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On orbits of the automorphism group on an affine toric variety

Research Article

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Abstract: Let X be an affine toric variety. The total coordinates on X provide a canonical presentation $\overline{X} \to X$ of X as a quotient of a vector space $\overline{X}$ by a linear action of a quasitorus. We prove that the orbits of the connected component of the automorphism group $\mathrm{Aut}(X)$ on X coincide with the Luna strata defined by the canonical quotient presentation.

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Introduction

Every algebraic variety X carries a canonical stratification by orbits of the automorphism group $\mathrm{Aut}(X)$. The aim of this paper is to give several characterizations of this stratification when X is an affine toric variety.

In the case of a complete toric variety X the group $\mathrm{Aut}(X)$ is a linear algebraic group. It admits an explicit description in terms of combinatorial data defining X; see [7, 9]. The orbits of the connected component $\mathrm{Aut}(X)^0$ on the variety X are described in [6]. It is proved there that two points $x, x' \in X$ are in the same $\mathrm{Aut}(X)^0$-orbit if and only if the semigroups in the divisor class group $\mathrm{Cl}(X)$ generated by classes of prime torus invariant divisors that do not contain x and x' respectively, coincide.

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We obtain an analogue of this result for affine toric varieties. It turns out that in the affine case one may replace the semigroups mentioned above by the groups generated by the same classes. When X is non-degenerate, this relates $\mathrm{Aut}(X)^0$-orbits on X with stabilizers of points in fibres of the canonical quotient presentation $\pi\colon \overline{X} \to \overline{X}/\!\!/H_X \cong X$, where H_X is a quasitorus with the character group identified with $\mathrm{Cl}(X)$ and $\overline{X}$ is a finite-dimensional H_X-module whose coordinate ring is the total coordinate ring (or the Cox ring) of X; see [7], [8, Chapter 5], or Section 4 for details. More precisely, our main result (Theorem 5.1) states that the collection of $\mathrm{Aut}(X)^0$-orbits on X coincides with the Luna stratification of X as the quotient space of the linear H_X-action on $\overline{X}$. In particular, in our settings the Luna stratification is intrinsic in the sense of [14]. For connections between quotient presentations of an arbitrary affine variety, the Luna stratification, and Cox rings, see [1].

In contrast to the complete case, the automorphism group of an affine toric variety is usually infinite-dimensional. An explicit description of the automorphism group of an affine toric surface in terms of free amalgamated products is given in [5]. Starting from dimension three such a description is unknown. Another difference is that in the affine case the open orbit of $\mathrm{Aut}(X)^0$ coincides with the smooth locus of X [4, Theorem 2.1], while for a smooth complete toric variety X the group $\mathrm{Aut}(X)^0$ acts on X transitively if and only if X is a product of projective spaces [6, Theorem 2.8].

In Section 1 we recall basic facts on automorphisms of algebraic varieties and define the connected component $\mathrm{Aut}(X)^0$. Section 2 contains some background on affine toric varieties. We consider one-parameter unipotent subgroups in $\mathrm{Aut}(X)$ normalized by the acting torus (root subgroups). They are in one-to-one correspondence with the so-called Demazure roots of the cone of the variety X. Also we recall the technique developed in [4], which will be used later. In Section 3 we discuss the Luna stratification on the quotient space of a rational G-module, where G is a reductive group. Necessary facts on Cox rings and canonical quotient presentations of affine toric varieties are collected in Section 4. We define the Luna stratification of an arbitrary affine toric variety and give a characterization of strata in terms of some groups of classes of divisors (Proposition 4.7). Our main result is proved in Section 5. We illustrate it by an example. It shows that although the group $\mathrm{Aut}(X)^0$ acts (infinitely) transitively on the smooth locus X^{reg}, it may be non-transitive on the set of smooth points of the singular locus X^{sing}, even when X^{sing} is irreducible. Finally, in Section 6 we prove collective infinite transitivity on X along the orbits of the subgroup of $\mathrm{Aut}(X)$ generated by root subgroups and their replicas (Theorem 6.1). Here we follow the approach developed in [3].

1. Automorphisms of algebraic varieties

Let X be a normal algebraic variety over an algebraically closed field $\mathbb{K}$ of characteristic zero and $\mathrm{Aut}(X)$ be the automorphism group. At the moment we consider $\mathrm{Aut}(X)$ as an abstract group and our aim is to define the connected component of $\mathrm{Aut}(X)$ following [18].

Definition 1.1.

A family $\{\phi_b\}_{b\in B}$ of automorphisms of a variety X, where the parametrizing set B is an algebraic variety, is an *algebraic family* if the map $B\times X \to X$ given by $(b,x)\mapsto \phi_b(x)$ is a morphism.

If G is an algebraic group and $G\times X \to X$ is a regular action, then we may take $B = G$ and consider the algebraic family $\{\phi_g\}_{g\in G}$, where $\phi_g(x) = gx$. So any automorphism defined by an element of G is included in a natural algebraic family.

Definition 1.2.

The *connected component* $\mathrm{Aut}(X)^0$ of the group $\mathrm{Aut}(X)$ is the subgroup of automorphisms that occur as member of an algebraic family $\{\phi_b\}_{b\in B}$ with an (irreducible) rational curve as a base B such that $\phi_{b_0} = \mathrm{id}_X$ for some $b_0 \in B$.

Remark 1.3.

It is also natural to consider arbitrary irreducible base B in Definition 1.2, but for our purposes related to toric varieties rational curves as bases are more suitable.

It is easy to check that $\mathrm{Aut}(X)^0$ is indeed a subgroup; see [18].

