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# Noncommutative extensions of elliptic integrable Euler–Arnold tops and Painlevé VI equation

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## Abstract

In this paper we suggest generalizations of elliptic integrable tops to matrix-valued variables. Our consideration is based on the  $R$ -matrix description which provides Lax pairs in terms of quantum and classical  $R$ -matrices. First, we prove that for relativistic (and non-relativistic) tops, such Lax pairs with spectral parameters follow from the associative Yang–Baxter equation and its degenerations. Then we proceed to matrix extensions of the models and find out that some additional constraints are required for their construction. We describe a matrix version of the  $\mathbb{Z}_2$  reduced elliptic top and verify that the latter constraints are fulfilled in this case. The construction of matrix extensions is naturally generalized to the monodromy preserving equation. In this way we get matrix extensions of the Painlevé VI equation and its multidimensional analogues written in the form of non-autonomous elliptic tops. Finally, it is mentioned that the matrix valued variables can be replaced by elements of noncommutative associative algebra. At the end of the paper we also describe special elliptic Gaudin models which can be considered as matrix extensions of the ( $\mathbb{Z}_2$  reduced) elliptic top.

Keywords: elliptic integrable systems, noncommutative integrable systems, Euler–Arnold tops, Painlevé VI equation

### 1. Introduction and summary

Noncommutative generalizations of integrable systems have a long history that started from the non-abelian generalization of the Toda model proposed by Polyakov<sup>6</sup>. The incomplete list of papers devoted to this subject is [13, 17, 33, 36, 37] and references therein. The generalization means a passage in the equations of motion to the variables taking values in associative algebras, possibly with additional structures. This can be treated as quantization of the original system. On the other hand, in this way one can pass from the classical finite-dimensional Hamiltonian systems to corresponding field theories. Our construction of the noncommutative integrable systems is based on the associative Yang–Baxter equation for (quantum)  $R$ -matrices. We will show that the existence of this equation governs the integrability of the related top-like system. Then it is mentioned that any such  $R$ -matrix can be simply generalized to that corresponding to the matrix-valued extension of the initial top. Finally, we prove that this extension is indeed integrable under the additional reduction procedure.

In this paper we describe a noncommutative generalization of the integrable Euler–Arnold tops related to the group  $SL(N, \mathbb{C})$ . The simplest example of the latter is given by the Euler top:

$$\dot{S} = [S, J(S)], \tag{1.1}$$

$$S = \sum_{\alpha=1}^3 \frac{1}{2\iota} \sigma_{\alpha} S_{\alpha}, \quad J(S) = \sum_{\alpha=1}^3 \frac{1}{2\iota} \sigma_{\alpha} S_{\alpha} J_{\alpha}, \tag{1.2}$$

where  $\sigma_{\alpha}$  are the Pauli matrices,  $\iota = \sqrt{-1}$ ,  $J_1, J_2, J_3$ —arbitrary constants (inverse components of the inertia tensor written in principle axes) and  $(S_1, S_2, S_3)$ —the dynamical variables (components of the angular momentum vector). The model is Hamiltonian. Its phase space is parameterized by the  $S_{\alpha}$  variables treated as coordinates on  $\mathfrak{su}^*(2)$  Lie coalgebra, where the Poisson–Lie structure is defined:

$$\{S_{\alpha}, S_{\beta}\} = \varepsilon_{\alpha\beta\gamma} S_{\gamma}, \quad H = \frac{1}{2} \sum_{\alpha=1}^3 J_{\alpha} S_{\alpha}^2. \tag{1.3}$$

The Hamiltonian equations  $\dot{S}_{\alpha} = \{H, S_{\alpha}\}$  are equivalent to (1.1). In what follows we deal with the complexified version of the Euler equation and its generalizations, i.e.  $S_{\alpha} \in \mathbb{C}$ ,  $J_{\alpha} \in \mathbb{C}$  and  $\mathfrak{su}^*(2)$  is replaced by  $\mathfrak{sl}^*(2, \mathbb{C})$ .

The Euler–Arnold generalizations of (1.1) correspond to higher rank Lie algebras (or groups). It means that  $S = \sum_{\alpha} S_{\alpha} T_{\alpha}$ , where  $\{T_{\alpha}\}$ —some basis in the Lie algebra  $\mathfrak{g}$ . Such dynamical type systems were introduced by Arnold [1], and were shown to be Liouville integrable in some particular cases [11, 30, 34]. We focus on elliptic integrable systems which appeared originally for many-body systems of Calogero–Moser type [38]. The construction of its solutions [19] requires the Lax pair with spectral parameter  $z$  living on an elliptic curve  $\Sigma_{\tau} = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$  with moduli  $\tau$ ,  $\text{Im } \tau > 0$ . For the top-like systems, such Lax pairs were constructed for continuous and discrete XYZ models by Sklyanin [44, 45] and then were generalized to the Gaudin type models and to higher rank cases [9, 35, 42] using the Belavin–Drinfeld elliptic  $r$ -matrix [6]. Later, both types of elliptic models (the many-body systems and the elliptic tops) were unified [22] by the Symplectic Hecke correspondence (the classical analogue of the IRF-vertex correspondence [5]). The classification of general elliptic models, including those of mixed types for simple Lie groups, can be found in [21].

<sup>6</sup> See appendix by Krichever in paper [12].

**Elliptic  $sl_N$  top** is a generalization of the Euler one (1.1) for the case  $S \in \text{Mat}(N, \mathbb{C})$ :

$$\dot{S} = [S, J(S)], \quad S = \sum_{i,j=1}^N E_{ij} S_{ij} = \sum_{\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N; \alpha \neq 0} T_\alpha S_\alpha, \quad (1.4)$$

$$J(S) = \sum_{\alpha \neq 0} T_\alpha S_\alpha J_\alpha, \quad J_\alpha = -E_2(\omega_\alpha), \quad \omega_\alpha = \frac{\alpha_1 + \alpha_2 \tau}{N}, \quad (1.5)$$

where  $\alpha \neq 0$  is a short notation for  $\alpha = (\alpha_1, \alpha_2) \neq (0, 0)$ , the set  $\{T_\alpha\}$  is a higher rank analogue of the Pauli matrices basis in  $\text{Mat}(N, \mathbb{C})$  (see (A.2)–(A.8)), and  $E_2$  is the second Eisenstein elliptic function (A.13). The absence of  $\alpha = 0$  term in (1.4) means that  $\text{tr}S = 0$ .

The (inverse) inertia tensor  $J$  depends on only one complex parameter—the moduli  $\tau$  of the elliptic curve. In fact, one can multiply  $J(S)$  by an arbitrary constant and shift all the components  $J_\alpha$  by another one constant (the latter does not effect equations of motion). Thus we have three parameters, and in this sense the elliptic  $sl(2, \mathbb{C})$  top (in this case  $\{\omega_\alpha\}$  is the set of half-periods  $\{0, \tau/2, 1/2 + \tau/2, 1/2\}$ ) coincides with the complexified Euler top.

The Lax equations

$$\dot{L}(z, S) = [L(z, S), M(z, S)] \quad (1.6)$$

written for the Lax pair with the spectral parameter  $z$

$$L(z) = \sum_{\alpha \neq 0} T_\alpha S_\alpha \varphi_\alpha(z, \omega_\alpha), \quad M(z) = \sum_{\alpha \neq 0} T_\alpha S_\alpha f_\alpha(z, \omega_\alpha) \quad (1.7)$$

are equivalent to (1.4)–(1.5) identically in  $z$ . The functions entering (1.7) are given in (A.25), (A.26). Let us also write down equations of motion (1.4) in components  $S_\alpha$  (i.e. equations as coefficient behind  $T_\alpha$ )<sup>7</sup>:

$$\dot{S}_\alpha = \sum_{\beta, \gamma: \beta + \gamma = \alpha} (\kappa_{\beta, \gamma} - \kappa_{\gamma, \beta}) S_\beta S_\gamma J_\gamma, \quad \alpha \neq 0, \quad (1.8)$$

where  $\kappa_{\beta, \gamma}$  are structure constants defined by relations  $T_\beta T_\gamma = \kappa_{\beta, \gamma} T_{\beta + \gamma}$  (A.4).

**Relativistic elliptic  $gl_N$  top** is a deformation of (1.4)–(1.5). It generalizes the non-relativistic top in the same way as the (elliptic) Ruijsenaars–Schneider model [43] generalizes the Calogero–Moser model. The Lax equation (1.6) is written for the Lax pair

$$L^\eta(z) = \sum_{\alpha} T_\alpha S_\alpha \varphi_\alpha^\eta(z), \quad M^\eta(z) = -\sum_{\alpha \neq 0} T_\alpha S_\alpha \varphi_\alpha(z, \omega_\alpha), \quad S \in \text{Mat}(N, \mathbb{C}), \quad (1.9)$$

where  $\eta$  is the deformation parameter and  $\{\varphi_a^\eta(z)\}$  is the set of functions (A.23). It provides the equations of motion

$$\dot{S} = [S, J^\eta(S)], \quad (1.10)$$

$$J^\eta(S) = \sum_{\alpha \neq 0} T_\alpha S_\alpha J_\alpha^\eta, \quad J_\alpha^\eta = E_1(\eta + \omega_\alpha) - E_1(\omega_\alpha), \quad (1.11)$$

where  $E_1$ —is the first Eisenstein function (A.12). For the rank 1 matrix  $S$  this model is gauge equivalent to the elliptic Ruijsenaars–Schneider model. In the limit  $\eta \rightarrow 0$  (1.10)–(1.11) turn into (1.4)–(1.5). Similarly to (1.8) we have the following equations of motion written in components  $S_\alpha$ :

<sup>7</sup> In  $N = 2$  case equations (1.8) coincides with (1.1) up to redefinition (A.8) and the factor  $1/2t$  as in (1.2).

$$\dot{S}_\alpha = \sum_{\beta, \gamma: \beta + \gamma = \alpha} (\kappa_{\beta, \gamma} - \kappa_{\gamma, \beta}) S_\beta S_\gamma J_\gamma^\eta, \quad \alpha \neq 0; \quad \dot{S}_0 = 0. \quad (1.12)$$

The relativistic top also has an  $\eta$ -independent description, which at the level of equations of motion coincides with the non-relativistic one<sup>8</sup>. Substitution

$$S_\alpha \rightarrow S_\alpha \varphi_\alpha(\eta, \omega_\alpha) \quad \text{for } \alpha \neq 0 \text{ and } S_0 \rightarrow S_0 \quad (1.13)$$

transforms (1.12) into (1.8). It can be easily verified if one represents  $J_\gamma^\eta$  using (A.11) as  $J_\gamma^\eta = f_\gamma(\eta, \omega_\gamma)/\varphi_\gamma(\eta, \omega_\gamma)$ . Then  $\eta$  is cancelled out from the equations of motion in the same way as spectral parameter  $z$  is cancelled out from the Lax equations (1.6) providing (1.8).

