case 1. Consequently, $M = \mathscr{U}'$ and the foliation $(M, \mathscr{F})$ is covered by the bundle $\widehat{M} \to S^q$.

By the above, the foliation $(M, \mathscr{F})$ is covered by the bundle $\widehat{M} \to \mathbb{E}^q$ if and only if we are in Case 2, i.e., $(M, \mathscr{F})$ is a transversally similar foliation.

§ 6. Proofs of Theorems 4–6

6.1. Proof of Theorem 4. Assume that assertion (i) is fulfilled and $\mathfrak{M}$ is an Ehresmann connection for $(M, \mathscr{F})$. By Lemma 6, $(M, \mathscr{F})$ is a complete conformal foliation, that is, (i) $\Longrightarrow$ (ii).

Conversely, let $(M, \mathscr{F})$ be a complete conformal foliation of codimension $q \geqslant 3$. This means that there exists a q-dimensional transversal distribution $\mathfrak{M}$ on M, such that $(M, \mathscr{F})$ is an $\mathfrak{M}$-complete Cartan foliation. By Proposition 3, which we proved in [3], $\mathfrak{M}$ is an Ehresmann connection for the foliation $(M, \mathscr{F})$. This proves that (ii) $\Longrightarrow$ (i).

6.2. A global attractor of the global holonomy group. Let Ψ be a group of homeomorphisms of a topological space B. A nonempty closed invariant subset $\mathscr{M} \subset B$ is called an *attractor* of the group Ψ if there exists an open invariant subset $W \subset B$, such that the closure of orbit $\mathrm{Cl}(\Psi \cdot z)$ of any point $z \in W$ contains $\mathscr{M}$. An attractor W of the group Ψ is *global* if $W = B$.

We recall that a *minimal set of a group of homeomorphisms* Ψ of a topological space B is a nonempty closed invariant subset $\mathscr{K}$ of B containing no proper subset with this property. A finite minimal set is called *trivial*.

Note that the limit set $\Lambda(\Psi)$ of an arbitrary subgroup Ψ of the group $\mathrm{Conf}(S^q)$ coincides with intersection of closures of all non-one-point orbits of this group, that is, $\Lambda(\Psi) = \bigcap \mathrm{Cl}(\Psi \cdot z)$, where $z \in S^q$ and $\Psi \cdot z \neq z$. If the limit set $\Lambda(\Psi)$ is finite, then either it is empty or it consists of one or two points. In this case the group Ψ is called *elementary*.

Proposition 6. *Let Ψ be a countable essential subgroup of the group* $\mathrm{Conf}(S^q)$. *Then the group Ψ has a global attractor that is either one or two fixed points, or a nontrivial minimal set, which coincides with the limit set $\Lambda(\Psi)$ of the group Ψ.*

Proof. Using Zorn's Lemma, it is not hard to show that any group of homeomorphisms of a compact manifold has a minimal set.

Assume that the group Ψ has a fixed point a. Then $\mathscr{K} = \{a\}$ is a minimal set of Ψ. Since $\Psi_a = \Psi$ is an essential group, the proof of Lemma 6 implies that either a is a unique fixed point of the group Ψ, or there exists one more fixed point b, and $\Lambda(\Psi) = \{a\}$ or $\Lambda(\Psi) = \{a, b\}$, respectively.

Suppose now that the group Ψ has no fixed point. By assumption, there exists a transformation $\gamma \in \Psi$ which is essential at some point $v \in S^q$. We have shown in the proof of Lemma 6 that there exists a neighbourhood V of v such that the set $P = \bigcup_{n\in\mathbb{Z}} \gamma^n(V)$ coincides with either the standard sphere S^q or $S^q \setminus \{a\}$, where $a \in S^q$. Moreover, the neighbourhood V has the following property: the closure $\mathrm{Cl}(\Psi\cdot z)$ of the orbit $\Psi\cdot z$ of any point $z \in V$ contains v. Consequently, this property holds for all $z \in P$. Since the group Ψ has no fixed point, if $P = S^q \setminus a$, then the orbit $\Psi \cdot a$ contains a point of P. This yields the inclusion

