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Symplectic Degenerate Flag Varieties

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Abstract. A simple finite dimensional module V_λ of a simple complex algebraic group G is naturally endowed with a filtration induced by the PBW-filtration of $U(\operatorname{Lie} G)$. The associated graded space V_λ^a is a module for the group G^a, which can be roughly described as a semi-direct product of a Borel subgroup of G and a large commutative unipotent group $\mathbb{G}_a^M$. In analogy to the flag variety $\mathcal{F}_\lambda = G.[v_\lambda] \subset \mathbb{P}(V_\lambda)$, we call the closure $\overline{G^a.[v_\lambda]} \subset \mathbb{P}(V_\lambda^a)$ of the G^a-orbit through the highest weight line the degenerate flag variety $\mathcal{F}_\lambda^a$. In general this is a singular variety, but we conjecture that it has many nice properties similar to that of Schubert varieties. In this paper we consider the case of G being the symplectic group. The symplectic case is important for the conjecture because it is the first known case where, even for fundamental weights ω, the varieties $\mathcal{F}_\omega^a$ differ from $\mathcal{F}_\omega$. We give an explicit construction of the varieties $\operatorname{Sp}\mathcal{F}_\lambda^a$ and construct desingularizations, similar to the Bott–Samelson resolutions in the classical case. We prove that $\operatorname{Sp}\mathcal{F}_\lambda^a$ are normal locally complete intersections with terminal and rational singularities. We also show that these varieties are Frobenius split. Using the above mentioned results, we prove an analogue of the Borel–Weil theorem and obtain a q-character formula for the characters of irreducible Sp_{2n}-modules via the Atiyah–Bott–Lefschetz fixed points formula.

1 Introduction

Let G be a complex simple algebraic group and $\mathfrak{g}$ the corresponding Lie algebra. Let $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$ be the Cartan decomposition, where $\mathfrak{b}$ is a Borel subalgebra and $\mathfrak{n}^-$ is the nilpotent radical of the opposite Borel subalgebra. Let B and N^- be the subgroups in G corresponding to $\mathfrak{b}$ and $\mathfrak{n}^-$. The Lie algebra $\mathfrak{g}$ has a degeneration $\mathfrak{g}^a = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is the abelian Lie algebra with the underlying vector space $\mathfrak{n}^-$ (see [Fe1], [Fe2]). Here $(\mathfrak{n}^-)^a$ is an abelian ideal in $\mathfrak{g}^a$ and $\mathfrak{b}$ acts on $(\mathfrak{n}^-)^a$ via the adjoint action on the quotient $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$. The corresponding Lie group G^a is the semi-direct product $B \ltimes \mathfrak{g}/\mathfrak{b} \simeq B \ltimes \mathbb{G}_a^M$, where $\mathbb{G}_a$ is the additive group of the field and $M = \dim \mathfrak{n}^-$.

For a dominant integral weight λ, let V_λ be the corresponding irreducible highest weight $\mathfrak{g}$-module. Let $v_\lambda \in V_\lambda$ be a highest weight vector, recall that $V_\lambda = U(\mathfrak{n}^-)v_\lambda$. The (generalized) flag varieties G/P (P being a parabolic subgroup of G) are known to be embedded into the projective spaces $\mathbb{P}(V_\lambda)$ with λ chosen in such a way that P is the stabilizer of $\mathbb{C}v_\lambda$. Explicitly, the image is given by the G-orbit through the highest weight line $\mathbb{C}v_\lambda \in \mathbb{P}(V_\lambda)$. We denote the corresponding orbit by $\mathcal{F}_\lambda$.

Let V_λ^a be the degeneration of V_λ into a $\mathfrak{g}^a$-module. More precisely, V_λ^a is the associated graded space with respect to the PBW filtration on V_λ (see, *e.g.*, [FFL1], [FFL2] and Section 2.3). Denote by the same symbol v_λ its image in V_λ^a, then $V_\lambda^a = S^\bullet(\mathfrak{n}^-)v_\lambda$. The degenerate flag variety $\mathcal{F}_\lambda^a$ is defined as the closure of the orbit $G^a(\mathbb{C}v_\lambda)$

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inside $\mathbb{P}(V^a_\lambda)$. In contrast with the classical situation, the orbit itself is not closed (it is only an open cell inside $\mathcal{F}^a_\lambda$) and the closure is in general singular. We put forward the following conjecture.

Conjecture 1.1 $\mathcal{F}^a_\lambda$ are normal varieties, have rational singularities and an analogue of the classical Borel–Weil theorem holds. We also conjecture that each $\mathcal{F}^a_\lambda$ admits a desingularization by a tower of successive $\mathbb{P}^1$ fibrations, similar to the Bott–Samelson desingularization of a Schubert variety.

The varieties $\mathcal{F}^a_\lambda$ for $\mathfrak{g} = \mathfrak{sl}_n$ were studied in [Fe1], [Fe2], [FF]. It was shown that in this case the $\mathcal{F}^a_\lambda$ are singular projective algebraic varieties that are flat degenerations of the classical $\mathcal{F}_\lambda$. The varieties $\mathcal{F}^a_\lambda$ enjoy several nice properties: as in the classical case, $\mathcal{F}^a_\lambda$ depends only on the class of regularity of λ; they are irreducible normal Frobenius split locally complete intersections; they have a nice crepant desingularization isomorphic to a tower of successive $\mathbb{P}^1$ fibrations; the singularities of $\mathcal{F}^a_\lambda$ are rational. In addition, the analogue of the classical Borel–Weil theorem still holds in the degenerate case.

If ω is a co-minuscule fundamental weight, then it is easy to see that $\mathcal{F}^a_\omega \simeq \mathcal{F}_\omega$. This makes the case $\mathfrak{g} = \mathfrak{sl}_n$ very special, because as a consequence one can embed $\mathcal{F}^a_\lambda$ for arbitrary λ in a product of Grassmann varieties. In this paper we study the case of the symplectic group $G = \mathrm{Sp}_{2n}$. This is an important step in checking the validity of Conjecture 1.1, because it is the first time that even for fundamental weights we have $\mathcal{F}^a_\omega \neq \mathcal{F}_\omega$ in general.

Since we often use the connection between SL_{2n}-flag varieties and the Sp_{2n}-flag varieties, we use the notation $\mathrm{Sp}\,\mathcal{F}^a_\lambda$ to distinguish it from the type $\mathtt{A}$ case. In the introduction we describe only the case of the complete degenerate symplectic flag variety, which we denote by $\mathrm{Sp}\,\mathcal{F}^a_{2n}$. (In this case the highest weight λ has to be regular, but, as in the classical case, the orbit closure does not depend on a regular λ.) However, in the main body of the paper we work out the general case of degenerate parabolic (partial) flag varieties as well.

Let W be a $2n$-dimensional vector space with a basis $w_1, \dots, w_{2n}$. Let us equip W with a symplectic form which pairs non-trivially w_i and w_{2n+1-i}. We also denote by $pr_k\colon W \to W$, $k = 1, \dots, 2n$ the projection operators along w_k to the span of all the basis vectors different from w_k. Our first theorem is as follows.

Theorem 1.2 *$\mathrm{Sp}\,\mathcal{F}^a_{2n}$ can be realized inside the product $\prod_{k=1}^n \mathrm{Gr}_k(W)$ of Grassmannians as a subvariety of collections $(V_1, \dots, V_n)$, $V_k \in \mathrm{Gr}_k(W)$ satisfying the following conditions:*

$$pr_{k+1} V_k \subset V_{k+1}, \quad k = 1, \dots, n-1, \quad V_n \text{ is Lagrangian}.$$

These varieties are flat degenerations of the classical flag varieties Sp_{2n}/B.

Our next task is to study the singularities of $\mathrm{Sp}\,\mathcal{F}^a_{2n}$. As in the type $\mathtt{A}$ case, we construct a desingularization $\mathrm{Sp}\,\mathrm{R}_{2n}$ of $\mathrm{Sp}\,\mathcal{F}^a_{2n}$. We prove that $\mathrm{Sp}\,\mathrm{R}_{2n}$ is a Bott–Samelson type variety, *i.e.*, it is isomorphic to a tower of successive $\mathbb{P}^1$-fibrations. Using this desingularization, we prove our next theorem.

Theorem 1.3 *The varieties* $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ *are normal locally complete intersections (hence Gorenstein). The singularities of* $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ *are terminal (hence rational). The varieties* $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ *are Frobenius split over* $\overline{\mathbb{F}}_p$ *for all primes* p.

We note that an important difference compared to the type $\mathtt{A}$ case is that the resolution of singularities $\mathrm{Sp}\,\mathrm{R}_{2n} \to \mathrm{Sp}\,\mathcal{F}^a_{2n}$ is no longer crepant.

Finally, we make a connection between the geometry of $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ and the representations V^a_λ. Namely, we prove an analogue of the Borel–Weil theorem. Let us denote by $\imath_\lambda$ the natural map $\mathrm{Sp}\,\mathcal{F}^a_{2n} \to \mathbb{P}(V^a_\lambda)$. Then we have the following.

Theorem 1.4 $H^0\big(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \imath^*_\lambda\mathcal{O}(1)\big)^* = V^a_\lambda$, *and* $H^k\big(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \imath^*_\lambda\mathcal{O}(1)\big)^* = 0$ *for* $k > 0$.

Using the rationality of the singularities of $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ and the Atiyah–Bott–Lefschetz fixed points formula for $\mathrm{Sp}\,\mathrm{R}_{2n}$ we derive a formula for the q-character of V^a_λ (see [FFL2] for the combinatorial formula).

Our paper is organized as follows: in Section 2 we introduce notations, recall definitions and collect main results to be used in the main body of the paper. In Section 3 we establish a connection between the PBW filtrations on $\mathfrak{sp}_{2n}$-modules and on $\mathfrak{sl}_{2n}$-modules. In Section 4 we derive an explicit description for the degenerate symplectic Grassmannians and, more generally, for the symplectic degenerate flag varieties. We also prove that the $\mathrm{Sp}\,\mathcal{F}^a_\lambda$ are flat degenerations of their classical analogues $\mathrm{Sp}\,\mathcal{F}_\lambda$. In Section 5 we construct a resolution of singularities of $\mathrm{Sp}\,\mathcal{F}^a_\lambda$. In Section 6 we prove that all symplectic degenerate flag varieties are normal locally complete intersections. In Section 7 we show that the singularities of $\mathrm{Sp}\,\mathcal{F}^a_\lambda$ are terminal (hence rational). In Section 8 we prove that the varieties $\mathrm{Sp}\,\mathcal{F}^a_\lambda$ are Frobenius split. This allows us to prove the degenerate analogue of the Borel–Weil theorem. We also derive an Atiyah–Bott–Lefschetz type formula for the graded characters of V^a_λ.

2 Degenerations: Definitions and the Type A Case

In this section we fix the notation and recall the main results on the degenerate representations and degenerate flag varieties. We also collect important constructions and theorems from algebraic geometry, which we use in the paper.

2.1 Notations

Let $\mathfrak{g}$ be a simple Lie algebra. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Let $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$ be the corresponding Borel subalgebra. We denote by α_i, $i = 1, \dots, \mathrm{rk}(\mathfrak{g})$ the simple roots of $\mathfrak{g}$ and by ω_i the corresponding fundamental weights. Let Φ^+ be the set of positive roots for $\mathfrak{g}$. For a root α we often write $\alpha > 0$ instead of $\alpha \in \Phi^+$. In particular, for $\mathfrak{g} = \mathfrak{sl}_n$ the simple roots are $\alpha_1, \dots, \alpha_{n-1}$ and all positive roots are of the form

(2.1) $$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \le i \le j \le n-1.$$

For $\mathfrak{g} = \mathfrak{sp}_{2n}$ the simple roots are $\alpha_1, \dots, \alpha_n$ and all positive roots are of the form

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \le i \le j \le n, \tag{2.2}$$

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_n + \alpha_{n-1} + \cdots + \alpha_{2n-j}, \tag{2.3}$$
$$1 \le i \le n < j, i + j \le 2n.$$

For a positive root α we fix an element $f_\alpha \in \mathfrak{n}^-$ of weight $-\alpha$ and an element $e_\alpha \in \mathfrak{n}$ of weight α. The f_α, $\alpha \in \Phi^+$, form a basis of $\mathfrak{n}^-$ and so do the root vectors e_α in $\mathfrak{n}$. We use the shorthand notations $f_{i,j} = f_{\alpha_{i,j}}$.

Let $P^+ = \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} \mathbb{Z}_{\ge 0}\omega_i$ be the submonoid of the weight lattice for $\mathfrak{g}$ generated by the fundamental weights, the elements of P^+ are the dominant integral weights. For $\lambda \in P^+$ we denote by V_λ the $\mathfrak{g}$-module of highest weight λ. We also fix a highest weight vector $v_\lambda \in V_\lambda$. In particular, one has

$$\mathfrak{n} v_\lambda = 0, \quad h v_\lambda = \lambda(h) v_\lambda, h \in \mathfrak{h}, \quad V_\lambda = U(\mathfrak{n}^-) v_\lambda.$$

Let G be a simple complex algebraic group with the Lie algebra $\mathfrak{g}$. We denote by B, N, T, N^- the subgroups in G corresponding to the Lie subalgebras $\mathfrak{b}, \mathfrak{n}, \mathfrak{h}, \mathfrak{n}^-$. Let P be a parabolic subgroup of G; the quotient G/P is called a *generalized flag variety*. These varieties can be also realized as follows: for a dominant integral weight λ the group G acts naturally on the projective space $\mathbb{P}(V_\lambda)$. Assume that $P \supset B$ and $(\lambda, \omega_i) = 0$ if and only if f_{α_i} belongs to the Lie algebra $\mathfrak{p}$ of P. Then G/P is isomorphic to the G-orbit $G(\mathbb{C}v_\lambda) \subset \mathbb{P}(V_\lambda)$.

In what follows we denote the orbit $G(\mathbb{C}v_\lambda)$ by $\mathcal{F}_\lambda$. Note that $\mathcal{F}_\lambda$ depends only on the class of regularity of λ, *i.e.*, $\mathcal{F}_\lambda \simeq \mathcal{F}_\mu$ if and only if the supports of λ and μ coincide, *i.e.*, $(\lambda, \omega_i) \ne 0$ if and only if $(\mu, \omega_i) \ne 0$.

For example, for $G = \mathrm{SL}_n$ the Grassmann variety $\mathrm{Gr}_d(n)$ of d-dimensional subspaces in n-dimensional space can be realized either as the quotient of SL_n by a maximal parabolic subgroup or as an orbit in the projective space of the fundamental representation V_{ω_d}. The quotients SL_n/P, where $P \supseteq B$ is a parabolic subgroup, are partial flag varieties: let $1 \le d_1 < \cdots < d_k \le n$ be the integers such that $f_{\alpha_d} \in \mathfrak{p}$ if and only if $d = d_i$ for some i. Then G/P is known to coincide with the variety of collections $(V_i)_{i=1}^k$ of subspaces V_i in an n-dimensional space such that $\dim V_i = d_i$ and $V_i \subset V_{i+1}$. We note also that the same partial flag variety sits inside $\mathbb{P}(V_{\omega_{d_1}+\cdots+\omega_{d_k}})$ as the orbit of the highest weight line. We denote this variety by $\mathcal{F}_{\mathbf{d}}$, $\mathbf{d} = (d_1, \dots, d_k)$.

Another important example for us is $G = \mathrm{Sp}_{2n}$. The symplectic Grassmannians $\mathrm{Sp}\,\mathrm{Gr}_d(2n)$ (quotients of Sp_{2n} by maximal parabolic subgroups) are known to coincide with the varieties of isotropic d-dimensional subspaces in a $2n$-dimensional vector space equipped with a non-degenerate symplectic form. In general, as in the SL_{2n} case, let $1 \le d_1 < \cdots < d_k \le n$ be the numbers such that $f_{\alpha_d} \in \mathfrak{p}$ if and only if $d = d_i$ for some i ($\mathfrak{p}$ is the Lie algebra of a parabolic subgroup $P \subset \mathrm{Sp}_{2n}$). Then Sp_{2n}/P is known to coincide with the variety of collections $(V_i)_{i=1}^n$ of subspaces $V_i \in \mathrm{Sp}\,\mathrm{Gr}_{d_i}(2n)$ such that $V_i \subset V_{i+1}$. In addition, the same partial flag variety sits inside $\mathbb{P}(V_{\omega_{d_1}+\cdots+\omega_{d_k}})$ as the orbit of the highest weight line. We denote this variety by $\mathrm{Sp}\,\mathcal{F}_{\mathbf{d}}$, $\mathbf{d} = (d_1, \dots, d_k)$.

2.2 Algebraic Geometry

We recall the definition of a Frobenius split variety. Let X be an algebraic variety over an algebraically closed field of characteristic $p > 0$. Let $F\colon X \to X$ be the Frobenius morphism, *i.e.*, the identity map on the underlying space X and the p-th power map on the space of functions. Then X is called Frobenius split if there exists a projection $F_*\mathcal{O}_X \to \mathcal{O}_X$ such that the composition $\mathcal{O}_X \to F_*\mathcal{O}_X \to \mathcal{O}_X$ is the identity map. The Frobenius split varieties enjoy the following important property (see, *e.g.*, [MR, Proposition 1]).

Proposition 2.1 *Let X be a Frobenius split projective variety and let $\mathcal{L}$ be a line bundle on X. If $i \geq 0$ is such that $H^i(X, \mathcal{L}^m) = 0$ for all $m \gg 0$, then $H^i(X, \mathcal{L}^m) = 0$ for all $m \geq 1$.*

The following two statements are proved in [MR, Proposition 4 and Proposition 8].

Proposition 2.2 *Let $f\colon Z \to X$ be a proper morphism of algebraic varieties such that $f_*\mathcal{O}_Z = \mathcal{O}_X$. If Z is Frobenius split, then X is also Frobenius split.*

Theorem 2.3 *Let Z be a smooth projective variety of dimension M and let $Z_1, \ldots, Z_M$ be codimension one subvarieties satisfying the following conditions:*

(i) *$\forall I \subset \{1, \ldots, M\}$: the intersection $\bigcap_{i \in I} Z_i$ is smooth of codimension $\#I$.*

(ii) *There exists a global section s of the anti-canonical bundle ω_Z^{-1} on Z such that the zero divisor of s equals $\mathcal{O}(Z_1 + \cdots + Z_M + D)$ for some effective divisor D with $\bigcap_{i=1}^M Z_i \notin \operatorname{supp} D$.*

Then Z is Frobenius split, and for any subset $I \subset \{1, \ldots, M\}$ the intersection $Z_I = \bigcap_{i \in I} Z_i$ is Frobenius split as well.

Now we recall some results on the singularities of algebraic varieties. Let X be a projective algebraic variety. If X is smooth, then ω_X denotes its canonical line bundle of top degree forms on X. For a singular X one can define a dualizing complex of coherent sheaves DX, which for Cohen–Macaulay varieties reduces to a (cohomologically shifted) sheaf. If X is Gorenstein, then this sheaf is a line bundle denoted by ω_X. For example, if X is a locally complete intersection, then it is known that X is Gorenstein and thus ω_X is a line bundle.

Let $\pi\colon Y \to X$ be a resolution of singularities of X. The singularities of X are called rational if $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ and $R^k\pi_*\mathcal{O}_Y = 0$ for $k > 0$ (if this holds for some desingularization, then it holds for all). An important property is that, in this case, for any line bundle $\mathcal{L}$ on X and any k, one has $H^k(X, \mathcal{L}) = H^k(Y, \pi^*\mathcal{L})$.