Lemma 1.4.
Let G be a connected linear algebraic group and $G\times X\to X$ be a regular action. Then the image of G in $\mathrm{Aut}(X)$ is contained in $\mathrm{Aut}(X)^0$.

Proof. We have to prove that every $g\in G$ can be connected with the unit by a rational curve. By [13, Theorem 22.2], an element g is contained in a Borel subgroup of G. As any connected solvable linear algebraic group, a Borel subgroup is isomorphic (as a variety) to $(\mathbb{K}^\times)^r\times\mathbb{K}^m$ with some non-negative integers r and m. The assertion follows. □

We denote by $\mathrm{WDiv}(X)$ the group of Weil divisors on a variety X and by $\mathrm{PDiv}(X)$ the subgroup of principal divisors, i.e., $\mathrm{PDiv}(X)=\{\mathrm{div}(f): f\in\mathbb{K}(X)^\times\}$. Then the divisor class group of X is defined as $\mathrm{Cl}(X)=\mathrm{WDiv}(X)/\mathrm{PDiv}(X)$. The image of a divisor D in $\mathrm{Cl}(X)$ is denoted by $[D]$ and is called the *class* of D. Any automorphism $\phi\in\mathrm{Aut}(X)$ acts naturally on the set of prime divisors and thus on the group $\mathrm{WDiv}(X)$. Under this action a principal divisor goes to a principal one and we obtain an action of $\mathrm{Aut}(X)$ on $\mathrm{Cl}(X)$.

Recall that the local class group of X in a point x is the factor group

$$\mathrm{Cl}(X,x)=\mathrm{WDiv}(X)/\mathrm{PDiv}(X,x),$$

where $\mathrm{PDiv}(X,x)$ is the group of Weil divisors on X that are principal in some neighbourhood of the point x. We have a natural surjection $\mathrm{Cl}(X)\to\mathrm{Cl}(X,x)$. Let us denote by $\mathrm{Cl}_x(X)$ the kernel of this homomorphism, i.e., $\mathrm{Cl}(X,x)=\mathrm{Cl}(X)/\mathrm{Cl}_x(X)$. Equivalently, $\mathrm{Cl}_x(X)$ consists of classes that have a representative whose support does not contain x.

We obtain the following result.

Lemma 1.5.
Assume that an automorphism $\phi\in\mathrm{Aut}(X)$ acts on $\mathrm{Cl}(X)$ trivially. Then $\mathrm{Cl}_x(X)=\mathrm{Cl}_{\phi(x)}(X)$ for any $x\in X$.

2. Affine toric varieties and Demazure roots

A *toric variety* is a normal algebraic variety X containing an algebraic torus T as a dense open subset such that the action of T on itself extends to a regular action of T on X. Let N be the lattice of one-parameter subgroups $\lambda\colon\mathbb{K}^\times\to T$ and $M=\mathrm{Hom}(N,\mathbb{Z})$ be the dual lattice. We identify M with the lattice of characters $\chi\colon T\to\mathbb{K}^\times$, and the pairing $N\times M\to\mathbb{Z}$ is given by

$$(\lambda,\chi)\mapsto\langle\lambda,\chi\rangle,\qquad\text{where}\quad\chi(\lambda(t))=t^{\langle\lambda,\chi\rangle}.$$

Let us recall a correspondence between affine toric varieties and rational polyhedral cones. Let σ be a polyhedral cone in the rational vector space $N_\mathbb{Q}=N\otimes_\mathbb{Z}\mathbb{Q}$ and $\sigma^\vee$ be the dual cone in $M_\mathbb{Q}$. Then the affine variety

$$X_\sigma=\mathrm{Spec}\,\mathbb{K}[\sigma^\vee\cap M]$$

is toric and any affine toric variety arises this way, see [8, 11]. The T-orbits on X_σ are in order-reversing bijection with faces of the cone σ. If $\sigma_0\preceq\sigma$ is a face, then we denote the corresponding T-orbit by $\mathcal{O}_{\sigma_0}$. In particular, $\mathcal{O}_\sigma$ is a closed T-orbit and for the minimal face $0\preceq\sigma$, the corresponding orbit is $\mathcal{O}_0=T\subseteq X_\sigma$.

An affine variety X is called *non-degenerate* if any regular invertible function on X is constant. If X is toric, this condition means that there are no non-trivial decompositions $T=T_1\times T_2$ and $X=X_0\times T_2$, where X_0 is an affine toric variety with acting torus T_1. If $X=X_\sigma$, then X is non-degenerate if and only if the cone σ spans $N_\mathbb{Q}$ or, equivalently, the cone $\sigma^\vee$ is pointed.

Let us consider for a moment the case of an arbitrary toric variety X. Recall that X contains a finite number of prime T-invariant divisors $D_1,\dots,D_m$. (In the affine case they are in bijection with rays of the cone σ.) It is well known that

the group $\mathrm{Cl}(X)$ is generated by classes of these divisors. Let us associate with a T-orbit $\mathcal{O}$ on X the set of all prime T-invariant divisors $D(\mathcal{O}) = \{D_{i_1}, \dots, D_{i_k}\}$ that do not contain $\mathcal{O}$. Denote by $G(\mathcal{O})$ the subgroup of $\mathrm{Cl}(X)$ generated by the classes of divisors from $D(\mathcal{O})$.

Proposition 2.1.
Let X be a toric variety and $x \in X$. Then $\mathrm{Cl}_x(X) = G(T \cdot x)$.