$$\Psi \cdot v \subset \mathrm{Cl}(\Psi \cdot z) \qquad \forall\, z \in S^q.$$

The orbit $\Psi\cdot v$ may consist of two points v and a. Then $\Psi\cdot v$ is a trivial minimal set, which is a global attractor. Otherwise, $\mathscr{M} = \mathrm{Cl}(\Psi \cdot v)$ is a nontrivial minimal set, which is a global attractor, and $\mathscr{M} = \Lambda(\Psi)$. Since v is an arbitrary point with an essential isotropy group, the statement is proved.

Remark 3. The case $\Lambda(\Psi) = S^q$ is not excluded from Proposition 6.

6.3. Proof of Theorem 5. If $(M, \mathscr{F})$ is a complete Riemannian foliation, then (see, for example, [3], Corollary 8) the closure of every leaf is a minimal set, so that 1) holds.

If $(M, \mathscr{F})$ is a complete transversally similar foliation, then 2) follows from the results in our article [3].

Now assume that $(M, \mathscr{F})$ is a complete conformal foliation that is not transversally similar. Theorem 4, Lemma 6 and Theorem 2 imply that there exists a regular covering map $f\colon \widehat{M} \to M$ such that the induced foliation $\widehat{\mathscr{F}} = f^*\mathscr{F}$ is formed by fibres of a locally trivial bundle $r\colon \widehat{M} \to S^q$. Moreover, there is an isomorphism $\rho\colon \Phi(f) \to \Psi$ from the group of covering transformations $\Phi(f)$ to the global holonomy group $\Psi \subset \mathrm{Conf}(S^q)$ of the foliation $(M, \mathscr{F})$. Here Ψ is an essential group of conformal transformations.

1. Assume that the group Ψ is elementary. Then part (i) in 3) follows from Proposition 6.

2. If the group Ψ is nonelementary, then Proposition 6 implies that the foliation $(M, \mathscr{F})$ has a global attractor $\mathscr{M} := f(r^{-1}(\Lambda(\Psi)))$, which is a nontrivial minimal set of this foliation. Moreover, $\mathscr{M}$ is the closure of any leaf L with essential holonomy group, as in case 2). So part (ii) in 3) is proved.

In cases 2) and 3), $M_0 := M \setminus f(r^{-1}(\Lambda(\Psi)))$ is an open submanifold of M that contains no leaf with essential holonomy group. By Theorem 3, $(M_0, \mathscr{F}_{M_0})$ is a Riemannian foliation with Ehresmann connection $\mathfrak{M}_{M_0}$. Theorem 1 implies that the closures of leaves of the Riemannian foliation $(M_0, \mathscr{F}_{M_0})$ form a Riemannian foliation with singularities $(M_0, \mathscr{F}_0)$. Thus the closure $\mathfrak{L}$ in M_0 of any leaf L from M_0 is an embedded submanifold of M_0 and M. We conclude that the closure $\overline{L}$ in M of any leaf L from M_0 satisfies $\overline{L} = \mathfrak{L} \cup \mathscr{M}$.

6.4. Proof of Theorem 6. It is known (see [12]) that a minimal set of any foliation is either a closed leaf or the closure of a non-proper leaf. Hence if a proper foliation has a minimal set $\mathscr{M}$, then $\mathscr{M}$ is a closed leaf. Thus 1) and 2) follow from parts 1) and 2) of Theorem 5. Now assume that $(M, \mathscr{F})$ is not a transversally similar foliation. The foliation $(M, \mathscr{F})$ is proper, so its global holonomy group Ψ is a discrete subgroup of the group $\mathrm{Conf}(S^q)$. Thus Ψ is an elementary Kleinian group and we have case 3), (i) of Theorem 5. Consequently, the union M_0 of nonclosed leaves is a connected, open, everywhere dense subset of M. Moreover, $(M_0, \mathscr{F}_{M_0})$ is a Riemannian foliation, where all leaves are closed. Since $\mathfrak{M}_{M_0}$ is an Ehresmann connection for this foliation, Theorem 1 implies that the space of leaves $M_0/\mathscr{F}_{M_0}$ is a smooth connected q-dimensional orbifold. So $\overline{L}_\alpha$, the closure in M of an arbitrary leaf L_α in M_0, equals $L_\alpha \cup \mathscr{M}$. A similar statement holds for a transversally similar foliation.