An irreducible divisor $Z \subset Y$ is called an *exceptional* divisor if $\dim \pi(Z) < \dim Z$. Assume that X is $\mathbb{Q}$-Gorenstein, ω_X is its canonical line bundle, and

$$\omega_Y = \pi^*\omega_X \otimes \bigotimes_{i=1}^{N} \mathcal{O}(a_i Z_i),$$

where $Z_1, \ldots, Z_N$ are the exceptional divisors in Y. If $a_i \geq 0$ for all i, then the singularities of X are called *canonical*. If all the a_i are positive, then the singularities are called *terminal*. We will use the following theorem from [E, Theorem 1] (see also [F, 1.3]).

Theorem 2.4 *If X is a normal Gorenstein variety with canonical singularities, then it has rational singularities.*

2.3 Degenerate Representations

Consider the PBW filtration $U(\mathfrak{n}^-)_s$ on the universal enveloping algebra $U(\mathfrak{n}^-)$:

$$U(\mathfrak{n}^-)_s = \mathrm{span}(x_1 \cdots x_l : x_i \in \mathfrak{n}^-, l \le s).$$

The associated graded algebra is isomorphic to the symmetric algebra $S^\bullet(\mathfrak{n}^-)$. For any $\lambda \in P^+$ consider the induced PBW filtration $F_s = U(\mathfrak{n}^-)_s v_\lambda$ on the space V_λ. We denote the associated graded space by V_λ^a, so V_λ^a carries an additional grading:

$$V_\lambda^a = \bigoplus_{s \ge 0} V_\lambda^a(s) = \bigoplus_{s \ge 0} F_s/F_{s-1}.$$

Note that the operators from $\mathfrak{n}^-$ acting on V_λ induce an action of the abelian algebra $(\mathfrak{n}^-)^a$ on V_λ^a, where $(\mathfrak{n}^-)^a$ is isomorphic to $\mathfrak{n}^-$ as a vector space. Thus $V_\lambda^a = S^\bullet(\mathfrak{n}^-)v_\lambda$, where (slightly abusing notations) we denote by $v_\lambda \in V_\lambda^a$ the image of the highest weight vector in V_λ. More precisely, $V_\lambda^a \simeq \mathbb{C}[f_\alpha]_{\alpha>0}/I_\lambda$, where I_λ is an ideal.

We note that $(\mathfrak{n}^-)^a$ comes equipped with the natural structure of a $\mathfrak{b}$-module. In fact, we have an isomorphism of vector spaces $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$ and $\mathfrak{b}$ acts on the right hand side via the adjoint action. This gives a $\mathfrak{b}$-module structure on $S^\bullet(\mathfrak{n}^-)$. We denote this $\mathfrak{b}$-action by $\circ$. By construction, the ideals I_λ are $\mathfrak{b}$-invariant and thus $\mathfrak{b}$ acts on all modules V_λ^a. It is convenient to combine the actions of $(\mathfrak{n}^-)^a$ and $\mathfrak{b}$. Consider the Lie algebra $\mathfrak{g}^a \simeq \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is an abelian ideal and the Borel subalgebra $\mathfrak{b}$ acts on $(\mathfrak{n}^-)^a$ via the induced adjoint action on the quotient space $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$. Then each V_λ^a carries the structure of a $\mathfrak{g}^a$-module (see [Fe1], [Fe2]). In what follows we will need an explicit description of the ideals I_λ for $\mathfrak{g} = \mathfrak{sl}_n$ (see [FFL1]), and $\mathfrak{g} = \mathfrak{sp}_{2n}$ (see [FFL2]).

Theorem 2.5 *Let* $\mathfrak{g} = \mathfrak{sl}_n$, $\lambda = \sum_{i=1}^{n-1} m_i\omega_i$. *Then*

$$I_\lambda = S(\mathfrak{n}^-)\big(U(\mathfrak{b}) \circ f_{\alpha_{i,j}}^{m_i+\cdots+m_j+1} \bigm| 1 \le i \le j \le n-1\big).$$

Let $\mathfrak{g} = \mathfrak{sp}_{2n}$, $\lambda = \sum_{i=1}^{n} m_i\omega_i$. *Then*

$$I_\lambda = S(\mathfrak{n}^-)\begin{pmatrix} U(\mathfrak{b}) \circ f_{\alpha_{i,j}}^{m_i+\cdots+m_j+1} \mid 1 \le i \le j < n, \\ U(\mathfrak{b}) \circ f_{\alpha_{i,2n-i}}^{m_i+\cdots+m_n+1} \mid 1 \le i \le n. \end{pmatrix}$$

We will also use monomial bases in V_λ^a (see [FFL1], [FFL2], [V]). Given an element $\mathbf{s} = (s_\alpha)_{\alpha>0}$, $s_\alpha \in Z_{\ge 0}$, we define $f^{\mathbf{s}} = \prod_{\alpha>0} f_\alpha^{s_\alpha}$. We need the notion of a Dyck path inside the set of positive roots for $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$.

Definition 2.6 We call a sequence $\mathbf{p} = \big(\beta(0), \beta(1), \ldots, \beta(k)\big)$ of positive roots of $\mathfrak{sl}_n$ a *Dyck path* if $\beta(0) = \alpha_i$, $\beta(k) = \alpha_j$ for some $1 \le i \le j \le n-1$ and

$$\text{if } \beta(s) = \alpha_{p,q} \quad \text{then } \beta(s+1) = \alpha_{p,q+1} \text{ or } \beta(s+1) = \alpha_{p+1,q}.$$

We call a sequence $\mathbf{p} = \big(\beta(0), \beta(1), \ldots, \beta(k)\big)$ of positive roots of $\mathfrak{sp}_{2n}$ a (symplectic) Dyck path if $\beta(0) = \alpha_i$, $\beta(k) = \alpha_j$ or $\beta(k) = \alpha_{j,2n-j}$ for some $1 \le i \le j \le n-1$ and

$$\text{if } \beta(s) = \alpha_{p,q} \quad \text{then } \beta(s+1) = \alpha_{p,q+1} \text{ or } \beta(s+1) = \alpha_{p+1,q}.$$

For a dominant $\mathfrak{sl}_n$-weight $\lambda = \sum_{i=1}^{n-1} m_i\omega_i$ let $\mathbf{P}_{\mathfrak{sl}_n}(\lambda) \subset \mathbb{R}_{\ge 0}^{\frac{1}{2}n(n-1)}$ be the polytope consisting of collections $(r_\alpha)_{\alpha\in\Phi^+_{\mathfrak{sl}_n}}$ such that for any $\mathfrak{sl}_n$ Dyck path $\mathbf{p}$ with $\beta(0) = \alpha_i$, $\beta(k) = \alpha_j$ one has

$$r_{\beta(0)} + \cdots + r_{\beta(k)} \le m_i + \cdots + m_j.$$

Let $S_{\mathfrak{sl}_n}(\lambda)$ be the set of integral points in $\mathbf{P}_{\mathfrak{sl}_n}(\lambda)$.

Similarly, for a dominant $\mathfrak{sp}_{2n}$-weight $\lambda = \sum_{i=1}^{n} m_i\omega_i$ let $\mathbf{P}(\lambda) \subset \mathbb{R}_{\ge 0}^{n^2}$ be the polytope consisting of collections $(r_\alpha)_{\alpha\in\Phi^+_{\mathfrak{sp}_{2n}}}$ such that for any $\mathfrak{sp}_{2n}$ Dyck path $\mathbf{p}$ with $\beta(0) = \alpha_i$, $\beta(k) = \alpha_j$ one has

$$r_{\beta(0)} + \cdots + r_{\beta(k)} \le m_i + \cdots + m_j$$

and for any $\mathfrak{sp}_{2n}$ Dyck path $\mathbf{p}$ with $\beta(0) = \alpha_i$, $\beta(k) = \alpha_{j,2n-j}$ one has

$$r_{\beta(0)} + \cdots + r_{\beta(k)} \le m_i + \cdots + m_n.$$

Let $S_{\mathfrak{sp}_{2n}}(\lambda)$ be the set of integral points in $\mathbf{P}_{\mathfrak{sp}_{2n}}(\lambda)$.

Theorem 2.7 *For a dominant $\mathfrak{sl}_n$-weight λ the monomials $f^{\mathbf{s}}v_\lambda$, $\mathbf{s} \in S_{\mathfrak{sl}_n}(\lambda)$, form a basis of $V^a_{\lambda,\mathfrak{sl}_n}$. For a dominant $\mathfrak{sp}_{2n}$-weight λ the monomials $f^{\mathbf{s}}v_\lambda$, $\mathbf{s} \in S_{\mathfrak{sp}_{2n}}(\lambda)$, form a basis of $V^a_{\lambda,\mathfrak{sp}_{2n}}$.*

Finally, for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{g} = \mathfrak{sp}_{2n}$ the representations V^a_λ have the following important property (similar to the classical situation): for two dominant weights λ and μ the representation $V^a_{\lambda+\mu}$ is embedded into the tensor product $V^a_\lambda \otimes V^a_\mu$ as the highest weight component, *i.e.*, there exists a unique injective homomorphism of $\mathfrak{g}^a$-modules:

$$V^a_{\lambda+\mu} \hookrightarrow V^a_\lambda \otimes V^a_\mu \quad \text{such that } v_{\lambda+\mu} \mapsto v_\lambda \otimes v_\mu. \tag{2.4}$$

2.4 Degenerate Flag Varieties

As in [Fe1], [Fe2] and [FF], let $(N^-)^a$ be the product of $\dim \mathfrak{n}^-$ copies of the one dimensional additive group $\mathbb{G}_a$ of the field $\mathbb{C}$, so $\mathrm{Lie}(N^-)^a = (\mathfrak{n}^-)^a$. The action of B on $(\mathfrak{n}^-)^a$ $(= \mathfrak{g}/\mathfrak{b})$ induces a natural action of B on $(N^-)^a$. We denote by G^a the semi-direct product of B and $(N^-)^a$, it follows immediately that $\mathrm{Lie}\, G^a = \mathfrak{g}^a$. For a given dominant integral weight λ the group G^a acts naturally on $\mathbb{P}(V^a_\lambda)$. By definition, the degenerate flag variety $\mathcal{F}^a_\lambda$ is the orbit closure of the highest weight line, *i.e.*,

$$\mathcal{F}^a_\lambda = \overline{G^a(\mathbb{C}v_\lambda)} \subset \mathbb{P}(V^a_\lambda).$$

Note that $\mathcal{F}^a_\lambda = \overline{(N^-)^a \cdot (\mathbb{C}v_\lambda)}$, *i.e.*, the group acts on $\mathcal{F}^a_\lambda$ with an open orbit isomorphic to an affine space. The varieties $\mathcal{F}^a_\lambda$ are hence $\mathbb{G}_a^{\dim \mathfrak{n}}$-varieties; see [HT], [A], [AS] for more information. The variety $\mathcal{F}^a_\lambda$ is not a homogeneous G^a-variety, in contrast to the classical situation.

The existence of the embeddings (2.4) implies two important properties of the varieties $\mathcal{F}^a_\lambda$ in types A and C. First, for two dominant weights λ and μ one has the embedding of varieties $\mathcal{F}^a_{\lambda+\mu} \hookrightarrow \mathcal{F}^a_\lambda \times \mathcal{F}^a_\mu$ sending the highest weight line $\mathbb{C}v_{\lambda+\mu}$ to the product $\mathbb{C}v_\lambda \times \mathbb{C}v_\mu$. Secondly, $\mathcal{F}^a_\lambda \simeq \mathcal{F}^a_\mu$ provided $(\lambda, \alpha_i) = 0$ if and only if $(\mu, \alpha_i) = 0$, *i.e.*, the supports of λ and μ coincide.

By definition, the group G^a acts on the variety $\mathcal{F}^a_\lambda$. The group G^a has a natural one-dimensional extension still acting on $\mathcal{F}^a_\lambda$. Consider the Lie algebra $\mathfrak{g}^a \oplus \mathbb{C}d$, where $[d, \mathfrak{b}] = 0$ and $[d, f_\alpha] = f_\alpha$ for all $\alpha > 0$. Then the extended algebra acts on V^a_λ and d acts as "a PBW-degree operator", *i.e.*, on $V^a_\lambda(s)$ the operator d acts as a scalar s. Let $G^a \rtimes \mathbb{C}^*$ be the corresponding extended Lie group. Note that the dimension of the torus of the extended group is increased by one.

Assume now $G = \mathrm{SL}_n$. Then the varieties $\mathcal{F}^a_\lambda$ are known to be flat degenerations of the corresponding classical flag varieties. They have analogues of the Plücker embeddings cut out by the degenerate Plücker relations. The varieties $\mathcal{F}^a_\lambda$ share some important properties with their classical analogues, and enjoy the following explicit description. Let W be an n-dimensional vector space with a basis $w_1, \dots, w_n$. We define the projection operators $\mathrm{pr}_k\colon W \to W$, $k = 1, \dots, n$ by the formula $\mathrm{pr}_k(\sum_{i=1}^n c_i w_i) = \sum_{i\neq k} c_i w_i$. Fix a collection of numbers $1 \le d_1 < \cdots < d_k \le n$. Then $\mathcal{F}^a_{\omega_{d_1}+\cdots+\omega_{d_k}}$ is isomorphic to the variety of collections $(V_i)_{i=1}^k$ of subspaces $V_i \subset W$ satisfying for $i = 1, \dots, k-1$

$$\dim V_i = d_i, \quad \mathrm{pr}_{d_i+1} \cdots \mathrm{pr}_{d_{i+1}} V_i \subset V_{i+1}.$$

In what follows, we sometimes denote this variety by $\mathcal{F}^a_{(d_1,\dots,d_k)}$, or simply by $\mathcal{F}^a_{\mathbf{d}}$ where $\mathbf{d} = (d_1, \dots, d_k)$. For instance, $\mathcal{F}^a_{(d)} \simeq \mathcal{F}_{\omega_d} \simeq \mathrm{Gr}_d(n)$.

The varieties $\mathcal{F}^a_{\mathbf{d}}$ are in general singular projective algebraic varieties with rational singularities. They come equipped with line bundles $\mathcal{L}_\mu$, where $(\mu, \alpha_d) \ge 0$, and $(\mu, \alpha_d) = 0$ unless $d \in \mathbf{d}$. The line bundle $\mathcal{L}_\mu$ is the pullback of the line bundle $\mathcal{O}(1)$ under the map $\mathcal{F}^a_{\mathbf{d}} \to \mathbb{P}(V^a_\mu)$. One of the main tools for the study of algebro-geometric properties of the degenerate flag varieties is an explicit construction of a resolution of singularities of $\mathcal{F}^a_\lambda$. Namely, given a collection $\mathbf{d} = (d_1, \dots, d_n)$ we denote by $\mathfrak{P}_{\mathbf{d}}$ the set of positive roots in the radical of the parabolic subalgebra of $\mathfrak{g}$ containing (exactly) the simple roots $\alpha_{d_1}, \dots, \alpha_{d_k}$. We define $R_{\mathbf{d}}$ as the variety of collections of subspaces $V_{i,j}$, $1 \le i \le j \le n-1$, $\alpha_{i,j} \in \mathfrak{P}_{\mathbf{d}}$, satisfying the following conditions:

$$\dim V_{i,j} = i, \quad V_{i,j} \subset \mathrm{span}(w_1, \dots, w_i, w_{j+1}, \dots, w_n),$$
$$V_{i,j} \subset V_{i+1,j}, \quad V_{i,j} \subset V_{i,j+1} \oplus \mathbb{C}w_{j+1}.$$

The projection $\pi_{\mathbf{d}}\colon R_{\mathbf{d}} \to \mathcal{F}^a_{\mathbf{d}}$ sends $(V_{i,j})$ to $(V_{i,i})$. Using this desingularization, the following theorem was proved in [FF].

Theorem 2.8

(i) *The resolution $\pi_{\mathbf{d}}\colon R_{\mathbf{d}} \to \mathcal{F}^a_{\mathbf{d}}$ is crepant.*

(ii) *The varieties $\mathcal{F}^a_{\mathbf{d}}$ are normal locally complete intersections (thus Cohen–Macaulay and Gorenstein).*

(iii) *The varieties $\mathcal{F}^a_{\mathbf{d}}$ have rational singularities and are Frobenius split.*

(iv) *For a dominant μ such that $(\mu, \alpha_d) = 0$ unless $d \in \mathbf{d}$, the cohomology groups $H^m(\mathcal{F}^a_{\mathbf{d}}, \mathcal{L}_\mu)$ vanish unless $m = 0$, and the zero cohomology is isomorphic to $(V^a_\mu)^*$.*

3 Filtrations: $\mathfrak{sp}_{2n}$ vs. $\mathfrak{sl}_{2n}$

Let $w_1, \dots, w_{2n}$ be a basis of a $2n$-dimensional vector space W. We fix a non-degenerate sympletic form $\langle\cdot,\cdot\rangle$ defined by the conditions $\langle w_i, w_{2n+1-i}\rangle = 1$ for $1 \le i \le n$ and $\langle w_i, w_j\rangle = 0$ for all $1 \le i, j \le 2n$, $j \ne 2n+1-i$. We realize the symplectic group Sp_{2n} as the group of automorphisms of W leaving the form invariant. The diagonal matrices

$$T = \left\{ t = \begin{pmatrix} t_1 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \;\middle|\; t_1, \dots, t_n \in \mathbb{C}^* \right\}$$

form a maximal torus $T \subset \mathrm{Sp}_{2n}$, and the subgroup $B \subset \mathrm{Sp}_{2n}$ of upper triangular matrices is a Borel subgroup for Sp_{2n}. In such a realization the root vectors of $\mathfrak{sp}_{2n} = \mathrm{Lie}\,\mathrm{Sp}_{2n}$ are explicitly given by the formulas

$$f_{i,j} = f_{\alpha_{i,j}} = \begin{cases} E_{j+1,i} - E_{2n+1-i,2n-j}, & 1 \le i \le j < n, \\ E_{j+1,i} + E_{2n+1-i,2n-j}, & j \ge n, i + j < 2n, \\ E_{2n+1-i,i}, & 1 \le i \le n. \end{cases} \tag{3.1}$$

As usually, $E_{j,i}$ is the matrix having zero entries everywhere except for the entry 1 in the j-th row, i-th column.

Given a dominant weight $\lambda = a_1\omega_1 + \dots + a_n\omega_n$ for the group Sp_{2n}, we can consider this also as a dominant weight $\widetilde{\lambda} := a_1\omega_1 + \dots + a_n\omega_n$ for the larger group $\mathrm{SL}_{2n} \supset \mathrm{Sp}_{2n}$ (*i.e.*, the coefficients of $\omega_{n+1}, \dots, \omega_{2n-1}$ all vanish). In fact, let $V_{\widetilde{\lambda},\mathfrak{sl}_{2n}}$ be the corresponding irreducible $\mathfrak{sl}_{2n}$-representation and fix a highest weight vector $v_{\mathfrak{sl}_{2n}} \in V_{\widetilde{\lambda},\mathfrak{sl}_{2n}}$. We identify $V_{\lambda,\mathfrak{sp}_{2n}}$ with the irreducible $\mathfrak{sp}_{2n}$-submodule of $V_{\widetilde{\lambda},\mathfrak{sl}_{2n}}$ generated by $v_{\mathfrak{sl}_{2n}}$.