Proof. Take a class $[D] \in \mathrm{Cl}(X)$ with a T-invariant representative D. By definition, $[D]$ is contained in $\mathrm{Cl}_x(X)$ if and only if it contains a representative $D' \in \mathrm{WDiv}(X)$ whose support does not pass through x. In particular, $D' = D + \mathrm{div}(h)$ for some $h \in \mathbb{K}(X)$. Consider the decomposition $D' = D'_+ - D'_-$, where D'_+ and D'_- are effective. This allows to deal with only the case where D' is effective.

Suppose that $[D] \in \mathrm{Cl}_x(X)$. We claim that there exists a T-invariant effective divisor $D'' \in [D]$ whose support does not pass through x. Assume this is not the case and consider the vector space

$$H^0(X, D') = \left\{ f \in \mathbb{K}(X)^\times : D' + \mathrm{div}(f) \geq 0 \right\} \cup \{0\}.$$

Then $H^0(X, D) = hH^0(X, D')$ and the subspace $H^0(X, D)$ in $\mathbb{K}(X)$ is invariant with respect to the action $(t \cdot f)(x) = f(t^{-1} \cdot x)$. It is well known that $H^0(X, D)$ is a rational T-module. We can transfer the structure of rational T-module to $H^0(X, D')$ by the formula

$$t \circ f = h^{-1} t \cdot (hf) \qquad \text{for any} \quad t \in T, \quad f \in H^0(X, D').$$

Then a function $f \in H^0(X, D')$ is T-semiinvariant if and only if the divisor $D' + \mathrm{div}(f)$ is T-invariant. Since D' is effective, the support of $D' + \mathrm{div}(f)$ passes through x if and only if $f(x) = 0$. By our assumption, this is the case for all T-semiinvariants in $H^0(X, D')$. But any vector in $H^0(X, D')$ is a sum of semiinvariants. Thus the support of any effective divisor in $[D]$ contains x. This is a contradiction, because the support of D' does not pass through x.

Since D'' is a sum of prime T-invariant divisors not passing through x, the class $[D]$, and thus the group $\mathrm{Cl}_x(X)$, is contained in the group $G(T \cdot x)$. The opposite inclusion is obvious. □

Lemma 2.2.
Let X be an affine toric variety. Then $\mathrm{Aut}(X)^0$ is in the kernel of the action of $\mathrm{Aut}(X)$ on $\mathrm{Cl}(X)$.

Proof. Let $\{\phi_b\}_{b \in B}$ be an algebraic family of automorphisms with $\phi_{b_0} = \mathrm{id}_X$ for some $b_0 \in B$. We may assume that B is an affine rational curve. In particular, $\mathrm{Cl}(B) = 0$. Consider the morphism $\Phi\colon B \times X \to X$ given by $(b, x) \mapsto \phi_b(x)$. We have to show that for any divisor D in X the intersections of $\Phi^{-1}(D)$ with fibres $\{b\} \times X$ are linearly equivalent to each other. The torus T acts on the variety $B \times X$ via the action on the second component. It is well known that every divisor on $B \times X$ is linearly equivalent to a T-invariant one. Every prime T-invariant divisor on $B \times X$ is either vertical, i.e., is a product of a prime divisor in B and X, or horizontal, i.e., a product of B and a prime T-invariant divisor in X. Every vertical divisor is principal. So the divisor $\Phi^{-1}(D)$ plus some principal divisor $\mathrm{div}(f)$ is a sum of horizontal divisors. Restricting the rational function f to every fibre $\{b\} \times X$ we obtain that the intersections of $\Phi^{-1}(D)$ with fibres are linearly equivalent to each other. □

Our next aim is to present several facts on automorphisms of affine toric varieties. Denote by $\sigma(1)$ the set of rays of a cone σ and by v_τ the primitive lattice vector on a ray τ.

Definition 2.3.
An element $e \in M$ is a called a *Demazure root* of a polyhedral cone σ in $N_{\mathbb{Q}}$ if there is a $\tau \in \sigma(1)$ such that $\langle v_\tau, e \rangle = -1$ and $\langle v_{\tau'}, e \rangle \geq 0$ for all $\tau' \in \sigma(1) \setminus \{\tau\}$.

Let $\mathcal{R} = \mathcal{R}(\sigma)$ be the set of all Demazure roots of a cone σ. For any root $e \in \mathcal{R}$ denote by τ_e (resp. v_e) the ray τ (resp. primitive vector v_τ) with $\langle v_\tau, e\rangle = -1$. Let $\mathcal{R}_\tau$ be the set of roots e with $\tau_e = \tau$. Then

$$\mathcal{R} = \bigcup_{\tau \in \sigma(1)} \mathcal{R}_\tau.$$

One can easily check that every set $\mathcal{R}_\tau$ is infinite.

Example 2.4.

Let σ be the positive orthant in the space $N_\mathbb{Q} = \mathbb{Q}^n$, where N is the lattice $\mathbb{Z}^n$. For every ray τ_i generated by the ith standard basis vector the set $\mathcal{R}_{\tau_i}$ consists of the vectors $(a_1, \dots, a_{i-1}, -1, a_{i+1}, \dots, a_n)$, $a_j \in \mathbb{Z}_{\geq 0}$, in the dual lattice $M = \mathbb{Z}^n$.

With any root e one associates a one-parameter subgroup H_e in the group $\mathrm{Aut}(X)$ such that $H_e \cong (\mathbb{K}, +)$ and H_e is normalized by T, see [9, 16] or [4, Section 2] for an explicit form of H_e. Moreover, every one-parameter unipotent subgroup of $\mathrm{Aut}(X)$ normalized by T has the form H_e for some root e. The following result is obtained in [4, Proposition 2.1].