§ 7. Suspension foliations and their applications

7.1. Suspension foliations. Haefliger introduced the construction of suspension foliations. Let B and T be smooth connected manifolds, and $\rho\colon \pi_1(B,b) \to \mathrm{Diff}(T)$ be a group homomorphism. Let $G := \pi_1(B,b)$ and $\Psi := \rho(G)$. Consider a universal covering map $\widehat{p}\colon \widehat{B} \to B$. Define a right action of the group G on the product of manifolds $\widehat{B} \times T$ as

$$\Theta\colon \widehat{B} \times T \times G \to \widehat{B} \times T\colon (x,t,g) \to (x \cdot g, \rho(g^{-1})(t)),$$

where $\widehat{B} \to \widehat{B}\colon x \to x \cdot g$ is a covering transformation of the covering $\widehat{p}$ induced by an element $g \in G$, which acts on $\widehat{B}$ on the right.

The map $p\colon M\colon = (\widehat{B} \times T)/G \to B = \widehat{B}/G$ is a locally trivial bundle over B with standard fibre T. It is associated with the principal bundle $\widehat{p}\colon \widehat{B} \to B$ with structure group G. Let $\Theta_g := \Theta|_{\widehat{B} \times \{t\} \times \{g\}}$. Since

$$\Theta_g(\widehat{B} \times \{t\}) = \widehat{B} \times \rho(g^{-1})(t) \qquad \forall\, t \in T,$$

the action of the discrete group G preserves the trivial foliation

$$F := \{\widehat{B} \times \{t\} \mid t \in T\}$$

on the product $\widehat{B} \times T$. Therefore the quotient map

$$f_0\colon \widehat{B} \times T \to (\widehat{B} \times T)/G = M$$

induces a smooth foliation $\mathscr{F}$ on M whose leaves are transversal to fibres of the bundle $p\colon M \to B$. The pair $(M, \mathscr{F})$ is called a *suspension foliation* and is denoted by $\mathbf{Sus}(T, B, \rho)$. The group of diffeomorphisms $\Psi := \rho(G)$ of the manifold T is said to be the *global holonomy group* of the suspension foliation $(M, \mathscr{F})$.

7.2. Proof of Theorem 7. Let Ψ be an arbitrary essential subgroup of the conformal group $\mathrm{Conf}(S^q)$, $q \geqslant 3$, with a finite family of generators $\{\psi_1, \dots, \psi_m\}$.

We let S^2_m denote a two-dimensional sphere with m handles. The fundamental group of S^2_m equals $\langle a_i, b_i,\ i = 1, \dots, m \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_m b_m a_m^{-1} b_m^{-1} \rangle$. We set $B = S^2_m$, $T := S^q$ and define a group homomorphism $\rho\colon \pi_1(B,b) \to \mathrm{Conf}(S^q)$, which is given on generators by $\rho(a_i) := \psi_i$, $i = 1, \dots, m$, where $\rho(b_i) := \mathrm{Id}_{S^2_m}$ is the unit of the group Ψ. Then the suspension foliation $(M, \mathscr{F}) := \mathbf{Sus}(S^q, S^2_m, \rho)$ is a two-dimensional non-Riemannian conformal foliation of codimension $q \geqslant 3$. It is covered by the trivial bundle $\mathbb{R}^2 \times S^q \to S^q$ and has the global holonomy group Ψ. By a property of a suspension foliation, the $(q+2)$-manifold M is the total space of a locally trivial bundle with standard fibre S^q over the base S^2_m. Since S^q and S^2_m are compact, M is compact as well.

