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Gelfand–Tsetlin algebras and cohomology rings of Laumon spaces

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To the memory of Izrail Moiseevich Gelfand

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Abstract Laumon moduli spaces are certain smooth closures of the moduli spaces of maps from the projective line to the flag variety of GL_n. We calculate the equivariant cohomology rings of the Laumon moduli spaces in terms of Gelfand–Tsetlin subalgebra of $U(gl_n)$ and formulate a conjectural answer for the small quantum cohomology rings in terms of certain commutative shift of argument subalgebras of $U(gl_n)$.

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1 Introduction

1.1 Cohomology of Laumon spaces

The moduli spaces $\mathcal{Q}_{\underline{d}}$ were introduced by G. Laumon in [15] and [14]. They are certain compactifications of the moduli spaces of degree $\underline{d}$ maps from $\mathbb{P}^1$ to the flag

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variety $\mathcal{B}_n$ of GL_n. The original motivation of G. Laumon was to study the geometric Eisenstein series, but later the Laumon moduli spaces proved useful also in the computation of quantum cohomology and K-theory of $\mathcal{B}_n$, see e.g. [2,10]. The aim of the present note is to calculate the cohomology rings of the Laumon moduli spaces and to formulate a conjectural answer for the quantum cohomology rings.

The main tool is the action of the universal enveloping algebra $U(\mathfrak{gl}_n)$ by correspondences [8] on the direct sum (over all degrees) of cohomology of $\mathcal{Q}_{\underline{d}}$. More precisely, we consider the localized equivariant cohomology $B := \bigoplus_{\underline{d}} H^\bullet_{GL_n\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}}) \otimes_{H^\bullet_{GL_n\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{GL_n\times\mathbb{C}^*}(pt))$ where $\mathbb{C}^*$ acts as "loop rotations" on the source $\mathbb{P}^1$, while GL_n acts naturally on the target $\mathcal{B}_n$. We also consider a "local version" $\mathfrak{Q}_{\underline{d}}$ of the Laumon moduli space, which is a certain closure of the moduli space of *based* maps of degree $\underline{d}$ from $\mathbb{P}^1$ to $\mathcal{B}_n$. This local version does not carry the action of the whole group $GL_n \times \mathbb{C}^*$, but only of the Cartan torus $\widetilde{T} \times \mathbb{C}^*$. Accordingly, we consider the equivariant cohomology (resp. localized equivariant cohomology) $'V = \bigoplus_{\underline{d}} H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$ (resp. $V = \bigoplus_{\underline{d}} H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt))$).

According to [1] (cf. also [20] and our Theorem 2.7), the above action of $U(\mathfrak{gl}_n)$ identifies V with the universal Verma module $\mathfrak{V}$. Similarly, B carries the action of *two copies* of $U(\mathfrak{gl}_n)$ by correspondences and can be identified with the tensor square $\mathfrak{B}$ of $\mathfrak{V}$ (Theorem 5.8). The nonlocalized cohomology $'V$ is identified with a certain integral form $\underline{\mathfrak{V}}$ of $\mathfrak{V}$, a version of the universal *dual* Verma module (Theorem 3.5). We were unable to describe the *nonlocalized* equivariant cohomology of $\bigsqcup_{\underline{d}} \mathcal{Q}_{\underline{d}}$ as a $U(\mathfrak{gl}_n)^2$-module, but we propose a conjecture 5.14 in this direction; it is an equivariant generalization of Conjecture 6.4 of [8].

1.2 Gelfand–Tsetlin algebra

The description of the cohomology rings is given in representation theoretic terms. Namely, the universal enveloping algebra of $\mathfrak{gl}_n$ contains the *Gelfand–Tsetlin subalgebra* $\underline{\mathfrak{G}}$ (a maximal commutative subalgebra). For a given degree $\underline{d}$, the equivariant cohomology $'V_{\underline{d}}$ is identified with the weight subspace $\underline{\mathfrak{V}}_{\underline{d}}$. The identity element $1_{\underline{d}}$ of the cohomology ring goes to the weight component $\mathfrak{v}_{\underline{d}}$ of the Whittaker vector $\mathfrak{v} \in \underline{\mathfrak{V}}$. It turns out that the vector $1_{\underline{d}} \in {}'V$ is cyclic for $\underline{\mathfrak{G}}$; hence, the equivariant cohomology ring $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$ is identified with a quotient of the Gelfand–Tsetlin subalgebra (Corollary 3.7). Similar results hold for the localized equivariant cohomology of $\mathfrak{Q}_{\underline{d}}$ and $\mathcal{Q}_{\underline{d}}$ (Propositions 2.17 and 5.12).

The proof uses two ingredients. First, the localized equivariant cohomology has a natural basis of classes of the torus-fixed points. We check that under the identification $V \simeq \mathfrak{V}$, this basis goes to the Gelfand–Tsetlin basis of $\mathfrak{V}$. Also, the cohomology of $\mathfrak{Q}_{\underline{d}}$ contains the (Künneth components of the) Chern classes of the universal tautological vector bundles on $\mathfrak{Q}_{\underline{d}} \times \mathbb{P}^1$. The operators of multiplication by these Chern classes are diagonal in the fixed point basis, and by comparison to the Gelfand–Tsetlin basis, it is possible to identify these operators with the action of certain generators of $\underline{\mathfrak{G}}$. Finally, since the diagonal class of $\mathfrak{Q}_{\underline{d}}$ is decomposable, the above Chern classes generate the cohomology ring of $\mathfrak{Q}_{\underline{d}}$.

In a similar vein, in Proposition 6.7, we compute the localized equivariant K-ring of $\mathfrak{Q}_{\underline{d}}$ in terms of the "quantum Gelfand–Tsetlin algebra".

1.3 Quantum cohomology of Laumon spaces

The Picard group of the local Laumon space $\mathfrak{Q}_{\underline{d}}$ is free of rank $n-2$ iff all the entries of $\underline{d}$ are nonzero. It possesses the set of distinguished generators: the classes of determinant line bundles $\mathcal{D}_2, \ldots, \mathcal{D}_{n-1}$. Let T be a torus with the cocharacter lattice $\mathrm{Pic}(\mathfrak{Q}_{\underline{d}})$, and let q_i, $2 \le i \le n-1$, be the coordinates on T corresponding to $\mathcal{D}_i$. We conjecture a formula for the operator $M_{\mathcal{D}_i}$ of *quantum* multiplication by the first Chern class $c_1(\mathcal{D}_i)$. *A priori* this operator lies in $\mathrm{End}(V_{\underline{d}})[[q_2, \ldots, q_{n-1}]]$, but according to Conjecture 4.6, it is the Taylor expansion of a rational $\mathrm{End}(V_{\underline{d}})$-valued function on T. Moreover, this function arises from the action of a *universal* element $QC_i \in U(\mathfrak{gl}_n)$ (depending on $q_2, \ldots, q_{n-1}$) on the weight space $V_{\underline{d}}$ of the universal Verma module. The commutant of the collection of all such elements $\{QC_i(q_2, \ldots, q_{n-1})\}$ is a *shift of argument subalgebra* $\mathcal{A}_q \subset U(\mathfrak{gl}_n)$ (a maximal commutative subalgebra, see [23]).

We consider the flat $\mathrm{End}(V_{\underline{d}})$-valued connection on T : $\nabla = \sum_{i=2}^{n-1} q_i \frac{\partial}{\partial q_i} + QC_i$ (the *quantum connection*). Conjecture 4.6 (recently proved by A. Negut) implies that ∇ is induced by the *Casimir connection* [4,7,13,17] on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_n$ under an embedding $\mathsf{T} \hookrightarrow \mathfrak{h}$. In particular, ∇ has regular singularities, and its monodromy factors through the action of the pure braid group PB_n (fundamental group of the complement in $\mathfrak{h}$ to the root hyperplanes) on the weight space $V_{\underline{d}}$ by the "quantum Weyl group operators".

1.4 Acknowledgments

This note is a result of discussions with many people. In particular, the "restricted" correspondences $\mathfrak{E}_{\underline{d},\alpha_{ij}}$, $\mathcal{E}^0_{\underline{d},\alpha_{ij}}$, $\mathcal{E}^\infty_{\underline{d},\alpha_{ij}}$ (see 3.3, 5.13) and the action of $U(\mathfrak{gl}_n) \ltimes \mathbb{C}[\mathfrak{gl}_n]$ on nonequivariant cohomology of $\bigsqcup_{\underline{d}} \mathcal{Q}_{\underline{d}}$ were introduced by A. Kuznetsov in 1998. The idea to consider the action of correspondences on the equivariant cohomology of Laumon spaces was proposed by V. Schechtman in 1997. I. F. would like to thank A. Licata and A. Marian for many useful discussions of representation theory based on Laumon's spaces. In fact, some of the calculations associated with the double $\mathfrak{gl}_n$ action of Sect. 5 were carried out independently by A. Licata and A. Marian, also in a nonequivariant setting. Furthermore, we learnt from A. Marian that the Chern classes of the tautological bundles generate the cohomology ring of $\mathfrak{Q}_{\underline{d}}$. A. I. Molev provided us with references on Gelfand–Tsetlin bases. Above all, our interest in quantum cohomology of Laumon spaces was inspired by R. Bezrukavnikov, A. Braverman, A. Negut, A. Okounkov and V. Toledano Laredo during the beautiful special year 2007/2008 at IAS organized by R. Bezrukavnikov. We are deeply grateful to all of them. Finally, we are obliged to A. Tsymbaliuk for the careful reading of the first version of our note and spotting several mistakes. B. F. was partially supported by the grants RFBR 05-01-01007, RFBR 05-01-01934, and NSh-6358.2006.2. I. F. was supported by the NSF grant DMS-0457444. M. F. was partially supported by the Oswald Veblen Fund, RFBR grant 09-01-00242, the Science Foundation of the SU-HSE awards No.T3-62.0 and 10-09-0015, and the Ministry of Education and Science of Russian Federation, grant No. 2010-1.3.1-111-017-029. The work of L. R. was partially supported by RFBR grants 07-01-92214-CNRSL-a, 05-01-02805-CNRSL-a, 09-01-00242, the Science Foundation of the SU-HSE awards No.T3-62.0 and 10-09-0015, and the Ministry of Education and Science of Russian Federation, grant No. 2010-1.3.1-111-017-029. He gratefully acknowledges the support from Deligne 2004 Balzan prize in mathematics.

2 Local Laumon spaces

2.1 Laumon spaces

We recall the setup of [2,8]. Let $\mathbf{C}$ be a smooth projective curve of genus zero. We fix a coordinate z on $\mathbf{C}$, and consider the action of $\mathbb{C}^*$ on $\mathbf{C}$ such that $v(z) = v^{-2}z$. We have $\mathbf{C}^{\mathbb{C}^*} = \{0, \infty\}$.

We consider an n-dimensional vector space W with a basis $w_1, \ldots, w_n$. This defines a Cartan torus $T \subset G = GL_n \subset Aut(W)$. We also consider its 2^n-fold cover, the bigger torus $\widetilde{T}$, acting on W as follows: for $\widetilde{T} \ni \underline{t} = (t_1, \ldots, t_n)$ we have $\underline{t}(w_i) = t_i^2 w_i$. We denote by $\mathcal{B}$ the flag variety of G.

Given an $(n-1)$-tuple of nonnegative integers $\underline{d} = (d_1, \ldots, d_{n-1})$, we consider the Laumon's quasiflags' space $\mathcal{Q}_{\underline{d}}$, see [14], 4.2. It is the moduli space of flags of locally free subsheaves

$$0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W} = W \otimes \mathcal{O}_{\mathbf{C}}$$

such that $\operatorname{rank}(\mathcal{W}_k) = k$, and $\deg(\mathcal{W}_k) = -d_k$.

It is known to be a smooth projective variety of dimension $2d_1 + \cdots + 2d_{n-1} + \dim \mathcal{B}$, see [15], 2.10.

We consider the following locally closed subvariety $\mathfrak{Q}_{\underline{d}} \subset \mathcal{Q}_{\underline{d}}$ (quasiflags based at $\infty \in \mathbf{C}$) formed by the flags

$$0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W} = W \otimes \mathcal{O}_{\mathbf{C}}$$

such that $\mathcal{W}_i \subset \mathcal{W}$ is a vector subbundle in a neighborhood of $\infty \in \mathbf{C}$, and the fiber of $\mathcal{W}_i$ at ∞ equals the span $\langle w_1, \ldots, w_i \rangle \subset W$.

It is known to be a smooth quasiprojective variety of dimension $2d_1 + \cdots + 2d_{n-1}$.

2.2 Fixed points

The group $G \times \mathbb{C}^*$ acts naturally on $\mathcal{Q}_{\underline{d}}$, and the group $\widetilde{T} \times \mathbb{C}^*$ acts naturally on $\mathfrak{Q}_{\underline{d}}$. The set of fixed points of $\widetilde{T} \times \mathbb{C}^*$ on $\mathfrak{Q}_{\underline{d}}$ is finite; we recall its description from [8], 2.11.

Let $\widetilde{\underline{d}}$ be a collection of nonnegative integers $(d_{ij}),\ i \geq j$, such that $d_i = \sum_{j=1}^{i} d_{ij}$, and for $i \geq k \geq j$ we have $d_{kj} \geq d_{ij}$. Abusing notation we denote by $\widetilde{\underline{d}}$ the corresponding $\widetilde{\widetilde{T}} \times \mathbb{C}^*$-fixed point in $\mathfrak{Q}_{\underline{d}}$:
$\mathcal{W}_1 = \mathcal{O}_{\mathbf{C}}(-d_{11} \cdot 0)w_1,$
$\mathcal{W}_2 = \mathcal{O}_{\mathbf{C}}(-d_{21} \cdot 0)w_1 \oplus \mathcal{O}_{\mathbf{C}}(-d_{22} \cdot 0)w_2,$
$\dots\dots\dots,$
$\mathcal{W}_{n-1} = \mathcal{O}_{\mathbf{C}}(-d_{n-1,1} \cdot 0)w_1 \oplus \mathcal{O}_{\mathbf{C}}(-d_{n-1,2} \cdot 0)w_2 \oplus \dots \oplus \mathcal{O}_{\mathbf{C}}(-d_{n-1,n-1} \cdot 0)w_{n-1}.$

2.3 Correspondences

For $i \in \{1, \dots, n-1\}$, and $\underline{d} = (d_1, \dots, d_{n-1})$, we set $\underline{d} + i := (d_1, \dots, d_i + 1, \dots, d_{n-1})$. We have a correspondence $\mathcal{E}_{\underline{d},i} \subset \mathcal{Q}_{\underline{d}} \times \mathcal{Q}_{\underline{d}+i}$ formed by the pairs $(\mathcal{W}_\bullet, \mathcal{W}'_\bullet)$ such that for $j \neq i$ we have $\mathcal{W}_j = \mathcal{W}'_j$, and $\mathcal{W}'_i \subset \mathcal{W}_i$, see [8], 3.1. In other words, $\mathcal{E}_{\underline{d},i}$ is the moduli space of flags of locally free sheaves

$$0 \subset \mathcal{W}_1 \subset \dots \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \dots \subset \mathcal{W}_{n-1} \subset \mathcal{W}$$

such that $\operatorname{rank}(\mathcal{W}_k) = k$, and $\deg(\mathcal{W}_k) = -d_k$, while $\operatorname{rank}(\mathcal{W}'_i) = i$, and $\deg(\mathcal{W}'_i) = -d_i - 1$.