Proposition 2.5.

Let $e \in \mathcal{R}$. For every point $x \in X \setminus X^{H_e}$ the orbit $H_e \cdot x$ meets exactly two T-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$. Moreover, $\mathcal{O}_2 \subseteq \overline{\mathcal{O}_1}$ and $\dim \mathcal{O}_1 = 1 + \dim \mathcal{O}_2$.

A pair of T-orbits $(\mathcal{O}_1, \mathcal{O}_2)$ as in Proposition 2.5 is called *H_e-connected*. The next result is [4, Lemma 2.2].

Lemma 2.6.

Let $\mathcal{O}_{\sigma_1}$ and $\mathcal{O}_{\sigma_2}$ be two T-orbits corresponding to faces $\sigma_1, \sigma_2 \preceq \sigma$. Then the pair $(\mathcal{O}_{\sigma_1}, \mathcal{O}_{\sigma_2})$ is H_e-connected if and only if

$$e\restriction_{\sigma_2} \leq 0 \qquad \text{and} \qquad \sigma_1 = \sigma_2 \cap e^\perp \quad \text{is a facet of cone } \sigma_2.$$

Let $\mathrm{AT}(X)$ be the subgroup of $\mathrm{Aut}(X)$ generated by subgroups T and H_e, $e \in \mathcal{R}$. Clearly, two T-orbits $\mathcal{O}$ and $\mathcal{O}'$ on X are contained in the same $\mathrm{AT}(X)$-orbit if and only if there is a sequence $\mathcal{O} = \mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k = \mathcal{O}'$ such that for any i either the pair $(\mathcal{O}_i, \mathcal{O}_{i+1})$ or the pair $(\mathcal{O}_{i+1}, \mathcal{O}_i)$ is H_e-connected for some $e \in \mathcal{R}$.

This statement admits a purely combinatorial reformulation. Let $\Gamma(\mathcal{O})$ be the semigroup in $\mathrm{Cl}(X)$ generated by classes of the elements of $D(\mathcal{O})$. The following result is given in [6, Lemmas 2.2-4].

Proposition 2.7.

Two T-orbits $\mathcal{O}$ and $\mathcal{O}'$ on X lie in the same $\mathrm{AT}(X)$-orbit if and only if $\Gamma(\mathcal{O}) = \Gamma(\mathcal{O}')$.

3. The Luna stratification

In this section we recall basic facts on the Luna stratification introduced in [15], see also [17, Section 6]. Let G be a reductive affine algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic zero and V be a rational finite-dimensional G-module. Denote by $\mathbb{K}[V]$ the algebra of polynomial functions on V and by $\mathbb{K}[V]^G$ the subalgebra of G-invariants. Let $V/\!\!/G$ be the spectrum of the algebra $\mathbb{K}[V]^G$. The inclusion $\mathbb{K}[V]^G \subseteq \mathbb{K}[V]$ gives rise to a morphism $\pi\colon V \to V/\!\!/G$ called the *quotient morphism* for the G-module V. It is well known that the morphism π is a categorical quotient for the action of the group G on V in the category of algebraic varieties, see [17, 4.6]. In particular, π is surjective.

The affine variety $X = V/\!\!/G$ is irreducible and normal. It is smooth if and only if the point $\pi(0)$ is smooth on X. In the latter case the variety X is an affine space. Every fibre of the morphism π contains a unique closed G-orbit. For any

closed G-invariant subset $A \subseteq V$ its image $\pi(A)$ is closed in X. These and other properties of the quotient morphism may be found in [17, 4.6].

By Matsushima's criterion, if an orbit $G \cdot v$ is closed in V, then the stabilizer $\mathrm{Stab}(v)$ is reductive, see [17, 4.7]. Moreover, there exists a finite collection $\{H_1, \ldots, H_r\}$ of reductive subgroups in G such that if an orbit $G \cdot v$ is closed in V, then $\mathrm{Stab}(v)$ is conjugate to one of these subgroups. This implies that every fibre of the morphism π contains a point whose stabilizer coincides with some H_i.

For every stabilizer H of a point in a closed G-orbit in V the subset

$$V_H = \left\{ w \in V : \text{there exists } v \in V \text{ such that } \overline{G \cdot w} \supset G \cdot v = \overline{G \cdot v} \text{ and } \mathrm{Stab}(v) = H \right\}$$

is G-invariant and locally closed in V. The image $X_H = \pi(V_H)$ turns out to be a smooth locally closed subset of X. In particular, X_H is a smooth quasiaffine variety.

Definition 3.1.

The stratification $X = \bigsqcup_{i=1}^{r} X_{H_i}$ is called the *Luna stratification* of the quotient space X.

Two points $x_1, x_2 \in X$ are in the same Luna stratum if and only if the stabilizers of points from the closed G-orbits in $\pi^{-1}(x_1)$ and $\pi^{-1}(x_2)$ are conjugate. In particular, if G is a quasitorus, these stabilizers should coincide.

There is a unique open dense stratum called the *principal stratum* of X. The closure of any stratum is a union of strata. Moreover, a stratum X_{H_i} is contained in the closure of a stratum X_{H_j} if and only if the subgroup H_i contains a subgroup conjugate to H_j. This induces a partial ordering on the set of strata compatible with the (reverse) ordering on the set of conjugacy classes of stabilizers.

4. Cox rings and quotient presentations

Let X be a normal algebraic variety with finitely generated divisor class group $\mathrm{Cl}(X)$. Assume that any regular invertible function $f \in \mathbb{K}[X]^\times$ is constant. Roughly speaking, the *Cox ring* of X may be defined as

$$R(X) = \bigoplus_{[D] \in \mathrm{Cl}(X)} H^0(X, D).$$

In order to obtain a multiplicative structure on $R(X)$ some technical work is needed, especially when the group $\mathrm{Cl}(X)$ has torsion. We refer to [2, Section 4] for details.