**R-matrix formulation.** The (non)relativistic classical tops can be described in terms of quantum  $R$ -matrices [23]. In the elliptic case<sup>9</sup> we deal with the Baxter–Belavin  $GL_N$   $R$ -matrix [4] written in the form:

$$R_{12}^{\hbar}(z_1, z_2) = R_{12}^{\hbar}(z_1 - z_2) = \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi_a^{\hbar}(z_1 - z_2) T_a \otimes T_{-a} \in \text{Mat}(N, \mathbb{C})^{\otimes 2}, \quad (1.14)$$

It satisfies the quantum Yang–Baxter equation

$$R_{12}^{\hbar}(z_1, z_2) R_{13}^{\hbar}(z_1, z_3) R_{23}^{\hbar}(z_2, z_3) = R_{23}^{\hbar}(z_2, z_3) R_{13}^{\hbar}(z_1, z_3) R_{12}^{\hbar}(z_1, z_2) \quad (1.15)$$

and the unitarity condition which for (1.14) is as follows:

$$R_{12}^{\hbar}(z_1, z_2) R_{21}^{\hbar}(z_2, z_1) = N^2 (\wp(N\hbar) - \wp(z_1 - z_2)) 1 \otimes 1. \quad (1.16)$$

The construction of the (non)relativistic tops uses coefficients of local expansions near  $\hbar = 0$  (the classical limit)

$$R_{12}^{\hbar}(z) = \frac{1}{\hbar} 1 \otimes 1 + r_{12}(z) + \hbar m_{12}(z) + O(\hbar^2) \quad (1.17)$$

and near  $z = 0$ :

$$R_{12}^{\hbar}(z) = \frac{N}{z} P_{12} + R_{12}^{\hbar, (0)} + z R_{12}^{\hbar, (1)} + O(z^2), \quad (1.18)$$

$$r_{12}(z) = \frac{N}{z} P_{12} + r_{12}^{(0)} + O(z), \quad (1.19)$$

where  $P_{12}$  is the permutation operator (A.6). The coefficient  $r_{12}(z)$  from expansion (1.17) is the classical Belavin–Drinfeld  $r$ -matrix [6] (B.2). Explicit expressions for the coefficients are given in (B.2)–(B.6).

The elliptic (1.4)–(1.7) is formulated in terms of  $R$ -matrix data as follows:

$$J(S) = \text{tr}_2(m_{12}(0)S_2), \quad S_2 = 1 \otimes S, \quad \text{tr}(S) = 0, \quad (1.20)$$

$$L(z, S) = \text{tr}_2(r_{12}(z)S_2), \quad M(z, S) = \text{tr}_2(m_{12}(z)S_2). \quad (1.21)$$

Similarly, for the relativistic elliptic top (1.9)–(1.11) we have:

$$J^\eta(S) = \text{tr}_2((R_{12}^{\eta, (0)} - r_{12}^{(0)})S_2), \quad (1.22)$$

$$L^\eta(z, S) = \text{tr}_2(R_{12}^\eta(z)S_2), \quad M^\eta(z, S) = -\text{tr}_2(r_{12}(z)\bar{S}_2), \quad (1.23)$$

<sup>8</sup> This is because the relativistic top is a quasi-classical version of the one site spin chain, due to the fact that the elliptic top admits a bi-Hamiltonian structure consisting of linear and quadratic Poisson  $r$ -matrix structures. See details in [16, 23].

<sup>9</sup> See [24] and references therein for the rational and trigonometric cases.

where  $\bar{S}$  is a traceless part of  $S$ . Details can be found in [23, 24].  $M^\eta$  has no explicit dependence on  $\eta$ . We keep this notation to emphasize that it is the  $M$ -matrix of the relativistic model.

**$\mathbb{Z}_2$  reductions in elliptic tops.** To pass to the noncommutative version of the defined above elliptic tops, we will need to impose some constraints. They can be described for the elliptic tops as the  $\mathbb{Z}_2$  reduction. The idea of reduction provided by some finite groups in the classical integrable systems was proposed by Aleksander Mikhailov [32]. It allows one to construct non-trivial integrable systems starting from some trivial or known integrable systems.

The  $\mathbb{Z}_2$  reduction under consideration is simply written in terms of coordinates on the phase space  $S_\alpha$ . The corresponding constraints are

$$S_\alpha = S_{-\alpha} \text{ for all } \alpha \tag{1.24}$$

for non-relativistic tops and

$$\frac{S_\alpha}{\varphi_\alpha(\eta, \omega_\alpha)} = \frac{S_{-\alpha}}{\varphi_{-\alpha}(\eta, -\omega_\alpha)}, \text{ for all } \alpha \neq 0 \tag{1.25}$$

in the relativistic case. Some details of the reduction are given in the appendix. Let us just mention here that in the  $N = 2$  case (which is the Euler top (1.1)–(1.3)) the reduction is trivial since the constraints (1.24) and (1.25) are identities. Indeed,  $T_\alpha \equiv T_{-\alpha} = \sigma_\alpha$  and  $S_\alpha \equiv S_{-\alpha}$ . The arguments  $\omega_\alpha$  are half-periods  $\tau/2$ ,  $(\tau + 1)/2$ ,  $1/2$ , therefore, using (5.4) and (5.5), it is easy to show that  $\varphi_\alpha(\eta, \omega_\alpha) = \varphi_{-\alpha}(\eta, -\omega_\alpha)$ . As we will see below in the reduced case, one can replace commuting variables by non commuting variables.

The classical  $r$ -matrix structure on the reduced phase space turns into the classical reflection equation [46]. Two important examples of such type reduction were described in [51] and [28]. The first one is the BC<sub>1</sub> Calogero–Inozemtsev model [15] described by equation

$$\frac{d^2u}{dt^2} = \sum_{a=0}^3 \nu_a^2 \wp'(u + \omega_a). \tag{1.26}$$

The second example is the Zhukovsky–Volterra gyrostat [49]. It generalizes the Euler top (1.1) to a non-zero external field

$$\partial_t S = [S, J(S)] + [S, \nu'], \tag{1.27}$$

where  $\nu' = \sum_{\alpha=1}^3 \nu'_\alpha \sigma_\alpha$ , and  $(\nu'_1, \nu'_2, \nu'_3)$  plays the role of a constant external field (gyrostatic momentum in the classical case and magnetic field in the quantum case). The Lax pair for (1.27) generalizes (1.7) in the following way:

$$\begin{aligned} L^{ZV}(z) &= \frac{1}{2t} \sum_{\alpha=1}^3 \sigma_\alpha \left( S_\alpha \varphi_\alpha(z, \omega_\alpha) + \frac{\nu'_\alpha}{\varphi_\alpha(z, \omega_\alpha)} \right), \\ M^{ZV}(z) &= -\frac{1}{2t} \sum_{\alpha=1}^3 \sigma_\alpha S_\alpha \frac{\varphi_1(z, \omega_1) \varphi_2(z, \omega_2) \varphi_3(z, \omega_3)}{\varphi_\alpha(z, \omega_\alpha)}. \end{aligned} \tag{1.28}$$

The models (1.26) and (1.27) are gauge equivalent at the level of Lax pairs. An explicit change of variables  $S_a = S_a(\dot{u}, u, \nu_0, \dots, \nu_3)$  was obtained in [28]. The constants  $\nu'_\alpha$  from (1.27) are linear combinations of  $\nu_\alpha$  from (1.26) with  $\tau$ -dependent coefficients (see (1.32) and (1.33)). The fourth (missing) constant in (1.27) appears as the value of (Casimir function)  $\nu_0'^2 = S_1^2 + S_2^2 + S_3^2$ .

**Painlevé VI equation as a non-autonomous top.** The Painlevé VI equation is the top equation in the hierarchy of the classification of non-linear ODE of second order possessing the Painlevé property. It depends on four constants and can be defined as the monodromy preserving condition for a linear differential system with meromorphic coefficients defined on  $\mathbb{C}P^1$ . Equivalently, it can be formulated in elliptic form [31, 39]. Then it takes the form of a non-autonomous version of the Calogero–Inozemtsev system  $BC_1$  (1.26):

$$\frac{d^2u}{d\tau^2} = \sum_{a=0}^3 \nu_a^2 \wp'(u + \omega_a), \quad (1.29)$$

while the monodromy preserving condition is of the form:

$$\partial_\tau L(w) - \frac{1}{2\pi i} \partial_w M(w) = [L(w), M(w)]. \quad (1.30)$$

Equation (1.29) is non-autonomous since  $\wp'(u + \omega_a)$  depends on moduli  $\tau$  in both explicit (through its dependence on  $\omega_a$ ) and implicit (through definition (A.13) of  $\wp$ -function) ways. Similarly, one can define the non-autonomous version of the Zhukovsky–Volterra gyrostat (1.27)

$$\partial_\tau S = [S, J(S)] + [S, \nu']. \quad (1.31)$$

The latter model is non-autonomous due to  $\tau$ -dependence of the components of (inverse) inertia tensor  $J_\alpha$  (1.5) and the  $\tau$ -dependence entering  $\nu'_\alpha$ :

$$\nu'_\alpha = c_\alpha(\tau) \tilde{\nu}_\alpha, \quad c_\alpha(\tau) = \varphi_\alpha(z, \omega_\alpha) \varphi_\alpha(z - \omega_\alpha, \omega_\alpha) = -\exp(-2\pi i \omega_a \partial_\tau \omega_a) \left( \frac{\wp'(0)}{\wp(\omega_a)} \right)^2, \quad (1.32)$$

$$\begin{aligned} \tilde{\nu}_0 &= \frac{1}{2}(\nu_0 + \nu_1 + \nu_2 + \nu_3), & \tilde{\nu}_1 &= \frac{1}{2}(\nu_0 + \nu_1 - \nu_2 - \nu_3), \\ \tilde{\nu}_2 &= \frac{1}{2}(\nu_0 - \nu_1 + \nu_2 - \nu_3), & \tilde{\nu}_3 &= \frac{1}{2}(\nu_0 - \nu_1 - \nu_2 + \nu_3), \end{aligned} \quad (1.33)$$

where the set of  $\nu_a$  consists of  $\tau$ -independent constants from (1.29). Equations (1.29) and (1.31) are again (as in the autonomous case) gauge equivalent. The corresponding change of variables  $S_a = S_a(\dot{u}, u, \tau, \nu_0, \dots, \nu_3)$  is given in [28]. In this sense equation (1.31) is also a form of the Painlevé VI equation<sup>10</sup>. The Lax pair generating (1.31) through (1.30) is (almost) the same as in the autonomous case:

$$\begin{aligned} L^{PVI}(w) &= L^{ZV}(w) \stackrel{(1.32)}{=} \frac{1}{2i} \sum_{\alpha=1}^3 \sigma_\alpha (S_\alpha \varphi_\alpha(w, \omega_\alpha) + \tilde{\nu}_\alpha \varphi_\alpha(w - \omega_\alpha, \omega_\alpha)), \\ M^{PVI}(w) &= M^{ZV}(w) + E_1(w) L^{ZV}(w). \end{aligned} \quad (1.34)$$

It is an example of the so-called classical Painlevé–Calogero correspondence [20], claiming that properly defined Lax pairs for elliptic non-relativistic models describe both integrable mechanics through the Lax equation (1.6) and the monodromy preserving equation through (1.30). The proof of this fact is based on the heat equation (A.17) for the Kronecker function. In a general (Euler–Arnold) case the heat equation holds for  $R$ -matrices (1.17):

$$2\pi i \partial_\tau R_{12}^{\hbar}(z) = \partial_z \partial_{\hbar} R_{12}^{\hbar}(z), \quad 2\pi i \partial_\tau r_{12}(z) = \partial_z m_{12}(z). \quad (1.35)$$

In the  $sl_N$  case, substitution of the Lax pair (1.7) into (1.8) leads to the non-autonomous Euler–Arnold top

<sup>10</sup> See also [29] for interrelations between elliptic forms of the Painlevé VI.

$$\partial_\tau S = [S, J(S)], \tag{1.36}$$

which can be considered as a multidimensional analogue of Painlevé equations.

**Purpose of paper:**

**1. Lax equations from associative Yang–Baxter equation.** The quantum Baxter–Belavin  $R$ -matrix (1.14)–(1.16) can be interpreted as the matrix generalization of the Kronecker function (A.9) [25–27, 40]. Similarly to this scalar function, the  $R$ -matrix satisfies relations which are matrix analogues of the elliptic function identities and properties. The most important for our purposes (see also appendix) are:

- the associative Yang–Baxter equation [40] (analogue of the Fay identity (A.18)):

$$R_{12}^{\hbar} R_{23}^{\eta} = R_{13}^{\eta} R_{12}^{\hbar-\eta} + R_{23}^{\eta-\hbar} R_{13}^{\hbar}, \quad R_{ab}^{\hbar} = R_{ab}^{\hbar}(z_a - z_b), \tag{1.37}$$

- skew-symmetry (analogue of  $\phi(\hbar, z) = -\phi(-\hbar, -z)$  and  $E_1(z) = -E_1(-z)$ ):

$$R_{12}^{\hbar}(z) = -R_{21}^{-\hbar}(-z), \quad r_{12}(z) = -r_{21}(-z), \quad m_{12}(z) = m_{21}(-z), \tag{1.38}$$

In section 2 it is shown that Lax equations with Lax pairs of the relativistic (1.23) or non-relativistic (1.21) top are equivalent to equations of motion (1.10) or (1.4) with the inverse inertia tensors (1.22) or (1.20) respectively. We do not explicitly use the elliptic function identities. Our derivation is valid for any  $R$ -matrices (1.14)–(1.19) satisfying also (1.37), (1.38).