According to [15], 2.10, $\mathcal{E}_{\underline{d},i}$ is a smooth projective algebraic variety of dimension $2d_1 + \dots + 2d_{n-1} + \dim \mathcal{B} + 1$.

We denote by $\mathbf{p}$ (resp. $\mathbf{q}$) the natural projection $\mathcal{E}_{\underline{d},i} \to \mathcal{Q}_{\underline{d}}$ (resp. $\mathcal{E}_{\underline{d},i} \to \mathcal{Q}_{\underline{d}+i}$). We also have a map $\mathbf{r}:\ \mathcal{E}_{\underline{d},i} \to \mathbf{C}$,

$$(0 \subset \mathcal{W}_1 \subset \dots \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \dots \subset \mathcal{W}_{n-1} \subset \mathcal{W}) \mapsto \operatorname{supp}(\mathcal{W}_i/\mathcal{W}'_i).$$

The correspondence $\mathcal{E}_{\underline{d},i}$ comes equipped with a natural line bundle $\mathcal{L}_i$ whose fiber at a point

$$(0 \subset \mathcal{W}_1 \subset \dots \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \dots \subset \mathcal{W}_{n-1} \subset \mathcal{W})$$

equals $\Gamma(\mathbf{C}, \mathcal{W}_i/\mathcal{W}'_i)$.

Finally, we have a transposed correspondence ${}^{\mathsf{T}}\mathcal{E}_{\underline{d},i} \subset \mathcal{Q}_{\underline{d}+i} \times \mathcal{Q}_{\underline{d}}$.

Restricting to $\mathfrak{Q}_{\underline{d}} \subset \mathcal{Q}_{\underline{d}}$ we obtain the correspondence $\mathfrak{E}_{\underline{d},i} \subset \mathfrak{Q}_{\underline{d}} \times \mathfrak{Q}_{\underline{d}+i}$ together with line bundle $\mathfrak{L}_i$ and the natural maps $\mathbf{p}:\ \mathfrak{E}_{\underline{d},i} \to \mathfrak{Q}_{\underline{d}},\ \mathbf{q}:\ \mathfrak{E}_{\underline{d},i} \to \mathfrak{Q}_{\underline{d}+i},\ \mathbf{r}:\ \mathfrak{E}_{\underline{d},i} \to \mathbf{C} - \infty$. We also have a transposed correspondence ${}^{\mathsf{T}}\mathfrak{E}_{\underline{d},i} \subset \mathfrak{Q}_{\underline{d}+i} \times \mathfrak{Q}_{\underline{d}}$. It is a smooth quasiprojective variety of dimension $2d_1 + \dots + 2d_{n-1} + 1$.

2.4 Equivariant cohomology

We denote by ${}'V$ the direct sum of equivariant (complexified) cohomology: ${}'V = \oplus_{\underline{d}} H^\bullet_{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$. It is a module over $H^\bullet_{\widetilde{T} \times \mathbb{C}^*}(pt) = \mathbb{C}[\mathfrak{t} \oplus \mathbb{C}] = \mathbb{C}[x_1, \dots, x_n, \hbar]$. Here $\mathfrak{t} \oplus \mathbb{C}$ is the Lie algebra of $\widetilde{T} \times \mathbb{C}^*$. We define $\hbar$ as twice the positive generator of $H^2_{\mathbb{C}^*}(pt,\mathbb{Z})$. Similarly, we define $x_i \in H^2_{\widetilde{T}}(pt,\mathbb{Z})$ in terms of the corresponding one-parametric subgroup. We define $V = {}'V \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt))$.

We have an evident grading
$V = \oplus_{\underline{d}} V_{\underline{d}},\ V_{\underline{d}} = H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt))$.

2.5 Universal Verma module

We denote by $\mathfrak{U}$ the universal enveloping algebra of $\mathfrak{gl}_n$ over the field $\mathbb{C}(\mathfrak{t}\oplus\mathbb{C})$. For $1 \le j,k \le n$ we denote by $E_{jk} \in \mathfrak{gl}_n \subset \mathfrak{U}$ the usual elementary matrix. The standard Chevalley generators are expressed as follows:

$$\mathfrak{e}_i := E_{i+1,i},\ \mathfrak{f}_i := E_{i,i+1},\ \mathfrak{h}_i := E_{i+1,i+1} - E_{ii}$$

(note that $\mathfrak{e}_i$ is represented by a *lower* triangular matrix). Note also that $\mathfrak{U}$ is generated by $E_{ii},\ 1 \le i \le n,\ E_{i,i+1}, E_{i+1,i},\ 1 \le i \le n-1$. We denote by $\mathfrak{U}_{\le 0}$ the subalgebra of $\mathfrak{U}$ generated by $E_{ii},\ 1 \le i \le n,\ E_{i,i+1},\ 1 \le i \le n-1$. It acts on the field $\mathbb{C}(\mathfrak{t}\oplus\mathbb{C})$ as follows: $E_{i,i+1}$ acts trivially for any $1 \le i \le n-1$, and E_{ii} acts by multiplication by $\hbar^{-1}x_i + i - 1$. We define the *universal Verma module* $\mathfrak{V}$ over $\mathfrak{U}$ as $\mathfrak{U}\otimes_{\mathfrak{U}_{\le 0}} \mathbb{C}(\mathfrak{t}\oplus\mathbb{C})$. The universal Verma module $\mathfrak{V}$ is an irreducible $\mathfrak{U}$-module.

2.6 The action of generators

The grading and the correspondences ${}^{\mathsf{T}}\mathfrak{E}_{\underline{d},i}$, $\mathfrak{E}_{\underline{d},i}$ give rise to the following operators on V (note that though $\mathbf{p}$ is not proper, $\mathbf{p}_*$ is well defined on the localized equivariant cohomology due to the finiteness of the fixed point sets and smoothness of $\mathfrak{E}_{\underline{d},i}$):

$E_{ii} = \hbar^{-1}x_i + d_{i-1} - d_i + i - 1 :\ V_{\underline{d}} \to V_{\underline{d}}$;
$\mathfrak{h}_i = \hbar^{-1}(x_{i+1} - x_i) + 2d_i - d_{i-1} - d_{i+1} + 1 :\ V_{\underline{d}} \to V_{\underline{d}}$;
$\mathfrak{f}_i = E_{i,i+1} = \mathbf{p}_*\mathbf{q}^* :\ V_{\underline{d}} \to V_{\underline{d}-i}$;
$\mathfrak{e}_i = E_{i+1,i} = -\mathbf{q}_*\mathbf{p}^* :\ V_{\underline{d}} \to V_{\underline{d}+i}$.

Theorem 2.7 *The operators $\mathfrak{e}_i = E_{i+1,i}, E_{ii}, \mathfrak{f}_i = E_{i,i+1}$ on V defined in 2.6 satisfy the relations in $\mathfrak{U}$, i.e. they give rise to the action of $\mathfrak{U}$ on V. There is a unique isomorphism Ψ of $\mathfrak{U}$-modules V and $\mathfrak{V}$ carrying $1 \in H^0_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_0) \subset V$ to the lowest weight vector $1 \in \mathbb{C}(\mathfrak{t}\oplus\mathbb{C}) \subset \mathfrak{V}$.*

The proof is entirely similar to the proof of Theorem 2.12 of [2]; cf. also [20].

2.8 Fixed point basis

According to the Localization theorem in equivariant cohomology (see e.g. [3]), restriction to the $\widetilde{T}\times\mathbb{C}^*$-fixed point set induces an isomorphism

$$H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)) \to H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}^{\widetilde{T}\times\mathbb{C}^*}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt))$$

The fundamental cycles $[\widetilde{\underline{d}}]$ of the $\widetilde{T}\times\mathbb{C}^*$-fixed points $\widetilde{\underline{d}}$ (see 2.2) form a basis in $\oplus_{\underline{d}} H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}^{\widetilde{T}\times\mathbb{C}^*})\otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)}\mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt))$. The embedding of a point $\widetilde{\underline{d}}$ into $\mathfrak{Q}_{\underline{d}}$ is a proper morphism, so the direct image in the equivariant cohomology is well defined, and we will denote by $[\widetilde{\underline{d}}]\in V_{\underline{d}}$ the direct image of the fundamental cycle of the point $\widetilde{\underline{d}}$. The set $\{[\widetilde{\underline{d}}]\}$ forms a basis of V.

The matrix coefficients of the operators $\mathfrak{e}_i,\mathfrak{f}_i$ in the basis $\{[\widetilde{\underline{d}}]\}$ were computed in [2]; cf. also [20] 8.2. The result is:

Proposition 2.9 *The matrix coefficients of the operators $\mathfrak{e}_i,\mathfrak{f}_i$ in the basis $\{[\widetilde{\underline{d}}]\}$ are as follows:*

$$\mathfrak{e}_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']}=-\hbar^{-1}\prod_{j\neq k\leq i}(x_j-x_k+(d_{i,k}-d_{i,j})\hbar)^{-1}\prod_{k\leq i-1}(x_j-x_k+(d_{i-1,k}-d_{i,j})\hbar)$$

if $d'_{i,j}=d_{i,j}+1$ for certain $j\leq i$;

$$\mathfrak{f}_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']}=\hbar^{-1}\prod_{j\neq k\leq i}(x_k-x_j+(d_{i,j}-d_{i,k})\hbar)^{-1}\prod_{k\leq i+1}(x_k-x_j+(d_{i,j}-d_{i+1,k})\hbar)$$

if $d'_{i,j}=d_{i,j}-1$ for certain $j\leq i$;

All the other matrix coefficients of $\mathfrak{e}_i,\mathfrak{f}_i$ vanish.

The proof is entirely similar to that of Corollary 2.20 of [2].

2.10 Gelfand–Tsetlin basis of the universal Verma module

We will follow the notations of [18] on the Gelfand–Tsetlin bases in representations of $\mathfrak{gl}_n$. To a collection $\widetilde{\underline{d}}=(d_{ij}),\ n-1\geq i\geq j$ we associate a *Gelfand–Tsetlin pattern* $\Lambda=\Lambda(\widetilde{\underline{d}}):=(\lambda_{ij}),\ n\geq i\geq j$ as follows: $\lambda_{nj}:=\hbar^{-1}x_j+j-1,\ n\geq j\geq 1;\ \lambda_{ij}:=\hbar^{-1}x_j+j-1-d_{ij},\ n-1\geq i\geq j\geq 1$. Now we define $\mathfrak{V}\ni\xi_{\widetilde{\underline{d}}}=\xi_\Lambda:=(-\hbar)^{-|d|}\Psi[\widetilde{\underline{d}}]$. According to Proposition 2.9, the matrix coefficients of the operators $\mathfrak{e}_i,\mathfrak{f}_i$ in the basis $\{\xi_{\widetilde{\underline{d}}}\}$ are as follows:

$$\mathfrak{e}_{i,\Lambda(\widetilde{\underline{d}}),\Lambda(\widetilde{\underline{d}}')}=\prod_{j\neq k\leq i}(x_j-x_k+(d_{i,k}-d_{i,j})\hbar)^{-1}\prod_{k\leq i-1}(x_j-x_k+(d_{i-1,k}-d_{i,j})\hbar)$$

if $d'_{i,j}=d_{i,j}+1$ for certain $j\leq i$;

$$\mathfrak{f}_{i,\Lambda(\widetilde{\underline{d}}),\Lambda(\widetilde{\underline{d}}')}=-\hbar^{-2}\prod_{j\neq k\leq i}(x_k-x_j+(d_{i,j}-d_{i,k})\hbar)^{-1}\prod_{k\leq i+1}(x_k-x_j+(d_{i,j}-d_{i+1,k})\hbar)$$

if $d'_{i,j}=d_{i,j}-1$ for certain $j\leq i$;

All the other matrix coefficients of $\mathfrak{e}_i,\mathfrak{f}_i$ vanish.

The above matrix coefficients, under appropriate specialization of $x_1, \dots, x_n$, coincide with the matrix coefficients of $\mathfrak{e}_i, \mathfrak{f}_i$ in the *Gelfand–Tsetlin* basis of an irreducible finite dimensional $\mathfrak{gl}_n$-module, cf. formulas (2.7), (2.6) of Theorem 2.3 of [18]. For this reason we suggest to call the basis $\{\xi_{\widetilde{\underline{d}}}\}$ (over all collections $\widetilde{\underline{d}}$) of $\mathfrak{V}$ the *Gelfand–Tsetlin basis*. Algebraically, $\xi_{\widetilde{\underline{d}}} = \xi_\Lambda \in \mathfrak{V}$ can be defined by the formulas (2.9)–(2.11) of [18] (where $\xi = \xi_0 = 1 \in \mathfrak{V}$). Up to proportionality, the Gelfand–Tsetlin basis can also be defined as an eigenbasis of the Gelfand–Tsetlin subalgebra of $\mathfrak{U}$.

For a future reference, let us formulate once again the relation between the fixed point base of V and the Gelfand–Tsetlin base of $\mathfrak{V}$:

Theorem 2.11 *The isomorphism $\Psi : V \xrightarrow{\sim} \mathfrak{V}$ of Theorem 2.7 takes $[\widetilde{\underline{d}}]$ to $(-\hbar)^{|\underline{d}|}\xi_{\widetilde{\underline{d}}}$ where $|\underline{d}| = d_1 + \cdots + d_{n-1}$.*

Remark 2.12 One can prove that the isomorphism $\Psi : V \xrightarrow{\sim} \mathfrak{V}$ of Theorem 2.7 takes $[\widetilde{\underline{d}}]$ to $\xi_{\widetilde{\underline{d}}}$ up to proportionality without explicitly computing the matrix coefficients. In effect, the Gelfand–Tsetlin basis is uniquely (up to proportionality) characterized by the property that the matrix coefficients of $\mathfrak{e}_k, \mathfrak{f}_k$ with respect to $\xi_\Lambda, \xi_{\Lambda'}$ vanish if $\lambda_{ij} \neq \lambda'_{ij}$ for some $i > k$. Now it is immediate to see that the matrix coefficients of $\mathfrak{e}_k, \mathfrak{f}_k$ with respect to $[\widetilde{\underline{d}}], [\widetilde{\underline{d}}']$ vanish if $d_{ij} \neq d'_{ij}$ for some $i > k$.