It is well known that if X is toric and non-degenerate, then $R(X)$ is a polynomial ring $\mathbb{K}[Y_1, \ldots, Y_m]$, where the variables Y_i are indexed by T-invariant prime divisors D_i on X and the $\mathrm{Cl}(X)$-grading on $R(X)$ is given by $\deg(Y_i) = [D_i]$; see [7] and [8, Chapter 5].

The affine space $\overline{X} = \operatorname{Spec} R(X)$ comes with a linear action of a quasitorus $H_X = \operatorname{Spec} \mathbb{K}[\mathrm{Cl}(X)]$ given by the $\mathrm{Cl}(X)$-grading on $R(X)$. The algebra of H_X-invariants on $R(X)$ coincides with the zero weight component $R(X)_0 = \mathbb{K}[X]$. Assume that X is a non-degenerate affine toric variety. Then we obtain a quotient presentation

$$\pi \colon \overline{X} \to \overline{X} /\!\!/ H_X \cong X.$$

Definition 4.1.

Let X be a non-degenerate affine toric variety and V be a rational module of a quasitorus H. The quotient morphism $\pi' \colon V \to V /\!\!/ H$ is called a *canonical quotient presentation* of X, if there are an isomorphism $\varphi \colon H_X \to H$ and a linear isomorphism $\psi \colon \overline{X} \to V$ such that $\psi(h \cdot y) = \varphi(h) \cdot \psi(y)$ for any $h \in H_X$ and $y \in \overline{X}$.

A canonical quotient presentation may be characterized in terms of the quasitorus action.

Definition 4.2.
An action of a reductive group F on an affine variety Z is said to be *strongly stable* if there exists an open dense invariant subset $U \subseteq Z$ such that

1. the complement $Z \setminus U$ is of codimension at least two in Z;
2. the group F acts freely on U;
3. for every $z \in U$ the orbit $F \cdot z$ is closed in Z.

The following proposition may be found in [2, Remark 6.4.2 and Theorem 6.4.3].

Proposition 4.3.
1. *Let X be a non-degenerate toric variety. Then the action of $H = \operatorname{Spec}(\mathbb{K}[\operatorname{Cl}(X)])$ on $\overline{X}$ is strongly stable.*
2. *Let H be a quasitorus acting linearly on a vector space V. Then the quotient space $X = V/\!\!/T$ is a non-degenerate affine toric variety. If the action of H on V is strongly stable, then the quotient morphism $\pi\colon V \to V/\!\!/T$ is a canonical quotient presentation of X. In particular, the group $\operatorname{Cl}(X)$ is isomorphic to the character group of H.*

A canonical quotient presentation allows to define a canonical stratification on X.

Definition 4.4.
The *Luna stratification* of a non-degenerate affine toric variety X is the Luna stratification of Definition 3.1 induced on X by the canonical quotient presentation $\pi\colon \overline{X} \to X$.

Proposition 4.5.
Let X be a non-degenerate affine toric variety. Then the principal stratum of the Luna stratification on X coincides with the smooth locus X^{reg}.

Proof. As was pointed out above, points of the principal stratum are smooth on X. Conversely, the fibre $\pi^{-1}(x)$ over a smooth point $x \in X$ consists of one H_X-orbit and H_X acts on $\pi^{-1}(x)$ freely; see [2, Proposition 6.1.6]. This shows that x is contained in the principal stratum. □

Now we assume that X is a degenerate affine toric variety. Let us fix a point x_0 in the open T-orbit on X and consider the closed subvariety $X_0 = \{x \in X : f(x) = f(x_0)\}$, where f runs through all invertible regular functions on X. Then X_0 is a non-degenerate affine toric variety with respect to a subtorus $T_1 \subseteq T$, and X_0 depends on the choice of x_0 only up to shift by an element of T. Moreover, $X \cong X_0 \times T_2$ for a subtorus $T_2 \subset T$ with $T = T_1 \times T_2$. We define a *Luna stratum* on X as $T \cdot Y$, where Y is a Luna stratum on X_0. This way we obtain a canonical stratification of X with open stratum being the smooth locus.

The following lemma is straightforward.

Lemma 4.6.
In notations as above every Luna stratum on X is isomorphic to $Y \times T_2$, where Y is a Luna stratum on X_0.

Now we present the first characterization of the Luna stratification.

Proposition 4.7.
Let X be an affine toric variety. Then two points $x, x' \in X$ are in the same Luna stratum if and only if $\operatorname{Cl}_x(X) = \operatorname{Cl}_{x'}(X)$.

Proof. By Lemma 4.6 we may assume that X is non-degenerate. Let $\pi\colon \overline{X} \to X$ be the canonical quotient presentation. For any point $v \in \overline{X}$ such that the orbit $H_X \cdot v$ is closed in $\overline{X}$ the stabilizer $\mathrm{Stab}(v)$ in H_X is defined by the surjection of the character groups $\mathbb{X}(H_X) \to \mathbb{X}(\mathrm{Stab}(v))$. By [2, Proposition 6.2.2] we may identify $\mathbb{X}(H_X)$ with $\mathrm{Cl}(X)$, $\mathbb{X}(\mathrm{Stab}(v))$ with $\mathrm{Cl}(X,x)$, and the homomorphism with the projection $\mathrm{Cl}(X) \to \mathrm{Cl}(X,x)$, where $x = \pi(v)$. Thus two points $v, v' \in \overline{X}$ with closed H_X-orbits have the same stabilizers in H_X, or, equivalently, the points $x = \pi(v)$ and $x' = \pi(v')$ lie in the same Luna stratum on X if and only if $\mathrm{Cl}_x(X) = \mathrm{Cl}_{x'}(X)$. □

5. Orbits of the automorphism group

The following theorem describes orbits of the group $\mathrm{Aut}(X)^0$ in terms of local divisor class groups and the Luna stratification of an affine toric variety X.

