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ON EVALUATION OF THE POWER INDICES WITH ALLOWANCE OF AGENTS' PREFERENCES IN THE ANONIMOUS GAMES

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In general, the complexity of the algorithm to calculate the power indices, both classical and depending on agents’ preferences, grows exponentially with the number of agents.

But in the important specific case when all players have the same number of votes it is possible to compute preference-based indices for the most types of these indices and for all voting bodies.

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Сложность вычисления индексов влияния – как классических, так и зависящих от предпочтений участников – экспоненциально растет с ростом числа участников. Но в важном частном случае, когда все игроки имеют равное количество голосов, возможно построение эффективных алгоритмов для расчета индексов влияния, зависящих от предпочтений участников для большинства типов этих индексов и практически всех реальных ситуаций.

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1 Introduction

In calculation of the power indices, both well-known (Banzhaf, Shapley-Shubik and others [3, 5, 6]) and new (depending on the agent preferences, [2]) indices, one generally has to enumerate almost all coalitions, that is, the subsets of the set of players, which makes calculations impossible if the number of players exceeds fifty.

Yet, if all players have an integer number of votes, there are players with the same number of votes, many coalitions have equal total number of votes or the sum of votes of all players is small, then the algorithms based on calculations using the generating functions become efficient. But these algoritms work only for classical power indices [4] and some particular types of the power indices based on agents' preferences [7].

In this paper we consider an important specific case when all players have the same number of votes. For classical power indices in this case all players have the same power, however it is not the case for the indices which allow the preferences of agents to coaless.

We introduce efficient algorithms for calculation of the latter indices for most types of these indices [2].

2 Main definitions

Definition 1. *A simple game is a pair (N, v), where $N = \{1, \ldots, n\}$ is a set of agents, $v : 2^n \to \{0, 1\}$ is a function, which maps any subset of N to 0 or 1. The function v satisfies monotonicity property, that is, if S and T are subsets of N, and S is a subset of T, then $v(T) \geq v(S)$.*

Coalition S is called the winning one if $v(S) = 1$, and losing otherwise. Due to monotonicity property, if S and T are winning coalitions and T is a subset of S, then S is a winning coalition as well.

Definition 2. *A weighted game is denoted as a vector of $n + 1$ positive numbers $w = (q; w_1, w_2, \ldots, w_n)$, where the first component is called a quota, and the other n components mean predefined number of votes, which each agent has. A weighted game w determines a simple game (N, v)*

$$N = \{1, \ldots, n\}$$

$$v(S) = \begin{cases} 1, \sum_{i \in S} w_i \geq q \\ 0, \mathit{otherwise} \end{cases}$$

Example 1. *Since the last election in 2011, four political parties are represented in the State Duma of the Russian Federation: United Russia (238 seats), Communist Party of the Russian Federation (92), A Just Russia (64) and Liberal Democratic Party of Russia (56). All factions have 450 seats in total and consitutional bills pass by two thirds of the total number of the deputies. So, it is the weighted game with* $w = (301; 238, 92, 64, 56)$.

To create a winnig coalition the first and the second (or the third) agents should coalesce. So, the set of winning coalitios is

$$W = \{\{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\}.$$

Definition 3. *Player* $i \in S$ *is a swinger (or a pivotal player) in a coalition* S *if* S *is a winning one and* $S \setminus \{i\}$ *is the loosing coalition.*

In the previous example agent 1 is pivotal in any winning coalition, and agent 3 is a swinger in the coalitions $\{1,3\}$ and $\{1,3,4\}$.

A power index $\Phi : SG_n \to R^n$ associates with each simple game v a vector $\Phi(v)$. Its i-th component is interpreted as a measure of the power of player i. The best known power indices are Banzhaf and Shapley—Shubik [6] indices. Below we deal with only Banzhaf power index.

Banzhaf power index (BI) [3] is calculated under the assumption that power of a player is proportional to the number of coalitions in which she is a swinger.