Consider now the action of $\mathfrak{sl}_{2n}$ on $V_{\widetilde{\lambda},\mathfrak{sl}_{2n}}$. Let $\mathfrak{n}^+_{\mathfrak{sl}_{2n}}, \mathfrak{n}^-_{\mathfrak{sl}_{2n}} \subset \mathfrak{sl}_{2n}$ respectively be the Lie algebra of the unipotent radical of the Borel subgroup of upper triangular matrices $B_{\mathrm{SL}_{2n}} \subset \mathrm{SL}_{2n}$ and of the opposite Borel subgroup $B^-_{\mathrm{SL}_{2n}}$. Recall the group $\mathrm{SL}^a_{2n} = B_{\mathrm{SL}_{2n}} \ltimes N^-_{\mathrm{SL}_{2n}}$ and the degenerate flag varieties $\mathcal{F}^a_\lambda := \overline{\mathrm{SL}^a_{2n} \cdot v_{\mathfrak{sl}_{2n}}} \subset \mathbb{P}(V^a_{\lambda,\mathfrak{sl}_{2n}})$.

The inclusions $\mathrm{Sp}_{2n} \subset \mathrm{SL}_{2n}$, $B \subset B_{\mathrm{SL}_{2n}}$ and $\mathfrak{n}^- \subset \mathfrak{n}^-_{\mathfrak{sl}_{2n}}$ give rise to an action of $S(\mathfrak{n}^-)$ on $V^a_{\widetilde{\lambda},\mathfrak{sl}_{2n}}$. We want to compare this action with the action on $V^a_{\lambda,\mathfrak{sp}_{2n}}$. We consider the cyclic module $\mathfrak{C}_\lambda := S(\mathfrak{n}^-).v_{\mathfrak{sl}_{2n}} \subset V^a_{\widetilde{\lambda},\mathfrak{sl}_{2n}}$.

Proposition 3.1 *$\mathfrak{C}_\lambda \simeq V^a_{\lambda,\mathfrak{sp}_{2n}}$ as $S(\mathfrak{n}^-)$-module.*

Proof Let $\Phi^+_{\mathfrak{sl}_{2n}}$, $\Phi^+_{\mathfrak{sp}_{2n}}$ be the sets of positive roots for $\mathfrak{sl}_{2n}$ respectively $\mathfrak{sp}_{2n}$, see (2.1), (2.2), and (2.3). We define an injective map $\varphi\colon \Phi^+_{\mathfrak{sp}_{2n}} \to \Phi^+_{\mathfrak{sl}_{2n}}$ between the sets of positive roots by the formula $\alpha_{i,j} \mapsto \alpha_{i,j}$. For $\alpha = \alpha_{i,j} \in \Phi^+_{\mathfrak{sl}_{2n}}$ we write F_α for the root vector associated to $-\alpha$. Note that $F_\alpha = E_{j,i}$. For $\alpha \in \Phi^+_{\mathfrak{sp}_{2n}}$ we keep the usual notation f_α for the root vector in $\mathfrak{n}^-$ associated to $-\alpha$. The matrix corresponding to $f_{\alpha_{i,j}}$ is given in (3.1). We use the shorthand notations $F_{i,j} = F_{\alpha_{i,j}}$, $f_{i,j} = f_{\alpha_{i,j}}$. Note that since $E_{j,i}$ with $i > n$ acts trivially on $V^a_{\widetilde{\lambda},\mathfrak{sl}_{2n}}$ ($(\widetilde{\lambda}, \alpha_k) = 0$ for $k > n$), the image of f_α in $\operatorname{End} V^a_{\widetilde{\lambda},\mathfrak{sl}_{2n}}$ is equal to the image of $F_{\varphi(\alpha)}$.

The cyclic $\mathfrak{n}^-_{\mathfrak{sl}_{2n}}$-module $V^a_{\widetilde{\lambda},\mathfrak{sl}_{2n}}$ can be described in terms of generators and relations, see Theorem 2.5. Let us write $\mathfrak{C}_\lambda$ as a quotient $S(\mathfrak{n}^-)/I_{\mathfrak{C}_\lambda}$. Then the ideal $I_{\mathfrak{C}_\lambda}$ contains the ideal

$$I_\lambda = \left\langle \begin{matrix} U(\mathfrak{n}^+) \circ f_{i,j}^{a_i+\cdots+a_j+1} \mid 1 \le i \le j \le n-1, \\ U(\mathfrak{n}^+) \circ f_{i,2n-i}^{a_i+\cdots+a_n+1} \mid 1 \le i \le n. \end{matrix} \right\rangle$$

Theorem 2.5 implies that one has a natural $S(\mathfrak{n}^-)$-equivariant surjective map $V^a_{\lambda,\mathfrak{sp}_{2n}} \to \mathfrak{C}_\lambda$. To prove that the map is an isomorphism, it suffices to show that both modules have the same dimension. For the proof we use the bases of $V^a_{\lambda,\mathfrak{sp}_{2n}}$ and $V^a_{\widetilde{\lambda},\mathfrak{sl}_{2n}}$ constructed in Theorem 2.7. We define an injective map

$$\varphi\colon \mathbf{P}_{\mathfrak{sp}_{2n}}(\lambda) \to \mathbf{P}_{\mathfrak{sl}_{2n}}(\widetilde{\lambda}), \quad (c_\beta)_{\beta\in\Phi^+_{\mathfrak{sp}_{2n}}} \mapsto (d_\gamma)_{\gamma\in\Phi^+_{\mathfrak{sl}_{2n}}},$$

where $(d_\gamma)_{\gamma\in\Phi^+_{\mathfrak{sl}_{2n}}} := \varphi\big((c_\beta)_{\beta\in\Phi^+_{\mathfrak{sp}_{2n}}}\big)$ is defined by the rule

$$d_\gamma := \begin{cases} c_\beta & \text{if } \gamma = \varphi(\beta) \text{ for some } \beta \in \Phi^+_{\mathfrak{sp}_{2n}}, \\ 0 & \text{otherwise.} \end{cases}$$

It is now easy to see that

$$(d_\gamma)_{\gamma\in\Phi^+_{\mathfrak{sl}_{2n}}} = \varphi\big((c_\beta)_{\beta\in\Phi^+_{\mathfrak{sp}_{2n}}}\big) \in \mathbf{P}_{\mathfrak{sl}_{2n}}(\widetilde{\lambda})$$

is an element in the associated polytope for $\mathfrak{g} = \mathfrak{sl}_{2n}$. We know that the vectors

$$\left\{ \prod_{\beta\in\Phi^+_{\mathfrak{sp}_{2n}}} f_\beta^{c_\beta} v_{\mathfrak{sp}_{2n}} \;\middle|\; (c_\beta)_{\beta\in\Phi^+_{\mathfrak{sp}_{2n}}} \in \mathbf{P}_{\mathfrak{sp}_{2n}}(\lambda) \right\} \subset V^a_{\lambda,\mathfrak{sp}_{2n}}$$

form a basis. We want to show that the vectors

$$\left\{ \prod_{\beta\in\Phi^+_{\mathfrak{sp}_{2n}}} f_\beta^{c_\beta} v_{\mathfrak{sl}_{2n}} \;\middle|\; (c_\beta)_{\beta\in\Phi^+_{\mathfrak{sp}_{2n}}} \in \mathbf{P}_{\mathfrak{sp}_{2n}}(\lambda) \right\} \subset \mathfrak{C}_\lambda \subset V^a_{\widetilde{\lambda},\mathfrak{sl}_{2n}}$$

are linearly independent. Recall that f_β acts on $V^a_{\lambda,\mathfrak{sl}_{2n}}$ in the same way as $F_{\varphi(\beta)}$ for the roots $\beta = \alpha_{i,j}$, $1 \le i \le j < n$ and $\beta = \alpha_{i,2n-i}$, $1 \le i \le n$, and for $\beta = \alpha_{i,j}$ with $j \ge n$, $i + j < 2n$ f_β acts as $F_{i,j} + F_{2n-j,2n-i}$. Hence for $(d_\gamma)_{\gamma\in\Phi^+_{\mathfrak{sl}_{2n}}} = \varphi\big((c_\beta)_{\beta\in\Phi^+_{\mathfrak{sp}_{2n}}}\big)$ we get

(3.2)

$$
\begin{aligned}
b_{(c_\beta)} &:= \prod_{\beta\in\Phi^+_{\mathfrak{sp}_{2n}}} f_\beta^{c_\beta} v_{\mathfrak{sl}_{2n}} \\
&= \Big(\prod_{i+j<2n,\, j>n} (F_{i,j} + F_{2n-j,2n-i})^{d_{\alpha_{i,j}}}\Big)\Big(\prod_{1\le i\le j<n} F_{i,j}^{d_{\alpha_{i,j}}}\Big)\Big(\prod_{1\le i\le n} F_{i,2n-i}^{d_{\alpha_{i,2n-i}}}\Big) v_{\mathfrak{sl}_{2n}}.
\end{aligned}
$$

Let $\alpha_1, \ldots, \alpha_{2n-1}$ be the set of simple roots for the root system $\Phi_{\mathfrak{sl}_{2n}}$ (Bourbaki enumeration). We associate with a vector $v_{(\ell_\beta)} = \prod_{\beta\in\Phi^+_{\mathfrak{sl}_{2n}}} F_\beta^{\ell_\beta} v_{\mathfrak{sl}_{2n}}$ the collection

$$\zeta(v_{(\ell_\beta)}) = (r_1, \ldots, r_{2n-1}) \quad \text{such that} \sum_{\beta\in\Phi^+_{\mathfrak{sl}_{2n}}} \ell_\beta\beta = \sum_{i=1}^{2n-1} r_i\alpha_i,$$

called the *root weight*. We define a partial order on these vectors by:

$$v_{(\ell_\beta)} > v_{(k_\beta)} \quad \text{if} \quad \zeta(v_{(\ell_\beta)}) > \zeta(v_{(k_\beta)})$$

with respect to the induced lexicographic order on the $(2n-1)$-tuples. It follows for the vector $b_{(c_\beta)}$ in (3.2) that

$$b_{(c_\beta)} = v_{(d_\alpha)} + \text{ a sum} \sum x_{(\ell_\beta)} v_{(\ell_\beta)} \text{ of smaller terms,} \tag{3.3}$$

because if one chooses in a factor in (3.2) the factor $F_{2n-j,2n-i}$ instead of $F_{i,j}$, then the corresponding $(2n-1)$-tuple is strictly smaller with respect to the partial order above. It follows in particular that $b_{(c_\beta)} \neq 0$, because $v_{(d_\alpha)}$ is a basis vector for $V^a_{\tilde\lambda,\mathfrak{sl}_{2n}}$ by (3) and the smaller summands are weight vectors of different weights.

The linear independence of the vectors $\{b_{(c_\beta)} \mid (c_\beta) \in \mathbf{P}_{\mathfrak{sp}_{2n}}(\lambda)\}$, follows along the same lines: given a linear dependence relation for the $b_{(c_\beta)}$, choose a maximal element $(p_1, \ldots, p_{2n-1})$ among the $\zeta(v_{(\varphi(c_\beta))})$. By weight reasons the linear independence of the $b_{(c_\beta)}$ implies a linear independence relation between the summands of associated root weight $(p_1, \ldots, p_{2n-1})$. By maximality, a summand of $b_{(c_\beta)}$ as in (3.3) has root weight $(p_1, \ldots, p_{2n-1})$ if and only if $v_{(d_\alpha)}$ has root weight $(p_1, \ldots, p_{2n-1})$, so we obtain a linear dependence relation between the $v_{(d_\alpha)}$, which is not possible.

It follows that $\dim \mathfrak{C}_\lambda = \dim V^a_{\lambda,\mathfrak{sp}_{2n}}$ and the canonical map $V^{ab}_{\lambda,\mathfrak{sp}_{2n}} \to \mathfrak{C}_\lambda$ is an isomorphism. ■

Corollary 3.2 $\mathrm{Sp}\,\mathcal{F}^a_\lambda \subset \mathcal{F}^a_{\tilde\lambda}$ *for any dominant weight* λ.

Proof By Proposition 3.1, we can identify $\mathrm{Sp}\,\mathcal{F}^a_\lambda := \overline{\mathrm{Sp}^a_{2n} \cdot v_{\mathrm{Sp}_{2n}}} \subset \mathbb{P}(V^a_{\lambda,\mathrm{Sp}_{2n}})$ with $\overline{\mathrm{Sp}^a_{2n} \cdot v_{\mathfrak{sl}_{2n}}} \subset \mathbb{P}(\mathfrak{C}_\lambda)$ and hence

$$\mathrm{Sp}\,\mathcal{F}^a_\lambda = \overline{\mathrm{Sp}^a_{2n} \cdot v_{\mathfrak{sl}_{2n}}} \subset \overline{\mathrm{SL}^a_{2n} \cdot v_{\mathfrak{sl}_{2n}}} = \mathcal{F}^a_{\tilde\lambda} \subset \mathbb{P}(V^a_{\tilde\lambda,\mathfrak{sl}_{2n}}).$$ ■

4 Explicit Description

In this section we give an explicit description of the symplectic degenerate flag varieties in terms of linear algebra. We start with the case of symplectic Grassmannians.

4.1 The Degenerate Symplectic Grassmann Variety

Let V_{ω_k} be the irreducible fundamental Sp_{2n}-representation of highest weight ω_k (Bourbaki enumeration). In particular, $V_{\omega_1} \simeq W$ is the standard vector representation. Let $V_{\omega_k} \hookrightarrow \Lambda^k W$ be the canonical embedding defined by mapping a fixed highest weight vector to $w_1 \wedge \cdots \wedge w_k$. In the following we will identify V_{ω_k} with the image. By definition, the symplectic degenerate Grassmann variety $\mathrm{Sp}\,\mathrm{Gr}_k^a(W) \subset \mathbb{P}(V_{\omega_k}^a)$ is equal to $\mathrm{Sp}\,\mathcal{F}_{\omega_k}^a$.

Let $\mathrm{P}_k \subset \mathrm{Sp}_{2n}$ be the maximal parabolic subgroup associated to ω_k. Set $\mathfrak{p} = \mathrm{Lie}\,\mathrm{P}_k$ and let $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ be the decomposition such that $\mathfrak{l}$ is the Lie algebra of the Levi subgroup of P_k containing T, $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, and $\mathfrak{u}^-$ is the Lie algebra of the unipotent radical of the parabolic subgroup P_k^- opposite to P_k. The Lie algebra $\mathfrak{u}^-$ consists of matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ C & B^{tn} & -A^{nt} & 0 \end{pmatrix}, \tag{4.1}$$

where A, B are $(n-k) \times k$ matrices, X^{nt} denotes the transposed matrix with respect to the skew diagonal, and C is a $k \times k$ matrix such that $C^{nt} = C$.

We write $W = W_{k,1} \oplus W_{k,2} \oplus W_{k,3}$, where

$$W_{k,1} = \mathrm{span}(w_1, \ldots, w_k), \quad W_{k,2} = \mathrm{span}(w_{k+1}, \ldots, w_{2n-k}),$$
$$W_{k,3} = \mathrm{span}(w_{2n-k+1}, \ldots, w_{2n}).$$

Denote by $p_{1,3}$ the projection $p_{1,3}\colon W \to W_{k,1} \oplus W_{k,3}$, *i.e.*,

$$\mathrm{pr}_{1,3}(x_1, \ldots, x_{2n}) = (x_1, \ldots, x_k, 0, \ldots, 0, x_{2n-k+1}, \ldots, x_{2n}).$$

Proposition 4.1

$$\mathrm{Sp}\,\mathrm{Gr}_k^a(W) = \{U \in \mathrm{Gr}_k(W) \mid \mathrm{pr}_{1,3}(U) \textit{ is isotropic}\}.$$

Denote by Z_k the subvariety $\{U \in Gr_k(W) \mid \mathrm{pr}_{1,3}(U) \text{ is isotropic}\}$. The first simple observation in the proof of the proposition is the following.

Lemma 4.2 $\mathrm{Gr}_k(W_{k,1} \oplus W_{k,2}) \subset Z_k$.

Proof Since $\mathrm{pr}_{1,3}(U) \subset W_{k,1}$ for all $U \in \mathrm{Gr}_k(W_{k,1} \oplus W_{k,2})$, we have $\mathrm{pr}_{1,3}(U)$ isotropic and hence $U \in Z_k$. ∎

Denote by $\mathrm{Sp}_{2k} \subset \mathrm{Sp}_{2n}$ the symplectic subgroup acting only on the first and last k coordinates. The matrix for an element of this subgroup looks like

$$\begin{pmatrix} A & 0 & 0 & B \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ C & 0 & 0 & D \end{pmatrix},$$

where A, B, C, D are $k \times k$ matrices,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a $2k \times 2k$-sympletic matrix, and $\mathbb{1}$ is the $(n-k) \times (n-k)$ identity matrix. Further, let

$$\mathrm{P}'_k := \{g \in \mathrm{Sp}_{2k} \mid C = 0\},$$

then P'_k is the maximal parabolic subgroup of Sp_{2k} associated to the long simple root. The next simple observation is the following.

Lemma 4.3 $Z_k = \mathrm{Sp}_{2k} \cdot \mathrm{Gr}_k(W_{k,1} \oplus W_{k,2})$.

Proof Note first that $Y = \mathrm{Gr}_k(W_{k,1} \oplus W_{k,2})$ is stable under the action of P'_k, so $\mathrm{Sp}_{2k} \cdot Y$ is closed. Since Z_k is Sp_{2k}-stable, by Lemma 4.2 we have $\mathrm{Sp}_{2k} \cdot Y \subset Z_k$. Now assume $U \in Z_k$ and let $\bar{U} = \mathrm{pr}_{1,3}(U)$. This subspace of $W_{k,1} \oplus W_{k,3}$ is isotropic by assumption, so there exists a $g \in \mathrm{Sp}_{2k}$ such that $g \cdot \bar{U} \subset W_{k,1}$, and hence $g \cdot U \subset W_{k,1} \oplus W_{k,2}$. It follows $g \cdot U \in \mathrm{Gr}_k(W_{k,1} \oplus W_{k,2})$ and hence $Z_k = \mathrm{Sp}_{2k} \cdot \mathrm{Gr}_k(W_{k,1} \oplus W_{k,2})$. ∎

As an immediate consequence one sees that Z_k is the image of the canonical product map

$$\tilde{\pi}\colon\ \mathrm{Sp}_{2k} \times \mathrm{Gr}_k(W_{k,1} \oplus W_{k,2}) \to Z_k \subset \mathrm{Gr}_k(W), \quad (g, U) \mapsto g \cdot U,$$

and hence the following corollary.

Corollary 4.4 *Z_k is irreducible.*

Proof of Proposition 4.1 Recall the description of the Lie algebra $\mathfrak{u}^-$ in (4.1) and the inclusion of the abelianized modules described in Proposition 3.1. It follows that if $\gamma \in \mathfrak{u}^-$ is a matrix as in (4.1), then in $V^a_{\omega_k}$ we have $\exp \gamma \cdot [w_1 \wedge \cdots \wedge w_k]$ is the k-dimensional subspace having as basis the column vectors of the matrix

$$\begin{pmatrix} \mathbb{1} \\ A \\ B \\ C \end{pmatrix}. \tag{4.2}$$

The symmetry condition of the matrix C implies that these subspaces lie in Z_k. For $\underline{i} = (i_1, \ldots, i_k)$, $1 \le i_1 < \cdots < i_k \le 2n$, we set $w_{\underline{i}} = w_{i_1} \wedge \cdots \wedge w_{i_k} \in \Lambda^k W$, and we denote by $p_{\underline{i}}$ the corresponding Plücker coordinate, *i.e.*, $p_{\underline{i}}(w_{\underline{j}}) = \delta_{\underline{i},\underline{j}}$.