2.13 Determinant line bundles

We consider the line bundle $\mathcal{D}_k$ on $\mathfrak{Q}_{\underline{d}}$ whose fiber at the point $(\mathcal{W}_\bullet)$ equals $\det R\Gamma(\mathbf{C}, \mathcal{W}_k)$.

Lemma 2.14 *$\mathcal{D}_k$ is a $\widetilde{T} \times \mathbb{C}^*$-equivariant line bundle, and the character of $\widetilde{T} \times \mathbb{C}^*$ acting in the fiber of $\mathcal{D}_k$ at a point $\widetilde{\underline{d}} = (d_{ij})$ equals $\sum_{j \le k}(1 - d_{kj})x_j + \frac{d_{kj}(d_{kj}-1)}{2}\hbar$.*

Proof Straightforward. □

Let $Cas_k = \sum_{i,j=1}^{k} E_{ij}E_{ji}$ be the quadratic Casimir element of $U(\mathfrak{gl}_k)$ naturally embedded into $U(\mathfrak{gl}_n) \subset \mathfrak{U}$. The operator Cas_k is diagonal in the Gelfand–Tsetlin basis, and the eigenvalue of Cas_k on the basis vector $\xi_{\widetilde{\underline{d}}} = \xi_\Lambda$ is $\sum_{j \le k} \lambda_{kj}(\lambda_{kj} + k - 2j + 1)$. We define the following element of $\mathfrak{U}$:

$$\widetilde{Cas_k} := Cas_k + (2-k)\sum_{j=1}^{k} E_{jj} - \sum_{j=1}^{k} \hbar^{-1}x_j(\hbar^{-1}x_j - 1) + \frac{k(k-1)(k-2)}{3}.$$

The eigenvalue of this element on the basis vector $\xi_{\widetilde{\underline{d}}}$ is $\sum_{j \le k} 2(1 - d_{kj})x_j\hbar^{-1} + d_{kj}(d_{kj} - 1)$.

Corollary 2.15 a) *The operator of multiplication by the first Chern class $c_1(\mathcal{D}_k)$ in V is diagonal in the basis $\{[\widetilde{\underline{d}}]\}$, and the eigenvalue corresponding to $\widetilde{\underline{d}} = (d_{ij})$ equals $\sum_{j \le k}(1 - d_{kj})x_j + \frac{d_{kj}(d_{kj}-1)}{2}\hbar$.*

b) *The set* $\{c_1(\mathcal{D}_k) : k \geq 2,\ d_k \neq 0 \neq d_{k-1}\}$ *forms a basis in the nonequivariant cohomology* $H^2(\mathfrak{Q}_{\underline{d}})$.

c) *The isomorphism* $\Psi : V \stackrel{\sim}{\longrightarrow} \mathfrak{V}$ *carries the operator of multiplication by* $c_1(\mathcal{D}_k)$ *to the operator* $\frac{\hbar}{2}\widetilde{Cas_k}$.

Proof a) follows from Lemma 2.14.

b) It follows e.g. from [8] that $\dim H^2(\mathfrak{Q}_{\underline{d}}) = \sharp\{k \geq 2,\ d_k \neq 0 \neq d_{k-1}\}$. Now it is easy to see from Lemma 2.14 that the classes $\{[\mathcal{D}_k] : k \geq 2,\ d_k \neq 0 \neq d_{k-1}\}$ in $\mathrm{Pic}(\mathfrak{Q}_{\underline{d}})$ are linearly independent, and hence the classes $\{c_1(\mathcal{D}_k) : k \geq 2,\ d_k \neq 0 \neq d_{k-1}\}$ are linearly independent in $H^2(\mathfrak{Q}_{\underline{d}})$.

c) Straightforward from a) and formula for eigenvalue of $\widetilde{Cas_k}$ on ξ_Λ. □

2.16 Gelfand–Tsetlin subalgebra and cohomology rings

It is known that a completion $\widehat{\mathfrak{V}}$ of the universal Verma module $\mathfrak{V}$ contains a unique *Whittaker vector* $\mathfrak{v} = \sum_{\underline{d}} \mathfrak{v}_{\underline{d}}$ such that $\mathfrak{v}_0 = 1$ (the lowest weight vector), and $\mathfrak{f}_i\mathfrak{v} = \hbar^{-1}\mathfrak{v}$ for any $1 \leq i \leq n-1$. Let us denote by $1_{\underline{d}} \in H^0_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \subset V_{\underline{d}}$ the unit element of the cohomology ring. Then $\Psi(1_{\underline{d}}) = \mathfrak{v}_{\underline{d}}$. The proof is entirely similar to the proof of Proposition 2.31 of [2], and goes back to [1].

Recall that the Gelfand–Tsetlin subalgebra $\mathfrak{G} \subset \mathrm{End}(\mathfrak{V})$ is generated by the Harish-Chandra centers of the universal enveloping algebras $\mathfrak{gl}_1, \mathfrak{gl}_2, \ldots, \mathfrak{gl}_n$ (embedded into $\mathfrak{gl}_n$ as the upper left blocks) over the field $\mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$. We denote by $\mathfrak{J}_{\underline{d}} \subset \mathfrak{G}$ the annihilator ideal of the vector $\mathfrak{v}_{\underline{d}} \in \mathfrak{V}$, and we denote by $\mathfrak{G}_{\underline{d}}$ the quotient algebra of $\mathfrak{G}$ by $\mathfrak{J}_{\underline{d}}$. The action of $\mathfrak{G}$ on $\mathfrak{v}_{\underline{d}}$ gives rise to an embedding $\mathfrak{G}_{\underline{d}} \hookrightarrow \mathfrak{V}_{\underline{d}}$.

Proposition 2.17 a) $\mathfrak{G}_{\underline{d}} \stackrel{\sim}{\longrightarrow} \mathfrak{V}_{\underline{d}}$.

b) *The composite morphism* $\Psi : H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{\mathbb{C}[\mathfrak{t}\oplus\mathbb{C}]} \mathbb{C}(\mathfrak{t}\oplus\mathbb{C}) = V_{\underline{d}} \stackrel{\sim}{\longrightarrow} \mathfrak{V}_{\underline{d}} \stackrel{\sim}{\longrightarrow} \mathfrak{G}_{\underline{d}}$ *is an algebra isomorphism.*

c) *The algebra* $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{\mathbb{C}[\mathfrak{t}\oplus\mathbb{C}]} \mathbb{C}(\mathfrak{t}\oplus\mathbb{C})$ *is generated by* $\{c_1(\mathcal{D}_k) : k \geq 2,\ d_k \neq 0 \neq d_{k-1}\}$.

Proof c) The algebra $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{\mathbb{C}[\mathfrak{t}\oplus\mathbb{C}]} \mathbb{C}(\mathfrak{t}\oplus\mathbb{C})$ consists of operators on the space $V_{\underline{d}}$ which are diagonal in the basis of fixed points $[\widetilde{\underline{d}}]$. On the other hand, the operators $Cas_k \in \mathfrak{G}$, $k \geq 2$, are diagonal in the Gelfand–Tsetlin basis $\xi_{\widetilde{\underline{d}}}$ and have different joint eigenvalues on different $\xi_{\widetilde{\underline{d}}}$. Hence, the images of Cas_k in $\mathrm{End}(\mathfrak{V}_{\underline{d}})$ generate the algebra of operators which are diagonal in the Gelfand–Tsetlin basis, and in particular, the images of Cas_k, $k \geq 2$, in $\mathfrak{G}_{\underline{d}}$ generate $\mathfrak{G}_{\underline{d}}$. By Theorem 2.11, the isomorphism $\Psi : V_{\underline{d}} \to \mathfrak{V}_{\underline{d}}$ carries $[\widetilde{\underline{d}}]$ to $(-\hbar)^{|\underline{d}|}\xi_{\widetilde{\underline{d}}}$. By Corollary 2.15, $c_1(\mathcal{D}_k)$ is $\Psi^{-1}(\frac{\hbar}{2}Cas_k)$ up to an additive constant. Hence, the elements $c_1(\mathcal{D}_k) = \Psi^{-1}(\frac{\hbar}{2}Cas_k) + const \in H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{\mathbb{C}[\mathfrak{t}\oplus\mathbb{C}]} \mathbb{C}(\mathfrak{t}\oplus\mathbb{C})$ generate the algebra $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{\mathbb{C}[\mathfrak{t}\oplus\mathbb{C}]} \mathbb{C}(\mathfrak{t}\oplus\mathbb{C})$.

a-b) Since $\Psi(1_{\underline{d}}) = \mathfrak{v}_{\underline{d}}$, the (surjective) homomorphism $\Psi^{-1} : \mathbb{C}[Cas_2, \ldots, Cas_{n-1}] \to H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{\mathbb{C}[\mathfrak{t}\oplus\mathbb{C}]} \mathbb{C}(\mathfrak{t}\oplus\mathbb{C})$ factors through $\mathfrak{G}_{\underline{d}}$. Hence (a) and (b). □

3 Integral forms

3.1 Renormalized universal enveloping algebra

We denote by $\underline{\mathfrak{U}} \subset \mathfrak{U}$ the $\mathbb{C}[\mathfrak{t}\oplus\mathbb{C}]$-subalgebra generated by the set $\{\underline{E}_{ij} := \hbar E_{ij},\ 1 \leq i < j \leq n;\ E_{ij},\ 1 \leq j < i \leq n;\ E_{ii}' := E_{ii} - \hbar^{-1}x_i,\ i = 1, \ldots, n\}$. We denote by $\underline{\mathfrak{U}}_{\leq 0}$ the subalgebra of $\underline{\mathfrak{U}}$ generated by $\{E_{ii}',\ 1 \leq i \leq n;\ \underline{E}_{ij},\ 1 \leq i < j \leq n\}$. It acts on the ring $\mathbb{C}[\mathfrak{t} \oplus \mathbb{C}]$ as follows: $\underline{E}_{ij}$ acts trivially for any $i < j$, and E_{ii}' acts by multiplication by $i - 1$. We define the integral form of the universal Verma module $\underline{\mathfrak{V}} \subset \mathfrak{V}$ over $\underline{\mathfrak{U}}$ as $\underline{\mathfrak{V}} := \underline{\mathfrak{U}} \otimes_{\underline{\mathfrak{U}}_{\leq 0}} \mathbb{C}[\mathfrak{t}\oplus\mathbb{C}]$. We define the integral form of the universal dual Verma module $\underline{\mathfrak{V}}^* \subset \mathfrak{V}$ as $\underline{\mathfrak{V}}^* := \{u \in \mathfrak{V} :\ (u, u') \in \mathbb{C}[\mathfrak{t} \oplus \mathbb{C}]$ for any $u' \in \underline{\mathfrak{V}}\}$ (where (u, u') stands for the Shapovalov form). Clearly, $\underline{\mathfrak{V}}^*$ is a $\underline{\mathfrak{U}}$-module.

Note that the Whittaker vector $\mathfrak{v} \in \widehat{\mathfrak{V}}$ lies inside the completion of $\underline{\mathfrak{V}}^*$ and is uniquely characterized by the properties a) $\underline{\mathfrak{f}}_i\mathfrak{v} = \mathfrak{v}$ where $\underline{\mathfrak{f}}_i := \underline{E}_{i,i+1}$; b) the highest weight component of $\mathfrak{v}$ equals $1 \in \mathbb{C}[\mathfrak{t} \oplus \mathbb{C}]$.

Finally, we denote by $\underline{\mathfrak{G}} \subset \mathfrak{G}$ the integral form of the Gelfand–Tsetlin subalgebra, generated by the centers of the algebras $\underline{\mathfrak{U}}_1, \underline{\mathfrak{U}}_2, \ldots, \underline{\mathfrak{U}}_n = \underline{\mathfrak{U}}$ constructed from the Lie algebras $\mathfrak{gl}_1, \mathfrak{gl}_2, \ldots, \mathfrak{gl}_n$ (embedded into $\mathfrak{gl}_n$ as the upper left blocks) the same way as $\underline{\mathfrak{U}}$ is constructed from $\mathfrak{gl}_n$. Recall that the Harish-Chandra isomorphism identifies the center of $\mathfrak{U}(\mathfrak{gl}_k)$ with the ring of symmetric polynomials in k variables. Namely, to any symmetric polynomial P one assigns a central element $HC(P)$, whose PBW degree equals $\deg P$, acting on the Verma module with the highest weight $\lambda = (\lambda_1, \ldots, \lambda_k)$ as the scalar operator with the eigenvalue $P(\lambda_1, \ldots, \lambda_i - i + 1, \ldots, \lambda_k - k + 1)$. Clearly, the central element $\underline{HC}(P) := \hbar^{\deg P} HC(P)$ lies in $\underline{\mathfrak{U}}(\mathfrak{gl}_k)$. Moreover, the difference $\underline{HC}(P) - P(x_1, \ldots, x_k)$ is divisible by $\hbar$ in $\underline{\mathfrak{U}}(\mathfrak{gl}_k)$, hence $\hbar^{-1}(\underline{HC}(P) - P(x_1, \ldots, x_k))$ also lies in the center of $\underline{\mathfrak{U}}(\mathfrak{gl}_k)$.

We denote by $\underline{\mathfrak{I}}_d \subset \underline{\mathfrak{G}}$ the annihilator ideal of the vector $\mathfrak{v}_{\underline{d}} \in \underline{\mathfrak{V}}^*$, and we denote by $\underline{\mathfrak{G}}_{\underline{d}}$ the quotient algebra of $\underline{\mathfrak{G}}$ by $\underline{\mathfrak{I}}$. The action of $\underline{\mathfrak{G}}$ on $\mathfrak{v}_{\underline{d}}$ gives rise to an embedding $\underline{\mathfrak{G}}_{\underline{d}} \hookrightarrow \underline{\mathfrak{V}}^*_{\underline{d}}$.

Lemma 3.2 $\underline{\mathfrak{G}}_{\underline{d}} \xrightarrow{\sim} \underline{\mathfrak{V}}^*_{\underline{d}}$.