The total Banzhaf index TBz_i for player i is

$$TBz_i = |W_i|,$$

where W_i is a set of winnig coalitions, which contain the pivotal player i.

Banzhaf power index (Bz_i) is obtained from the general index by the normalizaton, i. e.

$$Bz_i = \frac{|W_i|}{\sum_{j=1}^{n} |W_j|}.$$

Let us calculate both these indices for players in our example. First player is a swinger in any winning coalition, second player is pivotal in two winning coalitions ($\{1,2\}$ and $\{1,2,4\}$), the third one is also pivotal in two coalitions and the last one is not a swinger in any coalition. So, the total Banzhaf index is equal to

$$TBz_1 = 6, TBz_2 = 2, TBz_3 = 2, TBz_4 = 0,$$

and normalized Bazhaf index is equal to

$$Bz_1 = 0.6, Bz_2 = 0.2, Bz_3 = 0.2, Bz_4 = 0$$

3 Power indices taking into account agents' preferences

Consider an example. Let $N = \{A, B, C\}$ be a set of agents, $w = (3; 2, 1, 1)$. Winning coalitions are $\{A, B\}, \{A, C\}, \{A, B, C\}$, Banzhaf index equals $Bz_A = 3/5$, $Bz_B = 1/5$, $Bz_C = 1/5$. Assume now that agents B and C decide not to coalesce with each other. Does it affect their capabilities in the voting? Obviously, coalition $\{A, B, C\}$ is impossible, but it reduces total Banzhaf index only for player A, because she is a swinger in that coalition. So, the values of Banzhaf index are $Bz_A = 1/2, Bz_B = 1/4, Bz_C = 1/4$. As we can see, the desire of two players not to coalesce with each other increases their capabilities in that voting. The same decision may also has an opposite effect, for example, if players A and B cannot be in one coalition. Then $Bz_A = 1/2$, $Bz_B = 0$, $Bz_C = 1/2$.

In order to evaluate a connection between a player i and a coalition S, an intensity function $f(i, S)$ was introduced in [1, 2]. It is assumed that the desire of the player i to cooperate with the player j can be represented as $p_{ij}, 0 \leq p_{ij} \leq 1$. If $p_{ij} > p_{ik}$, it means that agent i wants to coalesce with agent j more than with agent k. The construction of the intensity function is based on the predefined matrix $P = \|p_{ij}\|$. Then power indices are constructed as

$$\alpha(i, v, P) = \sum_{\substack{S, v(S)=1, \\ v(S \setminus \{i\})=0}} f(i, S),$$

and

$$N\alpha(i, v, P) = \frac{\alpha(i, v, P)}{\sum_{j=1}^{n} \alpha(j, v)}.$$

There are many ways of construction $f(i, S, P)$ suggested in [1].