Now suppose that $\Psi \subset \mathrm{Conf}(S^q)$ has a countable family of generators $\{\psi_i \mid i \in \mathbb{N}\}$. Let $B := \mathbb{R}^2 \setminus A$ be the plane with the discrete subset $A = \{(i,0) \in \mathbb{R}^2 \mid i \in \mathbb{N}\}$ removed, and let $b = (1,1) \in B$. Then $G = \pi_1(B,b) = \langle g_i, i \in \mathbb{N}\rangle$ is a free group with a countable family of generators. The conditions $\rho_\infty(g_i) := \psi_i$, where $i \in \mathbb{N}$, define a group homomorphism $\rho_\infty \colon \pi_1(B,b) \to \mathrm{Conf}(S^q)$. The suspension foliation $(M,\mathscr{F}) := \mathbf{Sus}(S^q, B, \rho_\infty)$ is a conformal foliation of codimension $q \geqslant 3$ with global holonomy group Ψ.

Note that $(M,\mathscr{F})$ is a Riemannian foliation if and only if Ψ is an inessential group of conformal transformations of the sphere S^q. It is known that there exists a Riemannian metric on S^q, which is conformally equivalent to the canonical metric and such that Ψ is a group of isometries. Since S^q is compact, this Riemannian metric is complete. Consequently, the suspension foliation $(M,\mathscr{F})$ is also complete.

If Ψ is an essential group of conformal transformations of the sphere S^q, then $(M,\mathscr{F})$ is a non-Riemannian conformal foliation, which admits an Ehresmann connection formed by tangent spaces to a transversal bundle $p\colon M \to B$. By Theorem 4, the conformal foliation $(M,\mathscr{F})$ is complete.

7.3. Exceptional and exotic minimal sets of diffeomorphism groups. Let $\mathscr{K}$ be a minimal set of a diffeomorphism group Ψ of a manifold B. The minimal set $\mathscr{K}$ with empty interior is called *exceptional* if $\mathscr{K}$ is the Cantor set, and *exotic* if $\mathscr{K}$ is not a totally disconnected topological subspace of B.

Corollary 3 follows from the equality $\mathscr{M} = f(r^{-1}(\Lambda(\Psi)))$ in Theorem 5.

Let B_k be a smooth closed 3-manifold homeomorphic to the connected sum $\#_{i=1}^k S^1 \times S^2$ of k copies of the product $S^1 \times S^2$. Then $\pi_1(B_k,b) = \langle g_1,\dots,g_k\rangle$ is a free group of rank k.

Example 2. Assume that we have a finite collection of disjoint closed balls $\mathscr{B}_1^+,\dots,\mathscr{B}_k^+,\mathscr{B}_1^-,\dots,\mathscr{B}_k^-$ on the sphere S^q and a conformal transformation $\psi_i \in \mathrm{Conf}(S^q)$, such that $\psi_i(\mathrm{int}(\mathscr{B}_i^+)) = \mathrm{ext}(\mathscr{B}_i^-)$. We suppose that for any $\mathscr{B}_i^+$ and $\mathscr{B}_i^-$ there exists a diffeomorphism of the sphere S^q sending these balls to round balls. The group Ψ with generators $\psi_1,\dots,\psi_k$ is called a *Schottky group*. It is known (see [19], for example) that the Schottky group Ψ is a free group of rank k, that is, $\Psi = \langle\psi_1,\dots,\psi_k\rangle$, and it has a minimal set $\Lambda(\Psi)$ homeomorphic to the Cantor subset of the segment $[0,1]$. Consequently, the topological dimension of $\Lambda(\Psi)$ equals zero.

Now we consider a group isomorphism $\rho_k \colon \pi_1(B_k,b) \to \Psi$ defined by $\rho_k(g_i) = \psi_i$, $i = 1,\dots,k$. The suspension foliation $(M_k,\mathscr{F}_k) := \mathbf{Sus}(S^q, B_k, \rho_k)$ is a complete conformal foliation with global holonomy group Ψ. By Corollary 3, the foliation $(M_k,\mathscr{F}_k)$ has a global attractor, which is an exceptional minimal set.