Consider in Z_k the open affine set

$$Z_{k(1,2,\dots,k)} = \{U \in Z_k \mid p_{(1,2,\dots,k)}(U) \neq 0\}.$$

The spaces having a basis as in (4.2) lie in $Z_{k(1,2,\dots,k)}$, so this set is non-empty and (since Z_k is irreducible) dense. Now given an element in $Z_{k(1,2,\dots,k)}$, one can find a basis corresponding to the columns of a matrix of the form

$$\begin{pmatrix} \mathbb{1} \\ A \\ B \\ C \end{pmatrix},$$

where A, B are $(n-k) \times k$ matrices, and the condition $\mathrm{pr}_{1,3}(U)$ is isotropic implies that $C = C^{nt}$. It follows that $Z_{k(1,2,\dots,k)} = \mathrm{Sp}^a_{2n} \cdot [w_1 \wedge \cdots \wedge w_k]$, and hence $Z_k = \mathrm{Sp}\,\mathrm{Gr}^a_k(W)$. ■

Remark 4.5 (a) It is easy to check that the canonical map

$$\pi\colon\ \mathrm{Sp}_{2k} \times_{\mathrm{P}'_k} \mathrm{Gr}_k(W_{k,1} \oplus W_{k,2}) \to \mathrm{Sp}\,\mathrm{Gr}^a_k(W)$$

is a desingularization.

(b) For $j \geq 0$, denote by Z_j the subset $Z_j := \{U \in \mathrm{Sp}\,\mathrm{Gr}^a_k(W) \mid \dim \mathrm{pr}_{1,3}(U) = j\}$ and define

$$\mathrm{Gr}^j_k(W_{k,1} \oplus W_{k,2}) := \{U \in \mathrm{Gr}_k(W_{k,1} \oplus W_{k,2}) \mid \dim \mathrm{pr}_{1,3}(U) = j\}.$$

The set Z_j (and similarly $\mathrm{Gr}^j_k(W_{k,1} \oplus W_{k,2})$) is not empty for $0 \leq k-j \leq 2(n-k)$. We have obviously a partition:

$$\mathrm{Sp}\,\mathrm{Gr}^a_k(W) = \bigcup_{0 \leq k-j \leq 2(n-k)} Z_j.$$

One can show that the group

$$H := \big(\mathrm{Sp}_{2k} \times \mathrm{GL}(W_{k,2})\big) \ltimes \big(\mathbb{1}_{\mathbb{C}^{2n}} \oplus \mathrm{Hom}(W_{k,1} \oplus W_{k,3}, W_{k,2})\big)$$

acts transitively on the non-empty Z_j's. The latter is hence smooth, of dimension

$$\dim Z_j = \frac{j}{2}(j+1) + 2j(k-j) + k(2n-3k+j),$$

and $\overline{Z_j} = \bigcup_{i \leq j} Z_i$. The desingularization map above is compatible with the partition and induces maps

$$\pi^j\colon\ \mathrm{Sp}_{2k} \times_{\mathrm{P}'_k} \mathrm{Gr}^j_k(W_{k,1} \oplus W_{k,2}) \to Z_j$$

where

$$\mathrm{Sp}_{2k} \times_{\mathrm{P}'_k} \mathrm{Gr}^j_k(W_{k,1} \oplus W_{k,2}) = \frac{1}{2}k(k+1) + j(k-j) + k(2n-3k+j).$$

The fibres of π^j are all of dimension $\frac{1}{2}(k-j)(k-j+1)$. In particular, the resolution map π satisfies the dimension condition for being semismall, but it is not a small resolution; the strong dimension condition is not satisfied for $j = k-1$.

4.2 Symplectic Degenerate Flag Varieties

Based on Proposition 4.1 we derive an explicit description for all symplectic degnerate flag varieties. For a regular dominant weight λ we denote the complete symplectic degenerate flag variety $\mathcal{F}^a_\lambda$ by $\mathrm{Sp}\,\mathcal{F}^a_{2n}$.

Recall the basis $w_1, \dots, w_{2n}$ of W. We denote by $\mathrm{pr}_i \colon W \to W$ the projections along the w_i, *i.e.*, $\mathrm{pr}_i(\sum_{j=1}^{2n} c_j w_j) = \sum_{j\neq i} c_j w_j$.

Theorem 4.6 *The degenerate symplectic flag variety* $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ *is naturally embedded into the product* $\prod_{i=1}^n \mathrm{Sp}\,\mathrm{Gr}^a_i(2n)$ *of degenerate symplectic Grassmannians. The image of the embedding is equal to the subvariety formed by the collections* $(V_i)_{i=1}^n$, $V_i \in \mathrm{Sp}\,\mathrm{Gr}^a_i(2n)$ *satisfying the conditions*

$$\mathrm{pr}_{i+1} V_i \subset V_{i+1}, i = 1, \dots, n-1.$$

Proof According to the proof of Proposition 4.1 we have

$$\mathrm{Sp}\,\mathcal{F}^a_{2n} \cap \prod_{i=1}^n Z_{i(1,\dots,i)} = \{(V_1, \dots, V_i) : V_i \in Z_{i(1,\dots,i)}, \mathrm{pr}_{i+1} V_i \subset V_{i+1}\}.$$

Moreover this subvariety of $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ coincides with the Sp^a_{2n}-orbit of the highest weight line. Thus we only need to show that the variety defined above is irreducible. This is proved in Corollary 5.6, using a desingularization of $\mathrm{Sp}\,\mathcal{F}^a_{2n}$. ■

For a subspace $V \subset W$ we denote by $V^\perp \subset W$ the orthogonal complement to V. Define an order two automorphism $\sigma \in \mathrm{Aut}\big(\prod_{i=1}^{2n-1} \mathrm{Gr}_i(2n)\big)$ by the formula

$$\sigma(V_i)_{i=1}^{2n-1} = (V_{2n-1}^\perp, V_{2n-2}^\perp, \dots, V_1^\perp).$$

Proposition 4.7 *The automorphism* σ *defines an order two automorphism of the complete degenerate flag variety* $\mathcal{F}^a_{2n}$. *The set of* σ*-fixed points* $(\mathcal{F}^a_{2n})^\sigma$ *is isomorphic to the complete symplectic degenerate flag variety* $\mathrm{Sp}\,\mathcal{F}^a_{2n}$.

Proof Follows from the definition. ■

Let $\mathbf{d} = (d_1, \dots, d_k)$ be a collection of integers such that $1 \le d_1 < \cdots < d_k \le n$. Let $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ be the parabolic degenerate flag variety, corresponding to the highest weight $\sum_{i=1}^k \omega_{d_i}$.

Theorem 4.8 *The parabolic degenerate symplectic flag variety* $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ *is naturally embedded into the product* $\prod_{i=1}^k \mathrm{Sp}\,\mathrm{Gr}^a_{d_i}(2n)$ *of degenerate symplectic Grassmannians. The image of the embedding is equal to the variety of collections* $(V_i)_{i=1}^k$, $V_i \in \mathrm{Sp}\,\mathrm{Gr}^a_{d_i}(2n)$ *satisfying the conditions*

$$\mathrm{pr}_{d_i+1} \dots \mathrm{pr}_{d_{i+1}} V_{d_i} \subset V_{d_{i+1}}, i = 1, \dots, k-1.$$

Proof As above, we only need to show the irreducibility of $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$. This is proved in Corollary 5.11. ■

Let us fix a collection $\mathbf{d} = (d_1, \ldots, d_k)$, $1 \le d_1 < \cdots < d_k \le n$). Assume $d_k < n$. Then we define an extended collection $\mathbf{D}$ by the formula

$$\mathbf{D} = (d_1, \ldots, d_k, 2n - d_k, \ldots, 2n - d_1).$$

If $d_k = n$, then we set $\mathbf{D} = (d_1, \ldots, d_k, 2n - d_{k-1}, \ldots, 2n - d_1)$. We define an order two automorphism $\sigma_{\mathbf{d}} \in \mathrm{Aut}(\prod_{d \in \mathbf{D}} \mathrm{Gr}_d(2n)$ by the formula

$$\sigma_{\mathbf{d}}(V_i)_{i=1}^{l} = (V_l^{\perp}, V_{l-1}^{\perp}, \ldots, V_1^{\perp}),$$

where $l = 2k$ if $d_k < n$ and $l = 2k - 1$ otherwise.

The following proposition is an immediate corollary of Theorem 4.8.

Proposition 4.9 *The automorphism $\sigma_{\mathbf{d}}$ defines an order two automorphism of the SL_{2n} parabolic degenerate flag variety $\mathcal{F}^a_{\mathbf{D}}$. The set of $\sigma_{\mathbf{d}}$-fixed points $(\mathcal{F}^a_{\mathbf{D}})^{\sigma_{\mathbf{d}}}$ is isomorphic to the parabolic symplectic degenerate flag variety $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{D}}$.* ■

4.3 The Degeneration

In this subsection we prove that the varieties $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ are flat degenerations of their classical analogues. Let J_s be the $2n \times 2n$-matrix

$$\begin{pmatrix} 0 & 0 & 0 & I_k \\ 0 & 0 & sI_{n-k} & 0 \\ 0 & -sI_{n-k} & 0 & 0 \\ -I_k & 0 & 0 & 0 \end{pmatrix}, \quad \text{where } I_l \text{ is an } l \times l \text{ matrix } \begin{pmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The matrix defines a non-degenerate symplectic form for $s \neq 0$, the group Sp_{2n} is the one leaving invariant the form for $s = 1$, and

$$\mathrm{Sp}\,\mathrm{Gr}_k(2n) = \{U \in \mathrm{Gr}_k(\mathbb{C}^{2n}) \mid U \text{ is isotropic with respect to } J_0\}.$$

Denote by η the following one-parameter subgroup:

$$\eta\colon \mathbb{C}^* \to D = \text{ diagonal matrices in } \mathrm{GL}_{2n}, \quad s \mapsto \begin{pmatrix} \mathbb{1}_k & 0 & 0 \\ 0 & s\mathbb{1}_{2(n-k)} & 0 \\ 0 & 0 & \mathbb{1}_k \end{pmatrix}.$$

Then $\eta(s)^t J_1 \eta(s) = J_{s^2}$, and it follows that if $U \in \mathrm{Sp}\,\mathrm{Gr}_k(\mathbb{C}^{2n})$ is a subspace isotropic with respect to J_1, then $\eta(s^{-1})(U)$ is isotropic with respect to J_{s^2}.

Recall that $\mathrm{Sp}_{2n}/\mathrm{P}_k$ is sitting in the Grassmann variety as the set of isotropic subspaces. Consider

$$Y := \overline{\left\{ \left(\eta(s^{-1})(U), s\right) \;\middle|\; s \in \mathbb{C}^*, U \in \mathrm{Sp}_{2n}/\mathrm{P}_k \right\}} \subset \mathrm{Gr}_k(\mathbb{C}^{2n}) \times \mathbb{A}^1 \tag{4.3}$$

together with the natural projection $\phi\colon Y \to \mathbb{A}^1$ onto the second factor.

Proposition 4.10 *The projection map ϕ is flat, $\phi^{-1}(s) \simeq \mathrm{Sp}_{2n}/\mathrm{P}_k$ for all $s \neq 0$ and $\phi^{-1}(0) = \mathrm{Sp}\,\mathrm{Gr}_k^a(\mathbb{C}^{2n})$.*

Proof The map is flat because Y is irreducible and ϕ is dominant [H, Chap. III, Proposition 9.7]. It remains to show that the fibres are the spaces described above. We assume without loss of generality: $k \geq 2$. Consider the map

$$\Psi\colon \Lambda^k\mathbb{C}^{2n} \times \mathbb{A}^1 \to \Lambda^{k-2}\mathbb{C}^{2n},$$
$$(v_1 \wedge \cdots \wedge v_k, s) \mapsto \sum_{\ell<m} Q_s(v_\ell, v_m)(-1)^{m+\ell-1} v_1 \wedge \cdots \wedge \hat{v}_\ell \wedge \cdots \wedge \hat{v}_m \wedge \cdots \wedge v_k$$

where $Q_s(v,w) = v^t J_s w$. The map is homogeneous with respect to $\Lambda^k\mathbb{C}^{2n}$, the preimage $\Psi^{-1}(0)$ defines hence a closed subset of $\mathbb{P}(\Lambda^k\mathbb{C}^{2n}) \times \mathbb{A}^1$, and the intersection with the Grassmann variety defines a closed subset $\tilde{Y} \subset \mathrm{Gr}_k(\mathbb{C}^{2n}) \times \mathbb{A}^1$. Let $\sigma\colon \tilde{Y} \to \mathbb{A}^1$ be the projection onto the second component. By the definition of Ψ, one has $(U,s) \in \sigma^{-1}(s) \subset \tilde{Y}$ if and only if $\Psi(v_1 \wedge \cdots \wedge v_k, s) = 0$ for a basis $\{v_1, \ldots, v_k\}$ of U, that is if and only if U is an isotropic subspace with respect to the form Q_s. It follows that $\tilde{Y}$ and Y have in common the subset

$$Y' = \left\{ \left(\eta(s^{-1})(U), s\right) \;\middle|\; s \in \mathbb{C}^*, U \in \mathrm{Sp}_{2n}/\mathrm{P}_k \right\} \subset Y,$$

so $Y = \overline{Y'}$ is an irreducible component of $\tilde{Y}$. Since $\sigma^{-1}(0) = \mathrm{Sp}\,\mathrm{Gr}_k^a(\mathbb{C}^{2n})$, it follows that σ has equidimensional and irreducible fibers, and hence $\tilde{Y}$ is irreducible, which implies that $Y = \tilde{Y}$ and $\sigma = \phi$. ∎

Let $\mathbf{d} = (d_1, \ldots, d_m)$.

Theorem 4.11 *There exists an irreducible variety $\mathbf{Sp}\mathcal{F}_{\mathbf{d}}$ together with a flat morphism $\varphi\colon \mathbf{Sp}\mathcal{F}_{\mathbf{d}} \to \mathbb{A}^1$ such that the zero fiber is isomorphic to $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ and all fibers over $\mathbb{A}^1 - \{0\}$ are isomorphic to $\mathrm{Sp}\,\mathcal{F}_{\mathbf{d}}$. In other words, the variety $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ is flat a degeneration of the classical analogue $\mathrm{Sp}\,\mathcal{F}_{\mathbf{d}}$.*

Proof Let us denote by Y_k the variety defined by (4.3). Recall the projections $\phi_k\colon Y_k \to \mathbb{A}^1$. Let $\mathbf{Sp}\mathcal{F}_{\mathbf{d}}$ be the fibered product of all Y_{d_i} over $\mathbb{A}^1$, *i.e.*,

$$\mathbf{Sp}\mathcal{F}_{\mathbf{d}} = \{(y_1, \ldots, y_m) : y_i \in Y_{d_i}, \phi_{d_1}(y_1) = \cdots = \phi_{d_m}(y_m)\}.$$

Recall (see [Fe1]) that there exists flat degeneration $\psi_{\mathbf{d}}\colon M_{\mathbf{d}} \to \mathbb{A}^1$ of the SL_{2n} flag variety $\mathcal{F}_{\mathbf{d}}$ into the degenerate version $\mathcal{F}^a_{\mathbf{d}}$, so $\psi_{\mathbf{d}}^{-1}(0) \simeq \mathcal{F}^a_{\mathbf{d}}$ and the general fiber is isomorphic to $\mathcal{F}_{\mathbf{d}}$. We note that both $M_{\mathbf{d}}$ and $\mathbf{Sp}\mathcal{F}_{\mathbf{d}}$ are subvarieties of the product $\mathbb{A}^1 \times \prod_{i=1}^m \mathrm{Gr}_{d_i}(\mathbb{C}^{2n})$. Consider the intersection $M_{\mathbf{d}} \cap \mathbf{Sp}\mathcal{F}_{\mathbf{d}}$ and the natural projection $\varphi_{\mathbf{d}}\colon M_{\mathbf{d}} \cap \mathbf{Sp}\mathcal{F}_{\mathbf{d}} \to \mathbb{A}^1$. Then the general fiber is isomorphic to the classical symplectic flag variety $\mathrm{Sp}\,\mathcal{F}_{\mathbf{d}}$ and $\varphi_{\mathbf{d}}^{-1}(0) \simeq \mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$. Since $M_{\mathbf{d}} \cap \mathbf{Sp}\mathcal{F}_{\mathbf{d}} = \overline{\varphi_{\mathbf{d}}^{-1}(\mathbb{A}^1 \setminus \{0\})}$, the left hand side is irreducible. Now [H, Chap. III, Proposition 9.7], implies the theorem. ∎

5 Resolution of Singularities

In this section we construct varieties $\operatorname{Sp} R_{\mathbf{d}}$ which serve as desingularizations for the symplectic degenerate flag varieties $\operatorname{Sp} \mathcal{F}^a_{\mathbf{d}}$. We start with the case of complete flags.

5.1 Complete Flag Varieties

Recall the $2n$-dimensional vector space W with a basis w_i, $i = 1, \dots, 2n$. Let $W_{i,j}$ be the linear span of the vectors $w_1, \dots, w_i, w_{j+1}, \dots, w_{2n}$.

We define $\operatorname{Sp} R_{2n} \subseteq \prod_{i \le j, i+j \le 2n} \operatorname{Gr}_i(W_{i,j})$ as the variety of collections of subspaces

$$V_{i,j}, \quad 1 \le i \le j \le 2n, \quad i + j \le 2n$$

subject to the conditions

$$\dim V_{i,j} = i, \quad V_{i,j} \subset W_{i,j}, \tag{5.1}$$

$$V_{i,j} \subset V_{i+1,j}, \quad \operatorname{pr}_{j+1} V_{i,j} \subset V_{i,j+1}, \tag{5.2}$$

$$V_{i,2n-i} \text{ are isotropic for } i = 1, \dots, n. \tag{5.3}$$

It is convenient to view the spaces $V_{i,j}$ as being attached to the positive roots $\alpha_{i,j}$ of $\mathfrak{sp}_{2n}$ (see (2.2), (2.3)). We fix an enumeration $\beta_1, \beta_2, \dots, \beta_{n^2}$ of the positive roots like in the following example for $\mathfrak{sp}_8$:

$$\begin{array}{ccccccc} \beta_{16} & \beta_{12} & \beta_9 & \beta_6 & \beta_4 & \beta_2 & \beta_1 \\ & \beta_{15} & \beta_{11} & \beta_8 & \beta_5 & \beta_3 & \\ & & \beta_{14} & \beta_{10} & \beta_7 & & \\ & & & \beta_{13} & & & \end{array} \quad = \quad \begin{array}{ccccccc} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{1,5} & \alpha_{1,6} & \alpha_{1,7} \\ & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & \alpha_{2,5} & \alpha_{2,6} & \\ & & \alpha_{3,3} & \alpha_{3,4} & \alpha_{3,5} & & \\ & & & \alpha_{4,4} & & & \end{array}$$

i.e.,

$$\begin{array}{ccccccc} \beta_1 = \alpha_{1,2n-1} & \beta_2 = \alpha_{1,2n-2} & \beta_3 = \alpha_{2,2n-2} & \beta_5 = \alpha_{2,2n-3} & \beta_7 = \alpha_{3,2n-3} & \dots & \beta_{n^2-n+1} = \alpha_{n,n} \\ & & \beta_4 = \alpha_{1,2n-3} & \beta_6 = \alpha_{1,2n-4} & \beta_8 = \alpha_{2,2n-4} & \dots & \vdots \\ & & & & \beta_9 = \alpha_{1,2n-5} & \dots & \vdots \\ & & & & & & \beta_{n^2} = \alpha_{1,1}. \end{array}$$

Using the correspondence above between subspaces and positive roots we sometimes denote $V_{i,j}$ and $W_{i,j}$ by $V_{\alpha_{i,j}}$ and $W_{\alpha_{i,j}}$ or V_{β_l} and W_{β_l} for $\beta_l = \alpha_{i,j}$. To simplify the notation we write $(i, j) \le \ell$ if $\alpha_{i,j} = \beta_k$ is such that $k \le \ell$, and we write $\big(i(k), j(k)\big)$ if $\beta_k = \alpha_{i(k),j(k)}$. Note that if $\big(i(k) - 1, j(k)\big)$ is an admissible indexing pair in the scheme above, then $\big(i(k) - 1, j(k)\big) < k$. Similarly we have $\big(i(k), j(k) + 1\big) < k$.