$$f^{+}(i,S,P)=\sum_{j\in S}\frac{p_{ij}}{s-1}, \tag{1}$$

$$f^{-}(i,S,P)=\sum_{j\in S}\frac{p_{ji}}{s-1}, \tag{2}$$

$$f(i,S,P)=(f^{+}(i,S,P)+f^{-}(i,S,P))/2, \tag{3}$$

$$f(S,P)=\frac{\sum_{i,j\in S}p_{ij}}{s(s-1)}, \tag{4}$$

$$f^{+}_{max}(i,S,P)=\max_{j\in S,j\neq i}p_{ij}, \tag{5}$$

$$f^{+}_{min}(i,S,P)=\min_{j\in S,j\neq i}p_{ij}, \tag{6}$$

$$f_{mf}(i,S,P)=(f^{+}_{max}(i,S,P)+f^{+}_{min}(i,S,P))/2, \tag{7}$$

$$f^{-}_{max}(i,S,P)=\max_{j\in S,j\neq i}p_{ji}, \tag{8}$$

$$f^{-}_{min}(i,S,P)=\min_{j\in S,j\neq i}p_{ji}, \tag{9}$$

$$f^{+}_{sm}(i,S,P)=\sqrt[d]{\frac{1}{s-1}\sum_{j\in S}p_{ij}^{d}}, \tag{10}$$

$$f^{-}_{sm}(i,S,P)=\sqrt[d]{\frac{1}{s-1}\sum_{j\in S}p_{ji}^{d}}, \tag{11}$$

$$f_{maxmin}(S,P)=\max_{i\in S}\min_{j\in S,j\neq i}p_{ij}, \tag{12}$$

$$f_{minmax}(S,P)=\min_{i\in S}\max_{j\in S,j\neq i}p_{ij}, \tag{13}$$

$$f_{mf}(S,P)=(f_{maxmin}(S,P)+f_{minmax}(S,P))/2. \tag{14}$$

Let us denote as $\alpha^{k}(i,v,P)$ an α-index based on the intensity function from formula (k).

Assume that any agent has only one vote. This means that any player has the same opportunity. Therefore, any original index, which does not take into account agents' preferences, will also be the same for anyone. This result is simply predictable and does not lead to any important conclusion. However, real possibilities of agents could be strongly distinctive. Consequently, in order to get the whole representation of this type of voting it is necessary to take into consideration agents' preferences.

4 Main results

On evaluation of the indices $\alpha^k(i,v,P)$

As we can see from formulas (1) – (14), in order to calculate $\alpha(i,v,P)$ we have to search for all winning coalitions, where agent i is pivotal. In weighted games i is a swinger in the coalition S, if $q > \sum_{j\in S\setminus\{i\}} w_j \geq q - w_i$. As any agent has only one vote, this inequality can be represented as

$$q > \sum_{j\in S\setminus\{i\}} 1 = |S\setminus\{i\}| = |S| - 1 = s - 1 \geq q - 1$$

or $s = q$. Thus, we should enumerate all coalitions S, $i \in S$ and $s = q$. But sometimes the number of these coalitions can be too large to do that. Let the number of agents be n and the quota q be approximately equal to $[n/2+1]$. So, using the Stirling approximation formula, the number of winning coalitions consisting of pivotal player i would be equal to

$$\binom{n-1}{q-1} = \frac{(n-1)!}{(q-1)!(n-q)!} \sim \frac{1}{2}\cdot\frac{n!}{(n/2)!(n/2)!} \sim \frac{1}{2}\cdot\frac{n^n\cdot\sqrt{\pi n}\cdot 2^n\cdot e^n}{e^n\cdot n^n\cdot(\pi n)} \sim \frac{2^{n-1}}{\sqrt{\pi n}}.$$

When the number of players is more than 500, this value is greater than 10^{148}.

In order to calculate power indices quickly, computational methods must allow parallel processing. This goal may be achieved by developing the same methods for power indices with the same structure. Then we can calculate α-indices deriving one from another.

Example 2. *If $\alpha^1(v)$ is already calculated, $\alpha^2(v)$ can be obtained by applying the same approach with transposed matrix P'*

$$f^-(i,S,P) = \sum_{j\in S}\frac{p_{ji}}{s-1} = \sum_{j\in S}\frac{p'_{ij}}{s-1} = f^+(i,S,P'),$$

$$\alpha^2(i,v,P) = \sum_{S,i\in S,s=q} f^-(i,S,P) = \sum_{S,i\in S,s=q} f^+(i,S,P') = \alpha^1(i,v,P').$$

It suggests us to classify these indices into several groups:
1. Linear type (1) – (3),
2. Symmetric linear type (4),
3. Max-type (5) – (9),
4. Minkowski type (10), (11),
5. Minimax-type (12) – (14).

In each group only one index should be evaluated, the other are obtained from it by applying three simple methods:

1) Use the same approach with transposed matrix P,
2) Replace each element p_{ij} of matrix P by $(1 - p_{ij})$,
3) Sum up denormalized indices.