We recall that $\mathrm{Cl}_x(X)$ denotes the group of classes of divisors which have a representative whose support does not contain x. Further, $G(T \cdot x)$ (resp. $\Gamma(T \cdot x)$) is the group (resp. the semigroup) in $\mathrm{Cl}(X)$ generated by classes of prime T-invariant divisors that do not contain the orbit $T \cdot x$.

Theorem 5.1.
Let X be an affine toric variety and $x, x' \in X$. Then the following conditions are equivalent.

1. *The $\mathrm{Aut}(X)^0$-orbits of the points x and x' coincide.*
2. $G(T \cdot x) = G(T \cdot x')$.
3. $\mathrm{Cl}_x(X) = \mathrm{Cl}_{x'}(X)$.
4. *The points x and x' lie in the same Luna stratum on X.*

Proof. Implication 1. ⇒ 3. follows from Lemma 1.5. Conditions 3. and 4. are equivalent by Proposition 4.7 and conditions 2. and 3. are equivalent by Proposition 2.1. So it remains to prove implication 2. ⇒ 1.

Proposition 5.2.
Let X be an affine toric variety and $x \in X$. Then $G(T \cdot x) = \Gamma(T \cdot x)$.

Proof. We begin with some generalities on quasitorus representations. Let K be a finitely generated abelian group. Consider a diagonal linear action of the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ on a vector space V of dimension m given by characters $\chi_1, \ldots, \chi_m \in K$. Then we have a weight decomposition $V = \bigoplus_{i=1}^m \mathbb{K} e_i$, where $h \cdot e_i = \chi_i(h) e_i$ for any $h \in H$. With any vector $v = x_1 e_1 + \cdots + x_m e_m$ one associates the set of characters $\Delta(v) = \{\chi_{i_1}, \ldots, \chi_{i_k}\}$ such that $x_{i_1} \neq 0, \ldots, x_{i_k} \neq 0$. It is well known that the orbit $H \cdot v$ is closed in V if and only if the cone generated by $\chi_{i_1} \otimes 1, \ldots, \chi_{i_k} \otimes 1$ in $K_{\mathbb{Q}} = K \otimes_{\mathbb{Z}} \mathbb{Q}$ is a subspace.

Below we make use of the following elementary lemma.

Lemma 5.3.
Let $\chi_1, \ldots, \chi_m$ be elements of a finitely generated abelian group K. If the cone generated by $\chi_1 \otimes 1, \ldots, \chi_m \otimes 1$ in $K_{\mathbb{Q}}$ is a subspace, then the semigroup generated by $\chi_1, \ldots, \chi_m$ in K is a group.

Let us return to the canonical quotient presentation $\pi\colon \overline{X} \to X$. By construction, the H_X-weights of the linear H_X-action on $\overline{X}$ are the classes $-[D_1], \ldots, -[D_m]$ in $\mathrm{Cl}(X)$. Moreover, for any point $v \in \overline{X}$ such that $H_X \cdot v$ is closed in $\overline{X}$ the set of weights $\Delta(v)$ coincides with the classes of the divisors from $-D(T \cdot x) = \{-D_{i_1}, \ldots, -D_{i_k}\}$, where $x = \pi(v)$. Since the orbit $H_X \cdot v$ is closed in $\overline{X}$, by Lemma 5.3 the semigroup $\Gamma(T \cdot x)$ generated by $[D_{i_1}], \ldots, [D_{i_k}]$ coincides with the group $G(T \cdot x)$ generated by $[D_{i_1}], \ldots, [D_{i_k}]$, and we obtain Proposition 5.2. ■

By Proposition 5.2, we have $\Gamma(T \cdot x) = \Gamma(T \cdot x')$. Further, Proposition 2.7 implies that x and x' lie in the same $\mathrm{AT}(X)$-orbit and thus in the same $\mathrm{Aut}(X)^0$-orbit. This completes the proof of Theorem 5.1. □

Remark 5.4.
Condition 2. of Theorem 5.1 is the most effective in practice. It is interesting to know for which wider classes of varieties equivalence of conditions 1. and 3. holds. The following observation is due to the referee. Let Y be a normal complete $\mathbb{K}^\times$-surface with $\mathrm{Cl}(Y) = \mathbb{Z}$ and a singular $\mathbb{K}^\times$-fixed point x_0; concrete examples may be found in [12, Theorem 3.23]. If X is a factorial $\mathbb{K}^\times$-invariant open affine neighborhood of x_0, then $\mathrm{Cl}_{x_0}(X) = 0 = \mathrm{Cl}_x(X)$ for any $x \in X$, but x_0 is a fixed point of $\mathrm{Aut}(X)^0$.

Remark 5.5.
It follows from properties of the Luna stratification that the $\mathrm{Aut}(X)^0$-orbit of a point x is contained in the closure of the $\mathrm{Aut}(X)^0$-orbit of a point x' if and only if $\mathrm{Cl}_x(X)$ is a subgroup of $\mathrm{Cl}_{x'}(X)$.