Definition 5.1 Denote by $\operatorname{Sp} R_{2n}(l) \subseteq \prod_{k=1}^{\ell} \operatorname{Gr}_{i(k)}(W_{\beta_k})$ the variety of collections of subspaces $V_{\beta_k} \subset W_{\beta_k}$ satisfying the conditions (5.1), (5.2), (5.3) for all pairs of indices $\big(i(k), j(k)\big)$, $k \le l$. In addition we set $\operatorname{Sp} R_{2n}(0) = pt$.

Note that $\operatorname{Sp} \mathrm{R}_{2n} = \operatorname{Sp} \mathrm{R}_{2n}(n^2)$. If $\ell > \ell'$, then the projection maps

$$\pi_{\ell,\ell'}\colon \prod_{k=1}^{\ell} \mathrm{Gr}_{i(k)}(W_{\beta_k}) \to \prod_{k=1}^{\ell'} \mathrm{Gr}_{i(k)}(W_{\beta_k})$$

induce natural maps $\pi_{\ell,\ell'}\colon \operatorname{Sp} \mathrm{R}_{2n}(l) \longrightarrow \operatorname{Sp} \mathrm{R}_{2n}(l')$.

Proposition 5.2

(i) *The projection maps $\pi_{\ell,\ell'}$ are surjective for all $\ell \geq \ell' \geq 1$.*

(ii) *For all $l = 1, \dots, n^2$, the map $\pi_{\ell,\ell-1}\colon \operatorname{Sp} \mathrm{R}_{2n}(l) \to \operatorname{Sp} \mathrm{R}_{2n}(l-1)$ is a $\mathbb{P}^1$-fibration.*

(iii) *The map $s_l\colon \operatorname{Sp} \mathrm{R}_{2n}(l-1) \to \operatorname{Sp} \mathrm{R}_{2n}(l)$, defined by $s_l(\mathbf{V})_{\beta_k} = V_{\beta_k}$ for $k \leq l-1$ and*

$$s_l(\mathbf{V})_l = \begin{cases} \mathbb{C}w_{j(l)+1}, & \text{if } i(l) = 1, \\ V_{i(l)-1,\, j(l)+1} \oplus \mathbb{C}w_{j(l)+1}, & \text{if } i(l) > 1. \end{cases}$$

defines a section to the map $\pi_{\ell,\ell-1}$.

As an immediate consequence there is the following.

Corollary 5.3 $\operatorname{Sp} \mathrm{R}_{2n}$ *is a smooth projective variety; more precisely, it is a tower of successive $\mathbb{P}^1$-fibrations.*

Proof To prove (i) and (ii), it is sufficient to show that $\pi_{\ell,\ell-1}$ is surjective for all $\ell = 1, \dots, n^2$, and $\pi_{\ell,\ell-1}$ is a fibration with fibres isomorphic to $\mathbb{P}^1$. We will only show that the fibre over each point is a $\mathbb{P}^1$, but the construction can be lifted to vector bundles obtained by extending the tautological bundles on $\mathrm{Gr}_{i(\ell)-1}(W_{i(\ell)-1,j(\ell)})$, respectively $\mathrm{Gr}_{i(\ell)}(W_{i(\ell),j(\ell)+1})$, to $\prod_{k=1}^{\ell-1} \mathrm{Gr}_{i(k)}(W_{\beta_k})$ and then the restrictions to $\operatorname{Sp} \mathrm{R}_{2n}(\ell-1)$, so this will indeed lead to a $\mathbb{P}^1$-fibration, locally trivial in the Zariski topology.

If $\ell = 1$, then $\operatorname{Sp} \mathrm{R}_{2n}(0) = pt$ and $\operatorname{Sp} \mathrm{R}_{2n}(1)$ is the variety of lines (which are automatically isotropic) in the subspace spanned by w_1 and w_{2n}. Hence $\operatorname{Sp} \mathrm{R}_{2n}(1)$ is isomorphic to $\mathbb{P}^1$, $\pi_{1,0}$ is surjective and a $\mathbb{P}^1$-fibration and $s_1(pt)$ is the line through w_{2n}.

Assume now $\ell > 1$, fix $\mathbf{V} = (V_{\beta_k})_{k=1}^{\ell-1} \in \operatorname{Sp} \mathrm{R}_{2n}(l-1) \subset \prod_{k=1}^{\ell-1} \mathrm{Gr}_{i(k)}(W_{\beta_k})$. Suppose first $i(\ell) = 1$ and let $V_{\beta_\ell} \in \mathrm{Gr}_1(W_{\beta_\ell})$ be a line. Then $(V_{\beta_k})_{k=1}^{\ell} \in \operatorname{Sp} \mathrm{R}_{2n}(l)$ if and only if $\mathrm{pr}_{j(\ell)+1}(V_{\beta_\ell}) \subseteq V_{i(\ell),j(\ell)+1}$. Let v be a generator of $V_{i(\ell),j(\ell)+1}$, then this condition is equivalent to V_{β_ℓ} is spanned by $av + bw_{j(\ell)+1}$ for some $[a:b] \in \mathbb{P}^1$. Hence $\pi_{\ell,\ell-1}$ is surjective, it is a $\mathbb{P}^1$-fibration and $s_\ell(\mathbf{V})$ is obtained from $\mathbf{V}$ by adding the line through $w_{j(\ell)+1}$ to the collection.

Suppose next $i(\ell) + j(\ell) = 2n$, we can assume $i(\ell) > 1$ because otherwise $\ell = 1$. Fix a point $\mathbf{V} = (V_{\beta_k})_{k=1}^{\ell-1} \in \operatorname{Sp} \mathrm{R}_{2n}(l-1)$ and let $V_{\beta_\ell} \in \mathrm{Gr}_{i(\ell)}(W_{\beta_\ell})$ be a subspace. Then $(V_{\beta_k})_{k=1}^{\ell} \in \operatorname{Sp} \mathrm{R}_{2n}(l)$ if and only if

(a) V_{β_ℓ} is isotropic, and

(b) $V_{i(\ell)-1,j(\ell)} \subseteq V_{\beta_\ell}$.

Since $\mathrm{pr}_{j(\ell)+1}(V_{i(\ell)-1,j(\ell)}) \subseteq V_{i(\ell)-1,j(\ell)+1}$ (the latter is isotropic because $i(\ell) + j(\ell) = 2n$) and $V_{i(\ell)-1,j(\ell)} \subset W_{i(\ell)-1,j(\ell)}$, $V_{i(\ell)-1,j(\ell)}$ has to be isotropic as a subspace of $W_{i(\ell),j(\ell)}$. So (a) and (b) imply $V_{i(\ell)-1,j(\ell)} \subseteq V_{\beta_\ell} \subseteq V^{\perp}_{i(\ell)-1,j(\ell)} \subseteq W_{i(\ell),j(\ell)}$. In fact, the possible choices for $V_{\beta_\ell} \subseteq W_{i(\ell),j(\ell)}$ are parametrized by the lines in the two dimensional space $V^{\perp}_{i(\ell)-1,j(\ell)}/V_{i(\ell)-1,j(\ell)}$. Again we see that $\pi_{\ell,\ell-1}$ is surjective, it is a $\mathbb{P}^1$-fibration, and $s_\ell(\mathbf{V})$ is obtained from $\mathbf{V}$ by adding the subspace spanned by $V_{i(\ell)-1,j(\ell)+1}$ and $w_{j(\ell)+1}$ to the collection.

Finally suppose $i(\ell) + j(\ell) < 2n$ and $i(\ell) > 1$. Fix $\mathbf{V} = (V_{\beta_k})_{k=1}^{\ell-1} \in \mathrm{Sp}\,\mathrm{R}_{2n}(l-1)$ and let $V_{\beta_\ell} \in \mathrm{Gr}_{i(\ell}(W_{\beta_\ell})$ be a subspace. Then $(V_{\beta_k})_{k=1}^{\ell} \in \mathrm{Sp}\,\mathrm{R}_{2n}(l)$ if and only if

(a) $V_{i(\ell)-1,j(\ell)} \subseteq V_{i(\ell),j(\ell)}$, and
(b) $\mathrm{pr}_{j(\ell)+1} V_{i(\ell),j(\ell)} \subseteq V_{i(\ell),j(\ell)+1}$.

So the possible choices for $V_{\beta_\ell} = V_{i(\ell),j(\ell)}$ are parametrized by the lines in the two dimensional space

$$\langle V_{i(\ell),j(\ell)+1}, w_{j(\ell)+1}\rangle / V_{i(\ell)-1,j(\ell)}.$$

Hence $\pi_{\ell,\ell-1}$ is surjective, it is a $\mathbb{P}^1$-fibration, and $s_\ell(\mathbf{V})$ is obtained from $\mathbf{V}$ by adding the subspace spanned by $V_{i(\ell)-1,j(\ell)+1}$ and $w_{j(\ell)+1}$ to the collection. ■

We fix now ℓ and $(i_0, j_0) = \big(i(\ell), j(\ell)\big)$. We associate to s_l $(= s_{i_0,j_0})$ a divisor in $\mathrm{Sp}\,\mathrm{R}_{2n}$:

$$Z_{i_0,j_0} = \{\mathbf{V} \in \mathrm{Sp}\,\mathrm{R}_{2n} : \exists \mathbf{V}' \in \mathrm{Sp}\,\mathrm{R}_{2n}(l-1) \text{ such that } s_l(\mathbf{V}')_{i_0,j_0} = V_{i_0,j_0}\}.$$

We also denote by $Z^\circ_{i_0,j_0} \subset Z_{i_0,j_0}$ the open part of Z_{i_0,j_0} that is the complement to the intersection of Z_{i_0,j_0} with all other divisors:

$$Z^\circ_{i_0,j_0} = Z_{i_0,j_0} \setminus \bigcup_{(i,j)\neq(i_0,j_0)} Z_{i,j}. \tag{5.4}$$

Denote by $\mathrm{Sp}\,\mathrm{R}^\circ_{2n} \subset \mathrm{Sp}\,\mathrm{R}_{2n}$ the complement in $\mathrm{Sp}\,\mathrm{R}_{2n}$ to the union of the divisors $Z_{i,.j}$.

We have fixed a basis $w_1, \dots, w_{2n}$. Let $\{w_{j_1} \wedge \cdots \wedge w_{j_i} \mid 1 \le j_1 < \cdots < j_i \le 2n\}$ be the corresponding basis for the i-th exterior product and denote by $p_{j_1,\dots,j_i}$ the corresponding Plücker coordinates.

Lemma 5.4 $\mathrm{Sp}\,\mathrm{R}^\circ_{2n}$ *is an open cell in* $\mathrm{Sp}\,\mathrm{R}_{2n}$ *(an open subvariety isomorphic to an affine space) enjoying the following explicit description:*

$$\mathrm{Sp}\,\mathrm{R}^\circ_{2n} = \{\mathbf{V} \in \mathrm{Sp}\,\mathrm{R}_{2n} : p_{1,\dots,i} V_{i,j} \neq 0 \textit{ for all } i, j\},$$

where $p_{1,\dots,i}$ is the corresponding Plücker coordinate.

Proof $\mathrm{Sp}\,\mathrm{R}^\circ_{2n}$ is, by definition, a tower of successive fibrations with one-dimensional affine spaces as fibers, and hence $\mathrm{Sp}\,\mathrm{R}^\circ_{2n}$ is isomorphic to an affine space, proving the first claim.

To prove the second statement, note that it is easy to see that if $p_{1,\dots,i}V_{i,j} \neq 0$ for all i, j, then $\mathbf{V}$ does not belong to any of the divisors Z_{i_0,j_0}. The reverse implication is more intricate. It suffices to show that

$$\dim \mathrm{pr}_{i+1} \cdots \mathrm{pr}_{2n} V_{i,i} = i, \quad i = 1, \dots, n.$$

In fact, let i be the minimal number with the following property: there exists j, $i+1 \le j \le 2n$ such that $\dim \mathrm{pr}_{i+1} \cdots \mathrm{pr}_j V_{i,i} = i-1$. For such an i let us take the minimal j with this property. First, let $j \le 2n-i$. Since $\mathrm{pr}_{i+1} \cdots \mathrm{pr}_j V_{i,i} \subset V_{i,j}$, it follows that $w_j \in V_{i,j-1}$. The choice of i being minimal with this property implies $w_j \notin V_{i-1,j-1}$ and hence $\dim \mathrm{pr}_j V_{i-1,j-1} = i-1$ and $V_{i-1,j} = \mathrm{pr}_j V_{i-1,j-1}$. In addition, $V_{i,j-1} = V_{i-1,j} \oplus \mathbb{C}w_j$. This implies $V_{i,j-1} = V_{i-1,j} \oplus \mathbb{C}w_j$, which means $\mathbf{V} \in Z_{i,j-1}$, which contradicts the assumption $\mathbf{V} \in \mathrm{Sp}\,\mathrm{R}^{\circ}_{2n}$. Second, let $j > 2n-i$. Then $\dim \mathrm{pr}_{i+1} \cdots \mathrm{pr}_{2n-i} V_{i,i} = i$ and hence $\mathrm{pr}_{i+1} \cdots \mathrm{pr}_{2n-i} V_{i,i} = V_{i,2n-i}$. The choice of i as above implies that $w_j \in V_{i,2n-i}$ and $p_{1,\dots,i-1} V_{i-1,2n-i} \neq 0$. But $p_{1,\dots,i-1} V_{i-1,2n-i} \neq 0$ implies that there exists a vector $u \in V_{i-1,2n-i}$ such that w_{2n-j+1} appears in u with a non-zero coefficient (recall $j > 2n-i$) and hence u is not orthogonal to w_j. Since $u \in V_{i-1,2n-i} \subset V_{i,2n-i}$ and $w_j \in V_{i,2n-i}$ we obtain a contradiction with the statement that $V_{i,2n-i}$ is isotropic subspace in $W_{i,2n-i}$. ■

We have a natural map $\pi_{2n}\colon \mathrm{Sp}\,\mathrm{R}_{2n} \to \mathrm{Sp}\,\mathcal{F}^a_{2n}$

$$\pi_{2n}(V_{i,j})_{i,j} = (V_{i,i})_{i=1}^n$$

forgetting the off-diagonal elements.

Lemma 5.5 *The map π_{2n} is a birational isomorphism.*

As an immediate consequence one sees the following.

Corollary 5.6 *The variety $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ is irreducible.*

Proof Let $(\mathrm{Sp}\,\mathcal{F}^a_{2n})^{\circ}$ be the open (and non-empty) subset of collections $\mathbf{V} = (V_i)_{i=1}^n \in \mathrm{Sp}\,\mathcal{F}^a_{2n}$ such that $p_{1,\dots,i} V_i \neq 0$ for all $i = 1, \dots, n$. Let $\mathbb{B} = \{b_1, \dots, b_i\}$ be a basis for V_i. Since the projections pr_j for $j > i$ do not change the entries of a vector in the first i rows, $\mathrm{pr}_j \cdots p_{i+1}(\mathbb{B})$ is still a set of linearly independent vectors. Set $\mathbf{U} = (U_{i,j})_{1 \le i \le n,\, i \le j \le 2n-i}$, where $U_{i,i} = V_i$ and $U_{i,j} = p_{i+1} \cdots p_j(V_i) \subset W_{i,j}$. Then $\mathbf{U} \in \mathrm{Sp}\,\mathrm{R}^{\circ}_{2n}$. Similarly: if $\mathbf{U} = (U_{i,j})_{1 \le i \le j \le n} \in \mathrm{Sp}\,\mathrm{R}^{\circ}_{2n}$, then $\pi_{2n}(\mathbf{U}) \in (\mathrm{Sp}\,\mathcal{F}^a_{2n})^{\circ}$ and $U_{i,j} = p_{i+1} \cdots p_j(U_{i,i})$. It follows that $\pi_{2n}\colon \mathrm{Sp}\,\mathrm{R}^{\circ}_{2n} \to (\mathrm{Sp}\,\mathcal{F}^a_{2n})^{\circ}$ is an isomorphism.

The irreducibility of $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ is now equivalent to the surjectivity of π_{2n}. Given $(V_i)_{i=1}^n \in \mathrm{Sp}\,\mathcal{F}^a_{2n}$, we need to construct $\mathbf{V} \in \mathrm{Sp}\,\mathrm{R}_{2n}$ such that $\pi_{2n}\mathbf{V} = (V_i)_{i=1}^n$. We construct the entries $V_{i,j}$ of $\mathbf{V}$ by increasing induction on j. In the course of the construction, we always check that $\mathrm{pr}_{j+1} \cdots \mathrm{pr}_{2n-i} V_{i,j}$ is isotropic (this is a necessary condition for $\mathbf{V} \in \mathrm{Sp}\,\mathrm{R}_{2n}$). For $j = 1$ we have the only space $V_{1,1}$, which has to be equal to V_1.

Now assume $j \le n$. We need to define $V_{i,j}$ with $i = 1, \dots, j$. We do this by decreasing induction on i. For $i = j$ we have $V_{j,j} = V_j$. If $V_{i+1,j}$ is already defined, then the conditions for $V_{i,j}$ are as follows:

$$\mathrm{pr}_j V_{i,j-1} \subset V_{i,j} \subset W_{i,j} \cap V_{i+1,j}, \quad \mathrm{pr}_{j+1} \cdots \mathrm{pr}_{2n-i} V_{i,j} \text{ is isotropic.}$$

If $\dim \mathrm{pr}_j V_{i,j-1} = i$, then we have $V_{i,j} = \mathrm{pr}_j V_{i,j-1}$. Assume that the dimension drops, *i.e.*, $w_j \in V_{i,j-1}$. Then, by induction $\mathrm{pr}_j V_{i,j-1} \subset W_{i,j} \cap V_{i+1,j}$. In addition $\dim(W_{i,j} \cap V_{i+1,j}) \ge i$. Therefore there exists an i-dimensional subspace in between

$\mathrm{pr}_j V_{i,j-1}$ and $W_{i,j} \cap V_{i+1,j}$. Now since $\mathrm{pr}_{j+1} \cdots \mathrm{pr}_{2n-i-1} V_{i+1,j}$ is isotropic and $V_{i,j} \subset W_{i,j} \cap V_{i+1,j}$, we obtain that $\mathrm{pr}_{j+1} \cdots \mathrm{pr}_{2n-i} V_{i,j}$ is also isotropic.