Next we will show how this approach works for any group.

4.1 Linear type

$$\alpha^1(i, v, P) = \sum_{S, i \in S, s=q} \frac{1}{q-1} \sum_{j \in S, j \neq i} p_{ij},$$

$$\alpha^2(i, v, P) = \sum_{S, i \in S, s=q} \sum_{j \in S} \frac{p_{ji}}{q-1} = \sum_{S, i \in S, s=q} \sum_{j \in S} \frac{p'_{ij}}{q-1} = \alpha^1(i, v, P'),$$

$$\alpha^3(i, v, P) = \sum_{S, i \in S, s=q} (f^+(i, S, P) + f^-(i, S, P))/2 =$$
$$(\alpha^1(i, v, P) + \alpha^1(i, v, P'))/2.$$

Let us emphasize four key features of this index

— it is the sum of elements $p_{ij}, j = 1, \ldots, n$ with appropriate weights (or non-negative coefficients),

— $p_{xy}, x, y \neq i$, is included in the sum with zero coefficient,

— as any player has one vote, number of coalitions including i and x is equal for any $x \neq i$. So, the coefficient for $p_{ix}, x \neq i$ does not depend on particular x,

— the sum of all coefficients (equal to one another) is

$$\sum_{S, i \in S, s=q} \frac{1}{q-1} \sum_{j \in S, j \neq i} 1 = \binom{n-1}{q-1} \cdot \frac{1}{q-1} \cdot (q-1) = \binom{n-1}{q-1},$$

so this sum (as well as any coefficient) does not depend on i.

Thus we first can evaluate $\alpha^1(i,v,P)$ as

$$c \cdot \sum_{j=1, j\neq i}^{n} p_{ij},$$

where c is a constant, which does not depend on particular i. Therefore, the normalized $\alpha^1(i,v,P)$ is equal to

$$N\alpha^1(i,v,P) = \frac{c \cdot \sum_{j=1,j\neq i}^{n} p_{ij}}{c \cdot \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} p_{ij}} = \frac{\sum_{j=1,j\neq i}^{n} p_{ij}}{\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} p_{ij}},$$

and

$$N\alpha^2(i,v,P) = \frac{\sum_{j=1,j\neq i}^{n} p_{ji}}{\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} p_{ji}}$$

As we have already mentioned, coefficient c does not depend on elements of matrix P, so this coefficient is the same for transposed matrix P'. Consequently, if $\alpha^3(i,v,P) = (\alpha^1(i,v,P) + \alpha^2(i,v,P))/2$, then

$$\sum_{i=1}^{n} \alpha^3(i,v,P) = \sum_{i=1}^{n} \frac{(\alpha^1(i,v,P) + \alpha^2(i,v,P))}{2} =$$
$$\frac{1}{2} \cdot (c \cdot \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} p_{ij} + c \cdot \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} p_{ji}) = c \cdot \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} p_{ij}.$$

As a result we obtain

$$N\alpha^3(i,v,P) = \frac{1}{2}(\frac{\alpha^1(i,v,P)}{c \cdot \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} p_{ij}} + \frac{\alpha^2(i,v,P)}{c \cdot \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} p_{ij}}) =$$
$$\frac{1}{2}(N\alpha^1(i,v,P) + N\alpha^2(i,v,P)).$$

4.2 Symmetric linear type

$$\alpha^4(i,v,P) = \sum_{S, i\in S, s=q} \frac{1}{q(q-1)} \sum_{j,l\in S, j\neq l} p_{jl} = \frac{1}{q(q-1)} \sum_{S, i\in S, s=q} \sum_{j,l\in S, j\neq l} p_{jl}$$