Proof By graded Nakayama lemma, it suffices to prove the surjectivity of $\underline{\mathfrak{G}}_{\underline{d}}/(x_1, \ldots, x_n, \hbar = 0) \to \underline{\mathfrak{V}}^*_{\underline{d}}/(x_1, \ldots, x_n, \hbar = 0)$. We denote by $\mathfrak{g}_{>0} \subset \mathfrak{gl}_n$ the Lie subalgebra spanned by the set $\{E_{ij},\ 1 \leq j < i \leq n\}$. We denote by $\mathfrak{g}_{\geq 0} \subset \mathfrak{gl}_n$ (resp. $\mathfrak{g}_{\leq 0} \subset \mathfrak{gl}_n$, $\mathfrak{g}_{>0} \subset \mathfrak{gl}_n$, $\mathfrak{g}_{<0} \subset \mathfrak{gl}_n$) the Lie subalgebra spanned by the set $\{E_{ij},\ 1 \leq j \leq i \leq n\}$ (resp. $\{E_{ij},\ 1 \leq i \leq j \leq n\}$, $\{E_{ij},\ 1 \leq j < i \leq n\}$, $\{E_{ij},\ 1 \leq i < j \leq n\}$). The Killing form identifies the vector space $\mathfrak{g}_{>0}$ with the dual of $\mathfrak{g}_{<0}$ and gives rise to an isomorphism $\mathrm{Sym}(\mathfrak{g}_{<0}) \simeq \mathbb{C}[\mathfrak{g}_{>0}]$. The universal enveloping algebra of $\mathfrak{g}_{\geq 0}$ over $\mathbb{C}[\mathfrak{t}]$ lies inside $\underline{\mathfrak{U}}$ and is denoted by $\underline{\mathfrak{U}}_{\geq 0}$. Evidently, $\underline{\mathfrak{U}}_{\geq 0} \simeq U(\mathfrak{g}_{\geq 0}) \otimes \mathbb{C}[\mathfrak{t}]$. We have $\underline{\mathfrak{U}}/(x_1, \ldots, x_n, \hbar = 0) \simeq \mathbb{C}[\mathfrak{g}_{>0}] \rtimes U(\mathfrak{g}_{\geq 0})$. Here, the semidirect product is taken with respect to the adjoint action of $\mathfrak{g}_{\geq 0}$ on $\mathfrak{g}_{>0}$ (and the induced action on the algebra of functions).

Let V denote the space of distributions on $\mathfrak{g}_{>0}$ supported at the origin, that is cohomology with support of the structure sheaf $H_{\{0\}}^{\frac{n(n-1)}{2}}(\mathfrak{g}_{>0}, \mathcal{O})$. The algebra $\mathbb{C}[\mathfrak{g}_{>0}] \rtimes U(\mathfrak{g}_{\geq 0})$ acts on V naturally. As a $\mathbb{C}[\mathfrak{g}_{>0}]$-module, V is cofree, and its completion is naturally isomorphic to $\mathbb{C}[\mathfrak{g}_{>0}]^*$. Clearly, $\mathfrak{V}^*_{\underline{d}}/(x_1,\ldots,x_n,\hbar=0)$ as a module over $\underline{\mathfrak{U}}/(x_1,\ldots,x_n,\hbar=0)\simeq\mathbb{C}[\mathfrak{g}_{>0}]\rtimes U(\mathfrak{g}_{\geq 0})$ is isomorphic to V. The value of the Whittaker vector $\mathfrak{v}|_{\hbar=0}$ in the completion of V is the functional $\chi:\mathbb{C}[\mathfrak{g}_{>0}]\to\mathbb{C}$ which sends $P\in\mathbb{C}[\mathfrak{g}_{>0}]$ to $P(\mathfrak{f})\in\mathbb{C}$, where $\mathfrak{f}=\sum_{i=1}^{n-1}\mathfrak{f}_i$ is the principal nilpotent element. The adjoint $G_{\geq 0}$-orbit of $\mathfrak{f}$ is dense in $\mathfrak{g}_{>0}$; hence, the submodule generated by the Whittaker vector is dense in the completion of V. This means that each weight space of V is generated by the component of the Whittaker vector.

Consider the Whittaker module W over $\mathbb{C}[\mathfrak{g}_{>0}]\rtimes U(\mathfrak{g}_{\geq 0})$, that is, induced from the character $\chi:\mathbb{C}[\mathfrak{g}_{>0}]\to\mathbb{C}$. The module W is free with respect to $U(\mathfrak{g}_{\geq 0})$ and hence has natural filtration coming from the PBW filtration on $U(\mathfrak{g}_{\geq 0})$. The associated graded $\mathrm{gr}W$ is naturally a $\mathbb{C}[\mathfrak{g}_{\geq 0}]\otimes S(\mathfrak{g}_{>0})=\mathbb{C}[\mathfrak{g}]$-module. It is easy to see that $\mathrm{gr}W=\mathbb{C}[\mathfrak{f}+\mathfrak{g}_{\geq 0}]$. The restriction of the Gelfand–Tsetlin subalgebra in $\mathbb{C}[\mathfrak{g}]$ to the affine subspace $\mathfrak{f}+\mathfrak{g}_{\geq 0}$ is known to be an isomorphism onto $\mathbb{C}[\mathfrak{f}+\mathfrak{g}_{\geq 0}]$ (see [12,24]). Thus, the module W is generated by the Whittaker vector as a $\underline{\mathfrak{G}}_{\underline{d}}/(x_1,\ldots,x_n,\hbar=0)$-module. Hence the $\underline{\mathfrak{G}}_{\underline{d}}/(x_1,\ldots,x_n,\hbar=0)$-submodule generated by the Whittaker vector $\mathfrak{v}$ is dense in the completion of V. This means that each weight space of V is generated by the component of the Whittaker vector with respect to the action of the Gelfand–Tsetlin subalgebra. □

3.3 Kuznetsov correspondences

We consider a correspondence $\underline{\mathfrak{E}}_{\underline{d},i}\subset\mathfrak{Q}_{\underline{d}}\times\mathfrak{Q}_{\underline{d}+i}$ defined as $\mathbf{r}^{-1}\{0\}$. In other words, $\underline{\mathfrak{E}}_{\underline{d},i}$ is a closed subvariety of $\mathfrak{E}_{\underline{d},i}$ where we impose a condition that the quotient flag is supported at $\{0\}\in\mathbf{C}$. It is a smooth quasiprojective variety of dimension $2d_1+\cdots+2d_{n-1}$. We denote by $\underline{\mathbf{p}}:\underline{\mathfrak{E}}_{\underline{d},i}\to\mathfrak{Q}_{\underline{d}}$, $\underline{\mathbf{q}}:\underline{\mathfrak{E}}_{\underline{d},i}\to\mathfrak{Q}_{\underline{d}+i}$ the natural projections. Note that both $\underline{\mathbf{p}}$ and $\underline{\mathbf{q}}$ are proper.

More generally, for $1\leq i<j\leq n$ we denote by $\underline{d}\pm\alpha_{ij}$ the sequence $(d_1,\ldots,d_{i-1},d_i\pm 1,\ldots,d_{j-1}\pm 1,d_j,\ldots,d_{n-1})$. We have a correspondence ${}^\circ\underline{\mathfrak{E}}_{\underline{d},\alpha_{ij}}\subset\mathfrak{Q}_{\underline{d}}\times\mathfrak{Q}_{\underline{d}+\alpha_{ij}}$ formed by the pairs $(\mathcal{W}_\bullet,\mathcal{W}'_\bullet)$ such that a) $\mathcal{W}'_k\subset\mathcal{W}_k$ for any $1\leq k\leq n-1$; b) The quotient $\mathcal{W}_\bullet/\mathcal{W}'_\bullet$ is supported at $\{0\}\in\mathbf{C}$; c) For $i\leq k<j$ the natural map $\mathcal{W}_k/\mathcal{W}'_k\to\mathcal{W}_{k+1}/\mathcal{W}'_{k+1}$ is an isomorphism (of one-dimensional vector spaces). We define a correspondence $\underline{\mathfrak{E}}_{\underline{d},\alpha_{ij}}\subset\mathfrak{Q}_{\underline{d}}\times\mathfrak{Q}_{\underline{d}+\alpha_{ij}}$ as the closure of ${}^\circ\underline{\mathfrak{E}}_{\underline{d},\alpha_{ij}}$. According to Lemma 5.2.1 of [8], $\underline{\mathfrak{E}}_{\underline{d},\alpha_{ij}}$ is irreducible of dimension $2d_1+\cdots+2d_{n-1}+j-i-1$. We denote by $\underline{\mathbf{p}}_{ij}:\underline{\mathfrak{E}}_{\underline{d},\alpha_{ij}}\to\mathfrak{Q}_{\underline{d}}$, $\underline{\mathbf{q}}_{ij}:\underline{\mathfrak{E}}_{\underline{d},\alpha_{ij}}\to\mathfrak{Q}_{\underline{d}+\alpha_{ij}}$ the natural projections. Note that both $\underline{\mathbf{p}}_{ij}$ and $\underline{\mathbf{q}}_{ij}$ are proper.

Also, we consider the correspondences $\mathfrak{E}_{\underline{d},\alpha_{ij}}\subset\mathfrak{Q}_{\underline{d}}\times\mathfrak{Q}_{\underline{d}+\alpha_{ij}}$ defined exactly as $\underline{\mathfrak{E}}_{\underline{d},\alpha_{ij}}\subset\mathfrak{Q}_{\underline{d}}\times\mathfrak{Q}_{\underline{d}+\alpha_{ij}}$ in 3.3 but with condition b) removed (i.e. we allow the quotient flag to be supported at an arbitrary point of $\mathbf{C}-\infty$). In particular, $\mathfrak{E}_{\underline{d},\alpha_{i,i+1}}=\mathfrak{E}_{\underline{d},i}$. We denote by $\mathbf{p}_{ij}:\mathfrak{E}_{\underline{d},\alpha_{ij}}\to\mathfrak{Q}_{\underline{d}}$, $\mathbf{q}_{ij}:\mathfrak{E}_{\underline{d},\alpha_{ij}}\to\mathfrak{Q}_{\underline{d}+\alpha_{ij}}$ the natural projections. Note that $\mathbf{q}_{ij}$ is proper, while $\mathbf{p}_{ij}$ is not.

3.4 The action of the renormalized universal enveloping algebra

Recall that ${}'V=\oplus_{\underline{d}}\,{}'V_{\underline{d}}:=\oplus_{\underline{d}}H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$. The grading and the correspondences $\underline{\mathfrak{E}}_{\underline{d},\alpha_{ij}}$ give rise to the following operators on ${}'V$:

$\underline{E}_{ii} = x_i + (d_{i-1} - d_i + i - 1)\hbar : {}'V_{\underline{d}} \to {}'V_{\underline{d}};$
$\mathfrak{h}_i = (x_{i+1} - x_i) + (2d_i - d_{i-1} - d_{i+1} + 1)\hbar : {}'V_{\underline{d}} \to {}'V_{\underline{d}};$
$\mathfrak{f}_i = \underline{E}_{i,i+1} = \underline{\mathbf{p}}_*\underline{\mathbf{q}}^* : {}'V_{\underline{d}} \to {}'V_{\underline{d}-i};$
$\mathfrak{e}_i = \underline{E}_{i+1,i} = -\underline{\mathbf{q}}_*\underline{\mathbf{p}}^* : {}'V_{\underline{d}} \to {}'V_{\underline{d}+i};$
$\underline{E}_{ij} = \underline{\mathbf{p}}_{ij*}\underline{\mathbf{q}}_{ij}^* : {}'V_{\underline{d}} \to {}'V_{\underline{d}-\alpha_{ij}}\ (1 \le i < j \le n);$
$\underline{E}_{ji} = (-1)^{j-i}\underline{\mathbf{q}}_{ij*}\underline{\mathbf{p}}_{ij}^* : {}'V_{\underline{d}} \to {}'V_{\underline{d}+\alpha_{ij}}\ (1 \le i < j \le n);$
$E_{ji} = (-1)^{j-i}\mathbf{q}_{ij*}\mathbf{p}_{ij}^* : {}'V_{\underline{d}} \to {}'V_{\underline{d}+\alpha_{ij}}\ (1 \le i < j \le n).$

Theorem 3.5 a) *The operators* $\{\underline{E}_{ij},\ 1 \le i \le j \le n;\ E_{ij},\ 1 \le j < i \le n\}$ *on* $'V$ *defined in 3.4 satisfy the relations in* $\underline{\mathfrak{U}}$*, i.e. they give rise to the action of* $\underline{\mathfrak{U}}$ *on* $'V$.
b) *There is a unique isomorphism* Φ *of* $\underline{\mathfrak{U}}$*-modules* $'V$ *and* $\underline{\mathfrak{V}}^*$ *carrying* $1 \in H^0_{\tilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_0) \subset {}'V$ *to the lowest weight vector* $1 \in \mathbb{C}[\mathfrak{t}\oplus\mathbb{C}] \subset \underline{\mathfrak{V}}^*$.

Proof a) We define the operators

$$E_{ij} = \mathbf{p}_{ij*}\mathbf{q}_{ij}^* : V_{\underline{d}} \to V_{\underline{d}-\alpha_{ij}}\ (1 \le i < j \le n). \tag{1}$$

It is clear that $\mathfrak{E}_{\underline{d},\alpha_{ij}} \simeq \underline{\mathfrak{E}}_{\underline{d},\alpha_{ij}} \times (\mathbf{C} - \infty)$. It follows that for any $1 \le i, j \le n$ we have $\underline{E}_{ij} = \hbar E_{ij}$. Furthermore, the operators $E_{i,i\pm1}$ are exactly those defined in 2.6, and they satisfy the relations of $\mathfrak{U}$ (and generate it) by Theorem 2.7. Finally, according to Proposition 5.6 of [8], the elements $E_{ij} \in \mathfrak{U},\ 1 \le i \ne j \le n$ act in V by the same named operators of (1) and 3.4. This proves a).

b) We have $'V \subset V \xrightarrow{\sim} \mathfrak{V} \supset \underline{\mathfrak{V}}^*$, so we have to check that $\Psi('V) = \underline{\mathfrak{V}}^*$, and then $\Phi = \Psi|_{'V}$. Recall that $\Psi(\mathfrak{v}_{\underline{d}}) = 1 \in H^0_{\tilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$. By the virtue of Lemma 3.2, it suffices to prove that $H^\bullet_{\tilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$ is generated by the action of the integral form $\underline{\mathfrak{G}}$ of the Gelfand–Tsetlin subalgebra on the vector $1 \in H^0_{\tilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$.