Finally, assume $j > n$. We need to define $V_{i,j}$ with $i = 1, \ldots, 2n - j$. Again, we do this by decreasing induction on i. Let us start with $i = 2n - j$. Then the conditions we have are

$$\mathrm{pr}_{j+1} V_{2n-j,j-1} \subset V_{2n-j,j} \subset W_{2n-j,j}, \quad V_{2n-j,j} \text{ is isotropic.}$$

If $\dim \mathrm{pr}_{j+1} V_{2n-j,j-1} = 2n - j$, then we are done, since $\mathrm{pr}_{j+1} V_{2n-j,j-1}$ is isotropic by induction. If $\dim \mathrm{pr}_{j+1} V_{2n-j,j-1} = 2n - j - 1$, then we need to show that $(\mathrm{pr}_{j+1} V_{2n-j,j-1})^{\perp} \cap W_{2n-j,j}$ is not zero. But the first space is $(2n-j-1)$-dimensional and the second one is $2(2n - j)$-dimensional and the restriction of the symplectic form to $W_{2n-j,j}$ is non-degenerate. Now assume that we have fixed $V_{i+1,j}$. Using the same arguments as above one can show that there exists $V_{i,j}$ satisfying

$$\mathrm{pr}_j V_{i,j-1} \subset V_{i,j} \subset V_{i+1,j} \cap W_{i,j}, \quad \mathrm{pr}_{j+1} \cdots \mathrm{pr}_{2n-i} V_{i,j} \text{ is isotropic.} \qquad \blacksquare$$

We now establish a connection between the SL_{2n} resolution R_{2n} and its symplectic analogue $\mathrm{Sp}\,\mathrm{R}_{2n}$. Define an involution $\sigma\colon R_{2n} \to R_{2n}$ by the following formula: let $\mathbf{V} = (V_{i,j})_{1 \le i \le j < 2n}$ be a point in R_{2n}. Then

$$(\sigma \mathbf{V})_{i,j} = V^{\perp}_{2n-j,2n-i} \cap W_{i,j}. \tag{5.5}$$

Proposition 5.7 *Formula (5.5) defines an order two automorphism of R_{2n}. The variety of σ fixed points R^{σ}_{2n} is isomorphic to $\mathrm{Sp}\,\mathrm{R}_{2n}$. For the resolution map $\pi_{2n}\colon \mathrm{Sp}\,\mathrm{R}_{2n} \to \mathrm{Sp}\,\mathcal{F}^a_{2n}$ one has $\pi_{2n}\sigma = \sigma\pi_{2n}$ (*i.e.*, π_{2n} is σ-equivariant).*

Proof First, we need to show that $\dim(V^{\perp}_{2n-j,2n-i} \cap W_{i,j}) = i$. We consider the case $i + j > 2n$ (the opposite case is very similar). We note that $\dim V_{2n-j,2n-i} = 2n - j$ and thus $\dim V^{\perp}_{i,j} = j$. Since $V_{2n-j,2n-i} \subset W_{2n-j,2n-i}$ we have

$$V^{\perp}_{2n-j,2n-i} \supset W^{\perp}_{2n-j,2n-i} \supset \{w_{i+1}, \ldots, w_j\}.$$

Since $W_{i,j} = \mathrm{span}(w_{i+1}, \ldots, w_j)$, we arrive at $\dim(V^{\perp}_{2n-j,2n-i} \cap W_{i,j}) = j - (j-i) = i$.

Second, we need to prove that all other conditions from the definition of R_{2n} are satisfied. This is a direct verification. $\blacksquare$

Recall that for a resolution of singularities $\pi\colon Y \to X$ an irreducible divisor $Z \subset Y$ is called exceptional if $\dim \pi(Z) < \dim Z$.

Proposition 5.8 *A divisor $Z_{i,j}$ is exceptional if and only if $j \ge n$ and $i + j < 2n$.*

Proof Fix a pair $i < j$ such that $j \ge n$ and $i + j < 2n$. We prove that the divisor $Z_{i,j}$ is exceptional. Fix a point $(V_i)^n_{i=1} \in \mathrm{Sp}\,\mathcal{F}^a_{2n}$ and a point $\mathbf{V} \in Z_{i,j}$ such that $\pi_{2n}\mathbf{V} = (V_i)^n_{i=1}$, *i.e.*, $V_{i,i} = V_i$. We prove that the preimage $\pi^{-1}_{2n}(V_i)^n_{i=1}$ is at least one-dimensional. Let us construct a one-dimensional family of elements $\mathbf{U} \in Z_{i,j}$ such that $\pi_{2n}\mathbf{U} = \pi_{2n}\mathbf{V}$. We set $U_{k,l} = V_{k,l}$ if $l \le j$. Now (as in the proof of

Lemma 5.5) let us try to extend the already fixed components of $\mathbf{U}$ to some element of $Z_{i,j}$. We start with $U_{2n-j-1,j+1}$. In order to guarantee that $\mathbf{U} \in \operatorname{Sp} \mathrm{R}_{2n}$, we need the following conditions:

$$\text{(5.6)} \qquad \operatorname{pr}_{j+1} U_{2n-j-1,j} \subset U_{2n-j-1,j+1} \subset W_{2n-j-1,j+1}, \quad U_{2n-j-1,j+1} \text{ is isotropic.}$$

Since $V_{i,j} \supset w_{j+1}$ ($\mathbf{V} \in Z_{i,j}$) we have $\dim \operatorname{pr}_{j+1} U_{2n-j-1,j} = 2n - j - 2$. The restriction of the symplectic form to $W_{2n-j-1,j+1}$ is non-degenerate, thus the set of solutions of (5.6) is isomorphic to $\mathbb{P}^1$. Now, using the same arguments as in Lemma 5.5, we define successively $U_{2n-j-2,j+1}, \dots, U_{i,j+1}$. Let us now try to define $U_{i-1,j+1}$. We want $U_{i-1,j+1}$ to be contained in the (already defined) $U_{i,j+1}$. We note that $U_{i,j} = V_{i,j} = V_{i-1,j+1} \oplus \mathbb{C}w_{j+1}$ and $\operatorname{pr}_{j+1} U_{i,j} \subset U_{i,j+1}$ (by induction). Therefore, we can set $U_{i-1,j+1} = V_{i-1,j+1}$, which leads to $U_{i,j} = V_{i,j} = U_{i-1,j+1} \oplus \mathbb{C}w_{j+1}$. We note that since we want $\mathbf{U} \in Z_{i,j}$ such a relation has to hold. Now as in the proof of Lemma 5.5, we can define the remaining $U_{k,l}$ (by increasing induction on l and decreasing induction on k) in such a way that $\mathbf{U} \in \operatorname{Sp} \mathrm{R}_{2n}$. Moreover, by the construction, $\mathbf{U} \in Z_{i,j}$.

Now let us fix $j < n$. We prove that the divisor $Z_{i,j}$ with $i \le j$ is non-exceptional. More precisely, we prove that π_{2n} restricted to $Z^{\circ}_{i,j}$ (see (5.4)) is one-to-one. So let $\mathbf{V} \in Z^{\circ}_{i,j}$.

First, we claim that if $w_{l+1} \in V_{k,l}$ then $l = j$ and $i \le k \le j$. In fact, assume that for a given l the number k is minimal with the property $w_{l+1} \in V_{k,l}$. Then $k \neq 1$ (unless $i = 1$) since $w_{l+1} \in V_{1,l}$ means $\mathbf{V} \in Z_{1,l}$. We know that $w_{l+1} \notin V_{k-1,l}$. Hence $\operatorname{pr}_{l+1} V_{k-1,l} = V_{k-1.l+1}$ and $V_{k,l} = V_{k-1,l} \oplus \mathbb{C}w_{l+1}$. Therefore, $V_{k,l} = V_{k-1,l+1} \oplus \mathbb{C}w_{l+1}$, implying $\mathbf{V} \in Z_{k,l}$. Since $\mathbf{V} \in Z^{\circ}_{i,j}$, we arrive at $(k, l) = (i, j)$.

Second, we note that the above claim implies that for any pair (k, l) except for $l = j$, $i \le k \le j$ we have

$$\text{(5.7)} \qquad \dim \operatorname{pr}_{l+1} V_{k,l} = V_{k,l+1}.$$

In fact, for such pairs (k, l) the left and right hand sides have the same dimensions and the left hand side is contained in the right hand side. From equation (5.7) we obtain that if a point $(V_i)_{i=1}^n \in \operatorname{Sp} \mathcal{F}^a_{2n}$ is fixed, the spaces $V_{k,l}$ with $k < i$ or $k > j$ or $l \le j$ are fixed as well. Also, if one proves that the spaces $V_{k,j+1}$ are also uniquely fixed by the choice of $(V_i)_{i=1}^n$, this would imply that the whole collection $\mathbf{V} \in Z^{\circ}_{i,j}$ with the property $\pi_{2n}\mathbf{V} = (V_i)_{i=1}^n$ is unique.

Third, we claim that $\mathbf{V} \in Z^{\circ}_{i,j}$ implies that for $l > j$ one has $p_{1,\dots,k} V_{k,l} \neq 0$ (where $p_{1,\dots,l}$ is the corresponding Plücker coordinate). In effect, first $V_{1,2n-1}$ is a one-dimensional space, not equal to $\mathbb{C}w_{2n}$ (since $\mathbf{V} \in Z^{\circ}_{i,j}$). Therefore, $p_1(V_{1,2n-1}) \neq 0$. Now the statement can be verified directly by decreasing induction on l and increasing induction on k.

Finally, we are left to show that the values of V_i and of $V_{k,l}$ with $l \le j$ fix $V_{k,j+1}$. First, $V_{j+1,j+1} = V_{j+1}$ is fixed. Now, we use the decreasing induction on k. The space $V_{k,j+1}$ has to satisfy

$$V_{k,j+1} \subset V_{k+1,j+1} \cap W_{k,j+1}.$$

We claim that the right hand side is (at most) k-dimensional. In effect, we have $p_{1,\dots,k+1}(V_{k+1,j+1}) \neq 0$. Since $w_{k+1} \notin W_{k,j+1}$, we have $\dim(V_{k+1,j+1} \cap W_{k,j+1}) \leq k$. Since $\dim V_{k,j+1} = k$, the space $V_{k,j+1}$ is uniquely fixed.

The last statement of the proposition is that the divisors $Z_{i,2n-i}$ are non-exceptional. This can be proved along the same lines as the above case. ■

5.2 Parabolic Case

We define a desingularization $\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}$ of $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ in the following way. Let $\mathfrak{P}_{\mathbf{d}}$ be the subset of the set of positive roots for $\mathfrak{sp}_{2n}$ corresponding to the radical of the parabolic subalgebra defined by the roots $\alpha_{d_1}, \dots, \alpha_{d_k}$. Explicitly, $\alpha \in \mathfrak{P}_{\mathbf{d}}$ if and only if there exists an $i = 1, \dots, k$ such that $(\alpha, \omega_{d_i}) > 0$. We sometimes write $(i, j) \in \mathfrak{P}_{\mathbf{d}}$ instead of $\alpha_{i,j} \in \mathfrak{P}_{\mathbf{d}}$. Now, a point in $\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}$ is a collection of subspaces $V_{i,j} \subset W$, $(i, j) \in \mathfrak{P}_{\mathbf{d}}$ subject to the conditions

$$\dim V_{i,j} = i, \quad V_{i,j} \subset W_{i,j},$$
$$V_{i,j} \subset V_{i+1,j}, \quad \mathrm{pr}_{j+1} V_{i,j} \subset V_{i,j+1},$$
$$V_{i,2n-i} \text{ are isotropic for } i = 1, \dots, n.$$

We note that there is a natural embedding $\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}} \subset \prod_{(i,j)\in\mathfrak{P}_{\mathbf{d}}} \mathrm{Gr}_i(W_{i,j})$.

Proposition 5.9 $\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}$ *is a tower of successive* $\mathbb{P}^1$*-fibrations.*

Proof The proof is very similar to the proof of Proposition 5.2 and we omit it. ■

Proposition 5.10 *The morphism* $(V_{i,j})_{(i,j)\in\mathfrak{P}_{\mathbf{d}}} \mapsto (V_{d_i,d_i})_{i=1}^k$ *is a surjective morphism from* $\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}$ *to* $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$*, generically one-to-one.*

Proof The proof is very similar to the proof of Lemma 5.5 and we omit it. ■

Corollary 5.11 *The varieties* $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ *are irreducible.*

In what follows we denote the desingularization map $\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}} \to \mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ by $\pi_{\mathbf{d}}$. As in the case of the complete flag varieties, for any pair $(i, j) \in \mathfrak{P}_{\mathbf{d}}$, we define the corresponding divisor $Z_{i,j} \subset \mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}$.

Proposition 5.12 *The non-exceptional divisors are exactly the* $Z_{i,j}$ *with* (i, j) *from the following list:*

$$(1, d_i - 1),\ \ i = 2, \dots, k; \quad (d_i + 1, d_j - 1),\ \ i \leq j - 2, j \leq k;$$
$$(1, 2n - 1); \quad (d_i + 1, 2n - d_i - 1),\ \ i = 1, \dots, k - 1.$$

Proof The proof is similar to the proof of Proposition 5.8 and we omit it. ■

6 Normal Locally Complete Intersections

Our next goal is to prove that all symplectic degenerate flag varieties are normal locally complete intersections (and thus Cohen–Macaulay and Gorenstein). We start with the case of the complete flags.

6.1 Complete Flag Varieties

As in [FF], we first define an affine scheme Q_{2n} which we prove to be a complete intersection. Let $W_1, \dots, W_n$ be a collection of vector spaces with $\dim W_i = i$. Also recall the space W_{2n} with a basis $w_1, \dots, w_{2n}$, a non-degenerate symplectic form and the projections pr_k along w_k. We now construct an affine scheme Q_{2n} as follows. A point of Q_{2n} is a collection of linear maps

$$A_i \colon W_i \to W_{2n}, \quad i = 1, \dots, n,$$
$$B_j \colon W_j \to W_{j+1}, \quad j = 1, \dots, n-1$$

subject to the relations

$$A_{i+1}B_i = \mathrm{pr}_{i+1} A_i, \quad i = 1, \dots, n-1,$$

and such that the image $A_n(W_n)$ is isotropic. The following picture illustrates the construction:

$$\begin{tikzcd}
W_{2n} \arrow[r, "\mathrm{pr}_2"] & W_{2n} & \cdots\cdots & W_{2n} \arrow[r, "\mathrm{pr}_n"] & W_{2n} \\
W_1 \arrow[u, "A_1"] \arrow[r, "B_1"] & W_2 \arrow[u, "A_2"'] & \cdots\cdots & W_{n-1} \arrow[u, "A_{n-1}"'] \arrow[r, "B_{n-1}"] & W_n \arrow[u, "A_n"']
\end{tikzcd}$$

We also consider the open subscheme $Q^\circ_{2n} \subset Q_{2n}$ consisting of collections (A_i, B_j) such that $\ker A_i = 0$ for all i. The group $\Gamma = \prod_{i=1}^n \mathrm{GL}(W_i)$ acts freely on Q°_{2n} via change of bases. Consider the map

$$Q^\circ_{2n} \to \mathrm{Sp}\,\mathcal{F}^a_{2n}, \quad (A_i, B_j) \mapsto (\Im A_1, \dots, \Im A_n).$$

Lemma 6.1 *The morphism $Q^\circ_{2n} \to \mathrm{Sp}\,\mathcal{F}^a_{2n}$ is a locally trivial Γ-torsor in the Zariski topology. The variety Q°_{2n} is irreducible of dimension $n^2 + 1^2 + 2^2 + \dots + n^2$.*

Proof Consider the embedding $\mathrm{Sp}\,\mathcal{F}^a_{2n} \hookrightarrow \prod_{d=1}^n \mathrm{Sp}\,\mathrm{Gr}_d(2n)$. For a point $p \in \mathrm{Sp}\,\mathcal{F}^a_{2n}$ let $U \ni p$ be an open part of $\prod_{d=1}^n \mathrm{Sp}\,\mathrm{Gr}_d(2n)$ such that all tautological bundles on Grassmannians are trivial on U. Let $U' = U \cap \mathrm{Sp}\,\mathcal{F}^a_{2n}$. Then on U' the map $Q^\circ_{2n} \to \mathrm{Sp}\,\mathcal{F}^a_{2n}$ has a section. Now using the Γ action on Q_{2n} we obtain that $Q^\circ_{2n} \to \mathrm{Sp}\,\mathcal{F}^a_{2n}$ is a Γ-torsor. In particular, $\dim Q^\circ_{2n} = \dim Q_{2n} = \dim \mathrm{Sp}\,\mathcal{F}^a_{2n} + \dim \Gamma$. ■

We note that Q_{2n} is a closed subscheme of the affine space

$$\prod_{i=1}^n \mathrm{Hom}(W_i, W_{2n}) \times \prod_{i=1}^{n-1} \mathrm{Hom}(W_i, W_{i+1}).$$

Lemma 6.2 *Q°_{2n} is a locally complete intersection.*

Proof We note that the condition $A_{i+1}B_i = \mathrm{pr}_{i+1} A_i$ produces $2n \times i$ equations (the number of equations is equal to $\dim \mathrm{Hom}(W_i, W_{2n})$). Also the condition that $A_n(W_n)$ is Lagrangian produces another $n(n-1)/2$ equations. Now our lemma follows from the equality

$$\dim Q_{2n} = \sum_{i=1}^{n} 2ni + \sum_{i=1}^{n-1} i(i+1) - \sum_{i=1}^{n-1} 2ni - \frac{n(n-1)}{2}. \qquad \blacksquare$$

Theorem 6.3 *The degenerate flag varieties* $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ *are normal locally complete intersections (in particular, Cohen–Macaulay and even Gorenstein).*

Proof Since $Q^\circ_{2n} \to \mathrm{Sp}\,\mathcal{F}^a_{2n}$ is a torsor locally trivial in Zariski topology, $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ inherits the lci and normality properties from Q°_{2n}. So it suffices to prove that Q°_{2n} is a normal reduced scheme (*i.e.*, a variety). Since Q°_{2n} is a locally complete intersection, the property of being reduced (resp. normality) of Q°_{2n} follows from the fact that Q°_{2n} is smooth in codimension one, by the virtue of Proposition 5.8.5 (resp. Theorem 5.8.6) of [EGA]. Using again that $Q^\circ_{2n} \to \mathrm{Sp}\,\mathcal{F}^a_{2n}$ is a torsor, it suffices to prove that $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ is smooth in codimension one. We prove this statement in a separate lemma. $\blacksquare$

Lemma 6.4 $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ *is smooth in codimension one.*

Proof We use the desingularization $\pi_{2n}\colon \mathrm{Sp}\,\mathrm{R}_{2n} \to \mathrm{Sp}\,\mathcal{F}^a_{2n}$. Since $\mathrm{Sp}\,\mathrm{R}_{2n}$ is smooth, it suffices to show that the morphism π_{2n} is an isomorphism over an open subvariety $M \subset \mathrm{Sp}\,\mathcal{F}^a_{2n}$ such that the codimension of the complement of M is at least two. We set M to be the union of $\pi_{2n}\,\mathrm{Sp}\,\mathrm{R}^\circ_{2n}$ (the open cell in $\mathrm{Sp}\,\mathrm{R}_{2n}$) and $\pi_{2n} Z^\circ_{i,j}$ for all non-exceptional divisors $Z_{i,j}$. $\blacksquare$

The following theorem can be proved along the same lines as Theorem 6.3.

Theorem 6.5 *The degenerate flag varieties* $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ *are normal locally complete intersections (in particular, Cohen–Macaulay and Gorenstein).*

7 The Singularities of $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ are Rational

7.1 The Complete Flag Varieties

In this section we prove that the singularities of $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ are terminal (hence canonical and rational). We use the sections $s_{i,j}$ as above in order to compute the relative canonical line bundle $\omega_{\mathrm{Sp}\,\mathrm{R}_{2n}} \otimes \pi^*_{2n}\omega^{-1}_{\mathrm{Sp}\,\mathcal{F}^a_{2n}}$.

We use the notation $\Omega_{i,j}$ for the determinant of the i-dimensional bundle on $\mathrm{Sp}\,\mathrm{R}_{2n}$, whose fiber at a point $\mathbf{V}$ equals $\Lambda^i(V_{i,j})^*$. Also, we denote by Ω_i the line bundle on $\mathrm{Sp}\,\mathcal{F}^a_{2n}$, whose fiber at a point $(V_1,\ldots,V_n)$ is equal to $\Lambda^i(V^*_i)$.