**2. Matrix extensions of tops.** A direct meaning of a matrix extension is that (the scalar,  $\mathbb{C}$ -valued) variables of a model are replaced by noncommutative  $\text{Mat}(M, \mathbb{C})$  matrices. See examples in [36]. When  $M = 1$  we come back to the initial system. The matrix extension can be thought of as a noncommutative version of a model. It is then described by non-commutative (double) Poisson brackets [17]. The appearance of matrix variables also provides additional  $\text{GL}_M$  symmetry: this group acts on all matrix variables by conjugation. The corresponding Poisson algebra and its quantization was studied in [3]. The double brackets formalism is not used in our paper. Our aim is to get the equations of motion for matrix extensions by generalizing the Lax pairs.

The set of variables (or the coordinates on the phase space) in the elliptic  $\mathfrak{gl}_N$  top  $\{S_\alpha \in \mathbb{C}, \alpha \in \mathbb{Z}_N \times \mathbb{Z}_N\}$  should be replaced by the set of matrices  $\{S_\alpha \in \text{Mat}(M, \mathbb{C})\}$ :

$$S_\alpha \rightarrow S_\alpha = \sum_{\tilde{\alpha} \in \mathbb{Z}_M \times \mathbb{Z}_M} \tilde{T}_{\tilde{\alpha}} S_\alpha^{\tilde{\alpha}} \in \text{Mat}(M, \mathbb{C}), \quad \alpha \in \mathbb{Z}_N \times \mathbb{Z}_N, \tag{1.39}$$

where  $\{\tilde{T}_{\tilde{\alpha}} \in \text{Mat}(M, \mathbb{C}), \tilde{\alpha} \in \mathbb{Z}_M \times \mathbb{Z}_M\}$  is the basis (A.2) in  $\text{Mat}(M, \mathbb{C})$ . We will use tildes for ‘matrix’ or ‘noncommutative’ space<sup>11</sup>. It becomes ‘scalar’ or ‘commutative’ when  $M = 1$ . The space  $\text{Mat}(N, \mathbb{C})$  is an auxiliary space. It coincides with the matrix space of Lax equations (1.6) or the matrix form of equations of motion (1.7) of initial (scalar) models.

A natural way to get generalizations of the construction of Lax pairs (1.23), (1.21) to matrix-valued variables is to consider the following  $\text{Mat}(NM, \mathbb{C})^{\otimes 2}$ -valued  $R$ -matrix:

<sup>11</sup> The noncommutativity means that  $S_\alpha S_\beta \neq S_\beta S_\alpha$ , and we do not imply any constraints for any  $S_\alpha$  inside noncommutative space.



$$R_{12, \tilde{1}\tilde{2}}^\eta(z) = R_{12}^\eta(z) \otimes \tilde{P}_{\tilde{1}\tilde{2}}, \tag{1.40}$$

where  $R_{12}^\eta(z)$  is the same  $\text{Mat}(N, \mathbb{C})^{\otimes 2}$ -valued  $R$ -matrix in auxiliary space as in (1.23), while  $\tilde{P}_{\tilde{1}\tilde{2}}$  is the permutation operator in noncommutative space. It is easy to see that  $R_{12, \tilde{1}\tilde{2}}^\eta(z)$  is indeed an  $R$ -matrix in the sense of the quantum Yang–Baxter equation (1.15) and unitarity condition (1.16). Moreover, it satisfies the associative Yang–Baxter equation (1.37) as well. However it has a different to (1.17) classical limit (it starts not from  $\hbar^{-1} 1_{NM} \times 1_{NM}$ ). For this reason the general construction of the Lax pairs does not work for matrix extensions in the same way as the scalar case. To overcome this, problem additional constraints are required. The first one is that

$$S_0 = 1_M S_0, \tag{1.41}$$

i.e. the matrix extension of the  $S_0$  variable should also be scalar. This condition obviously needs to be preserved by dynamics (equations of motion). The latter provides another constraint. With these constraints the generalization of the construction of Lax pairs works for the  $R$ -matrix (1.40) and provides equations of motion

$$\dot{S}_{1\bar{1}} = [S_{1\bar{1}}, J_1(S_{1\bar{1}})], \tag{1.42}$$

where the inertia tensor  $J$  acts in auxiliary space only.

We will show that the above mentioned constraints are fulfilled for matrix extensions of  $\mathbb{Z}_2$  reduced elliptic tops. It means that similarly to (1.24) we set  $S_\alpha = S_{-\alpha}$ . In this case one obtains

$$\dot{S}_\alpha = \sum_{\beta, \gamma: \beta + \gamma = \alpha} (\kappa_{\beta, \gamma} S_\beta S_\gamma - \kappa_{\gamma, \beta} S_\gamma S_\beta) J_\gamma, \alpha \neq 0. \tag{1.43}$$

where  $J_\gamma = -E_2(\omega_\gamma)$ . In the scalar case  $M = 1$  the latter equations coincide with (1.8). The same holds true for the relativistic top (1.12) and  $\mathbb{Z}_2$ -reduction constraints (1.25).

**Remark 1.** In this construction one can replace the algebra  $\text{Mat}(M, \mathbb{C})$  by an arbitrary associative noncommutative algebra with a well defined trace functional and the permutation operator acting on the basis of the algebra. For example, one can take the infinite group of the quantum torus, its trigonometric and rational degenerations, or their quasi-classical limits to the algebras of vector fields.

**Remark 2.** Our results allow one to define the equations of motion in the Hamiltonian form using double Poisson brackets [10, 18, 47], as was done in [2, 3, 36]. It is straightforward to construct the ‘classical’  $\tilde{r}$ -matrix by means of the classical  $r$ -matrix (1.17) and the permutation operator in  $\text{Mat}(M, \mathbb{C})$  and consider the classical reflection equation defined by  $\tilde{r}$ . It leads the double Poisson brackets for the Lax operators  $L^\eta(z, \mathbb{S})$  in terms of the  $\tilde{r}$ -matrix. Furthermore, it opens a way to quantize the noncommutative tops by means of the  $R$ -matrix (1.40) and the quantum reflection equation.

At the end of the paper we also describe a special elliptic Gaudin model with equations of motion

$$\dot{A}^\alpha = \sum_{\beta + \gamma = \alpha} [A^\beta, A^\gamma] J_\gamma \quad \alpha \neq 0, \tag{1.44}$$

where  $\{A^\alpha\}$  is a set of  $N^2 - 1$  matrices of size  $N \times N$  with constraints  $A^\alpha = A^{-\alpha}$ . Equation (1.44) reproduces the elliptic top equations of motion (1.8) via reduction  $A^\alpha \rightarrow T_\alpha S_\alpha$  ( $S_\alpha = S_{-\alpha}$ ).

**3. Matrix extensions of Painlevé equations.** Finally, we construct the noncommutative generalization of the Painlevé VI equation. The non-commutative generalizations of the Painlevé II–IV equations were considered before in [7, 8, 37, 41]. Here we identify the non-commutative Painlevé VI equation with the non-commutative non-autonomous Zhukovsky–Volterra gyrostat (1.31):

$$\frac{d}{d\tau} \mathbb{S}_\alpha = \frac{1}{2}(\mathbb{S}_\beta \mathbb{S}_\gamma + \mathbb{S}_\gamma \mathbb{S}_\beta)(E_2(\omega_\beta) - E_2(\omega_\gamma)) + \mathbb{S}_\beta \nu'_\gamma - \mathbb{S}_\gamma \nu'_\beta, \quad (1.45)$$

These equations take the form (1.31) for  $N = 1$ . Our construction allows one to define the Lax pair for the Painlevé VI equation using the same Lax operators as for the autonomous case.

## 2. Lax pairs from associative Yang–Baxter equation

In this section we do not use explicit forms of Lax pairs but only the properties of the underlying  $R$ -matrices. Our current purpose is to show that the Lax equations with the  $R$ -matrix forms of the Lax pairs of integrable tops (1.21), (1.23) are equivalent to equations of motion (1.4), (1.10) with the corresponding inertia tensors (1.20), (1.22) due to additional properties of the  $R$ -matrix (1.37), (1.38). Below we prove these statements for relativistic and non-relativistic tops separately.

**Proposition 2.1.** *Suppose that the quantum  $R$ -matrix entering the Lax pair of the relativistic top (1.23) satisfies not only (1.15)–(1.19) but also the associative Yang–Baxter equation (1.37) and the skew-symmetry property (1.38). Then the Lax equations (1.6) with the Lax pair (1.23) are equivalent to equations of motion of the relativistic top (1.10) with (inverse) inertia tensor  $J^\eta$  (1.22).*

**Proof.** Let us verify that the Lax equations

$$\dot{L}^\eta(z, S) = [L^\eta(z, S), M^\eta(z, S)] \quad (2.1)$$

with  $L^\eta$  and  $M^\eta$  (1.23) are fulfilled on equations of motion

$$\dot{S} = [S, J^\eta(S)], \quad J^\eta(S) = \text{tr}_2((R_{12}^{\eta(0)} - r_{12}^{(0)})S_2) \quad (2.2)$$

identically in spectral parameter  $z$ , i.e.

$$L^\eta(z, [S, J^\eta(S)]) = [L^\eta(z, S), M^\eta(z, S)], \quad \forall z. \quad (2.3)$$

The left-hand side (lhs) of (2.3) is equal to<sup>12</sup>

$$\begin{aligned} L_1^\eta(z, [S, J^\eta(S)]) &= \text{tr}_{2,3}\{R_{12}^\eta(z)[S_2, (R_{23}^{\eta(0)} - r_{23}^{(0)})S_3]\} \\ &= -\text{tr}_{2,3}\{[R_{12}^\eta(z), R_{23}^{\eta(0)}]S_2S_3\} - \text{tr}_{2,3}\{[R_{13}^\eta(z), r_{23}^{(0)}]S_2S_3\}, \end{aligned} \quad (2.4)$$

where we have used  $r_{23}^{(0)} = -r_{32}^{(0)}$  (B.8). To simplify the right-hand side (rhs) of (2.3), notice that  $\bar{S}$  in (1.23) can be replaced by  $S$  since the scalar part of  $M^\eta$  does not give any input to  $[L^\eta, M^\eta]$ . In fact  $\bar{S}$  in (1.23) is used in order to match the elliptic definition (1.9). This question will become nontrivial in the case of matrix valued variables.