For any $1 \le i \le n-1$ we will denote by $\underline{\mathcal{W}}_i$ the tautological i-dimensional vector bundle on $\mathfrak{Q}_{\underline{d}} \times \mathbf{C}$. By the Künneth formula, we have $H^\bullet_{\tilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}} \times \mathbf{C}) = H^\bullet_{\tilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes 1 \oplus H^\bullet_{\tilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes \tau$ where $\tau \in H^2_{\mathbb{C}^*}(\mathbf{C})$ is the first Chern class of $\mathcal{O}(1)$. Under this decomposition, for the Chern class $c_j(\underline{\mathcal{W}}_i)$, we have $c_j(\underline{\mathcal{W}}_i) =: c_j^{(j)}(\underline{\mathcal{W}}_i) \otimes 1 + c_j^{(j-1)}(\underline{\mathcal{W}}_i) \otimes \tau$ where $c_j^{(j)}(\underline{\mathcal{W}}_i) \in H^{2j}_{\tilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$, and $c_j^{(j-1)}(\underline{\mathcal{W}}_i) \in H^{2j-2}_{\tilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$.

The following Lemma goes back to [5][1]:

Lemma 3.6 *The equivariant cohomology ring* $H^\bullet_{\tilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$ *is generated by the classes* $c_j^{(j)}(\underline{\mathcal{W}}_i),\ c_j^{(j-1)}(\underline{\mathcal{W}}_i),\ 1 \le j \le i \le n-1$ *(over the algebra* $\mathbb{C}[\mathfrak{t}\oplus\mathbb{C}]$*).*

Proof By the graded Nakayama lemma, it suffices to prove that the nonequivariant cohomology ring $H^\bullet(\mathfrak{Q}_{\underline{d}})$ is generated by the Künneth components of the (nonequivariant) Chern classes $c_j^{(j)}(\underline{\mathcal{W}}_i),\ c_j^{(j-1)}(\underline{\mathcal{W}}_i),\ 1 \le j \le i \le n-1$. The locally closed embedding $\mathfrak{Q}_{\underline{d}} \hookrightarrow \mathcal{Q}_{\underline{d}}$ induces a surjection on the cohomology rings (see e.g. the computation of cohomology of $\mathcal{Q}_{\underline{d}}$ in [8]), so it suffices to prove that the cohomology

[1] We have learnt of it from A. Marian.

ring of the compact smooth variety $H^\bullet(\mathcal{Q}_{\underline{d}})$ is generated by the Künneth components of the Chern classes of the tautological bundles. But this follows from Theorem 2.1 of [5], since the fundamental class of the diagonal in $\mathcal{Q}_{\underline{d}} \times \mathcal{Q}_{\underline{d}}$ can be expressed via the Chern classes of the tautological vector bundles (cf. [21], Sect. 5). □

Returning to the proof of the theorem, if suffices to check that the operators of multiplication by $c_j^{(j)}(\underline{\mathcal{W}}_i),\ c_j^{(j-1)}(\underline{\mathcal{W}}_i),\ 1 \le j \le i \le n-1$, in the equivariant cohomology ring $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) = {}'V_{\underline{d}}$ lie in the integral form $\underline{\mathfrak{G}}$ of the Gelfand–Tsetlin subalgebra. To this end, we compute these operators explicitly in the fixed point basis $\{[\underline{\widetilde{d}}]\}$ (alias Gelfand–Tsetlin basis $\{\xi_{\underline{\widetilde{d}}}\}$) of $V_{\underline{d}} = \mathfrak{V}_{\underline{d}}$.

The set of eigenvalues of $\mathfrak{t} \oplus \mathbb{C}$ in the fiber of $\underline{\mathcal{W}}_i$ at a point $(\underline{\widetilde{d}}, \infty)$ (resp. $(\underline{\widetilde{d}}, 0)$) equals $\{-x_1, \ldots, -x_i\}$ (resp. $\{-x_1 + d_{i,1}\hbar, \ldots, -x_i + d_{i,i}\hbar\}$). For $1 \le j \le i$, let e^∞_{ji} (resp. $e^0_{ji}(\underline{\widetilde{d}})$) stand for the sum of products of the j-tuples of distinct elements of the set $\{-x_1, \ldots, -x_i\}$ (resp. of the set $\{-x_1+d_{i,1}\hbar, \ldots, -x_i+d_{i,i}\hbar\}$). Then the operator of multiplication by the Chern class $c_j(\underline{\mathcal{W}}_i)$ is diagonal in the basis $\{[\underline{\widetilde{d}}, \infty], [\underline{\widetilde{d}}, 0]\}$ with eigenvalues $\{e^\infty_{ji}, e^0_{ji}(\underline{\widetilde{d}})\}$. It follows that the operator of multiplication by $c_j^{(j)}(\underline{\mathcal{W}}_i)$ (resp. by $c_j^{(j-1)}(\underline{\mathcal{W}}_i),\ 1 \le j \le i \le n-1$) is diagonal in the basis $\{[\underline{\widetilde{d}}]\}$ with eigenvalues $\{\frac{1}{2}(e^\infty_{ji} + e^0_{ji}(\underline{\widetilde{d}}))\}$ (resp. $\{\frac{\hbar^{-1}}{2}(e^\infty_{ji} - e^0_{ji}(\underline{\widetilde{d}}))\}$). Note that $e^\infty_{ji} \in \mathbb{C}[\mathfrak{t} \oplus \mathbb{C}]$, and $e^0_{ji}(\underline{\widetilde{d}})$) is precisely the eigenvalue of the central element $\underline{HC}(e_{ji}) \in \underline{\mathfrak{U}}(\mathfrak{gl}_i)$ corresponding to the j-th elementary symmetric function e_j via the Harish-Chandra isomorphism, on the Verma module with highest weight $\{\lambda_{i1}\hbar, \ldots, \lambda_{ii}\hbar\}$. Hence, the operator of multiplication by $c_j^{(j)}(\underline{\mathcal{W}}_i)$ (with eigenvalues $\{\frac{1}{2}(e^\infty_{ji} + e^0_{ji}(\underline{\widetilde{d}}))\}$) lies in the integral form $\underline{\mathfrak{G}}$ of the Gelfand–Tsetlin subalgebra. Moreover, $e^\infty_{ji} - \underline{HC}(e_{ji})$ is divisible by $\hbar$ in $\underline{\mathfrak{U}}(\mathfrak{gl}_i)$. Hence, the operator of multiplication by $c_j^{(j-1)}(\underline{\mathcal{W}}_i)$ lies in $\underline{\mathfrak{G}}$ as well. □

Corollary 3.7 *The composition of isomorphisms* $\underline{\mathfrak{G}}_{\underline{d}} \xrightarrow{\sim} \underline{\mathfrak{V}}^*_{\underline{d}} \xleftarrow{\Phi^{-1}} H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$ *is an isomorphism of algebras.*

4 Speculation on equivariant quantum cohomology of $\mathfrak{Q}_{\underline{d}}$

4.1 Calabi-Yau property of Laumon spaces

According to Theorem 3 of [10], the variety $\mathfrak{Q}_{\underline{d}}$ is Calabi-Yau. For the reader's convenience, we recall a proof. First note that if $\underline{d} = (d_1, \ldots, d_{n-1})$, and $d_k = 0$, then $\mathfrak{Q}_{\underline{d}} = \mathfrak{Q}_{\underline{d}'} \times \mathfrak{Q}_{\underline{d}''}$ where $\underline{d}' = (d_1, \ldots, d_{k-1})$, and $\mathfrak{Q}_{\underline{d}'}$ is the corresponding Laumon moduli space for GL_k, while $\underline{d}'' = (d_{k+1}, \ldots, d_{n-1})$, and $\mathfrak{Q}_{\underline{d}''}$ is the corresponding Laumon moduli space for GL_{n-k}. Hence, we may assume that all the integers $d_1, \ldots, d_{n-1}$ are strictly positive.

For $1 \le i \le n-1$, we consider the locally closed subvariety of $\mathfrak{Q}_{\underline{d}}$ formed by all the quasiflags which have a defect of degree exactly i. We denote by $\mathfrak{D}_i \subset \mathfrak{Q}_{\underline{d}}$ the closure of this subvariety. It is a divisor. We denote by $[\mathcal{O}(\mathfrak{D}_i)]$ the class of the corresponding line bundle in $\mathrm{Pic}(\mathfrak{Q}_{\underline{d}})$.

Lemma 4.2 *Assume all the integers $d_1, \ldots, d_{n-1}$ are strictly positive. Then* $[\mathcal{O}(\mathfrak{D}_1)] = [\mathcal{D}_2], \ldots, [\mathcal{O}(\mathfrak{D}_i)] = [\mathcal{D}_{i+1}] - [\mathcal{D}_i], \ldots, [\mathcal{O}(\mathfrak{D}_{n-1})] = -[\mathcal{D}_{n-1}]$.

Proof Recall the morphism $\pi_{\underline{d}} : \mathfrak{Q}_{\underline{d}} \to Z_{\underline{d}}$ to Drinfeld's Zastava space (a small resolution of singularities). For $1 \le k \le n-2$, choose a point $s \in Z_{\underline{d}}$ with defect of degree exactly $k + (k+1)$. We denote by P_k the preimage $\pi_{\underline{d}}^{-1}(s) \subset \mathfrak{Q}_{\underline{d}}$. By a GL_3-calculation, P_k is a projective line. The fundamental classes $[P_1], \ldots, [P_{n-2}]$ form a basis of $H_2(\mathfrak{Q}_{\underline{d}}, \mathbb{C})$. It is easy to see that the restriction $\mathcal{D}_i|_{P_k}$ is trivial if $i \neq k+1$, while $\mathcal{D}_{k+1}|_{P_k} \simeq \mathcal{O}(1)$ (this is again a GL_3-calculation). Furthermore, it is easy to see that the restriction $\mathcal{O}(\mathfrak{D}_i)|_{P_k}$ is trivial for $i \neq k, k+1$, while $\mathcal{O}(\mathfrak{D}_k)|_{P_k} \simeq \mathcal{O}(1)$, and $\mathcal{O}(\mathfrak{D}_{k+1})|_{P_k} \simeq \mathcal{O}(-1)$ (once again a GL_3-calculation). The lemma is proved. □

Let ${}^\circ\mathfrak{Q}_{\underline{d}} \subset \mathfrak{Q}_{\underline{d}}$ denote the open subspace formed by all the quasiflags without defect. Recall the symplectic form Ω on ${}^\circ\mathfrak{Q}_{\underline{d}}$ constructed in [9]. Note that the complement $\mathfrak{Q}_{\underline{d}} \setminus {}^\circ\mathfrak{Q}_{\underline{d}}$ equals the union of divisors $\bigcup_{1 \le i \le n-1} \mathfrak{D}_i$. Thus the top power $\omega := \Omega^{top}$ is a meromorphic volume form on $\mathfrak{Q}_{\underline{d}}$ with poles at $\bigcup_{1 \le i \le n-1} \mathfrak{D}_i$. The formula for Ω given in Remark 3 of *loc. cit.* shows that the order of the pole of ω at $\mathfrak{D}_i$ is 2 for any $1 \le i \le n-1$. Hence, the canonical class of $\mathfrak{Q}_{\underline{d}}$ in $\mathrm{Pic}(\mathfrak{Q}_{\underline{d}})$ equals $2\sum_{1 \le i \le n-1} [\mathcal{O}(\mathfrak{D}_i)]$. By Lemma 4.2, the class $2\sum_{1 \le i \le n-1} [\mathcal{O}(\mathfrak{D}_i)] = 0$. We have proved

Corollary 4.3 *(Givental, Lee [10]) The canonical class of $\mathfrak{Q}_{\underline{d}}$ is trivial.*

4.4 Shift of argument subalgebras

To each regular element μ of the Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$, one can assign a space Q_μ of commuting quadratic elements of the universal enveloping algebra $U(\mathfrak{g})$. Namely,

$$Q_\mu = \left\{ \sum_{\alpha \in \Delta_+} \frac{\langle h, \alpha \rangle}{\langle \mu, \alpha \rangle} e_\alpha e_{-\alpha}, \ | \ h \in \mathfrak{h} \right\},$$

where Δ_+ is the set of positive roots, and $e_\alpha, \ e_{-\alpha}$ are nonzero elements of the root spaces such that $(e_\alpha, e_{-\alpha}) = 1$. Note that the space Q_μ does not change under dilations of μ, hence we have a family of spaces of commuting quadratic operators, parametrized by the regular part of $\mathbb{P}(\mathfrak{h})$.

These quadratic elements appear as the quasiclassical limit of the *Casimir* flat connection on the trivial bundle on the regular part of the Cartan subalgebra with the fiber $\mathfrak{V}_{\underline{d}}$, (cf. [4,7,13,17]). This connection is given by the formula

$$\nabla = d + \kappa \sum_{\alpha \in \Delta_+} e_\alpha e_{-\alpha} \frac{d\alpha}{\alpha},$$

where κ is a parameter. Since every element of $U(\mathfrak{g})$ of the form $e_\alpha e_{-\alpha}$ commutes with the Cartan subalgebra $\mathfrak{h}$, this connection remains flat after adding any closed $U(\mathfrak{h})$-valued 1-form.

The centralizer in $U(\mathfrak{g})$ of this space of commuting quadratic operators is the so-called *shift of argument subalgebra* $\mathcal{A}_\mu \subset U(\mathfrak{g})$, which is a free commutative subalgebra with $\frac{1}{2}(\dim\mathfrak{g} + \mathrm{rk}\,\mathfrak{g})$ generators. For $\mathfrak{g} = \mathfrak{sl}_n$, the family of commutative subalgebras $\mathcal{A}_\mu \subset U(\mathfrak{g})$ is an $(n-2)$-parametric deformation of the Gelfand–Tsetlin subalgebra (see [22,23]).

4.5 Conjecture on equivariant quantum cohomology

Consider the shift of argument subalgebra for $\mathfrak{g} = \mathfrak{g}_n$, $\mu = \sum_{i=1}^{n-1} q_{i+1}q_{i+2}\dots q_n\omega_i$ with ω_i being the fundamental weights of $\mathfrak{g}_n$. Since the shift of argument algebra does not change under dilations of μ, we can assume that $q_n = 1$. Taking $h_k = \sum_{i=1}^{k-1} q_iq_{i+1}\dots q_n\omega_i$, we find that the space Q_μ is generated by the elements

$$\sum_{\alpha\in\Delta_+}\frac{\langle h_k,\alpha\rangle}{\langle\mu,\alpha\rangle}e_\alpha e_{-\alpha} = \sum_{i<j\le k}E_{ij}E_{ji} + \sum_{i<k<j}\frac{\sum_{l=i+1}^{k} q_lq_{l+1}\dots q_n}{\sum\limits_{l=i+1}^{j} q_lq_{l+1}\dots q_n}E_{ij}E_{ji}$$

$$= \sum_{i<j\le k}E_{ij}E_{ji} + \sum_{i<k<j}\frac{\sum_{l=i+1}^{k} q_lq_{l+1}\dots q_{j-1}}{1+\sum\limits_{l=i+1}^{j-1} q_lq_{l+1}\dots q_{j-1}}E_{ij}E_{ji},$$

with $k = 2,\dots,n-1$.