Let us finish this section with a description of $\mathrm{Aut}(X)$-orbits on an affine toric variety X. Denote by $S(X)$ the image of the group $\mathrm{Aut}(X)$ in the automorphism group $\mathrm{Aut}(\mathrm{Cl}(X))$ of the abelian group $\mathrm{Cl}(X)$. The group $S(X)$ preserves the semigroup generated by the classes $[D_1], \ldots, [D_m]$ of prime T-invariant divisors. Indeed, this is the semigroup of classes containing an effective divisor. In particular, the group $S(X)$ preserves the cone in $\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by $[D_1]\otimes 1, \ldots, [D_m]\otimes 1$. This shows that $S(X)$ is finite.

The following proposition is a direct corollary of Theorem 5.1.

Proposition 5.6.
Let X be an affine toric variety. Two points $x, x' \in X$ are in the same $\mathrm{Aut}(X)$-orbit if and only if there exists $s \in S(X)$ such that $s(\mathrm{Cl}_x(X)) = \mathrm{Cl}_{x'}(X)$.

If X is an affine toric variety, then the group $\mathrm{Aut}(X)^0$ acts on the smooth locus X^{reg} transitively; see [4, Theorem 2.1]. Let $X^{\mathrm{sing}} = X_1 \cup \ldots \cup X_r$ be the decomposition of the singular locus into irreducible components. One may expect that the group $\mathrm{Aut}(X)^0$ acts transitively on every subset $X_i^{\mathrm{reg}} \setminus \bigcup_{j\neq i} X_j$ too. The following example shows that this is not the case.

Example 5.7.
Consider a two-dimensional torus T^2 acting linearly on the vector space $\mathbb{K}^7$:

$$(t_1, t_2)\cdot(z_1, z_2, z_3, z_4, z_5, z_6, z_7) = \left(t_1 z_1, t_1 z_2, t_1^{-1} z_3, t_2 z_4, t_2 z_5, t_1^{-1} t_2^{-1} z_6, t_1^{-1} t_2^{-1} z_7\right).$$

One can easily check that this action is strongly stable, and by Proposition 4.3 the quotient morphism $\pi\colon \mathbb{K}^7 \to \mathbb{K}^7/\!\!/T^2$ is the canonical quotient presentation of a five-dimensional non-degenerate affine toric variety $X = \mathbb{K}^7/\!\!/T^2$. Looking at closed T^2-orbits on $\mathbb{K}^7$, one obtains that there are three Luna strata

$$X = (X \setminus Z) \cup (Z \setminus \{0\}) \cup \{0\},$$

where $Z = \pi(W)$ and W is a subspace in $\mathbb{K}^7$ given by $z_4 = z_5 = z_6 = z_7 = 0$. In particular, Z is the singular locus of X. Clearly, Z is isomorphic to an affine plane with coordinates $z_1 z_3$ and $z_2 z_3$. So, Z is irreducible and smooth, but the groups $\mathrm{Aut}(X)^0$ (and $\mathrm{Aut}(X)$) has two orbits on Z, namely, $Z \setminus \{0\}$ and $\{0\}$.

6. Collective infinite transitivity

Let X be a non-degenerate affine toric variety of dimension ≥ 2. It is shown in [4, Theorem 2.1] that for any positive integer s and any two tuples of smooth pairwise distinct points $x_1, \ldots, x_s$ and $x_1', \ldots, x_s'$ on X there is an automorphism $\phi \in \mathrm{Aut}(X)^0$ such that $\phi(x_i) = x_i'$ for $i = 1, \ldots, s$. In other words, the action of $\mathrm{Aut}(X)^0$ on the smooth locus X^{reg} is *infinitely transitive*.

Our aim is to generalize this result and to prove collective infinite transitivity along different orbits of some subgroup of the automorphism group. Let us recall some general notions from [3]. Consider a one-dimensional algebraic group $H \cong (\mathbb{K}, +)$ and a regular action $H \times X \to X$ on an affine variety $X = \operatorname{Spec} A$. Then the associated derivation ∂ of A is locally nilpotent, i.e., for every $a \in A$ we can find $n \in \mathbb{N}$ such that $\partial^n(a) = 0$. Any derivation of A may be viewed as a vector field on X. So we may speak about locally nilpotent vector fields ∂. We use notation $H = H(\partial) = \exp(\mathbb{K}\partial)$. It is immediate that for every $f \in \operatorname{Ker}\partial$ the derivation $f\partial$ is again locally nilpotent [10, 1.4, Principle 7]. A one-parameter subgroup of the form $H(f\partial)$ for some $f \in \operatorname{Ker}\partial$ is called a *replica* of $H(\partial)$.

A set $\mathcal{N}$ of locally nilpotent vector fields on X is said to be *saturated* if it satisfies the following two conditions.

1. $\mathcal{N}$ is closed under conjugation by elements of G, where G is the subgroup of $\operatorname{Aut}(X)$ generated by all subgroups $H(\partial)$, $\partial \in \mathcal{N}$.
2. $\mathcal{N}$ is closed under taking replicas, i.e., for all $\partial \in \mathcal{N}$ and $f \in \operatorname{Ker}\partial$ we have $f\partial \in \mathcal{N}$.

If $X = X_\sigma$ is toric, we define $\mathcal{A}(X)$ as the subgroup of $\operatorname{Aut}(X)$ generated by H_e, $e \in \mathcal{R}$, and all their replicas. As $\mathcal{N}$ one can take locally nilpotent vector fields corresponding to replicas of H_e and all their conjugates.