Let us write down (B.11) with  $z_3 = 0$ , which is a consequence of the associative Yang–Baxter equation (1.37):

<sup>12</sup> The index 1 in (2.4) is the number of the tensor component corresponding to the matrix space  $\text{Mat}(N, \mathbb{C})$  of equation (2.3). We sometimes omit this index where it is obvious (for example, in the lhs of (1.20)–(1.23)). The components 2,3 are under trace.

$$[R_{13}^\eta(z_1), r_{12}(z_1 - z_2)] = [R_{12}^\eta(z_1 - z_2), R_{23}^\eta(z_2)] + [R_{13}^\eta(z_1), r_{23}(z_2)] \quad (2.5)$$

and consider the limit  $z_2 \rightarrow 0$  (together with renaming  $z_1 := z$ ):

$$[R_{13}^\eta(z), r_{12}(z)] = [R_{12}^\eta(z), R_{23}^{\eta(0)}] + [R_{13}^\eta(z), r_{23}^{(0)}] - [\partial_z R_{12}^\eta(z), NP_{23}] \quad (2.6)$$

The simple pole at  $z_2 = 0$  cancels out due to  $[R_{12}^\eta(z), P_{23}] + [R_{13}^\eta(z), P_{23}] = 0$  by definition of the permutation operator. From (1.23) and (2.6) we conclude that

$$\begin{aligned} [L^\eta(z, S), M^\eta(z, S)]_1 &= -\text{tr}_{2,3}\{[R_{13}^\eta(z), r_{12}(z)] S_2 S_3\} \\ &= -\text{tr}_{2,3}\{[R_{12}^\eta(z), R_{23}^{\eta(0)}] S_2 S_3\} - \text{tr}_{2,3}\{[R_{13}^\eta(z), r_{23}^{(0)}] S_2 S_3\} \stackrel{(2.4)}{=} L_1^\eta(z, [S, J^\eta(S)]). \end{aligned} \quad (2.7)$$

Here we used that  $\text{tr}_{2,3}\{[\partial_z R_{12}^\eta(z), NP_{23}] S_2 S_3\} = 0$  since  $[P_{23}, S_2 S_3] = 0$ . In this way we finished the proof of (2.3) as an identity in  $z$  on the equations of motion (2.2). Conversely, (following [23]) one can easily obtain the equations of motion (2.2) from the Lax equations (2.1) by taking the residue of both parts of (2.1) at  $z = 0$ .  $\square$

Let us remark that in [23] we did not prove (2.3), i.e. the Lax equations are identities in the spectral parameter on the equations of motion. Instead, the following indirect argument was used: we know that (2.3) holds true in the elliptic case. Other cases are degenerations of the elliptic one. A degeneration procedure can be performed at the level of the Lax equation as well as at the level of the equations of motion. That is, we used explicit elliptic formulae to argue that the Lax equations are identities in  $z$ . The above given proof (2.1)–(2.7) is more general. It does not use any explicit form. It is direct and based on the associative Yang–Baxter equation only.

Let us now prove a similar statement for the non-relativistic top.

**Proposition 2.2.** *Suppose that quantum R-matrix entering (through expansion (1.17)) the Lax pair of the non-relativistic top (1.21) satisfies not only (1.15)–(1.19) but also the associative Yang–Baxter equation (1.37) and the skew-symmetry property (1.38). Then the Lax equations (1.6) with the Lax pair (1.21) (with  $\text{tr}S = 0$ ) are equivalent to equations of motion of the non-relativistic top (1.4) with (inverse) inertia tensor  $J$  (1.20).*

**Proof.** In this case  $\text{tr}S = NS_0 = 0$ . Let us verify that the Lax equations

$$\dot{L}(z, S) = [L(z, S), M(z, S)] \quad (2.8)$$

with  $L$  and  $M$  (1.21) are fulfilled on the equations of motion

$$\dot{S} = [S, J(S)], \quad J(S) = \text{tr}_2(m_{12}(0)S_2) \quad (2.9)$$

identically in spectral parameter  $z$ , i.e.

$$L(z, [S, J(S)]) = [L(z, S), M(z, S)], \quad \forall z. \quad (2.10)$$

The lhs of (2.10) equals

$$L(z, [S, J(S)])_1 = \text{tr}_{2,3}\{r_{12}(z)[S_2, m_{23}(0)S_3]\} = \text{tr}_{2,3}\{[m_{23}(0), r_{12}(z)] S_2 S_3\}. \quad (2.11)$$

To simplify the rhs of (2.10) we use (B.12). Write it down for  $z_3 = 0$

$$\begin{aligned} [m_{13}(z_1), r_{12}(z_1 - z_2)] &= [r_{12}(z_1 - z_2), m_{23}(z_2)] \\ &\quad + [m_{12}(z_1 - z_2), r_{23}(z_2)] + [m_{13}(z_1), r_{23}(z_2)] \end{aligned}$$

and consider the limit  $z_2 \rightarrow 0$  (with renaming  $z_1 := z$ ). The simple pole at  $z_2 = 0$  cancels out due to  $[m_{12}(z), P_{23}] + [m_{13}(z), P_{23}] = 0$  and we have:

$$\begin{aligned}
 [m_{13}(z), r_{12}(z)] &= [r_{12}(z), m_{23}(0)] - [\partial_z m_{12}(z), NP_{23}] \\
 &+ [m_{12}(z), r_{23}^{(0)}] + [m_{13}(z), r_{23}^{(0)}].
 \end{aligned}
 \tag{2.12}$$

Now we can compute

$$\begin{aligned}
 [L(z, S), M(z, S)]_1 &= \text{tr}_{2,3}\{[r_{12}(z), m_{13}(z)] S_2 S_3\} \stackrel{(2.12)}{=} \text{tr}_{2,3}\{[m_{23}(0), r_{12}(z)] S_2 S_3\} \\
 &\stackrel{(2.11)}{=} L(z, [S, J(S)])_1.
 \end{aligned}
 \tag{2.13}$$

In the equality via (2.12) we used that  $\text{tr}_{2,3}\{[\partial_z m_{12}(z), NP_{23}] S_2 S_3\} = 0$  due to  $[P_{23}, S_2 S_3] = 0$  and

$$\text{tr}_{2,3}\{([m_{12}(z), r_{23}^{(0)}] + [m_{13}(z), r_{23}^{(0)}]) S_2 S_3\} = 0$$

because the expression  $[m_{12}(z), r_{23}^{(0)}] + [m_{13}(z), r_{23}^{(0)}]$  is skew symmetric with respect to  $2 \leftrightarrow 3$  due to the property  $r_{23}^{(0)} = -r_{32}^{(0)}$ .

Conversely, we can obtain the equations of motion (2.9) from the Lax equations (2.8) by taking the residue at  $z = 0$  of both its sides.  $\square$

### 3. Matrix valued tops

In the paragraph below, we argue why the construction of section 2 cannot be directly generalized to matrix extensions of the tops models. It appears that matrix variables are not arbitrary but satisfy some constraints. Then we mention that these constraints are fulfilled for  $\mathbb{Z}_2$  reduced models and describe their matrix extensions.

**General construction and constraints.** A general idea of matrix extension is to replace scalar variables  $S_\alpha \in \mathbb{C}$  by matrix valued variables  $\mathbb{S}_\alpha \in \text{Mat}(M, \mathbb{C})$  (1.39). The initial scalar variables of a top model  $S_\alpha$  were themselves arranged into the matrix valued variable  $S = \sum_\alpha T_\alpha S_\alpha \in \text{Mat}(N, \mathbb{C})$  (the residue of the Lax matrix). Therefore, we deal with the following matrix variable:

$$S \in \text{Mat}(N, \mathbb{C}) \rightarrow \mathbb{S} = \mathbb{S}_{1\tilde{1}} = \sum_\alpha \sum_{\tilde{\alpha}} T_\alpha \otimes \tilde{T}_{\tilde{\alpha}} \mathbb{S}_\alpha^{\tilde{\alpha}} = \sum_\alpha T_\alpha \otimes \mathbb{S}_\alpha \in \text{Mat}(NM, \mathbb{C}), \tag{3.1}$$

where indices  $1, \tilde{1}$  stand for  $\text{Mat}(N, \mathbb{C})$  and  $\text{Mat}(M, \mathbb{C})$  tensor components respectively; likewise it is used in  $R$ -matrix notations.

Recall that the Lax matrix of the integrable top was defined as (1.21)  $L^\eta(z, S) = \text{tr}_2(R_{12}^\eta(z) S_2)$ . The latter means that for a given  $R$ -matrix written in standard basis of  $\text{Mat}(N, \mathbb{C})$  as

$$R_{12}^\eta(z) = \sum_{i,j,k,l=1}^N E_{ij} \otimes E_{kl} R_{ijkl}(z, \eta) \tag{3.2}$$

the corresponding Lax matrix (1.23) is of the form:

$$L^\eta(z, S) = \sum_{i,j,k,l=1}^N E_{ij} S_{kl} R_{ijkl}(z, \eta). \tag{3.3}$$

A natural way to get a matrix generalization is to consider the following expression:

$$R_{12, \bar{1}\bar{2}}^\eta(z) = R_{12}^\eta(z) \otimes \tilde{P}_{\bar{1}\bar{2}} = \sum_{i,j,k,l=1}^N \sum_{m,n=1}^M E_{ij} \otimes E_{kl} \otimes \tilde{E}_{mn} \otimes \tilde{E}_{mn} R_{ijkl}(z, \eta), \quad (3.4)$$

where  $\tilde{E}_{mn}$  is the standard basis in  $\text{Mat}(M, \mathbb{C})$  and  $\tilde{P}_{\bar{1}\bar{2}}$  is the permutation operator in  $\text{Mat}(M, \mathbb{C})^{\otimes 2}$ .

First, notice that this expression is again a quantum  $R$ -matrix. It satisfies the quantum Yang–Baxter equation

$$R_{12, \bar{1}\bar{2}}^{\hbar}(z_1, z_2) R_{13, \bar{1}\bar{3}}^{\hbar}(z_1, z_3) R_{23, \bar{2}\bar{3}}^{\hbar}(z_2, z_3) = R_{23, \bar{2}\bar{3}}^{\hbar}(z_2, z_3) R_{13, \bar{1}\bar{3}}^{\hbar}(z_1, z_3) R_{12, \bar{1}\bar{2}}^{\hbar}(z_1, z_2) \quad (3.5)$$

due to the Yang–Baxter equation for  $R_{12}^\eta(z)$  (1.15) and  $\tilde{P}_{\bar{1}\bar{2}} \tilde{P}_{\bar{1}\bar{3}} \tilde{P}_{\bar{2}\bar{3}} = \tilde{P}_{\bar{2}\bar{3}} \tilde{P}_{\bar{1}\bar{3}} \tilde{P}_{\bar{1}\bar{2}}$ . The unitarity condition (1.16) is fulfilled as well:

$$R_{12, \bar{1}\bar{2}}^{\hbar}(z_1, z_2) R_{21, \bar{2}\bar{1}}^{\hbar}(z_2, z_1) = N^2(\wp(N\hbar) - \wp(z_1 - z_2)) \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_M \otimes \mathbf{1}_M. \quad (3.6)$$

Moreover, the  $R$ -matrix (3.4) satisfies the associative Yang–Baxter equation (1.37):

$$R_{12, \bar{1}\bar{2}}^{\hbar} R_{23, \bar{2}\bar{3}}^\eta = R_{13, \bar{1}\bar{3}}^\eta R_{12, \bar{1}\bar{2}}^{\hbar-\eta} + R_{23, \bar{2}\bar{3}}^{\eta-\hbar} R_{13, \bar{1}\bar{3}}^{\hbar}, \quad R_{ab, \bar{a}\bar{b}}^{\hbar} = R_{ab, \bar{a}\bar{b}}^{\hbar}(z_a - z_b) \quad (3.7)$$

because of (1.37) and  $\tilde{P}_{\bar{1}\bar{2}} \tilde{P}_{\bar{2}\bar{3}} = \tilde{P}_{\bar{1}\bar{3}} \tilde{P}_{\bar{1}\bar{2}} = \tilde{P}_{\bar{2}\bar{3}} \tilde{P}_{\bar{1}\bar{3}}$ . Such type quantum and classical  $R$ -matrix structures were considered in [3] and [14].

Second, similarly to (3.3), the Lax matrix corresponding to the  $R$ -matrix (3.4)

$$L^\eta(z, \mathbb{S}) = \text{tr}_{2, \bar{2}}(R_{12, \bar{1}\bar{2}}^\eta(z) \mathbb{S}_{2\bar{2}}) = \sum_{i,j,k,l=1}^N E_{ij} \otimes \mathbb{S}_{kl} R_{ijkl}(z, \eta), \quad \mathbb{S}_{kl} = \sum_{m,n=1}^M \mathbb{S}_{kl}^{mn} \tilde{E}_{mn} \quad (3.8)$$

is exactly the matrix generalization of (3.3).