We consider the equivariant (small) quantum cohomology ring of $\mathfrak{Q}_{\underline{d}}$ which depends on $n-2$ quantum parameters $q_2,\dots,q_{n-1}$ corresponding to the Chern classes of the determinant bundles. Note that $\sum_{i<j\le k}E_{ij}E_{ji}$ is Cas_k up to some Cartan term. Hence, the shift of argument subalgebra contains the following (commuting) elements

$$QC_k := \widetilde{Cas_k} + \sum_{i<k<j}\frac{\sum_{l=i+1}^{k} q_lq_{l+1}\dots q_{j-1}}{1+\sum_{l=i+1}^{j-1} q_lq_{l+1}\dots q_{j-1}}E_{ij}E_{ji}.$$

Conjecture 4.6 [2] *The isomorphism $\Psi : V_{\underline{d}} \to \mathfrak{V}_{\underline{d}}$ carries the operator $M_{\mathcal{D}_k}$ of quantum multiplication by $c_1(D_k)$ to the operator $\frac{\hbar}{2}QC_k$.*

Corollary 4.7 *The localized equivariant quantum cohomology ring of $\mathfrak{Q}_{\underline{d}}$ is isomorphic to the quotient of the shift of argument subalgebra $\mathcal{A}_\mu$ by the annihilator of $\mathfrak{v}_{\underline{d}}$.*

Let $\underline{\mathcal{A}}_\mu$ denote the integral form $\mathcal{A}_\mu \cap \underline{\mathfrak{U}}$.

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[2] It was recently proved by A. Negut.

Conjecture 4.8 *The equivariant quantum cohomology ring of $\mathfrak{Q}_{\underline{d}}$ is isomorphic to the quotient of $\underline{\mathcal{A}}_\mu$ by the annihilator of $\mathfrak{v}_{\underline{d}}$.*

Remark 4.9 It is natural to expect 4.8 since the analogue of Lemma 3.2 is valid for $\underline{\mathcal{A}}_\mu$ as well (and the proof is the same).

The map $(q_2, \dots, q_{n-1}) \mapsto \mu = \sum_{i=1}^{n-1} q_{i+1}q_{i+2} \dots q_{n-1}\omega_i$ embeds the torus $\mathsf{T} = \mathbb{C}^{*(n-2)}$ with coordinates $q_2, \dots, q_{n-1}$ into the Cartan subalgebra $\mathfrak{h} = \mathbb{C}^{n-1}$ of $\mathfrak{sl}_n$ as an open subset of an affine hyperplane. Restricting the Casimir connection to T and adding an appropriate Cartan term, we obtain the following flat connection on T in the coordinates q_i:

$$\nabla = d + \kappa \sum_{k=2}^{n-1} QC_k \frac{dq_k}{q_k}.$$

On the other hand, the trivial vector bundle with the fiber $V_{\underline{d}}$ over the space of quantum parameters T is equipped with the *quantum connection* $d + \sum_{k=2}^{n-1} M_{\mathcal{D}_k} \frac{dq_k}{q_k}$.

Corollary 4.10 *The isomorphism Ψ carries the quantum connection to the Casimir connection with $\kappa = \frac{\hbar}{2}$.*

Remark 4.11 According to Vinberg [25], the family of subspaces Q_μ form an open subset in the moduli space of $(n-1)$-dimensional spaces of commuting linear combinations of $e_\alpha e_{-\alpha}$ in $U(\mathfrak{g})$. Thus, it is natural to expect that the operators $M_{\mathcal{D}_k}$ span the space Q_μ for some μ depending on $q_2, \dots, q_{n-1}$ (but unfortunately, we have no idea how to prove that the operators $M_{\mathcal{D}_k}$ are quadratic expressions in the correspondences). Moreover, since the unit in the cohomology ring remains unit in the quantum cohomology ring, the quantum correction has to annihilate the vector $\mathfrak{v}_{\underline{d}}$. Therefore, the operator $\Psi(M_{\mathcal{D}_k})$ is QC_k up to a change of parametrization. Finally, the flatness of the quantum connection $d + \sum_{k=2}^{n-1} M_{\mathcal{D}_k} \frac{dq_k}{q_k}$ is a very restrictive condition on the parametrization $\mu(q_2, \dots, q_{n-1})$ — this leaves no other choice for $\Psi(M_{\mathcal{D}_k})$ but to coincide with $\frac{\hbar}{2} QC_k$.

5 Global Laumon spaces

5.1 Correspondences

Recall the setup of 2.3. We define two versions of the correspondences $\mathcal{E}_{\underline{d},i} \subset \mathcal{Q}_{\underline{d}} \times \mathcal{Q}_{\underline{d}+i}$, namely, $\mathcal{E}^0_{\underline{d},i} := \mathbf{r}^{-1}\{0\}$, $\mathcal{E}^\infty_{\underline{d},i} := \mathbf{r}^{-1}\{\infty\}$. Their projections to $\mathcal{Q}_{\underline{d}}$ (resp. $\mathcal{Q}_{\underline{d}+i}$) will be denoted by $\mathbf{p}^0, \mathbf{p}^\infty$ (resp. $\mathbf{q}^0, \mathbf{q}^\infty$). The projection of $\mathcal{E}_{\underline{d},i}$ to $\mathcal{Q}_{\underline{d}}$ (resp. $\mathcal{Q}_{\underline{d}+i}$) will be denoted by $\mathbf{p}^{\mathbf{C}}$ (resp. $\mathbf{q}^{\mathbf{C}}$).

We denote by $'A$ (resp. $'B$) the direct sum of equivariant (complexified) cohomology: $'A = \oplus_{\underline{d}} H^\bullet_{\tilde{T}\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}})$ (resp. $'B = \oplus_{\underline{d}} H^\bullet_{G\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}})$). We define $A = {}'A \otimes_{H^\bullet_{\tilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\tilde{T}\times\mathbb{C}^*}(pt))$, and $B = {}'B \otimes_{H^\bullet_{G\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{G\times\mathbb{C}^*}(pt))$.

We have an evident grading $A = \oplus_{\underline{d}} A_{\underline{d}},\ A_{\underline{d}} = H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt))$; similarly, $B = \oplus_{\underline{d}} B_{\underline{d}},\ B_{\underline{d}} = H^\bullet_{G\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}}) \otimes_{H^\bullet_{G\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{G\times\mathbb{C}^*}(pt))$.

Note that $H^\bullet_{G\times\mathbb{C}^*}(pt) = \mathbb{C}[e_1, \ldots, e_n, \hbar]$ where e_i is the i-th elementary symmetric function of $x_1, \ldots, x_n$.

5.2 Fixed points

The set of $\widetilde{T} \times \mathbb{C}^*$-fixed points in $\mathcal{Q}_{\underline{d}}$ is finite; it is described in [8], 2.11. Recall that this fixed point set is in bijection with the set of collections $\{\widehat{\underline{d}}\}$ of the following data: a) a permutation $\sigma \in S_n$; b) a matrix (d^0_{ij}) (resp. d^∞_{ij}), $i \geq j$ of nonnegative integers such that for $i \geq j \geq k$ we have $d^0_{kj} \geq d^0_{ij}$ (resp. $d^\infty_{kj} \geq d^\infty_{ij}$) such that $d_i = \sum_{j=1}^{i}(d^0_{ij} + d^\infty_{ij})$. Abusing notation we denote by $\widehat{\underline{d}}$ the corresponding $\widetilde{T} \times \mathbb{C}^*$-fixed point in $\mathcal{Q}_{\underline{d}}$:

$\mathcal{W}_1 = \mathcal{O}_{\mathbf{C}}(-d^0_{11} \cdot 0 - d^\infty_{11} \cdot \infty) w_{\sigma(1)}$,

$\mathcal{W}_2 = \mathcal{O}_{\mathbf{C}}(-d^0_{21} \cdot 0 - d^\infty_{21} \cdot \infty) w_{\sigma(1)} \oplus \mathcal{O}_{\mathbf{C}}(-d^0_{22} \cdot 0 - d^\infty_{22} \cdot \infty) w_{\sigma(2)}$,

$\ldots \ldots \ldots$,

$\mathcal{W}_{n-1} = \mathcal{O}_{\mathbf{C}}(-d^0_{n-1,1} \cdot 0 - d^\infty_{n-1,1} \cdot \infty) w_{\sigma(1)} \oplus \mathcal{O}_{\mathbf{C}}(-d^0_{n-1,2} \cdot 0 - d^\infty_{n-1,2} \cdot \infty) w_{\sigma(2)} \oplus$
$\oplus \cdots \oplus \mathcal{O}_{\mathbf{C}}(-d^0_{n-1,n-1} \cdot 0 - d^\infty_{n-1,n-1} \cdot \infty) w_{\sigma(n-1)}$.

Also, we will write $\widehat{\underline{d}} = (\sigma, \widetilde{\underline{d}}^0, \widetilde{\underline{d}}^\infty)$.

The localized equivariant cohomology A is equipped with the basis of direct images of the fundamental classes of the fixed points; by an abuse of notation, this basis will be denoted $\{[\widehat{\underline{d}}]\}$.

5.3 Chern classes of tautological bundles

As in the proof of Theorem 3.5 and Corollary 3.7, we see that the $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)$-algebra $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}})$ is generated by the Künneth components of the Chern classes $c_j^{(j)}(\underline{\mathcal{W}}_i^{\mathbf{C}}),\ c_j^{(j-1)}(\underline{\mathcal{W}}_i^{\mathbf{C}}),\ 1 \leq j \leq i \leq n-1$, of the tautological vector bundles $\underline{\mathcal{W}}_i^{\mathbf{C}}$ on $\mathcal{Q}_{\underline{d}} \times \mathbf{C}$. We compute the operators of multiplication by $c_j^{(j)}(\underline{\mathcal{W}}_i^{\mathbf{C}}),\ c_j^{(j-1)}(\underline{\mathcal{W}}_i^{\mathbf{C}}),\ 1 \leq j \leq i \leq n-1$, in the equivariant cohomology ring $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}})$ in the fixed point basis $\{[\widehat{\underline{d}}]\}$.

We introduce the following notation. For a function $f(x_1, \ldots, x_n, \hbar) \in \mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$ and a permutation $\sigma \in S_n$, we set $f^\sigma(x_1, \ldots, x_n, \hbar) := f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \hbar)$. Also, we set $\bar{f}(x_1, \ldots, x_n, \hbar) := f(x_1, \ldots, x_n, -\hbar)$. For $1 \leq j \leq i$, let $e^0_{ji}(\widehat{\underline{d}})$ (resp. $e^\infty_{ji}(\widehat{\underline{d}})$) stand for the sum of products of the j-tuples of distinct elements of the set $\{-x_1 + d^0_{i,1}\hbar, \ldots, -x_i + d^0_{i,i}\hbar\}$ (resp. of the set $\{-x_1 + d^\infty_{i,1}\hbar, \ldots, -x_i + d^\infty_{i,i}\hbar\}$).

Lemma 5.4 *The operator of multiplication by $c_j^{(j)}(\underline{\mathcal{W}}_i^{\mathbf{C}})$ (resp. by $c_j^{(j-1)}(\underline{\mathcal{W}}_i^{\mathbf{C}}),\ 1 \leq j \leq i \leq n-1$) is diagonal in the basis $\{[\widehat{\underline{d}}] = [(\sigma, \widetilde{\underline{d}}^0, \widetilde{\underline{d}}^\infty)]\}$ with eigenvalues $\{\frac{1}{2}(e^0_{ji}(\widehat{\underline{d}}) + e^\infty_{ji}(\widehat{\underline{d}}))^\sigma\}$ (resp. $\{\frac{\hbar^{-1}}{2}(e^\infty_{ji}(\widehat{\underline{d}}) - e^0_{ji}(\widehat{\underline{d}}))^\sigma\}$).*

Proof The same argument as in the proof of Theorem 3.5. □

Corollary 5.5 *The operator of multiplication by $c_1^{(1)}(\underline{\mathcal{W}}_i^{\mathbf{C}})$ is diagonal in the basis $\{[\widehat{\underline{d}}] = [(\sigma, \widetilde{\underline{d}}^0, \widetilde{\underline{d}}^\infty)]\}$ with eigenvalues $-(x_1 + \cdots + x_i)^\sigma + d_i\hbar$.*

5.6 The double universal enveloping algebra

We denote by $\mathfrak{U}^2$ the universal enveloping algebra of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ over the field $\mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$. For $1 \leq i, j \leq n$ we denote by $E_{ij}^{(1)}$ (resp. $E_{ij}^{(2)}$) the element $(E_{ij}, 0) \in \mathfrak{gl}_n \oplus \mathfrak{gl}_n \subset \mathfrak{U}^2$ (resp. the element $(0, E_{ij}) \in \mathfrak{gl}_n \oplus \mathfrak{gl}_n \subset \mathfrak{U}^2$). We denote by $\mathfrak{U}^2_{\leq 0}$ the subalgebra of $\mathfrak{U}^2$ generated by $E_{ii}^{(1)}, E_{ii}^{(2)},\ 1 \leq i \leq n,\ E_{i,i+1}^{(1)}, E_{i,i+1}^{(2)},\ 1 \leq i \leq n-1$. It acts on the field $\mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$ as follows: $E_{i,i+1}^{(1)}, E_{i,i+1}^{(2)}$ act trivially for any $1 \leq i \leq n-1$, and $E_{ii}^{(1)}$ (resp. $E_{ii}^{(2)}$) acts by multiplication by $\hbar^{-1}x_i + i - 1$ (resp. $-\hbar^{-1}x_i + i - 1$). We define the *universal Verma module* $\mathfrak{V}$ over $\mathfrak{U}^2$ as $\mathfrak{U}^2 \otimes_{\mathfrak{U}^2_{\leq 0}} \mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$. The universal Verma module $\mathfrak{V}$ is an irreducible $\mathfrak{U}^2$-module.

For a permutation $\sigma = (\sigma(1), \ldots, \sigma(n)) \in S_n$ (the Weyl group of $G = GL_n$), we consider a new action of $\mathfrak{U}^2_{\leq 0}$ on $\mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$ defined as follows: $E_{i,i+1}^{(1)}, E_{i,i+1}^{(2)}$ act trivially for any $1 \leq i \leq n-1$, and $E_{ii}^{(1)}$ (resp. $E_{ii}^{(2)}$) acts by multiplication by $\hbar^{-1}x_{\sigma(i)} + i - 1$ (resp. $-\hbar^{-1}x_{\sigma(i)} + i - 1$). We define a module $\mathfrak{V}^\sigma$ over $\mathfrak{U}^2$ as $\mathfrak{U}^2 \otimes_{\mathfrak{U}^2_{\leq 0}} \mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$ (with respect to the new action of $\mathfrak{U}^2_{\leq 0}$ on $\mathbb{C}(\mathfrak{f} \oplus \mathbb{C})$). Finally, we define $\mathfrak{A} := \bigoplus_{\sigma \in S_n} \mathfrak{V}^\sigma$.