Consider the following general situation. Let $\rho\colon E \to B$ be a $\mathbb{P}^1$-fibration with a section $s\colon B \to E$. Let $Z = s(B) \subset E$ be the corresponding divisor. Then for any line bundle $\mathcal{F}$ on E such that the restriction of $\mathcal{F}$ to a fiber of ρ is isomorphic to $\mathcal{O}(k)$ one has

$$\mathcal{F} = \mathcal{O}(kZ) \otimes \rho^*\big(\mathcal{F}|_Z \otimes \mathcal{O}(-kZ)|_Z\big). \tag{7.1}$$

We consider the 2-dimensional vector bundle $\mathcal{L}_2 := \rho_*\mathcal{O}_E(Z)$ on B, and its line subbundle $\mathcal{L}_1 := \rho_*\mathcal{O}_E = \mathcal{O}_B$. Then $E = \mathbb{P}(\mathcal{L}_2)$ and $Z = \mathbb{P}(\mathcal{L}_1)$. Let Ω be a line bundle such that the restriction of Ω to a fiber of ρ is isomorphic to $\mathcal{O}(1)$. The following lemma is nothing but [Ra, Lemma 3].

Lemma 7.1 $\omega_E^{-1} = \mathcal{O}(Z) \otimes \Omega \otimes \rho^*(\Omega^{-1}|_Z \otimes \omega_B^{-1})$.

Proof First, (7.1) with $\mathcal{F} = \Omega$ gives

$$\Omega = \mathcal{O}(Z) \otimes \rho^*\left(\Omega|_Z \otimes \mathcal{O}(-Z)|_Z\right). \tag{7.2}$$

Second, (7.1) with $\mathcal{F} = \omega_E^{-1}$ gives

$$\omega_E^{-1} = \mathcal{O}(2Z) \otimes \rho^*\left(\omega_E^{-1}|_Z \otimes \mathcal{O}(-2Z)|_Z\right). \tag{7.3}$$

We note that

$$\rho^*\left(\Omega^{-1}|_Z \otimes \mathcal{O}(-2Z)|_Z\right) = \rho^*\left(\omega_B^{-1}|_Z \otimes \mathcal{O}(-Z)|_Z\right).$$

Therefore, combining (7.2) and (7.3), we arrive at

$$\omega_E^{-1} = \mathcal{O}(Z) \otimes \Omega \otimes \rho^*(\Omega^{-1}|_Z \otimes \omega_B^{-1}). \qquad \blacksquare$$

We now consider the $\mathbb{P}^1$-fibration $\rho_l\colon \operatorname{Sp}\mathrm{R}_{2n}(l) \to \operatorname{Sp}\mathrm{R}_{2n}(l-1)$ and the corresponding divisors $Z_{i,j}$.

Lemma 7.2

$$\mathcal{O}(Z_{i,j}) = \begin{cases} \Omega_{1,j} \otimes \Omega^*_{1,j+1}, & \text{if } i = 1, \\ \Omega_{i,j} \otimes \Omega^*_{i-1,j} \otimes \Omega^*_{i,j+1} \otimes \Omega_{i-1.j+1}, & \text{if } i > 1 \text{ and } i+j < 2n, \\ \Omega_{i,j} \otimes (\Omega^*_{i-1,j})^{\otimes 2} \otimes \Omega_{i-1.j+1}, & \text{if } i > 1 \text{ and } i+j = 2n. \end{cases}$$

Proof The first two cases are worked out in [FF]. The only new case is the last one $i + j = 2n$. Consider the space $V_{i,2n-i}$. We know that

$$V_{i-1,2n-i} \subset V_{i,2n-i} \subset V^{\perp}_{i-1,2n-i} \cap W_{i,2n-i}.$$

Let $\Omega^{\perp}_{i,j}$ be the line bundle on $\operatorname{Sp}\mathrm{R}_{2n}$, whose fiber at a point $\mathbf{V}$ is equal to the top wedge power of $(V^{\perp}_{i,j})^*$. Then one shows that $\det \Omega^{\perp}_{i,j} = \det \Omega^*_{i,j}$. Therefore, using (7.1) we arrive at

$$\Omega_{i,2n-i} = \mathcal{O}(Z_{i,2n-i}) \otimes \Omega^*_{i-1,2n-i+1} \otimes \Omega^{\otimes 2}_{i-1,2n-1}. \qquad \blacksquare$$

Theorem 7.3 $\omega_{\operatorname{Sp}\mathrm{R}_{2n}} = \bigotimes_{i=1}^n (\Omega^*_{i,i})^{\otimes 2} \otimes \prod_{i+j<2n, j\geq n} \mathcal{O}(Z_{i,j})$.

Proof From Lemma 7.1 we obtain

$$\omega^{-1}_{\mathrm{Sp}\,\mathrm{R}_{2n}} = \bigotimes_{i,j} \mathcal{O}(Z_{i,j}) \otimes \bigotimes_{i,j} \Omega_{i,j} \otimes \bigotimes_{j-i>2} \Omega^*_{i,j}.$$

Now using Lemma 7.2 we are left to show that

$$\bigotimes_{i,j} \mathcal{O}(Z_{i,j}) \otimes \bigotimes_{i=1}^{n} \Omega_{i,i} \otimes \bigotimes_{i=1}^{n-1} \Omega_{i,i+1} = \bigotimes_{i=1}^{n} \Omega_{i,i}^{\otimes 2} \otimes \bigotimes_{j\geq n, i+j<2n} \mathcal{O}(-Z_{i,j}).$$

This equality follows from Lemma 7.2 again by writing both sides in terms of $\Omega_{i,j}$ only. ■

Corollary 7.4 $\omega_{\mathrm{Sp}\,\mathcal{F}^a_{2n}} = \bigotimes_{i=1}^{n} (\Omega_i^*)^{\otimes 2}$.

Proof Recall the open cell $\mathrm{Sp}\,\mathrm{R}^\circ_{2n} \subset \mathrm{Sp}\,\mathrm{R}_{2n}$ with the property that the restriction of π_{2n} to $\mathrm{Sp}\,\mathrm{R}^\circ_{2n}$ is one-to-one. Consider the open subvariety $M \subset \mathrm{Sp}\,\mathrm{R}_{2n}$ defined by

$$M = \mathrm{Sp}\,\mathrm{R}^\circ_{2n} \cup \bigcup_{j<n} Z^\circ_{i,j} \cup \bigcup_{i+j=2n} Z^\circ_{i,j}.$$

Then the restriction of π_{2n} to $\mathrm{Sp}\,\mathrm{R}_{2n} \backslash M$ is one-to-one and $\pi_{2n}(\mathrm{Sp}\,\mathrm{R}_{2n} \backslash M)$ has codimension at least two in $\mathrm{Sp}\,\mathcal{F}^a_{2n}$. Because of Theorem 7.3, the canonical line bundle ω on M is equal to $\bigotimes_{i=1}^{n} (\Omega^*_{i,i})^{\otimes 2}$. Since we also have $\pi^*_{2n}\omega_{\mathrm{Sp}\,\mathcal{F}^a_{2n}} = \omega_{\mathrm{Sp}\,\mathrm{R}_{2n}}$ on the image of M, we obtain $\omega_{\mathrm{Sp}\,\mathcal{F}^a_{2n}} = \bigotimes_{i=1}^{n} (\Omega^*_i)^{\otimes 2}$. ■

Theorem 7.5 *The singularities of* $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ *are terminal and hence rational.*

Proof Using Corollary 7.4 and Theorem 7.3 we obtain

$$\omega_{\mathrm{Sp}\,\mathrm{R}_{2n}} = \pi^*_{2n}\omega_{\mathrm{Sp}\,\mathcal{F}^a_{2n}} \otimes \prod_{i+j<2n, j\geq n} \mathcal{O}(Z_{i,j}).$$

Thus the singularities are terminal (and hence canonical) by Theorem 7.3 (all exceptional divisors appear in the discrepancy with coefficient one). Now rationality follows from [E, Theorem 1] (see Theorem 2.4), since the varieties $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ are normal locally complete intersections and so Gorenstein. ■

7.2 Rationality for Parabolic Flag Varieties

In this subsection we work out the case of partial degenerate flag varieties. Our goal is to compute $\omega_{\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}} \otimes \pi^*_{\mathbf{d}}\omega^{-1}_{\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}}$ (note that $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ is singular but Gorenstein by Theorem 6.5, thus its dualizing complex is a line bundle). We will prove that the expression for $\omega_{\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}}$ coincides with the one in the classical situation. So let $\mathrm{Sp}\,\mathcal{F}_{\mathbf{d}}$ be the classical parabolic flag variety.

Lemma 7.6 $\omega^{-1}_{\mathrm{Sp}\,\mathcal{F}_{\mathbf{d}}} = \Omega_{d_1}^{\otimes d_2} \otimes \prod_{i=2}^{k-1} \Omega_{d_i}^{\otimes(d_{i+1}-d_{i-1})} \otimes \Omega_{d_k}^{\otimes(2n+1-d_k-d_{k-1})}$.

Proof Since the canonical line bundle is the top wedge power of the cotangent bundle, its weight is given by $-2\rho_{Sp_{2n}} + 2\rho_{L_{\mathbf{d}}}$, where $2\rho_{Sp_{2n}}$ is the sum of all positive roots of $\mathfrak{sp}_{2n}$ and $2\rho_{L_{\mathbf{d}}}$ is the sum of all positive roots of the Levi subalgebra corresponding to $\mathbf{d}$. Now a direct computation gives the desired formula. ■

Now our strategy is as follows. We define a line bundle $\tilde{\omega}$ on $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ by the formula

$$\tilde{\omega}^{-1} = \Omega_{d_1}^{\otimes d_2} \otimes \prod_{i=2}^{k-1} \Omega_{d_i}^{\otimes(d_{i+1}-d_{i-1})} \otimes \Omega_{d_k}^{\otimes(2n+1-d_k-d_{k-1})}.$$

We will show that the discrepancy $\omega_{\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}} \otimes \pi^*_{\mathbf{d}}(\tilde{\omega}^{-1})$ is given by a product of exceptional divisors with positive coefficients. Since the images of the exceptional divisors have codimension at least two, this will prove that in the degenerate case the expression for the canonical class coincides with the one in the classical situation.

We start with the following lemma.

Lemma 7.7 *We have the equality*

$$\omega^{-1}_{\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}} = \bigotimes_{(i,j)\in\mathfrak{P}_{\mathbf{d}}} \mathcal{O}(Z_{i,j}) \otimes \bigotimes_{(i,j)\in B_{\mathbf{d}}} \Omega_{i,j},$$

where $B_{\mathbf{d}} \subset \mathfrak{P}_{\mathbf{d}}$ is the subset of pairs from the following list:

$$\begin{gathered}(1,d_1),(2,d_1),\dots,(d_1,d_1),\\ (d_1,d_1+1),(d_1,d_1+2),\dots,(d_1,d_2),\\ (d_1+1,d_2),(d_1+2,d_2),\dots,(d_2,d_2),\\ \vdots\\ (d_{k-1}+1,d_k),(d_{k-1}+2,d_k),\dots,(d_k,d_k),\\ (d_k,d_k+1),(d_k,d_k+2),\dots,(d_k,2n-d_k).\end{gathered}$$

Proof This proof follows from Lemma 7.1. In fact the sections $s_{i,j}$ are constructed in such a way that the space $V_{i,j}$ at a point of the section is given by $V_{i-1,j+1}$ plus some constant vector. The set $B_{\mathbf{d}}$ is exactly the set of pairs (i_0, j_0) which cannot be witten as $i_0 = i-1$, $j_0 = j+1$ for some $(i,j) \in \mathfrak{P}_{\mathbf{d}}$. ■

We now prove the main theorem of this section.

Theorem 7.8 *There exist non-negative integers $a_{i,j}$, $(i,j) \in \mathfrak{P}_{\mathbf{d}}$ such that*

$$\omega_{\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}} \otimes \pi^*_{\mathbf{d}}(\tilde{\omega}^{-1}) = \bigotimes_{(i,j)\in\mathfrak{P}_{\mathbf{d}}} \mathcal{O}(a_{i,j}Z_{i,j}).$$

In addition, $a_{i,j} = 0$ if and only if $Z_{i,j}$ is not exceptional.

Proof We use the formula from Lemma 7.2:

$$(7.4)\qquad \mathcal{O}(Z_{i,j}) = \begin{cases} \Omega_{1,j} \otimes \Omega^*_{1,j+1}, & \text{if } i = 1, \\ \Omega_{i,j} \otimes \Omega^*_{i-1,j} \otimes \Omega^*_{i,j+1} \otimes \Omega_{i-1.j+1}, & \text{if } i > 1 \text{ and } i + j < 2n, \\ \Omega_{i,j} \otimes (\Omega^*_{i-1,j})^{\otimes 2} \otimes \Omega_{i-1.j+1}, & \text{if } i > 1 \text{ and } i + j = 2n. \end{cases}$$

Using Lemma 7.7 we obtain

$$\omega_{\mathrm{Sp\,R}_{\mathbf{d}}} \otimes \pi^*_{\mathbf{d}}(\tilde{\omega}^{-1}) = \bigotimes_{(i,j)\in\mathfrak{P}_{\mathbf{d}}} \mathcal{O}(-Z_{i,j}) \otimes \bigotimes_{(i,j)\in B_{\mathbf{d}}} \Omega^{-1}_{i,j} \otimes \pi^*_{\mathbf{d}}(\tilde{\omega}^{-1}).$$

Our goal is to find strictly positive integers $b_{i,j} = a_{i,j} + 1$ such that

$$(7.5)\quad \bigotimes_{(i,j)\in B_{\mathbf{d}}} \Omega^{-1}_{i,j} \otimes \Omega^{\otimes d_2}_{d_1} \otimes \prod_{i=2}^{k-1} \Omega^{\otimes(d_{i+1}-d_{i-1})}_{d_i} \otimes \Omega^{\otimes(2n+1-d_k-d_{k-1})}_{d_k} = \bigotimes_{(i,j)\in\mathfrak{P}_{\mathbf{d}}} \mathcal{O}(b_{i,j} Z_{i,j}).$$

In addition, we want $b_{i,j} = 1$ if and only if $Z_{i,j}$ is non-exceptional.

First, formula (7.5) implies $b_{d_1,d_1} = d_2 - 1$, $b_{d_i,d_i} = d_{i+1} - d_{i-1} - 1$ for $i = 2, \dots, k-1$ and $b_{d_k,d_k} = 2n - d_k - d_{k-1}$. Now formula (7.4) gives a unique way to determine other $b_{i,j}$ in the following order:

$$b_{d_1,d_1}, b_{d_1-1,d_1}, \dots, b_{1,d_1},$$
$$b_{d_1,d_1+1}, b_{d_1-1,d_1+1}, \dots, b_{1,d_1+1},$$
$$\dots$$
$$b_{d_1,d_2-1}, b_{d_1-1,d_2-1}, \dots, b_{1,d_2-1},$$
$$b_{d_2,d_2}, b_{d_2-1,d_2}, \dots, b_{1,d_2},$$
$$\dots$$
$$b_{d_2,d_3-1}, b_{d_2-1,d_3-1}, \dots, b_{1,d_3-1},$$
$$\vdots$$
$$b_{d_k,d_k}, b_{d_k-1,d_k}, \dots, b_{1,d_k},$$
$$b_{d_k,d_k+1}, b_{d_k-1,d_k+1}, \dots, b_{1,d_k+1},$$
$$\dots$$
$$b_{d_k,2n-d_k}, b_{d_k-1,2n-d_k}, \dots, b_{1,2n-d_k},$$
$$b_{d_k-1,2n-d_k+1}, b_{d_k-2,2n-d_k+1}, \dots, b_{1,2n-d_k+1},$$
$$\dots$$
$$b_{2,2n-2}, b_{1,2n-2},$$
$$b_{1,2n-1}.$$

We now write down the values of the solutions $b_{i,j}$ (we assume $d_0 = 0$):
Let $d_1 \le j \le d_2 - 1$ and $1 \le i \le d_1$. Then

$$b_{i,j} = d_2 - j + i - 1.$$

Let $d_{s-1} \le j \le d_s - 1$ and $d_l + 1 \le i \le d_{l+1}$ for $0 \le l < s < k$. Then

$$b_{i,j} = d_s - d_l - j + i - 1.$$

Let $d_k \le j < 2n - d_k$ and $d_l + 1 \le i \le d_{l+1}$ for $0 \le l < k$. Then

$$b_{i,j} = 2n - d_k - d_l - j + i.$$

Let $2n - d_s \le j < 2n - d_{s-1} - 1$ and $i = 2n - j$ for $1 \le s \le k$. Then

$$b_{i,j} = 2n - j - d_{s-1}.$$

Let $2n - d_s \le j \le 2n - d_{s-1} - 1$ and $d_s - 1 \ge i \ge d_{s-1} + 1$ for $1 \le s \le k$. Then

$$b_{i,j} = 2n - 2d_{s-1} - j + i.$$

Let $2n - d_s \le j \le 2n - d_{s-1} - 1$ and $d_l \ge i \ge d_{l-1} + 1$ for $1 \le s \le k$, $s - 1 \ge l \ge 1$. Then

$$b_{i,j} = 2n - d_{s-1} - d_{l-1} - j + i.$$

In particular, one sees that $b_{i,j} \ge 1$ and $b_{i,j} = 1$ if and only if (i, j) is from the following list

$$(1, d_i - 1),\ \ i = 2, \dots, k;\quad (d_i + 1, d_j - 1),\ \ i \le j - 2, j \le k;$$
$$(1, 2n - 1);\quad (d_i + 1, 2n - d_i - 1),\ \ i = 1, \dots, k - 1.$$

Now Proposition 5.12 gives the desired result. ■

Corollary 7.9 $\omega_{\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}} = \tilde{\omega}$.

Proof Similar to the proof of Corollary 7.4. ■

Corollary 7.10 $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ *has terminal and thus canonical and rational singularities.*

8 Frobenius Splitting, the BW-type Theorem and Graded Character Formula

In this section we derive several applications of the results in the previous sections.

8.1 Frobenius Splitting

In the following let $\mathbf{k}$ be an arbitrary field. On $W = \mathbf{k}^{2n}$ let $\langle \cdot, \cdot \rangle$ be the same form as defined in Section 3, and let $\mathrm{Sp}_{2n} \subseteq \mathrm{GL}_{2n}(\mathbf{k})$ be the subgroup of automorphisms leaving the form invariant. Using Proposition 4.1 and Theorem 4.8 as a definition, the degenerate symplectic flag varieties make sense in every characteristic. To get the connection with a PBW-filtration in positive characteristic, for a dominant weight let $V_{\mathbf{k}}(\lambda)$ denote the Weyl module of highest weight λ obtained from the lattice $U_{\mathbb{Z}}(\mathfrak{n}^-).v_\lambda = V_{\mathbb{Z}}(\lambda) \subset V_{\mathbb{Q}}(\lambda)$ via reduction mod p. For the construction of the PBW-filtration we replace the enveloping algebra by the algebra of distributions or hyperalgebra [J], the varieties $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ and $\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}$ can then defined in the same way.

Theorem 8.1 *The varieties* $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ *and* $\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}$ *over* $\overline{\mathbb{F}}_p$ *are Frobenius split for all primes* p.