Theorem 6.1.
Let X be a non-degenerate affine toric variety. Suppose that $x_1, \dots, x_s$ and $x'_1, \dots, x'_s$ are points on X with $x_i \neq x_j$ and $x'_i \neq x'_j$ for $i \neq j$ such that for each i the orbits $\mathcal{A}(X) \cdot x_i$ and $\mathcal{A}(X) \cdot x'_i$ are equal and of dimension ≥ 2. Then there exists an element $\phi \in \mathcal{A}(X)$ such that $\phi(x_i) = x'_i$ for $i = 1, \dots, s$.

The proof of Theorem 6.1 is based on the following results. Let G be a subgroup of $\operatorname{Aut}(X)$ generated by subgroups $H(\partial)$, $\partial \in \mathcal{N}$, for some set $\mathcal{N}$ of locally nilpotent vector fields and $\Omega \subseteq X$ be a G-invariant subset. We say that a locally nilpotent vector field ∂ satisfies the *orbit separation property* on Ω if there is an $H(\partial)$-stable subset $U(H) \subseteq \Omega$ such that

1. for each G-orbit O contained in Ω, the intersection $U(H) \cap O$ is open and dense in O;
2. the global H-invariants $\mathbb{K}[X]^H$ separate all one-dimensional H-orbits in $U(H)$.

Similarly we say that a set of locally nilpotent vector fields $\mathcal{N}$ satisfies the *orbit separation property* on Ω if it holds for every $\partial \in \mathcal{N}$.

Theorem 6.2 ([3, Theorem 3.1]).
Let X be an irreducible affine variety and $G \subseteq \operatorname{Aut}(X)$ be a subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields, which has the orbit separation property on a G-invariant subset $\Omega \subseteq X$. Suppose that $x_1, \dots, x_s$ and $x'_1, \dots, x'_s$ are points in Ω with $x_i \neq x_j$ and $x'_i \neq x'_j$ for $i \neq j$ such that for each i the orbits $G \cdot x_i$ and $G \cdot x'_i$ are equal and of dimension ≥ 2. Then there exists an element $g \in G$ such that $g \cdot x_i = x'_i$ for $i = 1, \dots, s$.

Proof of Theorem 6.1. Let us take $G = \mathcal{A}(X)$ and $\Omega = X$. In order to apply Theorem 6.2, we have to check the orbit separation property for one-parameter subgroups in $\mathcal{A}(X)$. By [3, Lemma 2.8], it suffices to check it for the subgroups H_e, $e \in \mathcal{R}$.

Proposition 6.3.
Let e be a root of a cone $\sigma \subseteq N_{\mathbb{Q}}$ of full dimension and $X = X_\sigma$ be the corresponding affine toric variety. Then for any two one-dimensional H_e-orbits C_1 and C_2 there is an invariant $f \in \mathbb{K}[X]^{H_e}$ with $f\restriction_{C_1} = 0$ and $f\restriction_{C_2} = 1$.

Proof. Let R_e be a one-parameter subgroup of T represented by the vector $v_e \in N$. Then $\mathbb{K}[X]^{H_e} = \mathbb{K}[X]^{R_e}$; see [4, Section 2.4]. Moreover, the subgroup R_e normalizes but not centralizes H_e in $\operatorname{Aut}(X)$ and every one-dimensional H_e-orbit $C \cong \mathbb{A}^1$ is the closure of an R_e-orbit; see [4, Proposition 2.1]. In particular, every one-dimensional H_e-orbit contains a unique R_e-fixed point. Since the group R_e is reductive, every two R_e-fixed points can be separated by an invariant from $\mathbb{K}[X]^{R_e}$. This shows that any two one-dimensional H_e-orbits can be separated by an invariant from $\mathbb{K}[X]^{H_e}$. ■

This completes the proof of Theorem 6.1. □

It follows from the proof of [4, Theorem 2.1] that the group $\mathcal{A}(X)$ acts (infinitely) transitively on the smooth locus of a non-degenerate affine toric variety X. In particular, the open orbits of $\mathcal{A}(X)$ and $\mathrm{Aut}(X)^0$ on X coincide. The example below shows that this is not the case for smaller orbits.

Example 6.4.

Let X_σ be the affine toric threefold defined be the cone

$$\sigma = \mathrm{cone}(\tau_1, \tau_2, \tau_3), \qquad v_{\tau_1} = (1,0,0), \qquad v_{\tau_2} = (1,2,0), \qquad v_{\tau_3} = (0,1,2).$$

We claim that all points on one-dimensional T-orbits of X are $\mathcal{A}(X)$-fixed. Indeed, suppose that a point x on a one-dimensional T-orbit is moved by the subgroup H_e for some root e. Then x belongs to a union of two H_e-connected T-orbits. Assume, for example, that the T-orbit of x corresponds to the face $\mathrm{cone}(\tau_1, \tau_2)$ and the pair of H_e-connected T-orbits includes the T-fixed point on X. By Lemma 2.6, we have

$$\langle (1,0,0), e\rangle = 0, \qquad \langle (1,2,0), e\rangle = 0, \qquad \langle (0,1,2), e\rangle = -1.$$

These conditions imply $\langle (0,0,1), e\rangle = -1/2$, a contradiction. If the pair of H_e-connected T-orbits includes the two-dimensional orbit, then either

$$\langle (1,0,0), e\rangle = 0, \qquad \langle (1,2,0), e\rangle = -1, \qquad \text{or} \qquad \langle (1,0,0), e\rangle = -1, \qquad \langle (1,2,0), e\rangle = 0.$$

In both cases we have $\langle (0,1,0), e\rangle = \pm 1/2$, a contradiction. Other possibilities may be considered in the same way.

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