5.7 The action of the double universal enveloping algebra

The grading and the correspondences $\mathcal{E}_{\underline{d},i}, \mathcal{E}^0_{\underline{d},i}, \mathcal{E}^\infty_{\underline{d},i}$ give rise to the following operators on A, B:

$$\mathfrak{f}_i^{(1)} = E_{i,i+1}^{(1)} = \hbar^{-1}\mathbf{p}^0_*\mathbf{q}^{0*} : A_{\underline{d}} \to A_{\underline{d}-i} \text{ and } B_{\underline{d}} \to B_{\underline{d}-i};$$
$$\mathfrak{f}_i^{(2)} = E_{i,i+1}^{(2)} = -\hbar^{-1}\mathbf{p}^\infty_*\mathbf{q}^{\infty*} : A_{\underline{d}} \to A_{\underline{d}-i} \text{ and } B_{\underline{d}} \to B_{\underline{d}-i};$$
$$\mathfrak{e}_i^{(1)} = E_{i+1,i}^{(1)} = -\hbar^{-1}\mathbf{q}^0_*\mathbf{p}^{0*} : A_{\underline{d}} \to A_{\underline{d}+i} \text{ and } B_{\underline{d}} \to B_{\underline{d}+i};$$
$$\mathfrak{e}_i^{(2)} = E_{i+1,i}^{(2)} = \hbar^{-1}\mathbf{q}^\infty_*\mathbf{p}^{\infty*} : A_{\underline{d}} \to A_{\underline{d}+i} \text{ and } B_{\underline{d}} \to B_{\underline{d}+i};$$
$$\mathfrak{f}_i^{\Delta} = \mathbf{p}^{\mathbf{C}}_*\mathbf{q}^{\mathbf{C}*} : A_{\underline{d}} \to A_{\underline{d}-i} \text{ and } B_{\underline{d}} \to B_{\underline{d}-i};$$
$$\mathfrak{e}_i^{\Delta} = -\mathbf{q}^{\mathbf{C}}_*\mathbf{p}^{\mathbf{C}*} : A_{\underline{d}} \to A_{\underline{d}+i} \text{ and } B_{\underline{d}} \to B_{\underline{d}+i}.$$

Furthermore, we define the action of $\mathbb{C}[\mathfrak{t} \oplus \mathbb{C}]$ on B as follows: for $1 \leq i \leq n-1$, x_i acts on $B_{\underline{d}}$ by multiplication by $c_1^{(1)}(\underline{\mathcal{W}}_{i-1}^{\mathbf{C}}) - c_1^{(1)}(\underline{\mathcal{W}}_i^{\mathbf{C}}) - (d_i - d_{i-1})\hbar$ (cf. Corollary 5.5); and x_n acts by multiplication by $e_1 - x_1 - \cdots - x_{n-1}$ (recall that e_1 is the generator of $H^2_G(pt)$).

Theorem 5.8 a) *The operators $\mathfrak{e}_i^{(1)} = E_{i+1,i}^{(1)}, \mathfrak{e}_i^{(2)} = E_{i+1,i}^{(2)}, \mathfrak{f}_i^{(1)} = E_{i,i+1}^{(1)}, \mathfrak{f}_i^{(2)} = E_{i,i+1}^{(2)}$ along with the action of $\mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$ on B defined in 5.7 satisfy the relations in $\mathfrak{U}^2$, i.e. they give rise to the action of $\mathfrak{U}^2$ on B;*

b) *There is a unique isomorphism Ψ^2 of $\mathfrak{U}^2$-modules B and $\mathfrak{V}$ carrying $1 \in H^0_{\widetilde{T} \times \mathbb{C}^*}(\mathcal{Q}_0) \subset B$ to the lowest weight vector $1 \in \mathbb{C}(\mathfrak{t} \oplus \mathbb{C}) \subset \mathfrak{V}$.*

Proof We describe the matrix coefficients of $\mathfrak{e}_i^{(1)} = E^{(1)}_{i+1,i}, \mathfrak{e}_i^{(2)} = E^{(2)}_{i+1,i}, \mathfrak{f}_i^{(1)} = E^{(1)}_{i,i+1}, \mathfrak{f}_i^{(2)} = E^{(2)}_{i,i+1}$ in the fixed point basis $\{[\widehat{\underline{d}}]\}$. Recall the matrix coefficients $\mathfrak{e}_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']}, \mathfrak{f}_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']}$ computed in Proposition 2.9. The proof of the following easy lemma is omitted.

Lemma 5.9 *Let* $\widehat{\underline{d}} = (\sigma, \widetilde{\underline{d}}^0, \widetilde{\underline{d}}^\infty)$*, and* $\widehat{\underline{d}'} = (\sigma', \widetilde{\underline{d}}^{0\prime}, \widetilde{\underline{d}}^{\infty\prime})$*. The matrix coefficients are as follows:* $\mathfrak{e}^{(1)}_{i[\widehat{\underline{d}},\widehat{\underline{d}}']} = \delta_{\sigma,\sigma'}(\mathfrak{e}_{i[\widetilde{\underline{d}}^0,\widetilde{\underline{d}}^{0\prime}]})^\sigma$; $\mathfrak{e}^{(2)}_{i[\widehat{\underline{d}},\widehat{\underline{d}}']} = \delta_{\sigma,\sigma'}(\bar{\mathfrak{e}}_{i[\widetilde{\underline{d}}^\infty,\widetilde{\underline{d}}^{\infty\prime}]})^\sigma$; $\mathfrak{f}^{(1)}_{i[\widehat{\underline{d}},\widehat{\underline{d}}']} = \delta_{\sigma,\sigma'}(\mathfrak{f}_{i[\widetilde{\underline{d}}^0,\widetilde{\underline{d}}^{0\prime}]})^\sigma$; $\mathfrak{f}^{(2)}_{i[\widehat{\underline{d}},\widehat{\underline{d}}']} = \delta_{\sigma,\sigma'}(\bar{\mathfrak{f}}_{i[\widetilde{\underline{d}}^\infty,\widetilde{\underline{d}}^{\infty\prime}]})^\sigma$.

It follows that the operators $\mathfrak{e}_i^{(1)} = E^{(1)}_{i+1,i}, \mathfrak{e}_i^{(2)} = E^{(2)}_{i+1,i}, \mathfrak{f}_i^{(1)} = E^{(1)}_{i,i+1}, \mathfrak{f}_i^{(2)} = E^{(2)}_{i,i+1}$ on A defined in 5.7 satisfy the relations in $\mathfrak{U}^2$, i.e. they give rise to the action of $\mathfrak{U}^2$ on A. Moreover, it follows that the $\mathfrak{U}^2$-module A is isomorphic to $\mathfrak{A}$. In order to describe the image of the basis $\{[\widehat{\underline{d}}]\}$ under this isomorphism, we introduce the following notation. First, we introduce a $\mathfrak{U}$-module $\mathfrak{V}^\sigma := \mathfrak{U} \otimes_{\mathfrak{U}_{\le 0}} \mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$ where $\mathfrak{U}_{\le 0}$ acts on $\mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$ as follows: $E_{i,i+1}$ acts trivially for any $1 \le i \le n-1$, and E_{ii} acts by multiplication by $\hbar^{-1}x_{\sigma(i)} + i - 1$. Similarly, we introduce a $\mathfrak{U}$-module $\bar{\mathfrak{V}}^\sigma := \mathfrak{U} \otimes_{\mathfrak{U}_{\le 0}} \mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$ where $\mathfrak{U}_{\le 0}$ acts on $\mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$ as follows: $E_{i,i+1}$ acts trivially for any $1 \le i \le n-1$, and E_{ii} acts by multiplication by $-\hbar^{-1}x_{\sigma(i)} + i - 1$. Note that $\mathfrak{B}^\sigma \simeq \mathfrak{V}^\sigma \otimes_{\mathbb{C}(\mathfrak{t}\oplus\mathbb{C})} \bar{\mathfrak{V}}^\sigma$.

Now to a collection $\widetilde{\underline{d}} = (d_{ij})$, and a permutation $\sigma \in S_n$, we associate a Gelfand–Tsetlin pattern $\Lambda^\sigma = \Lambda^\sigma(\widetilde{\underline{d}}) := (\lambda^\sigma_{ij}),\ n \ge i \ge j$, as follows: $\lambda^\sigma_{nj} := \hbar^{-1}x_{\sigma(j)} + j - 1,\ n \ge j \ge 1$; $\lambda^\sigma_{ij} := \hbar^{-1}x_{\sigma(j)} + j - 1 - d_{ij},\ n-1 \ge i \ge j \ge 1$. We define $\xi^\sigma_{\widetilde{\underline{d}}} = \xi_{\Lambda^\sigma} \in \mathfrak{V}^\sigma$ by the formulas (2.9)–(2.11) of [18] (where $\xi = \xi_0 = 1 \in \mathfrak{V}^\sigma$). Similarly, to a collection $\widetilde{\underline{d}} = (d_{ij})$, and a permutation $\sigma \in S_n$, we associate a Gelfand–Tsetlin pattern $\bar{\Lambda}^\sigma = \bar{\Lambda}^\sigma(\widetilde{\underline{d}}) := (\bar{\lambda}^\sigma_{ij}),\ n \ge i \ge j$, as follows: $\bar{\lambda}^\sigma_{nj} := -\hbar^{-1}x_{\sigma(j)} + j - 1,\ n \ge j \ge 1$; $\bar{\lambda}^\sigma_{ij} := -\hbar^{-1}x_{\sigma(j)} + j - 1 - d_{ij},\ n-1 \ge i \ge j \ge 1$. We define $\bar{\xi}^\sigma_{\widetilde{\underline{d}}} = \xi_{\bar{\Lambda}^\sigma} \in \bar{\mathfrak{V}}^\sigma$ by the formulas (2.9)–(2.11) of [18] (where $\xi = \xi_0 = 1 \in \bar{\mathfrak{V}}^\sigma$). Finally, to a collection $\widehat{\underline{d}} = (\sigma, \widetilde{\underline{d}}^0, \widetilde{\underline{d}}^\infty)$, we associate an element $\xi_{\widehat{\underline{d}}} := \xi^\sigma_{\widetilde{\underline{d}}^0} \otimes \bar{\xi}^\sigma_{\widetilde{\underline{d}}^\infty} \in \mathfrak{V}^\sigma \otimes_{\mathbb{C}(\mathfrak{t}\oplus\mathbb{C})} \bar{\mathfrak{V}}^\sigma = \mathfrak{B}^\sigma$.

Theorem 2.11 and Lemma 5.9 implies that under the above isomorphism $A \simeq \mathfrak{A}$ a basis element $[\widehat{\underline{d}}]$ goes to $(-\hbar)^{|d|}\xi_{\widehat{\underline{d}}}$.

We are ready to finish the proof of the theorem. Since the action of $\widetilde{T} \times \mathbb{C}^*$ on $\mathcal{Q}_{\underline{d}}$ extends to the action of $G \times \mathbb{C}^*$, the equivariant cohomology $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}})$ is equipped with the action of the Weyl group S_n, and $H^\bullet_{G\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}}) = H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}})^{S_n}$. It follows $B = A^{S_n}$. Since B is closed with respect to the action of the operators $\mathfrak{e}_i^{(1)} = E^{(1)}_{i+1,i}, \mathfrak{e}_i^{(2)} = E^{(2)}_{i+1,i}, \mathfrak{f}_i^{(1)} = E^{(1)}_{i,i+1}, \mathfrak{f}_i^{(2)} = E^{(2)}_{i,i+1}$ on B, part a) follows. Note, however, that the action of S_n on A *does not* commute with the action of the above operators.

To prove part b), we describe the action of S_n on A explicitly in the basis $\{[\widehat{\underline{d}}]\}$. For $\widehat{\underline{d}} = (\sigma, \widetilde{\underline{d}}^0, \widetilde{\underline{d}}^\infty),\ \sigma' \in S_n,\ f \in \mathbb{C}(\mathfrak{t} \oplus \mathbb{C})$, we have $\sigma'(f[\widehat{\underline{d}}]) = f^{\sigma'}[(\sigma'\sigma, \widetilde{\underline{d}}^0, \widetilde{\underline{d}}^\infty)]$. We conclude that for $\widehat{\underline{d}} = (1, \widetilde{\underline{d}}^0, \widetilde{\underline{d}}^\infty)$ (so that $\xi_{\widehat{\underline{d}}} \in \mathfrak{B}^1 = \mathfrak{B}$), we have $(\Psi^2)^{-1}(f\xi_{\widehat{\underline{d}}}) = \sum_{\sigma\in S_n} f^\sigma[(\sigma, \widetilde{\underline{d}}^0, \widetilde{\underline{d}}^\infty)]$. This completes the proof of the theorem. □

Remark 5.10 The operators $\mathfrak{f}_i^\Delta, \mathfrak{e}_i^\Delta$ of 5.7 were introduced in [8]. It is easy to see that $\mathfrak{f}_i^\Delta = \mathfrak{f}_i^{(1)} + \mathfrak{f}_i^{(2)},\ \mathfrak{e}_i^\Delta = \mathfrak{e}_i^{(1)} + \mathfrak{e}_i^{(2)}$.

5.11 The double Gelfand–Tsetlin algebra

A completion of $\mathfrak{B} = \mathfrak{V}\otimes_{\mathbb{C}(\mathfrak{t}\oplus\mathbb{C})}\mathfrak{V}$ contains the Whittaker vector $\sum_{\underline{d}} \mathfrak{b}_{\underline{d}} = \mathfrak{b} := \mathfrak{v}\otimes\mathfrak{v}$. It follows from the proof of Theorem 5.8 that for the unit element of the cohomology ring $1^{\underline{d}} \in H^0_{G\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}})$ we have $\Psi^2(1^{\underline{d}}) = \mathfrak{b}_{\underline{d}}$. The double Gelfand–Tsetlin subalgebra $\mathfrak{G}^2 := \mathfrak{G}\otimes\mathfrak{G}$ acts by endomorphisms of $\mathfrak{B}$. We denote by $\mathfrak{J}^2_{\underline{d}} \subset \mathfrak{G}^2$ the annihilator ideal of the vector $\mathfrak{b}_{\underline{d}} \in \mathfrak{B}$, and we denote by $\mathfrak{G}^2_{\underline{d}}$ the quotient of $\mathfrak{G}^2$ by $\mathfrak{J}^2_{\underline{d}}$. The action of $\mathfrak{G}^2$ on $\mathfrak{b}_{\underline{d}}$ gives rise to an embedding $\mathfrak{G}^2_{\underline{d}} \hookrightarrow \mathfrak{B}_{\underline{d}}$. The same way as in Proposition 2.17.a) one proves that this embedding is an isomorphism $\mathfrak{G}^2_{\underline{d}} \xrightarrow{\sim} \mathfrak{B}_{\underline{d}}$.