Proof Lemma 7.7 says that

$$\omega^{-1}_{\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}} = \bigotimes_{(i,j)\in\mathfrak{P}_{\mathbf{d}}} \mathcal{O}(Z_{i,j}) \otimes \bigotimes_{(i,j)\in B_{\mathbf{d}}} \Omega_{i,j}.$$

Now since $\bigotimes_{(i,j)\in B_{\mathbf{d}}} \Omega_{i,j}$ is base point free, the Frobenius splitting for $\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}$ and $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ follows from the Mehta–Ramanathan criterion Theorem 2.3 [MR, Proposition 8]. ∎

8.2 The BW-type Theorem

Let $\mathcal{L}_\lambda$ be a line bundle on $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ which is the pullback of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(V^a_\lambda)$ for a dominant $\mathfrak{sp}_{2n}$-weight λ. We prove an analogue of the Borel–Weil theorem.

Theorem 8.2 *We have*

$$H^0(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \mathcal{L}_\lambda)^* \simeq H^0(\mathrm{Sp}\,\mathrm{R}_{2n}, \pi^*_{2n}\mathcal{L}_\lambda)^* \simeq V^a_\lambda,$$
$$H^{>0}(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \mathcal{L}_\lambda) = H^{>0}(\mathrm{Sp}\,\mathrm{R}_{2n}, \pi^*_{2n}\mathcal{L}_\lambda) = 0.$$

Proof First, we note that since $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ has rational singularities, we have the equalities

$$H^k(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \mathcal{L}_\lambda) \simeq H^k(\mathrm{Sp}\,\mathrm{R}_{2n}, \pi^*_{2n}\mathcal{L}_\lambda)$$

for all $k \geq 0$.

Second, we prove that all non-zero cohomologies $H^k(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \mathcal{L}_\lambda)$ vanish. In fact, first assume λ is regular. Then since the map $\mathrm{Sp}\,\mathcal{F}^a_{2n} \to \mathbb{P}(V^a_\lambda)$ is an embedding, the line bundle $\mathcal{L}_\lambda$ is very ample. Therefore, for any k and big enough N one has $H^k(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \mathcal{L}^{\otimes N}_\lambda) = 0$. This implies $H^k(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \mathcal{L}_\lambda) = 0$, because $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ is Frobenius split over $\overline{\mathbb{F}}_p$ for any p.

Now consider a non-regular λ. Let $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ be the corresponding degenerate parabolic flag variety, which is embedded into $\mathbb{P}(V^a_\lambda)$. Then we have the following commutative diagram of projections:

$$\begin{tikzcd}
\mathrm{Sp}\,\mathrm{R}_{2n} \arrow[r, "\eta"] \arrow[d, "\pi_{2n}"'] & \mathrm{Sp}\,\mathrm{R}_{\mathbf{d}} \arrow[d, "\pi_{\mathbf{d}}"] \\
\mathrm{Sp}\,\mathcal{F}^a_{2n} \arrow[r, "\phi"] & \mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}
\end{tikzcd}$$

Let $\mathcal{L}'_\lambda$ be the line bundle on $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ which is the pullback of the bundle $\mathcal{O}(1)$ on $\mathbb{P}(V^a_\lambda)$. Then $\mathcal{L}_\lambda = \phi^*\mathcal{L}'_\lambda$. Since $\mathcal{L}'_\lambda$ is very ample, and $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ is Frobenius split over $\overline{\mathbb{F}}_p$ for any p, $H^k(\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}, \mathcal{L}'_\lambda) = 0$ (for positive k). Since $\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}$ has rational singularities, $H^k(\mathrm{Sp}\,\mathrm{R}_{\mathbf{d}}, \pi^*_{\mathbf{d}}\mathcal{L}'_\lambda) = H^k(\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}, \mathcal{L}'_\lambda)$ ($= 0$ for positive k). Now since η is a fibration with the fibers being towers of successive $\mathbb{P}^1$-fibrations (Lemma 5.2), the higher direct images $R^{>0}\eta_*\mathcal{O}_{\mathrm{Sp}\,\mathrm{R}_{2n}}$ are equal to 0 and we obtain $H^k(\mathrm{Sp}\,\mathrm{R}_{2n}, \eta^*\pi^*_{\mathbf{d}}\mathcal{L}'_\lambda) = H^k(\mathrm{Sp}_{\mathbf{d}}, \pi^*_{\mathbf{d}}\mathcal{L}'_\lambda)$ ($= 0$ for positive k). Finally, since $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ has rational singularities, and $\eta^*\pi^*_{\mathbf{d}}\mathcal{L}'_\lambda = \pi^*_{2n}\mathcal{L}_\lambda$, we arrive at

$$H^k(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \mathcal{L}_\lambda) = H^k(\mathrm{Sp}\,\mathrm{R}_{2n}, \pi^*_{2n}\mathcal{L}_\lambda) = H^k(\mathrm{Sp}\,\mathrm{R}_{2n}, \eta^*\pi^*_{\mathbf{d}}\mathcal{L}'{}_\lambda),$$

which vanishes for $k > 0$.

Third, we note that there exists an embedding $(V^a_\lambda)^* \hookrightarrow H^0(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \mathcal{L}_\lambda)$. In fact take an element $v \in (V^a_\lambda)^* \simeq H^0\big(\mathbb{P}(V^a_\lambda), \mathcal{O}(1)\big)$. Then restricting to the embedded variety $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ we obtain a section of $\mathcal{L}_\lambda$. Assume that it is zero. Then v vanishes on the open cell $(N^-)^a \cdot \mathbf{k}v_\lambda$. But the linear span of the elements of this cell coincides with the whole representation V^a_λ. Therefore, the restriction map $(V^a_\lambda)^* \to H^0(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \mathcal{L}_\lambda)$ is an embedding.

Finally, we recall that the varieties $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ are flat degenerations of the classical flag varieties. Since the higher cohomologies of $\mathcal{L}_\lambda$ vanish, we arrive at the equality of the dimensions of $H^0(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \mathcal{L}_\lambda)$ and of V_λ. Therefore, the embedding $(V^a_\lambda)^* \to H^0(\mathrm{Sp}\,\mathcal{F}^a_{2n}, \mathcal{L}_\lambda)$ is an isomorphism. ∎

Similarly one proves a parabolic version of the BW-type theorem.

Theorem 8.3 *Let λ be a $\mathbf{d}$-dominant weight,* i.e., $(\lambda, \omega_d) > 0$ *implies $d \in \mathbf{d}$. Then there exists a map $\iota_\lambda\colon \mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}} \to \mathbb{P}(V^a_\lambda)$. We have*

$$H^0\big(\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}, \iota^*_\lambda\mathcal{O}(1)\big)^* \simeq V^a_\lambda, \quad H^{>0}\big(\mathrm{Sp}\,\mathcal{F}^a_{\mathbf{d}}, \iota^*_\lambda\mathcal{O}(1)\big) = 0.$$

8.3 The q-character Formula

We now compute the q-character (PBW-graded character) of the modules V_λ^a (for a combinatorial formula see Theorem 2.7). For this we use the Atiyah–Bott–Lefschetz fixed points formula applied to the variety $\mathrm{Sp}\,\mathrm{R}_{2n}$. We first describe the fixed points explicitly.

Lemma 8.4 *The T-fixed points on $\mathrm{Sp}\,\mathrm{R}_{2n}$ are labeled by the collections $\mathbf{S} = (S_{i,j})$, $1 \le i \le j < 2n$, $i + j \le 2n$, where $S_{i,j}$ are subsets of $\{1, \ldots, 2n\}$ satisfying the following properties:*

(i) $S_{i,j} \subset \{1, \ldots, i, j+1, \ldots, n\}$, $\# S_{i,j} = i$,
(ii) $S_{i,j} \subset S_{i+1,j} \subset S_{i+1,j+1} \cup \{j+1\}$,
(iii) *For any $i = 1, \ldots, n$ if $k \in S_{i,2n-i}$, then $2n + 1 - i \notin S_{i,2n-i}$.*

Proof Obviously, a collection $\mathbf{V} \in \mathrm{Sp}\,\mathrm{R}_{2n}$ is a T-fixed point if and only if each $V_{i,j}$ is the linear span of some basis vectors w_l. Now each collection $\mathbf{S}$ as above determines $\mathbf{V}$ by the formula $V_{i,j} = \mathrm{span}(w_l : l \in S_{i,j})$. ■

We call a collection $\mathbf{S}$ satisfying the conditions as above admissible. For an admissible $\mathbf{S}$ let $p(\mathbf{S}) \in \mathrm{Sp}\,\mathrm{R}_{2n}$ be the corresponding fixed point and let $p(S_{i,j}) \in \mathrm{Gr}_i(W_{i,j})$ be its (i, j)-th component.

Recall the extended degenerate group $G^a \rtimes \mathbb{C}^*$.

Lemma 8.5 *The actions of the group G^a and its extension $G^a \rtimes \mathbb{C}^*$ on $\mathrm{Sp}\,\mathcal{F}^a_{2n}$ lift to $\mathrm{Sp}\,\mathrm{R}_{2n}$.*

Proof Recall the embeddings $\mathrm{Sp}\,\mathcal{F}^a_{2n} \hookrightarrow \mathcal{F}^a_{2n}$ and $\mathrm{Sp}\,\mathrm{R}_{2n} \hookrightarrow R_{2n}$ (see Proposition 4.7). Since the analogue of our lemma for SL_{2n} holds (see [FF]), we obtain the desired result for Sp_{2n} as well. ■

In order to state the theorem we prepare some notations. Let $\mathbb{C}[e^{\omega_1}, \ldots, e^{\omega_n}, e^d]$ be the group algebra of the weight lattice of the extended Lie algebra $\mathfrak{g}^a \oplus \mathbb{C}d$. We sometimes use the notations $z_i = e^{\omega_i}$, $q = e^d$. For an element $\mu = md + \sum_{i=1}^n m_i\omega_i$ we write $e^\mu = q^m \prod_{i=1}^n z_i^{m_i}$. Also for a homogeneous vector $v \in V_\lambda^a$ we denote by $\mathrm{wt}_q(v)$ the extended weight of v.

Recall the Atiyah–Bott–Lefschetz formula (see [AB], [T]): let X be a smooth projective algebraic M-dimensional variety and let $\mathcal{L}$ be a line bundle on X. Let T be an algebraic torus acting on X with a finite set F of fixed points. Assume further that $\mathcal{L}$ is T-equivariant. Then for each $p \in F$ the fiber $\mathcal{L}_p$ is T-stable. We note also that since $p \in F$, the tangent space T_pX carries a natural T-action. Let $\gamma_1^p, \ldots, \gamma_M^p$ be the weights of the eigenvectors of T-action on T_pX. Then the Atiyah–Bott–Lefschetz formula gives the following expression for the character of the Euler characteristics:

$$\sum_{k\ge 0}(-1)^k \mathrm{ch} H^k(X, \mathcal{L}) = \sum_{p\in F} \frac{\mathrm{ch}\mathcal{L}_p}{\prod_{l=1}^M (1 - e^{-\gamma_l^p})}. \tag{8.1}$$

We apply this formula for $X = \mathrm{Sp}\,\mathrm{R}_{2n}$, $\mathcal{L} = \pi_{2n}^*\mathcal{L}_\lambda$ with the action of the extended torus $T \cdot \mathbb{C}^*$. Since $H^{>0}(R_n, \pi_n^*\mathcal{L}_\lambda) = 0$, the Euler characteristics coincides with the character of the zeroth cohomology, *i.e.*, with the character of $(V_\lambda^a)^*$. Therefore, for each admissible $\mathbf{S}$ we need to compute the character of $\pi_{2n}^*\mathcal{L}_\lambda$ at $p(\mathbf{S})$ and the eigenvalues of the torus action in $T_{p(\mathbf{S})}\,\mathrm{Sp}\,\mathrm{R}_{2n}$.

Let $\imath_\lambda\colon\ \mathrm{Sp}\,\mathcal{F}_{2n}^a \to \mathbb{P}(V_\lambda^a)$ be the standard map (which is an embedding for regular λ). Then $\mathrm{ch}(\pi_{2n}^*\mathcal{L}_\lambda)_{p(\mathbf{S})} = e^{-\,\mathrm{wt}_q(\imath_\lambda p(\mathbf{S}))}$ (the minus sign comes from the fact that a fiber of $\mathcal{O}(1)$ is a dual line). We note that the weight of $\imath_\lambda p(\mathbf{S})$ depends only on the diagonal entries $S_{i,i}$.

Now let us compute the eigenvalues of the tangent action of the torus at a point $p_{\mathbf{S}}$. For each pair (i, j), $1 \le i \le j < 2n$, $i + j \le 2n$ define a collection $S'_{i,j}$ as follows.

First, let $i+j < 2n$. Given the sets $S_{i-1,j}$ and $S_{i,j+1}$, let us look at the possible values of $S_{i,j}$ keeping $\mathbf{S}$ admissible. We denote such a possible collection by $\bar{S}_{i,j}$ in order to distinguish it from the already fixed component $S_{i,j}$. The definition of admissibility says that there exist exactly two variants for $\bar{S}_{i,j}$, namely

$$\bar{S}_{i,j} = S_{i-1,j} \cup \{a\} \quad \text{or} \quad \bar{S}_{i,j} = S_{i-1,j} \cup \{b\},$$

where $\{a, b\} = S_{i,j+1} \cup \{j+1\} \setminus S_{i-1,j}$. Given a collection $\mathbf{S}$ we denote the numbers a, b as above by $a_{i,j}^{\mathbf{S}}$ and $b_{i,j}^{\mathbf{S}}$. We have:

$$S_{i,j} = S_{i-1,j} \cup \{a_{i,j}^{\mathbf{S}}\}, \quad S_{i,j+1} \setminus S_{i-1,j} = \{a_{i,j}^{\mathbf{S}}, b_{i,j}^{\mathbf{S}}\}.$$

We denote by $S'_{i,j}$ the set $S_{i,j} \setminus \{a_{i,j}^{\mathbf{S}}\} \cup \{b_{i,j}^{\mathbf{S}}\}$.

Second, assume $i + j = 2n$. Given the set $S_{i-1,j}$, let us look at the possible values of $\bar{S}_{i,j}$ keeping $\mathbf{S}$ admissible. The definition of admissibility says that there exist exactly two variants for $\bar{S}_{i,j}$, namely

$$\bar{S}_{i,j} = S_{i-1,j} \cup \{a\} \quad \text{or} \quad \bar{S}_{i,j} = S_{i-1,j} \cup \{b\},$$

where $\{a, b\} = \{1, \dots, i, 2n-i+1, \dots, 2n\} \setminus \{2n+1-l : l \in S_{i-1,j}\}$. We also denote the numbers a, b by $a_{i,j}^{\mathbf{S}}$ and $b_{i,j}^{\mathbf{S}}$. We set $S_{i,j} = S_{i-1,j} \cup \{a_{i,j}^{\mathbf{S}}\}$ and we denote by $S'_{i,j}$ the set $S_{i,j} \setminus \{a_{i,j}^{\mathbf{S}}\} \cup \{b_{i,j}^{\mathbf{S}}\}$.

Recall that the variety $\mathrm{Sp}\,\mathcal{F}_{2n}^a$ sits inside the product of Grassmann varieties $\prod \mathrm{Gr}_i(W_{i,j})$. Each $\bigwedge^i(W_{i,j})$ is acted upon by $\mathfrak{g}^a \oplus \mathbb{C}d$ and therefore each Grassmannian carries a natural action of the group $G^a \rtimes \mathbb{C}^*$ (the additional $\mathbb{C}^*$ part corresponds to the PBW-grading operator). Thus for each collection $S_{i,j}$ we have the corresponding weight $\mathrm{wt}_q\, p(\mathbf{S}_{i,j})$, which is the weight of the corresponding point in $\bigwedge^i(W_{i,j})$.

Theorem 8.6 *The q-character of the representation V_λ^a is given by the sum over all admissible collections $\mathbf{S}$ of the summands*

$$\frac{e^{\mathrm{wt}_q(\imath_\lambda p(\mathbf{S}))}}{\prod_{\substack{i+j\le 2n\\ 1\le i\le j<2n}} \left(1 - e^{\mathrm{wt}_q\, p(S'_{i,j}) - \mathrm{wt}_q\, p(S_{i,j})}\right)}. \tag{8.2}$$

Proof Recall that $\mathrm{Sp}\,\mathrm{R}_{2n}$ can be constructed as a tower of successive $\mathbb{P}^1$-fibrations $\mathrm{Sp}\,\mathrm{R}_{2n}(l) \to \mathrm{Sp}\,\mathrm{R}_{2n}(l-1)$. Fix an admissible $\mathbf{S}$. Then the surjections $\mathrm{Sp}\,\mathrm{R}_{2n} \to$

$\operatorname{Sp}\mathrm{R}_{2n}(l)$ define the T-fixed points $p\big(\mathbf{S}(l)\big)$ in each $\operatorname{Sp}\mathrm{R}_{2n}(l)$ (note that $\mathbf{S}(l)$ consists of $S_{i,j}$ such that for $\beta_k = \alpha_{i,j}$ one has $k \le l$). For each $l = 1, \dots, M$ we denote by $v_l \in T_{p(\mathbf{S}(l))} \operatorname{Sp}\mathrm{R}_{2n}(l)$ a nonzero tangent vector to the fiber of the map $\operatorname{Sp}\mathrm{R}_{2n}(l) \to \operatorname{Sp}\mathrm{R}_{2n}(l-1)$ at the point $p\big(\mathbf{S}(l-1)\big)$. Then it is easy to see that the weights of the eigenvectors of the T action in $T_{p(\mathbf{S})} \operatorname{Sp}\mathrm{R}_{2n}$ are exactly the weights of the vectors v_l, $l = 1, \dots, M$.

So let us fix l, $1 \le l \le M$ and i, j with $\alpha_{i,j} = \beta_l$. Let us denote by Y_l the set of all pairs (k, m) such that for the root $\alpha_{k,m} = \beta_r$ one has $r \le l$. Then the fiber $\mathbb{P}^1$ of the map $\operatorname{Sp}\mathrm{R}_{2n}(l) \to \operatorname{Sp}\mathrm{R}_{2n}(l-1)$ at the point $p\big(\mathbf{S}(l-1)\big)$ consists of all collections $(V_{k,m})$ with $(k, m) \in Y_l$ subject to the following conditions:

(i) $V_{k,m} = p(S_{k,m})$ if $\alpha_{k,m} \ne \beta_l$,
(ii) $V_{i,j} \supset p(S_{i-1,j})$,
(iii) $V_{i,j} \subset p(S_{i-1,j}) \oplus \mathbb{C}w_{a^{\mathbf{S}}_{i,j}} \oplus \mathbb{C}w_{b^{\mathbf{S}}_{i,j}}$.

Now it is easy to see that the character of the tangent vector to this fiber at the point $p\big(\mathbf{S}(l-1)\big)$ is equal to $e^{\mathrm{wt}_q\, p(S'_{i,j}) - \mathrm{wt}_q\, p(S_{i,j})}$ (recall $a^{\mathbf{S}}_{i,j} \in S_{i,j}$ and $S'_{i,j} = S_{i,j} \setminus \{a^{\mathbf{S}}_{i,j}\} \cup \{b^{\mathbf{S}}_{i,j}\}$). ■

Remark 8.7 We note that the Euler characteristic

$$\sum_{k\ge 0} (-1)^k \mathrm{ch} H^k(\operatorname{Sp}\mathrm{R}_{2n}, \pi^*_{2n}\mathcal{L}_\lambda)$$

is equal to $\mathrm{ch}(V^a_\lambda)^*$. But in each summand (8.2) both numerator and denominator differ from the corresponding summand in the Atiyah–Bott–Lefschetz formula (8.1) by the change of variables $z_i \to z_i^{-1}$ and $q \to q^{-1}$. Via this change we pass from the character of $(V^a_\lambda)^*$ to the character of V^a_λ.

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