This index satisfies the first and the third key features of the previous index. In addition, as appropriate coefficient for element p_{xy}, $x \neq y$, is a number of winning coalitions consist of players x and y (and, obviously, the pivotal player i), therefore both elements p_{yx} and p_{xy} have the same coefficients. They are equal to $\binom{n-3}{q-3}$, if $x, y \neq i$, or $\binom{n-2}{q-2}$, otherwise. Let $c = \frac{1}{q-2}\binom{n-3}{q-3}$, $\binom{n-3}{q-3} = (q-2) \cdot c$ and $\binom{n-2}{q-2} = (n-2) \cdot c$. Tnen $\alpha^4(i, v, P)$ can be represented as

$$\alpha^4(i, v, P) = \frac{1}{q(q-1)} \left((n-2) \cdot c \cdot \sum_{j=1, j \neq i}^{n} (p_{ij} + p_{ji}) + \right.$$

$$\left. (q-2) \cdot c \cdot \sum_{j,l=1, j,l \neq i, j \neq l}^{n} p_{jl} \right).$$

$$C = \frac{1}{q(q-1)} \cdot c, (n-2) = (n-q+q-2) = (n-q) + (q-2),$$

$$\alpha^4(i, v, P) = C\Bigg((n-q) \cdot \sum_{j=1, j \neq i}^{n} (p_{ij} + p_{ji}) + (q-2) \cdot \Big(\sum_{j=1, j \neq i}^{n} (p_{ij} + p_{ji}) +$$

$$\sum_{j,l=1, j,l \neq i, j \neq l}^{n} p_{jl} \Big)\Bigg) = C\Bigg((n-q) \cdot \sum_{j=1, j \neq i}^{n} (p_{ij} + p_{ji}) + (q-2) \cdot \sum_{j,l=1, j \neq l}^{n} p_{jl} \Bigg),$$

and

$$N\alpha^4(i, v, P) = \frac{C \cdot \left((n-q) \cdot \sum_{j=1, j \neq i}^{n} (p_{ij} + p_{ji}) + (q-2) \cdot \sum_{j,l=1, j \neq l}^{n} p_{jl} \right)}{C \cdot \left((n-q) \cdot \sum_{i,j=1, j \neq i}^{n} (p_{ij} + p_{ji}) + n \cdot (q-2) \cdot \sum_{j,l=1, j \neq l}^{n} p_{jl} \right)}.$$

As

$$\sum_{i,j=1, j \neq i}^{n} p_{ij} = \sum_{i,j=1, j \neq i}^{n} p_{ji} = \sum_{j,l=1, j \neq l}^{n} p_{jl}$$

and

$$(n-q)+(n-q)+n(q-2)=2n-2q+nq-2n=q(n-2),$$

then

$$N\alpha^4(i,v,P)=\frac{(n-q)\cdot\sum_{j=1,j\neq i}^{n}(p_{ij}+p_{ji})+(q-2)\cdot\sum_{j,l=1,j\neq l}^{n}p_{jl}}{q(n-2)\cdot\sum_{j,l=1,j\neq l}^{n}p_{jl}}=$$

$$\frac{2(n-q)}{q(n-2)}N\alpha^3(i,v,P)+\frac{q-2}{q(n-2)}=\frac{2(n-q)}{q(n-2)}\cdot N\alpha^3(i,v,P)+\frac{(q-2)n}{q(n-2)}\cdot\frac{1}{n}.$$

As we have mentioned, Banzhaf power index equals $\frac{1}{n}$ in this kind of game, so we can represent $N\alpha^4(i,v,P)$ as

$$N\alpha^4(i,v,P)=\frac{2(n-q)}{q(n-2)}\cdot N\alpha^3(i,v,P)+\frac{(q-2)n}{q(n-2)}\cdot Bz_i=$$

$$\frac{n}{n-2}\cdot\frac{2}{q}\cdot N\alpha^3(i,v,P)-\frac{2}{n-2}\cdot N\alpha^3(i,v,P)+\frac{n}{n-2}\cdot\frac{q-2}{q}\cdot Bz_i.$$

Let $n\gg 1$, so $\frac{n}{n-2}\approx 1$, $\frac{2}{n-2}\approx 0$ and

$$N\alpha^4(i,v,P)=\frac{2}{q}\cdot N\alpha^3(i,v,P)+(1-\frac{2}{q})\cdot Bz_i.$$

As we can see from this formula, $N\alpha^4(i,v,P)$ is approximately equals Banzhaf power index with large q.