Example 3. First we recall the definition of a Menger curve. Let $S = I \times I$ be the unit square. We divide it into nine squares of size $1/3 \times 1/3$ and remove the open square $(1/3,2/3) \times (1/3,2/3)$ from S. We repeat this procedure with each of the remaining eight squares and then proceed by induction. As a result, we remove a countable family of open squares from S and obtain a compact subset $\mathscr{S}$

called the Sierpinski carpet. Let $Q = I \times I \times I$ be the unit cube, where each of its faces P_j, $j = 1, \dots, 6$, contains a copy $\mathscr{S}_j$ of the Sierpinski carpet. Let $p_j \colon Q \to P_j$ be the orthogonal projection. Then the set $\mathscr{M} := \bigcap p_j^{-1}(\mathscr{S}_j)$ is called the Menger curve. The topological dimension of the Menger curve $\mathscr{M}$ equals 1.

Bourdon proved (see [20]) that for any even positive integer $k \geqslant 6$ there exists an isomorphism χ_k of the group

$$\Gamma_k := \langle s_i, i \in \mathbb{Z}/k\mathbb{Z} \mid s_i^3, [s_i, s_{i+1}] \rangle$$

to some Kleinian subgroup $\Psi \subset \mathrm{Conf}(S^4)$, whose limit set $\Lambda(\Psi)$ is homeomorphic to the Menger curve. Consequently, $\Lambda(\Psi)$ is an exotic minimal set of the group Ψ in S^4.

Suppose that a group homomorphism $\rho_k \colon \pi_1(B_k, b) \to \Psi$ is given on generators as $\rho_k(g_i) := \chi_k(s_i)$, $i = 1, \dots, k$. Then we have a suspension foliation

$$(M_k, \mathscr{F}_k) := \mathbf{Sus}(S^4, B_k, \rho_k)$$

with global holonomy group Ψ. Let $f_k \colon \widehat{M_k} \to M_k$ be a regular covering map that satisfies the conditions of Theorem 2. Then the foliation $(M_k, \mathscr{F}_k)$ is covered by the bundle $r_k \colon \widehat{M_k} \to S^4$. By Corollary 3, $\mathscr{M}_k = f_k(r_k^{-1}(\Lambda(\Psi)))$ is a global attractor of the foliation $(M_k, \mathscr{F}_k)$, which is an exotic minimal set.

Remark 4. Let $(M_k, \mathscr{F}_k)$ be the foliation of codimension q constructed in Examples 2 or 3. Here $q \geqslant 2$ for the foliation $(M_k, \mathscr{F}_k)$ in Example 2 and $q = 4$ for $(M_k, \mathscr{F}_k)$ in Example 3. If $f_k \colon \widehat{M_k} \to M_k$ is a regular covering map satisfying the conditions of Theorem 2, then the induced foliation $\widehat{\mathscr{F}_k} = f_k^* \mathscr{F}_k$ is formed by fibres of a locally trivial bundle $r_k \colon \widehat{M_k} \to S^q$. Moreover, as we indicated above, M_k is the quotient space $\widehat{M_k}/G$, where $G \cong \pi_1(B_k, b)$ is a free group of rank k. The group Ψ acts properly discontinuously on the open everywhere dense subset $S^q \setminus \Lambda(\Psi)$ in S^q, so $M_k^0 := f_k(r_k^{-1}(S^q \setminus \Lambda(\Psi)))$ is an open saturated everywhere dense subset in the compact manifold M_k, and the induced foliation $(M_k^0, \mathscr{F}_{M_k^0})$ is Riemannian. The space of its leaves is a smooth q-dimensional orbifold $(S^q \setminus \Lambda(\Psi))/\Psi$.

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N. I. Zhukova
Nizhni Novgorod State University
E-mail: n.i.zhukova@rambler.ru

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