Therefore, we could expect to have a direct generalization (to the matrix case) of the Lax pairs construction via the associative Yang–Baxter equation described in section 2. However, we will see that it does not work in the same way. The reason is that the  $R$ -matrix (3.4) does not satisfy the local expansion of the classical limit (1.17). Indeed, near  $\hbar = 0$

$$R_{12, \bar{1}\bar{2}}^{\hbar}(z) = \frac{1}{\hbar} \mathbf{1}_N \otimes \mathbf{1}_N \otimes \tilde{P}_{\bar{1}\bar{2}} + r_{12, \bar{1}\bar{2}}(z) + O(\hbar), \quad (3.9)$$

i.e. in contrast to (1.17) the first coefficient of the expansion (3.9) is not  $\mathbf{1}_{NM} \otimes \mathbf{1}_{NM}$ . It causes a problem in the following way. The proofs of equivalence of the Lax equations and equations of motion given in section 2 did not use the associative Yang–Baxter equation itself but its degenerations (2.6) or (2.12) which appeared from (B.11). Equation (B.11) in its turn was obtained by subtracting (B.10) from (B.9). For the  $R$ -matrix (3.4) instead of (B.9), (B.10) we have

$$R_{12, \bar{1}\bar{2}}^{\hbar} R_{23, \bar{2}\bar{3}}^{\hbar} = R_{13, \bar{1}\bar{3}}^{\hbar} r_{12, \bar{1}\bar{2}} + r_{23, \bar{2}\bar{3}} R_{13, \bar{1}\bar{3}}^{\hbar} - \partial_{\hbar} R_{13, \bar{1}\bar{3}}^{\hbar} \tilde{P}_{\bar{1}\bar{2}}, \quad (3.10)$$

$$R_{23, \bar{2}\bar{3}}^{\hbar} R_{12, \bar{1}\bar{2}}^{\hbar} = R_{13, \bar{1}\bar{3}}^{\hbar} r_{23, \bar{2}\bar{3}} + r_{12, \bar{1}\bar{2}} R_{13, \bar{1}\bar{3}}^{\hbar} - \partial_{\hbar} R_{13, \bar{1}\bar{3}}^{\hbar} \tilde{P}_{\bar{2}\bar{3}}. \quad (3.11)$$

The difference between (3.10) and (3.11) contains the unwanted term  $\partial_{\hbar} R_{13, \bar{1}\bar{3}}^{\hbar} (\tilde{P}_{\bar{2}\bar{3}} - \tilde{P}_{\bar{1}\bar{2}})$  which equals zero in the scalar case  $M = 1$ . Then we need to require that

$$\text{tr}_{2,3,\bar{2},\bar{3}}\{\partial_{\hbar}R_{13,\bar{1}\bar{3}}^{\hbar}(\tilde{P}_{\bar{2}\bar{3}} - \tilde{P}_{\bar{1}\bar{2}})\mathbb{S}_{\bar{2}\bar{2}}\mathbb{S}_{\bar{3}\bar{3}}\} = \text{tr}_{2,3,\bar{2},\bar{3}}\{\partial_{\hbar}R_{13}^{\hbar}(\tilde{P}_{\bar{1}\bar{3}}\tilde{P}_{\bar{2}\bar{3}} - \tilde{P}_{\bar{1}\bar{3}}\tilde{P}_{\bar{1}\bar{2}})\mathbb{S}_{\bar{2}\bar{2}}\mathbb{S}_{\bar{3}\bar{3}}\} = 0, \quad (3.12)$$

i.e.

$$\text{tr}_{2,3,\bar{2},\bar{3}}\{\partial_{\hbar}R_{13}^{\hbar}[\tilde{P}_{\bar{1}\bar{2}}, \tilde{P}_{\bar{1}\bar{3}}]\mathbb{S}_{\bar{2}\bar{2}}\mathbb{S}_{\bar{3}\bar{3}}\} = \text{tr}_{2,3}\{\partial_{\hbar}R_{13}^{\hbar}[\mathbb{S}_{\bar{2}\bar{1}}, \mathbb{S}_{\bar{3}\bar{1}}]\} = 0 \quad (3.13)$$

and, therefore,

$$\text{tr}_2\mathbb{S}_{\bar{2}\bar{1}} \sim 1_M, \quad (3.14)$$

that is the *matrix analogue of the variable*  $\text{tr}S$  (or  $S_0$  in basis  $\{T_{\alpha}\}$ ) *should not be an arbitrary*  $\text{Mat}(M, \mathbb{C})$  *matrix but the one proportional to identity matrix*  $1_M$ . It is easy to see that the coefficient behind  $1_M$  in (3.14) should be a constant on the equations of motion (since it equals  $\text{tr}_{\bar{2}\bar{1}}\mathbb{S}_{\bar{2}\bar{1}}/M$ ). Therefore, the next set of constraints is generated by

$$\text{tr}_{\bar{1}}\dot{\mathbb{S}}_{\bar{1}\bar{1}} = 0 \quad (3.15)$$

which means that (3.14) should be preserved by the dynamics of the equations of motion.

**Equations of motion and Laxpairs.** On constraints (3.14), when (3.12) is true, we have the following equation obtained by subtracting (3.11) from (3.10):

$$\text{tr}_{2,3,\bar{2},\bar{3}}\{([R_{12,\bar{1}\bar{2}}^{\hbar}, R_{23,\bar{2}\bar{3}}^{\hbar}] - [R_{13,\bar{1}\bar{3}}^{\hbar}, r_{12,\bar{1}\bar{2}}] - [r_{23,\bar{2}\bar{3}}, R_{13,\bar{1}\bar{3}}^{\hbar}])\mathbb{S}_{\bar{2}\bar{2}}\mathbb{S}_{\bar{3}\bar{3}}\} = 0, \quad (3.16)$$

It is analogous to  $\text{tr}_{2,3}\{(equation(2.5))S_2S_3\}$ , which underlied the Lax equations in the scalar case. For a similar reason we obtain the following equations of motion in the relativistic case:

$$\dot{\mathbb{S}}_{\bar{1}\bar{1}} = [\mathbb{S}_{\bar{1}\bar{1}}, J_1^{\eta}(\mathbb{S}_{\bar{1}\bar{1}})], \quad (3.17)$$

where

$$\begin{aligned} J_1^{\eta}(\mathbb{S}_{\bar{1}\bar{1}}) &= \text{tr}_{\bar{2}\bar{2}}((R_{12,\bar{1}\bar{2}}^{\eta(0)} - r_{12,\bar{1}\bar{2}}^{(0)})\mathbb{S}_{\bar{2}\bar{2}}) = \text{tr}_{\bar{2}\bar{2}}((R_{12}^{\eta(0)} - r_{12}^{(0)}) \otimes \tilde{P}_{\bar{1}\bar{2}}\mathbb{S}_{\bar{2}\bar{2}}) \\ &= \text{tr}_2((R_{12}^{\eta(0)} - r_{12}^{(0)})\mathbb{S}_{\bar{2}\bar{1}}). \end{aligned} \quad (3.18)$$

In the scalar case  $M = 1$  the latter equation turns into (2.2). The Lax pair is given by

$$L^{\eta}(z, \mathbb{S}) = \text{tr}_{\bar{2},\bar{2}}(R_{12,\bar{1}\bar{2}}^{\eta}(z)\mathbb{S}_{\bar{2}\bar{2}}), \quad M^{\eta}(z, \mathbb{S}) = \text{tr}_{\bar{2},\bar{2}}(r_{12,\bar{1}\bar{2}}(z)\mathbb{S}_{\bar{2}\bar{2}}). \quad (3.19)$$

In the non-relativistic case, the equations of motion are

$$\dot{\mathbb{S}}_{\bar{1}\bar{1}} = [\mathbb{S}_{\bar{1}\bar{1}}, J_1(\mathbb{S}_{\bar{1}\bar{1}})], \quad (3.20)$$

where

$$J_1(\mathbb{S}_{\bar{1}\bar{1}}) = \text{tr}_{\bar{2}\bar{2}}(m_{12,\bar{1}\bar{2}}(0)\mathbb{S}_{\bar{2}\bar{2}}) = \text{tr}_{\bar{2}\bar{2}}(m_{12}(0) \otimes \tilde{P}_{\bar{1}\bar{2}}\mathbb{S}_{\bar{2}\bar{2}}) = \text{tr}_2(m_{12}(0)\mathbb{S}_{\bar{2}\bar{1}}). \quad (3.21)$$

The Lax pair is given by

$$L(z, \mathbb{S}) = \text{tr}_{\bar{2},\bar{2}}(r_{12,\bar{1}\bar{2}}(z)\mathbb{S}_{\bar{2}\bar{2}}), \quad M(z, \mathbb{S}) = \text{tr}_{\bar{2},\bar{2}}(m_{12,\bar{1}\bar{2}}(z)\mathbb{S}_{\bar{2}\bar{2}}). \quad (3.22)$$

Let us stress again that together with (3.17) or (3.20), the constraints (3.14), (3.15) should be fulfilled. Below we will see that these constraints are fulfilled for a special class of elliptic matrix tops.

**Matrix generalization of  $\mathbb{Z}_2$  reduced elliptic tops.** We start with the non-relativistic case. Similarly to (1.7) and following (3.22) we have the following Lax pair for the matrix elliptic top:

$$L(z, \mathbb{S}) = \sum_{\alpha \neq 0} T_\alpha \otimes \mathbb{S}_\alpha \varphi_\alpha(z, \omega_\alpha), \quad M(z, \mathbb{S}) = \sum_{\alpha \neq 0} T_\alpha \otimes \mathbb{S}_\alpha f_\alpha(z, \omega_\alpha). \quad (3.23)$$

The rhs of the Lax equation

$$\frac{d}{dt} L(z, \mathbb{S}) = [L(z, \mathbb{S}), M(z, \mathbb{S})] \quad (3.24)$$

is equal to

$$[L(z, \mathbb{S}), M(z, \mathbb{S})] = \sum_{\beta, \gamma \neq 0} T_\beta T_\gamma \otimes \mathbb{S}_\beta \mathbb{S}_\gamma \varphi_\beta(z) f_\gamma(z) - T_\beta T_\gamma \otimes \mathbb{S}_\beta \mathbb{S}_\gamma f_\beta(z) \varphi_\gamma(z) \quad (3.25)$$

By symmetrizing indices  $\beta$  and  $\gamma$  we get (here for short we use  $\varphi_\beta(z) = \varphi_\beta(z, \omega_\beta)$  and the same for  $f_\beta(z)$ )

$$\begin{aligned} &= \sum_{\beta, \gamma \neq 0} \frac{1}{2} T_\beta T_\gamma \otimes \mathbb{S}_\beta \mathbb{S}_\gamma \varphi_\beta(z) f_\gamma(z) + \frac{1}{2} T_\gamma T_\beta \otimes \mathbb{S}_\gamma \mathbb{S}_\beta \varphi_\gamma(z) f_\beta(z) \\ &\quad - \frac{1}{2} T_\beta T_\gamma \otimes \mathbb{S}_\beta \mathbb{S}_\gamma f_\beta(z) \varphi_\gamma(z) - \frac{1}{2} T_\gamma T_\beta \otimes \mathbb{S}_\gamma \mathbb{S}_\beta f_\gamma(z) \varphi_\beta(z) \quad (3.26) \\ &= \sum_{\beta, \gamma \neq 0} \frac{1}{2} (T_\beta T_\gamma \otimes \mathbb{S}_\beta \mathbb{S}_\gamma - T_\gamma T_\beta \otimes \mathbb{S}_\gamma \mathbb{S}_\beta) (\varphi_\beta(z) f_\gamma(z) - \varphi_\gamma(z) f_\beta(z)) \stackrel{(A.20)}{=} \\ &= \sum_{\beta, \gamma \neq 0} \frac{1}{2} T_{\beta+\gamma} \otimes (\kappa_{\beta, \gamma} \mathbb{S}_\beta \mathbb{S}_\gamma - \kappa_{\gamma, \beta} \mathbb{S}_\gamma \mathbb{S}_\beta) \varphi_{\beta+\gamma}(z) (E_2(\omega_\beta) - E_2(\omega_\gamma)) \\ &= \sum_{\beta, \gamma \neq 0} T_{\beta+\gamma} \otimes (\kappa_{\beta, \gamma} \mathbb{S}_\beta \mathbb{S}_\gamma - \kappa_{\gamma, \beta} \mathbb{S}_\gamma \mathbb{S}_\beta) \varphi_{\beta+\gamma}(z) J_\gamma, \quad (3.27) \end{aligned}$$

where  $J_\gamma = -E_2(\omega_\gamma)$  as in (1.5) and  $\kappa_{\gamma, \beta}$  are structure constants (A.4). Finally, equations of motion take the form

$$\dot{\mathbb{S}}_\alpha = \sum_{\beta, \gamma: \beta+\gamma=\alpha} (\kappa_{\beta, \gamma} \mathbb{S}_\beta \mathbb{S}_\gamma - \kappa_{\gamma, \beta} \mathbb{S}_\gamma \mathbb{S}_\beta) J_\gamma, \quad \alpha \neq 0; \quad J_\gamma = -E_2(\omega_\gamma). \quad (3.28)$$

In the scalar case  $M = 1$  the latter equations coincide with (1.8).