4.3 Max-type

$$\alpha^5(i,v,P)=\sum_{S,i\in S,s=q}\max_{j\in S,j\neq i}p_{ij},$$

$$\alpha^6(i,v,P)=\sum_{S,i\in S,s=q}\min_{j\in S,j\neq i}p_{ij}=$$

$$\sum_{S,i\in S,s=q}(1-\max_{j\in S,j\neq i}(1-p_{ij}))=\binom{n-1}{q-1}-\alpha^5(i,v,\mathbf{1}-P),$$

$$\alpha^7(i,v,P)=\sum_{S,i\in S,s=q}(f^+_{max}(i,S,P)+f^+_{min}(i,S,P))/2=$$

$$(\alpha^5(i,v,P)+\binom{n-1}{q-1}-\alpha^5(i,v,\mathbf{1}-P))/2,$$

$$\alpha^8(i,v,P) = \sum_{S, i\in S, s=q} f^-_{max}(i,S,P) = \sum_{S, i\in S, s=q} f^+_{max}(i,S,P') = \alpha^5(i,v,P'),$$

$$\alpha^9(i,v,P) = \sum_{S, i\in S, s=q} f^-_{min}(i,S,P) = \sum_{S, i\in S, s=q} f^+_{min}(i,S,P') = \binom{n-1}{q-1} - \alpha^5(i,v,\mathbf{1}-P'),$$

where $\mathbf{1}$ is a square matrix of order n consisting of ones.

Required index is a sum of elements p_{ij}, $j \neq i$, with different coefficients c_j. Let $p_{ij_1} = \max_{j\neq i} p_{ij}$ for some j_1. The number of winning coalitions S such that $i, j_1 \in S$, $s = q$, is $\binom{n-2}{q-2}$. All these coalitions include player j_1, so $f^+_{max}(i,S,P) = \max_{j\in S, j\neq i} p_{ij} = p_{ij_1}$ as well. Consequently, $c_{j_1} = \binom{n-2}{q-2}$. Then let j_2 be such that $p_{ij_2} = \max_{j\neq i, j_1} p_{ij}$. The number of winning coalitions S such that $i, j_2 \in S, j_1 \notin S$, $s = q$, is $\binom{n-3}{q-2}$. These coalitions include player j_2, but do not include player j_1, so $f^+_{max}(i,S,P) = p_{ij_2}$. This implies $c_{j_2} = \binom{n-3}{q-2}$. Similarly, $c_{j_k} = \binom{n-k-1}{q-2}$, where p_{j_k} is a k-th element in the i-th row of the matrix P, sorted in descending order, $k \leq n-q+1$. For $k > n-q+1$ it would be impossible to construct a winning coalition consisting of $n-(k-1) < q$ players, so the coefficient c_{j_k} is equal to zero.

To sum up, the method can be divided into two steps. First, we need to sort elements in row, and second, sum every element with the corresponding coefficient.

Let p_1 be the maximal element of $\{p_{i1}, p_{i2}, \ldots, p_{in}\}/\{p_{ii}\}$, p_2 be the second largest element, ..., p_{n-1} be the minimal element of this set. Then the corresponding α-index is equal to

$$\alpha^5(i,v,P) = \binom{n-2}{q-2}\cdot p_1 + \binom{n-3}{q-2}\cdot p_2 + \ldots + \binom{q-2}{q-2}\cdot p_{n-q+1}.$$

Then this index can be simply normalized.

Computational complexity of this approach is to sort elements in any row. The fastest method of sorting is 'heapsort', which sorts elements in one row in $O(n\log n)$, so it can put elements of matrix P in right order and calculate α-index in $O(n^2\log n)$.