In the above equations we did not include the  $\alpha = 0$  component into the Lax pair (3.23), i.e.  $\mathbb{S}_0 = 0$ , and therefore (3.14) is fulfilled. However (3.15) is not fulfilled. Indeed, for  $\alpha = \beta + \gamma = 0$  in (3.26) we need to use (A.22) instead of (A.20). It yields ( $\kappa_{\beta, -\beta} = 1$ ) the following explicit expression for (3.15):

$$0 = \dot{\mathbb{S}}_0 = \sum_{\beta \neq 0} [\mathbb{S}_\beta, \mathbb{S}_{-\beta}] E_2'(\omega_\beta). \quad (3.29)$$

It is nontrivial because  $E_2'(z)$  is an odd function. A natural way to fulfill this constraint is to set

$$\chi: \quad \mathbb{S}_\alpha = \mathbb{S}_{-\alpha}, \quad \text{for all } \alpha. \quad (3.30)$$

It is a matrix analogue of the  $\mathbb{Z}_2$  reduced elliptic top defined by (1.24).

The set of constraints (3.30) is preserved by dynamics (3.28):

$$\dot{\mathbb{S}}_{-\alpha}|_\chi = \dot{\mathbb{S}}_\alpha|_\chi \quad (3.31)$$

since  $\kappa_{-\beta, -\gamma} = \kappa_{\beta, \gamma}$  and  $J_\alpha = J_{-\alpha}$ . Therefore, we have a well defined matrix valued elliptic top given by the Lax pair (3.23), equations of motion (3.28) and  $\mathbb{Z}_2$  reduction constraints (3.30).

In the relativistic case we have the following direct generalization of (1.9):

$$L^\eta(z, \mathbb{S}) = \sum_{\alpha} T_{\alpha} \otimes \mathbb{S}_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha} + \eta), \quad M^\eta(z, \mathbb{S}) = - \sum_{\alpha \neq 0} T_{\alpha} \otimes \mathbb{S}_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha}). \quad (3.32)$$

The Lax equations lead to equations of motion (3.20) for matrix-variables

$$\dot{\mathbb{S}}_{\alpha} = \sum_{\beta, \gamma: \beta + \gamma = \alpha} (\kappa_{\beta, \gamma} \mathbb{S}_{\beta} \mathbb{S}_{\gamma} - \kappa_{\gamma, \beta} \mathbb{S}_{\gamma} \mathbb{S}_{\beta}) J_{\gamma}^{\eta}, \quad \alpha \neq 0; \quad J_{\alpha}^{\eta} = E_1(\eta + \omega_{\alpha}) - E_1(\omega_{\alpha}) \quad (3.33)$$

via (A.19). The constraints (3.14), (3.15) mean that

$$\mathbb{S}_0 = S_0 \mathbb{1}_M, \quad (3.34)$$

where  $\mathbb{1}_M$  is identity  $M \times M$  matrix, and

$$\frac{1}{\varphi_{\alpha}(\eta, \omega_{\alpha})} \mathbb{S}_{\alpha} = \frac{1}{\varphi_{-\alpha}(\eta, -\omega_{\alpha})} \mathbb{S}_{-\alpha}, \quad \alpha \neq 0. \quad (3.35)$$

Let us now mention that the derivation of the equations of motion from the Lax pairs (3.23) or (3.32) did not use  $\mathbb{S}$  as a matrix. In fact, we can perform the same calculation thinking of  $\mathbb{S}_{\alpha}$  as elements of associative and noncommutative algebra.

#### 4. Noncommutative Painlevé VI equation

As was explained in the introduction, the Lax pair (1.7) of the non-relativistic top (1.8) also satisfies the monodromy preserving condition (1.30) and provides in this way the non-autonomous version of the Euler–Arnold equations (1.36). This construction is straightforwardly generalized to the matrix extension of the elliptic top described by the Lax pair (3.23). Namely, we have the following statement.

**Proposition 4.1.** *The Lax pair*

$$L(z, \mathbb{S}) = \sum_{\alpha \neq 0} T_{\alpha} \otimes \mathbb{S}_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha}), \quad M(z, \mathbb{S}) = \sum_{\alpha \neq 0} T_{\alpha} \otimes \mathbb{S}_{\alpha} f_{\alpha}(z, \omega_{\alpha}).$$

where the  $\mathbb{Z}_2$  reduction condition  $\mathbb{S}_{\alpha} = \mathbb{S}_{-\alpha}$  satisfies the monodromy preserving condition

$$\frac{d}{d\tau} L(w, \mathbb{S}) - \frac{1}{2\pi i} \frac{\partial}{\partial w} M(w, \mathbb{S}) = [L(w, \mathbb{S}), M(w, \mathbb{S})]$$

and provides a non-autonomous version of the matrix top equations:

$$\frac{d}{d\tau} \mathbb{S}_{1\bar{1}} = [\mathbb{S}_{1\bar{1}}, J_1(\mathbb{S}_{1\bar{1}})], \quad (4.1)$$

or

$$\frac{d}{d\tau} \mathbb{S}_{\alpha} = \sum_{\beta, \gamma: \beta + \gamma = \alpha} (\kappa_{\beta, \gamma} \mathbb{S}_{\beta} \mathbb{S}_{\gamma} - \kappa_{\gamma, \beta} \mathbb{S}_{\gamma} \mathbb{S}_{\beta}) J_{\gamma}, \quad \alpha \neq 0; \quad J_{\gamma} = -E_2(\omega_{\gamma}). \quad (4.2)$$

As in the scalar case, the proof is based on the heat equation  $2\pi i \partial_{\tau} \varphi_{\alpha}(z, \omega_{\alpha}) = \partial_z f_{\alpha}(z, \omega_{\alpha})$ .

In the same way one can define the matrix extension of the non-autonomous version of the Zhukovsky–Volterra gyrost<sup>13</sup> in the  $N = 2$  case. The constants  $\nu'_1, \nu'_2, \nu'_3$  are kept scalar, i.e.

<sup>13</sup> The autonomous version is of course well defined also. One should just replace the  $\tau$ -derivative by the  $t$ -derivative.



$$[\mathbb{S}_\alpha, \nu'_\beta] = 0. \tag{4.3}$$

**Proposition 4.2.** *The Lax pair from  $\text{Mat}(2, \mathbb{C}) \otimes \text{Mat}(M, \mathbb{C})$*

$$\begin{aligned} L^{PVI}(w, \mathbb{S}) &= L^{ZV}(w) \stackrel{(1.32)}{=} \frac{1}{2t} \sum_{\alpha=1}^3 \sigma_\alpha \otimes (\mathbb{S}_\alpha \varphi_\alpha(w, \omega_\alpha) + \mathbf{1}_{M \times M} \tilde{\nu}_\alpha \varphi_\alpha(w - \omega_\alpha, \omega_\alpha)), \\ M^{PVI}(w, \mathbb{S}) &= -\frac{1}{2t} \sum_{\alpha=1}^3 \sigma_\alpha \otimes \mathbb{S}_\alpha \frac{\varphi_1(w, \omega_1) \varphi_2(w, \omega_2) \varphi_3(w, \omega_3)}{\varphi_\alpha(w, \omega_\alpha)} + E_1(w) L^{PVI}(w, \mathbb{S}). \end{aligned} \tag{4.4}$$

provides, through substitution into the monodromy preserving condition (1.30), the following equations

$$\begin{aligned} \frac{d}{d\tau} \mathbb{S}_\alpha &= \frac{1}{2} (\mathbb{S}_\beta \mathbb{S}_\gamma + \mathbb{S}_\gamma \mathbb{S}_\beta) (E_2(\omega_\beta) - E_2(\omega_\gamma)) + \mathbb{S}_\beta \nu'_\gamma - \mathbb{S}_\gamma \nu'_\beta, \\ \omega_1 &= \tau/2, \quad \omega_2 = (1 + \tau)/2, \quad \omega_3 = 1/2. \end{aligned} \tag{4.5}$$

where  $(\alpha, \beta, \gamma) = (1, 2, 3)$  up to cyclic permutations. In matrix form we have

$$\frac{d}{d\tau} \mathbb{S}_{1\bar{1}} = [\mathbb{S}_{1\bar{1}}, J_1(\mathbb{S}_{1\bar{1}}) + \nu' \otimes \mathbf{1}_{M \times M}]. \tag{4.6}$$

In the scalar ( $N = 1$ ) case, equations (4.5) or (4.6) turn into the non-autonomous Zhukovsky–Volterra gyrostat (1.31), which is known to be equivalent to the Painlevé VI equation. By this reason we call (4.5) or (4.6) the noncommutative Painlevé VI equation. Here we should repeat remark 1 from the end of the introduction that equation (4.5) keeps the same form if  $\mathbb{S}$  takes values in an arbitrary non-commutative associative algebra  $\mathcal{A}$ .

### 5. Special elliptic Gaudin models as matrix tops

Consider the following  $\mathfrak{gl}_N$  Lax pair given by  $N \times N$  matrices

$$L^G(z) = A^0 + \sum_{\alpha \neq 0} A^\alpha \varphi_\alpha(z, \omega_\alpha), \quad M^G(z) = \sum_{\alpha \neq 0} A^\alpha f_\alpha(z, \omega_\alpha), \tag{5.1}$$

where  $A^\alpha \in \text{Mat}(N, \mathbb{C})$  is a set of  $\mathfrak{gl}_N$ -valued matrices with constraints

$$A^0 = \mathbf{1}_N S_0, \tag{5.2}$$

$$A^\alpha = A^{-\alpha} \text{ for all } \alpha \neq 0, \tag{5.3}$$

which are similar to (3.30). It can be viewed as a special elliptic Gaudin model. Indeed, it follows from quasiperiodic properties

$$\phi(z + 1, u) = \phi(z, u), \quad \phi(z + \tau, u) = \exp(-2\pi i u) \phi(z, u) \tag{5.4}$$

that (for  $\alpha \neq 0$ )

$$\begin{aligned} \varphi_\alpha(z + 1, \omega_\alpha) &= \exp\left(2\pi i \frac{\alpha_2}{N}\right) \varphi_\alpha(z, \omega_\alpha), \\ \varphi_\alpha(z + \tau, \omega_\alpha) &= \exp\left(-2\pi i \frac{\alpha_1}{N}\right) \varphi_\alpha(z, \omega_\alpha). \end{aligned} \tag{5.5}$$

Therefore, functions (sections of bundles)  $\{\varphi_\alpha(z)\}$  are double-periodic on a ‘large’ torus  $\Sigma_{N, N\tau}$  generated by fundamental parallelogram with periods  $N, N\tau$ . The latter means that  $L^G(z)$  is a

double-periodic function on  $\Sigma_{N,N\tau}$  with  $N^2 - 1$  simple poles at points  $N\omega_\alpha = \alpha_1 + \alpha_2\tau$ ,  $\alpha \neq 0$ . The residues at these points are linear combinations of  $A^\beta$ :

$$\operatorname{Res}_{z=N\omega_\alpha} L^G(z) = \sum_{\beta \neq 0} \kappa_{\beta,\alpha}^2 A^\beta, \quad (5.6)$$

where  $\kappa_{\beta,\alpha}$  is given by (A.4). This is why we refer to this model as a Gaudin one. The Lax equations are equivalent to

$$\dot{A}^\alpha = \sum_{\beta+\gamma=\alpha} [A^\beta, A^\gamma] J_\gamma, \quad J_\gamma = -E_2(\omega_\gamma), \quad \alpha \neq 0. \quad (5.7)$$

These equations generalize the elliptic top equations of motion (1.8) in the following sense. Equations (1.8) are reproduced from (5.7) via reduction

$$A^\alpha = T_\alpha S_\alpha. \quad (5.8)$$

At the same time (5.3) reduces to (1.24), i.e. (5.7) can be viewed as the matrix generalization of the  $\mathbb{Z}_2$  reduced elliptic top.

As in (3.29), the constraints (5.3) fulfill the constraint

$$0 = \dot{A}^0 = \sum_{\beta} [A^\beta, A^{-\beta}] E_2'(\omega_\beta), \quad (5.9)$$

which appear from the ‘zero mode’ of the Lax equations. In the same way, similarly to (3.31)  $\dot{A}^\alpha = \dot{A}^{-\alpha}$  on constraints (5.3), i.e. these constraints are preserved by dynamics.

Similar to the results of the previous section, we can easily construct non-autonomous models generalizing (5.7) through the monodromy preserving condition (1.30). The answer is as follows:

$$\frac{d}{d\tau} A^\alpha = \sum_{\beta+\gamma=\alpha} [A^\beta, A^\gamma] J_\gamma, \quad J_\gamma = -E_2(\omega_\gamma), \quad \alpha \neq 0. \quad (5.10)$$

It is interesting to mention that in the  $N = 2$  case, these equations are equivalent to the Painlevé VI equation (1.29) after reduction by the coadjoint action of ‘common’  $GL(2, \mathbb{C})$ :  $A^\alpha \rightarrow gA^\alpha g^{-1}$ . See details in [29].

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### Appendix. Elliptic functions and $R$ -matrices

The Baxter–Belavin  $R$ -matrix as well as elliptic tops uses a special basis in  $\operatorname{Mat}(N, \mathbb{C})$ . Let

$$Q_{kl} = \delta_{kl} \exp\left(\frac{2\pi i}{N} k\right), \quad \Lambda_{kl} = \delta_{k-l+1=0 \pmod{N}}, \quad Q^N = \Lambda^N = 1_{N \times N}. \quad (A.1)$$

Then for

$$T_a = T_{a_1 a_2} = \exp\left(\frac{\pi t}{N} a_1 a_2\right) Q^{a_1} \Lambda^{a_2}, \quad a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N \quad (\text{A.2})$$

due to

$$\exp\left(\frac{2\pi t}{N} a_1 a_2\right) Q^{a_1} \Lambda^{a_2} = \Lambda^{a_2} Q^{a_1} \quad (\text{A.3})$$

we have

$$T_\alpha T_\beta = \kappa_{\alpha,\beta} T_{\alpha+\beta}, \quad \kappa_{\alpha,\beta} = \exp\left(\frac{\pi t}{N} (\beta_1 \alpha_2 - \beta_2 \alpha_1)\right), \quad (\text{A.4})$$

where  $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$ . The structure constant  $\kappa_{\alpha,\beta}$  satisfies

$$\sum_\alpha \kappa_{\alpha,\gamma}^2 = N^2 \delta_{\gamma,0}, \quad (\text{A.5})$$

which is equivalent to the identity  $P_{12}^2 = 1 \otimes 1$  for the permutation operator  $P_{12}$  given by

$$P_{12} = \frac{1}{N} \sum_{\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N} T_\alpha \otimes T_{-\alpha} = \sum_{i,j=1}^N E_{ij} \otimes E_{ji}. \quad (\text{A.6})$$

From (A.4) we obviously get

$$[T_\alpha, T_\beta] = C_{\alpha,\beta} T_{\alpha+\beta}, \quad C_{\alpha,\beta} = \kappa_{\alpha,\beta} - \kappa_{\beta,\alpha}, \quad (\text{A.7})$$

i.e. the set  $\{T_\alpha\}$  can be also considered as a basis in  $\mathfrak{gl}_N$  Lie algebra. It is also called the sin-algebra basis since  $C_{\alpha,\beta} = 2t \sin\left(\frac{\beta_1 \alpha_2 - \beta_2 \alpha_1}{2N}\right)$ . Being written in such a form, it has natural generalization to  $\mathfrak{gl}_\infty$ . From the point of view of integrable systems it corresponds to (Arnold's type) 2D hydrodynamics.

For  $N = 2$  we have

$$Q = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and, therefore,  $\{T_\alpha\}$  in this case is the set of Pauli matrices:

$$T_{00} = \sigma_0 = 1_{2 \times 2}, \quad T_{10} = -\sigma_3, \quad T_{01} = \sigma_1, \quad T_{11} = \sigma_2. \quad (\text{A.8})$$

### A.1. Elliptic functions

**The Kronecker and Eisenstein functions** [48]. The following set of elliptic functions<sup>14</sup> on elliptic curve  $\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$  with moduli  $\tau$  ( $\text{Im } \tau > 0$ ) is widely used in this paper:

The Kronecker function

$$\phi(\eta, z) = \frac{\vartheta'(0)\vartheta(\eta+z)}{\vartheta(\eta)\vartheta(z)} \quad (\text{A.9})$$

<sup>14</sup> To be exact, some of these function are not double-periodic. In this sense they are not functions but rather sections of bundles (the Kronecker functions) or components of connections ( $E_1$ -function). See the quasi-periodic properties, e.g. in [26].

is defined in terms of the odd Riemann theta-function

$$\vartheta(z) = \vartheta(z|\tau) = \sum_{k \in \mathbb{Z}} \exp\left(\pi i \tau \left(k + \frac{1}{2}\right)^2 + 2\pi i \left(z + \frac{1}{2}\right) \left(k + \frac{1}{2}\right)\right). \quad (\text{A.10})$$

In rational and trigonometric cases it equals  $1/\eta + 1/z$  and  $\coth(\eta) + \coth(z)$  respectively. The derivative of the Kronecker function

$$f(z, u) \equiv \partial_u \phi(z, u) = \phi(z, u)(E_1(z + u) - E_1(u)) \quad (\text{A.11})$$

uses the definition of the first Eisenstein function:

$$E_1(z) = \vartheta'(z)/\vartheta(z). \quad (\text{A.12})$$

It is odd. In rational and trigonometric cases it equals  $1/z$  and  $\coth(z)$  respectively. Its derivative

$$E_2(z) = -\partial_z E_1(z) = \wp(z) - \frac{1}{3} \frac{\vartheta'''(0)}{\vartheta'(0)}, \quad \left( \text{and } E_1(z) = \zeta(z) + \frac{z}{3} \frac{\vartheta'''(0)}{\vartheta'(0)} \right) \quad (\text{A.13})$$

is known as the second Eisenstein function. The functions  $\wp(z)$  and  $\zeta(z)$  are the Weierstrass  $\wp$ - and  $\zeta$ -functions.

The local expansion of the Kronecker and Eisenstein functions near  $z = 0$ :

$$\phi(z, u) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \wp(u)) + O(z^2), \quad (\text{A.14})$$

$$E_1(z) = \frac{1}{z} + \frac{z}{3} \frac{\vartheta'''(0)}{\vartheta'(0)} + O(z^3). \quad (\text{A.15})$$

In particular, we conclude from (A.14) that

$$f(0, u) = -E_2(u). \quad (\text{A.16})$$

The Kronecker function satisfies the heat equation

$$\partial_\tau \phi(u, w) - \frac{1}{2\pi i} \partial_u \partial_w \phi(u, w) = 0. \quad (\text{A.17})$$

Most of the Lax equations are due to the Fay trisecant identity

$$\phi(z, q)\phi(w, u) = \phi(z - w, q)\phi(w, q + u) + \phi(w - z, u)\phi(z, q + u) \quad (\text{A.18})$$

and its degenerations

$$\phi(z, q)\phi(w, q) = \phi(z + w, q)(E_1(z) + E_1(w) + E_1(q) - E_1(z + w + q)). \quad (\text{A.19})$$

$$\phi(z, x)f(z, y) - \phi(z, y)f(z, x) = \phi(z, x + y)(E_2(x) - E_2(y)), \quad (\text{A.20})$$

$$\phi(\hbar, z)\phi(\hbar, -z) = \wp(\hbar) - \wp(z) = E_2(\hbar) - E_2(z), \quad (\text{A.21})$$

$$\phi(z, x)f(z, -x) - \phi(z, -x)f(z, x) = E_2'(x) = \wp'(x). \quad (\text{A.22})$$

The definition (1.14) of the Baxter–Belavin  $R$ -matrix uses the set of  $N^2$  functions

$$\varphi_\alpha^\hbar(z) \equiv \varphi_\alpha(z, \omega_\alpha + \hbar) = \exp(2\pi i z \partial_\tau \omega_\alpha) \phi(z, \hbar + \omega_\alpha), \quad (\text{A.23})$$

where

$$\omega_\alpha = \frac{\alpha_1 + \alpha_2 \tau}{N}, \quad \partial_\tau \omega_\alpha = \frac{\alpha_2}{N}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_N \times \mathbb{Z}_N, \quad (\text{A.24})$$

The following notations are also used for  $\alpha \neq 0$  (i.e.  $(\alpha_1, \alpha_2) \neq (0, 0)$ ):

$$\varphi_\alpha(z, \omega_\alpha) = \varphi_\alpha^0(z) = \exp(2\pi i z \partial_\tau \omega_\alpha) \phi(z, \omega_\alpha), \quad (\text{A.25})$$

$$f_\alpha(z, \omega_\alpha) = \exp(2\pi i z \partial_\tau \omega_\alpha) f(z, \omega_\alpha). \quad (\text{A.26})$$

The index  $\alpha$  in  $\varphi_\alpha$  and  $f_\alpha$  reminds us about the exponential factor.

### A.2. R-matrix structures for elliptic tops

Let us list explicit formulae for the coefficients of expansions (1.17)–(1.19). First, write down again the Baxter–Belavin R-matrix (1.14) with both arguments in  $\varphi$ -functions (see notations (A.23)–(A.26)):

$$R_{12}^{\hbar}(z) = \sum_{\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi_\alpha(z, \omega_\alpha + \hbar) T_\alpha \otimes T_{-\alpha}. \quad (\text{B.1})$$

Using (A.14), (A.15) we obtain the classical Belavin–Drinfeld  $r$ -matrix

$$r_{12}(z) = E_1(z) 1 \otimes 1 + \sum_{\alpha \neq 0} \varphi_\alpha(z, \omega_\alpha) T_\alpha \otimes T_{-\alpha}. \quad (\text{B.2})$$

It satisfies the classical Yang–Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \quad r_{ab} = r_{ab}(z_a - z_b) \quad (\text{B.3})$$

due to the quantum one (1.15). The next term in (1.17):

$$\begin{aligned} m_{12}(z) &= \frac{1}{2} (E_1^2(z) - \wp(z)) 1 \otimes 1 + \sum_{\alpha \neq 0} f_\alpha(z, \omega_\alpha) T_\alpha \otimes T_{-\alpha} \\ &= \frac{1}{2} (r_{12}^2(z) - 1 \otimes 1 N^2 \wp(z)). \end{aligned} \quad (\text{B.4})$$

The second line follows from the unitarity condition (1.16).