4.4 Minkowski type

$$\alpha^{10}(i,v,P) = \sum_{S, i\in S, s=q} \sqrt[d]{\frac{1}{q-1}\sum_{j\in S, j\neq i} p_{ij}^d},$$

$$\alpha^{11}(i,v,P) = \sum_{S, i\in S, s=q} f_{sm}^{-}(i,S,P) = \sum_{S, i\in S, s=q} f_{sm}^{+}(i,S,P') = \alpha^{10}(i,v,P').$$

In general this problem is too complex to solve it fast. For instance, if all elements of matrix P are different, we need to search every set of q agents. Besides, we can use some approximate statistical methods, but it is difficult to estimate errors of these methods. To simplify the problem we assume that each element $p_{ij} \in M = \{0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m} = 1\}$, $|M| = m+1$. It means that preferences of agent i can be ranged with some scale providing comparison of any two categories. With this assumption we can deal with categories, not agents, cause any two agents in one category are indistinguishable. Let $r = (r_0, r_1, \ldots, r_m)$, r_j - number of players in category j ($p_{ix} = \frac{j}{m}$, if x from category j), and $l^s = (l_0^s, l_1^s, \ldots, l_m^s)$, where l_j^s is a quantity of players from category j ($l_j^s \leq r_j$) in the winning coalition S including pivotal player i ($\sum_{j=0}^m l_j^s = (q-1)$). These agents with i form the winning coalition S. So, for any vector l^s the intensity function can be calculated as

$$f(i,S,P) = f(i,l^s) = \sqrt[d]{\frac{1}{q-1}\sum_{j=1}^m l_j^s \cdot (\frac{j}{m})^d}.$$

The number of subsets of l_j^s distinct players from category j is equal to binomial coefficient

$$\binom{r_j}{l_j^s}.$$

The number of subsets of l_0^s agents from the category 0, l_1^s agents from the category 1, ..., l_m^s agents from the category m is equal to

$$c(l^s) = \binom{r_0}{l_0^s} \cdot \binom{r_1}{l_1^s} \cdots \binom{r_m}{l_m^s}.$$

As $\alpha^{10}(i,v,P)$ is the sum of intensity functions $f(i,S,P) = f(i,l^s)$, the value of $f(i,l^s)$ occurs in the sum $c(l^s)$ times. Then

$$\alpha^{10}(i,v,P) = \sum_{S, i\in S, s=q} \sqrt[d]{\frac{1}{q-1}\sum_{j\in S, j\neq i} p_{ij}^d} = \sum_{l^s} c(l^s) \cdot \sqrt[d]{\frac{1}{q-1}\sum_{j=1}^m l_j^s \cdot (\frac{j}{m})^d}$$

The number of different values of intensity function is equal to the number of vectors l^s. The latter (without assumption $l^s_j \leq r_j, j = 0, \ldots, m$) can be calculated as

$$|\{l^s\}| = \binom{q-1+m}{m}.$$

The real number of vectors l^s is less than this value.

Assume that calculation of intensity function takes $O(1)$. Then the quantity of required operations is proportional to the number of vectors l^s. Obviously, if $m = n$, the latter is equal to the number of winning coalitions S including pivotal player i. Therefore, this approach is reasonable to apply only with small values of m.

5 Conclusion

These algorithms were written to calculate the power indices for the council of the European union (near 700 agents) and for the Russian State Duma (450 agents).

The table below we give the complexity of evoluation of the main types of indices obtained above.

Index	$N\alpha^1$	$N\alpha^4$	$N\alpha^5$	$N\alpha^{10}$	$N\alpha^{12}$
Complexity	$O(n^2)$	$O(n^2)$	$O(n^2 \cdot \log n)$	$O(n \cdot n^m)$	$O(2^n)$

As it can be seen from the table, the algorithms for Minkowsky and minimax types of indices have exponential complexity.

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