Using local expansion (A.14) we obtain the terms from (1.18), (1.19):

$$R_{12}^{\hbar, (0)} = \sum_{\alpha} (E_1(\hbar + \omega_\alpha) + 2\pi i \partial_\tau \omega_\alpha) T_\alpha \otimes T_{-\alpha}, \quad (\text{B.5})$$

$$r_{12}^{(0)} = \sum_{\alpha \neq 0} (E_1(\omega_\alpha) + 2\pi i \partial_\tau \omega_\alpha) T_\alpha \otimes T_{-\alpha}. \quad (\text{B.6})$$

**Properties and identities.** The skew-symmetry (1.38) of the quantum R matrix (B.1) as well as the unitarity (1.16) leads to

$$r_{21}(-z) \equiv P_{12} r_{12}(-z) P_{12} = -r_{12}(z), \quad m_{12}(z) = m_{21}(-z), \quad (\text{B.7})$$

$$R_{12}^{\hbar, (0)} = -R_{21}^{-\hbar, (0)}, \quad r_{12}^{(0)} = -r_{21}^{(0)}. \quad (\text{B.8})$$

Various formulae relating to the coefficients follow from the associative Yang–Baxter equation (1.37) (and the original Yang–Baxter equation (1.15)). In particular, in the limit  $\eta \rightarrow \hbar$  it gives:

$$R_{12}^{\hbar} R_{23}^{\hbar} = R_{13}^{\hbar} r_{12} + r_{23} R_{13}^{\hbar} - \partial_{\hbar} R_{13}^{\hbar}. \quad (\text{B.9})$$

By changing indices  $1 \leftrightarrow 3$  (i.e. conjugating equation by  $P_{13}$  and renaming  $z_1 \leftrightarrow z_3$ ), changing also  $\hbar \rightarrow -\hbar$  and then using skew-symmetry (1.38) it transforms into

$$R_{23}^{\hbar} R_{12}^{\hbar} = R_{13}^{\hbar} r_{23} + r_{12} R_{13}^{\hbar} - \partial_{\hbar} R_{13}^{\hbar}. \quad (\text{B.10})$$

Subtracting (B.10) from (B.9) yields

$$[R_{12}^{\hbar}, R_{23}^{\hbar}] = [R_{13}^{\hbar}, r_{12}] - [R_{13}^{\hbar}, r_{23}]. \quad (\text{B.11})$$

Taking the limit  $\hbar \rightarrow 0$  and using (B.3) provides

$$[r_{12}, m_{13} + m_{23}] = [r_{23}, m_{12} + m_{13}] \quad (\text{B.12})$$

or (by interchanging  $1 \leftrightarrow 2$ )

$$[r_{12}, m_{13} + m_{23}] + [r_{13}, m_{12} + m_{23}] = 0. \quad (\text{B.13})$$

The latter identity was used in [25] for constructing the KZB connections. More identities for  $R$ -matrices can be found in [27] and [50].

### A.3. $\mathbb{Z}_2$ reduction in elliptic tops

In this paragraph we explain  $\mathbb{Z}_2$  reduction (1.24) in three ways. First, as an invariant flow of the equations of motion (1.4). Second, from the geometry of the Euler–Arnold tops, and finally, as a reduction of the Lax equations (1.6).

The first way is straightforward. Impose the constraints

$$S_{\alpha} = S_{-\alpha} \text{ for all } \alpha. \quad (\text{C.1})$$

for the non-relativistic top (1.4)–(1.8). These constraints are preserved by dynamics (1.8) because  $J_{\alpha} = J_{-\alpha}$  and  $\kappa_{-\beta, -\gamma} = \kappa_{\beta, \gamma}$ . Therefore, the constraints are well defined.

The Euler–Arnold equations (1.4) define a flow on a coadjoint orbit of the group  $\text{SL}(N, \mathbb{C})$ . One can pass to some  $\mathbb{Z}_2$ -invariant semi-simple subgroup  $G^{inv} \subset \text{SL}(N, \mathbb{C})$  and consider the Euler–Arnold equations on the coadjoint orbits in the Lie coalgebra  $(\mathfrak{g}^{inv})^* = \text{Lie}^*(G^{inv})$ . If the inverse inertia tensor  $J$  is also  $\mathbb{Z}_2$ -invariant then these orbits become invariant phase subspaces of the original phase space (1.4). In what follows we use the following subgroup

$$G^{inv} = \begin{cases} \text{SL}(N/2 + 1, \mathbb{C}) \times \text{SL}(N/2 - 1, \mathbb{C}) \times \mathbb{C}^*, & N - \text{ even,} \\ \text{SL}((N + 1)/2, \mathbb{C}) \times \text{SL}((N - 1)/2, \mathbb{C}) \times \mathbb{C}^*, & N - \text{ odd,} \end{cases} \quad (\text{C.2})$$

and  $N > 3$ . For  $N = 3$ ,  $G^{inv} = \text{SL}(2, \mathbb{C}) \times \mathbb{C}^*$ , and for  $N = 2$   $G^{inv} = \text{SL}(2, \mathbb{C})$ .

For the non-relativistic tops we consider the corresponding Lie algebras. The  $\mathbb{Z}_2$  reduction is provided by the second order automorphisms  $\varsigma$  of  $\mathfrak{sl}(N, \mathbb{C})$ . In terms of the generators  $T_{\alpha}$  (A.2)  $\varsigma$  acts as

$$\varsigma : T_{\alpha} \rightarrow T_{-\alpha}. \quad (\text{C.3})$$

Explicitly, it is defined by the conjugation by the matrix  $h$

$$\varsigma : x \rightarrow hxh^{-1}, \quad h = \mathcal{J}\Lambda^{-1}, \quad \mathcal{J}_{ij} = \delta_{i, N-j+1} \quad (\text{C.4})$$

where  $\Lambda$  is the one from (A.1). It follows from  $\mathcal{J}^2 = 1_N$  and

$$\mathcal{J}\Lambda\mathcal{J} = \Lambda^{-1}. \quad (\text{C.5})$$

that  $\varsigma$  is an involution  $\varsigma^2 = 1$ . For the matrix  $Q$  from (A.1) we also have

$$h Q h^{-1} = \mathcal{J} \Lambda^{-1} Q \Lambda \mathcal{J} \stackrel{(C.5)}{=} \exp\left(-\frac{2\pi i t}{N}\right) \mathcal{J} Q \mathcal{J} = Q^{-1}. \tag{C.6}$$

Therefore, for the matrices  $T_\alpha$  (A.2) we obtain (C.3)

$$h T_\alpha h^{-1} = T_{-\alpha} \quad \text{for all } \alpha. \tag{C.7}$$

Therefore, the invariant subalgebra has generators  $\frac{1}{2}(T_\alpha + T_{-\alpha})$ , and by imposing the constraints

$$S_\alpha = S_{-\alpha} \quad \text{for all } \alpha \tag{C.8}$$

we come to the invariant subalgebra

$$\mathfrak{g}^{inv} = \left\{ \frac{1}{2} \sum_\alpha S_\alpha (T_\alpha + T_{-\alpha}) \right\} = Lie(G^{inv}) \quad (C.2). \tag{C.9}$$

Since  $J$  (1.5) is also  $\mathbb{Z}_2$ -invariant the reduction to  $G^{inv}$  is consistent with the equations of motion.

To prove that  $G^{inv}$  has the form (C.2) we diagonalize  $h$  (C.4). The matrix  $h$  has  $m$  eigenvalues  $\lambda = 1$  and  $n$   $\lambda = -1$  ( $m + n = N$ ), where  $m = N/2 + 1$  for  $N$  even, and  $m = (N + 1)/2$  for  $N$  odd. Therefore, the subgroup of  $SL(N, \mathbb{C})$  commuting with  $h$  has the form (C.2).

As usual, to prove the integrability of the reduced system we represent the equations of motion in the Lax form (1.6). Consider the Lax operator  $L(z)$  (1.7). It is a meromorphic map from the complex plane  $\mathbb{C}$  to the Lie algebra  $\mathfrak{sl}(N, \mathbb{C})$  satisfying fixed quasi-periodicities with respect to the shifts on the lattice  $\mathbb{Z} \oplus \tau\mathbb{Z}$ . Consider the automorphism  $z \rightarrow -z$  of  $\mathbb{C}$ . It preserves the lattice  $\mathbb{Z} \oplus \tau\mathbb{Z}$  and in this way  $\Sigma_\tau$ . Consider the equivariant maps  $\mathbb{C} \rightarrow \mathfrak{sl}(N, \mathbb{C})$  with respect to the automorphisms  $\varsigma$  (C.3) and the automorphism  $z \rightarrow -z$ . It can be found that the combined actions of these automorphisms preserve the quasi-periodicity conditions. Define the Lax operator as an equivariant map<sup>15</sup>

$$h L^{inv}(S_\alpha, -z) h^{-1} = -L^{inv}(S_\alpha, z). \tag{C.10}$$

From (C.1) we find the equivariant Lax operator

$$L^{inv}(z) = \frac{1}{2} \sum_\alpha S_\alpha (\varphi_\alpha(z) T_\alpha + \varphi_{-\alpha}(z) T_{-\alpha}) = \frac{1}{2} \sum_\alpha (S_\alpha + S_{-\alpha}) \varphi_\alpha(z) T_\alpha. \tag{C.11}$$

The operator  $M(z)$  (1.7) is a map of 0-forms to  $\mathfrak{sl}(N, \mathbb{C})$  and due to (1.7), (A.11) and (A.26) are also the equivariant map. The equivariant maps form a Lie algebra. Therefore, the Lax equation being reduced on the equivariant operators  $L^{inv}, M^{inv}$  is equivalent to the equations of motion on the constrained surface.

Put it differently, we can say that the set of constraints (C.1) is generated by involution  $\varsigma$  (C.4) acting on the Lax matrix:

$$\varsigma(L(z, S)) = h L(-z, S) h^{-1}. \tag{C.12}$$

Indeed, it follows from (C.5)–(C.7) that the action of  $\varsigma$  (C.12) on the Lax matrix (1.7) is given as follows:

<sup>15</sup> In fact, the Lax operator is a one-form  $L(z)dz$  and the sign ‘-’ in the rhs of (C.10) is then absent.

$$h L(-z, S) h^{-1} = \sum_{\alpha \neq 0} T_{-\alpha} S_{\alpha} \varphi_{\alpha}(-z, \omega_{\alpha}) = - \sum_{\alpha \neq 0} T_{\alpha} S_{-\alpha} \varphi_{\alpha}(z, \omega_{\alpha}), \quad (\text{C.13})$$

where we used  $\varphi_{\alpha}(-z, \omega_{\alpha}) = -\varphi_{-\alpha}(z, -\omega_{\alpha})$ . Thus, condition

$$\varsigma(L(z, S)) = -L(z, S) \quad (\text{C.14})$$

is equivalent to (C.1).

In fact, the involution leads to decomposition

$$L^{\pm}(z, S) = \frac{1}{2}(L(z, S) \pm \varsigma(L(z, S))) = \frac{1}{2} \sum_{\alpha \neq 0} T_{\alpha} (S_{\alpha} \mp S_{-\alpha}) \varphi_{\alpha}(z, \omega_{\alpha}). \quad (\text{C.15})$$

Condition (C.14) or (C.1) is equivalent to  $L^{+}(z, S) = 0$ , and we are left with  $L^{-}(z, S) = L^{inv}(z)$  on the reduced phase space.

In the relativistic case we use relation to  $\eta$ -independent description, i.e. from (1.13) and (C.1) we get

$$\frac{S_{\alpha}}{\varphi_{\alpha}(\eta, \omega_{\alpha})} = \frac{S_{-\alpha}}{\varphi_{-\alpha}(\eta, -\omega_{\alpha})}, \quad \alpha \neq 0 \quad (\text{C.16})$$

and  $S_0$  is not changed. Then, similarly to the non-relativistic case, these constraints are preserved by dynamics (1.12).

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