Proposition 5.12 *The composite morphism* $\Psi^2 : H^\bullet_{G\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}})\otimes_{H^\bullet_{G\times\mathbb{C}^*}(pt)}\mathrm{Frac}(H^\bullet_{G\times\mathbb{C}^*}(pt)) = B_{\underline{d}} \xrightarrow{\sim} \mathfrak{B}_{\underline{d}} \xrightarrow{\sim} \mathfrak{G}^2_{\underline{d}}$ *is an algebra isomorphism.*

Proof As in the proof of Theorem 3.5 and Corollary 3.7, we see that the $H^\bullet_{G\times\mathbb{C}^*}(pt)$-algebra $H^\bullet_{G\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}})$ is generated by the Künneth components of the Chern classes $c_j^{(j)}(\underline{\mathcal{W}}_i^{\mathbf{C}}),\ c_j^{(j-1)}(\underline{\mathcal{W}}_i^{\mathbf{C}}),\ 1 \le j \le i \le n-1$, of the tautological vector bundles $\underline{\mathcal{W}}_i^{\mathbf{C}}$ on $\mathcal{Q}_{\underline{d}} \times \mathbf{C}$. In order to prove the proposition, it suffices to check that the operators of multiplication by $c_j^{(j)}(\underline{\mathcal{W}}_i^{\mathbf{C}}),\ c_j^{(j-1)}(\underline{\mathcal{W}}_i^{\mathbf{C}}),\ 1 \le j \le i \le n-1$, in the equivariant cohomology ring $H^\bullet_{G\times\mathbb{C}^*}(\mathcal{Q}_{\underline{d}})$ lie in $\mathfrak{G}^2_{\underline{d}}$. To this end, we compute these operators explicitly in the basis $\{(\Psi^2)^{-1}\xi_{\widehat{\underline{d}}},\ \widehat{\underline{d}} = (1, \widetilde{\underline{d}}^0, \widetilde{\underline{d}}^\infty)\}$ of $\mathfrak{B}$. Lemma 5.4 implies that the operator of multiplication by $c_j^{(j)}(\underline{\mathcal{W}}_i^{\mathbf{C}})$ (resp. by $c_j^{(j-1)}(\underline{\mathcal{W}}_i^{\mathbf{C}}),\ 1 \le j \le i \le n-1$) is diagonal in the basis $\{(\Psi^2)^{-1}\xi_{\widehat{\underline{d}}},\ \widehat{\underline{d}} = (1, \widetilde{\underline{d}}^0, \widetilde{\underline{d}}^\infty)\}$ with eigenvalues $\{e^0_{ji}(\widehat{\underline{d}}) + e^\infty_{ji}(\widehat{\underline{d}})\}$ (resp. $\{\hbar^{-1}(e^\infty_{ji}(\widehat{\underline{d}}) - e^0_{ji}(\widehat{\underline{d}}))\}$). As in the proof of Theorem 3.5, we see that $e^0_{ji}(\widehat{\underline{d}})$ is the eigenvalue of the element of $\mathfrak{G}_{\underline{d}} \otimes 1 \subset \mathfrak{G}^2_{\underline{d}}$, and $e^\infty_{ji}(\widehat{\underline{d}})$ is the eigenvalue of the same element in the other copy of the Gelfand–Tsetlin subalgebra $1 \otimes \mathfrak{G}_{\underline{d}} \subset \mathfrak{G}^2_{\underline{d}}$, hence $c_j^{(j)}(\underline{\mathcal{W}}_i^{\mathbf{C}})$ and $c_j^{(j-1)}(\underline{\mathcal{W}}_i^{\mathbf{C}})$ lie in $\mathfrak{G}^2_{\underline{d}}$. □

5.13 Integral forms

Recall the notations of 3.3. We consider the correspondences $\mathcal{E}^0_{\underline{d},\alpha_{ij}} \subset \mathcal{Q}_{\underline{d}} \times \mathcal{Q}_{\underline{d}+\alpha_{ij}}$ (resp. $\mathcal{E}^\infty_{\underline{d},\alpha_{ij}} \subset \mathcal{Q}_{\underline{d}} \times \mathcal{Q}_{\underline{d}+\alpha_{ij}}$) defined exactly as in *loc. cit.* (resp. replacing the condition in 3.3.b by the condition that $\mathcal{W}_\bullet/\mathcal{W}'_\bullet$ is supported at $\infty \in \mathbf{C}$). We denote by $\mathbf{p}^0_{ij} : \mathcal{E}^0_{\underline{d},\alpha_{ij}} \to \mathcal{Q}_{\underline{d}},\ \mathbf{q}^0_{ij} : \mathcal{E}^0_{\underline{d},\alpha_{ij}} \to \mathcal{Q}_{\underline{d}+\alpha_{ij}},\ \mathbf{p}^\infty_{ij} : \mathcal{E}^\infty_{\underline{d},\alpha_{ij}} \to \mathcal{Q}_{\underline{d}},\ \mathbf{q}^\infty_{ij} : \mathcal{E}^\infty_{\underline{d},\alpha_{ij}} \to \mathcal{Q}_{\underline{d}+\alpha_{ij}}$ the natural proper projections. We also consider the correspondences and projections $\mathcal{E}^{\mathbf{C}}_{\underline{d},\alpha_{ij}} \subset \mathcal{Q}_{\underline{d}} \times \mathcal{Q}_{\underline{d}+\alpha_{ij}},\ \mathbf{p}^{\mathbf{C}}_{ij},\ \mathbf{q}^{\mathbf{C}}_{ij}$ defined as above but without any restriction on the support of $\mathcal{W}_\bullet/\mathcal{W}'_\bullet$.

We consider the following operators on $'B$:

$\underline{E}_{ij}^{(1)} = \mathbf{p}_{ij*}^{0}\mathbf{q}_{ij}^{0*} : {}'B_{\underline{d}} \to {}'B_{\underline{d}-\alpha_{ij}};$

$\underline{E}_{ij}^{(2)} = -\mathbf{p}_{ij*}^{\infty}\mathbf{q}_{ij}^{\infty*} : {}'B_{\underline{d}} \to {}'B_{\underline{d}-\alpha_{ij}};$

$\underline{E}_{ji}^{(1)} = (-1)^{i-j}\mathbf{q}_{ij*}^{0}\mathbf{p}_{ij}^{0*} : {}'B_{\underline{d}} \to {}'B_{\underline{d}+\alpha_{ij}};$

$\underline{E}_{ji}^{(2)} = -(-1)^{i-j}\mathbf{q}_{ij*}^{\infty}\mathbf{p}_{ij}^{\infty*} : {}'B_{\underline{d}} \to {}'B_{\underline{d}+\alpha_{ij}};$

$E_{ij}^{\Delta} = \mathbf{p}_{ij*}^{\mathbf{C}}\mathbf{q}_{ij}^{\mathbf{C}*} : {}'B_{\underline{d}} \to {}'B_{\underline{d}-\alpha_{ij}};$

$E_{ji}^{\Delta} = (-1)^{i-j}\mathbf{q}_{ij*}^{\mathbf{C}}\mathbf{p}_{ij}^{\mathbf{C}*} : {}'B_{\underline{d}} \to {}'B_{\underline{d}+\alpha_{ij}}.$

We have $E_{ij}^{\Delta} = \hbar^{-1}(\underline{E}_{ij}^{(1)} + \underline{E}_{ij}^{(2)}),\ E_{ji}^{\Delta} = \hbar^{-1}(\underline{E}_{ji}^{(1)} + \underline{E}_{ji}^{(2)})$.

We define $\underline{\mathfrak{U}}^2 \subset \mathfrak{U}^2$ as the $\mathbb{C}[\mathfrak{t} \oplus \mathbb{C}]$-subalgebra generated by $\underline{y}^{(1)},\ \underline{y}^{(2)},\ y^{\Delta}$, $y \in \mathfrak{gl}_n$, where $\underline{y}^{(1)}$ (resp. $\underline{y}^{(2)}$) stands for $\hbar(y, 0)$ (resp. $\hbar(0, y)$), and y^{Δ} stands for (y, y). Then it is easy to see that the above operators give rise to the action of $\underline{\mathfrak{U}}^2$ on ${}'B$.

Also, it is easy to check that $\underline{\mathfrak{U}}^2/(\hbar = 0)$ is isomorphic to the algebra $\mathsf{U} := (\mathbb{C}[\mathfrak{gl}_n] \rtimes U(\mathfrak{gl}_n)) \otimes \mathbb{C}[\mathfrak{t}]$ (the semidirect product with respect to the adjoint action). Hence $\bar{B} := {}'B/(\hbar = 0) = \oplus_{\underline{d}} H_G^{\bullet}(\mathcal{Q}_{\underline{d}})$ inherits an action of U.

Conjecture 5.14 *The U-module $\bar{B}$ is isomorphic to $H_{\{\mathfrak{g}_{\leq 0}\}}^{\frac{n(n-1)}{2}}(\mathfrak{gl}_n, \mathcal{O})$. Under this isomorphism, the action of $H_G^{\bullet}(pt)$ on $\bar{B}$ corresponds to the action of $\mathbb{C}[\mathfrak{gl}_n]^G$ on $H_{\{\mathfrak{g}_{\leq 0}\}}^{\frac{n(n-1)}{2}}(\mathfrak{gl}_n, \mathcal{O})$.*

Conjecture 6.4 of [8] on the direct sum of *nonequivariant* cohomology of $\mathcal{Q}_{\underline{d}}$ is an immediate corollary of 5.14.

5.15 Relative Laumon spaces

We propose a generalization of Conjecture 6.4 of [8] in a different direction. Let $\underline{d} = (d_1, \ldots, d_n)$ be an n-tuple of integers (not necessarily positive). Let $\mathsf{Q}_{\underline{d}}$ be the moduli stack of flags of locally free sheaves $\mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W}_n$ on $\mathbf{C}$ such that $\mathrm{rk}(\mathcal{W}_i) = i,\ \deg(\mathcal{W}_i) = d_i$ (see [14]). We have a representable projective morphism $\pi : \mathsf{Q}_{\underline{d}} \to \mathrm{Bun},\ \mathcal{W}_\bullet \mapsto \mathcal{W}_n$, where Bun stands for the moduli stack of GL_n-bundles on $\mathbf{C}$. The fiber of π over the trivial GL_n-bundle (an open point of Bun) is $\mathcal{Q}_{\underline{d}}$. The correspondences $\mathcal{E}_{\underline{d},\alpha_{ij}}^{0}, \mathcal{E}_{\underline{d},\alpha_{ij}}^{\infty}, \mathcal{E}_{\underline{d},\alpha_{ij}}^{\mathbf{C}}$ of Subsection 5.13 make perfect sense for the stacks $\mathsf{Q}_{\underline{d}}$ in place of $\mathcal{Q}_{\underline{d}}$. As in *loc. cit.*, they give rise to the operators $\underline{E}_{ij}^{(1)}, \underline{E}_{ij}^{(2)}, E_{ij}^{\Delta}$, etc. on the constructible complex $\mathsf{B} := \bigoplus_{\underline{d} \in \mathbb{Z}^{n-1}} \pi_* \underline{\mathbb{C}_d}$ on Bun (where $\underline{\mathbb{C}_d}$ stands for the constant sheaf on $\mathsf{Q}_{\underline{d}}$). This constructible complex is the geometric Eisenstein series of [14]. The above operators give rise to the action of $\bar{\mathsf{U}} := \mathbb{C}[\mathfrak{gl}_n] \rtimes U(\mathfrak{gl}_n)$ on B: this follows from the results of 5.13 by the argument of [2], 3.8–3.11. In particular, $\bar{\mathsf{U}}$ acts on the stalks of B, and we propose a conjecture describing the resulting $\bar{\mathsf{U}}$-modules.

Recall that the isomorphism classes of GL_n-bundles on $\mathbf{C}$ are parametrized by the set X^+ of dominant weights of GL_n. For $\eta \in X^+$ we denote by B_η the corresponding stalk of B. Also, we will denote by $\mathcal{O}(\eta)$ the corresponding line bundle on the flag variety $\mathcal{B}_n$ of GL_n. We will keep the same notation for the lift of $\mathcal{O}(\eta)$ to the cotangent bundle $T^*\mathcal{B}_n$. We will denote by L_η the direct image of $\mathcal{O}(\eta)$ under the Springer resolution morphism $T^*\mathcal{B}_n \to \mathcal{N}$ to the nilpotent cone $\mathcal{N} \subset \mathfrak{gl}_n$. The cohomology of the coherent sheaf L_η carries a natural action of $\bar{\mathsf{U}}$.

Conjecture 5.16 *The $\bar{\mathsf{U}}$-module B_η is isomorphic to $H_{\mathfrak{g}_{<0}}^{\frac{n(n-1)}{2}}(\mathcal{N}, L_\eta)$ (cf. Conjecture 7.8 of [6]).*

6 Equivariant K-ring of $\mathfrak{Q}_{\underline{d}}$ and quantum Gelfand–Tsetlin algebra

6.1 Quantum universal enveloping algebra

We preserve the setup of [2] with the following slight changes of notation. Now $\widetilde{T}$ stands for a 2^n-cover of a Cartan torus of GL_n as opposed to SL_n in *loc. cit.* Now U' stands for the quantum universal enveloping algebra of $\mathfrak{gl}_n$ over the field $\mathbb{C}(\widetilde{T} \times \mathbb{C}^*)$, as opposed to the quantum universal enveloping algebra of $\mathfrak{sl}_n$ in 2.26 of *loc. cit.*

For the quantum universal enveloping algebra of $\mathfrak{gl}_n$, we follow the notations of Sect. 2 of [19]. Namely, U' has generators $t_{ij}, \bar{t}_{ij},\ 1 \le i, j \le n$, subject to relations (2.4) of *loc. cit.* The standard Chevalley generators are expressed via $t_{ij}, \bar{t}_{ij}$ as follows:

$$K_i = t_{i+1,i+1}\bar{t}_{ii},\ E_i = (v - v^{-1})^{-1}\bar{t}_{ii}t_{i+1,i},\ F_i = -(v - v^{-1})^{-1}\bar{t}_{i,i+1}t_{ii}$$

(note that this presentation differs from the one in (2.6) of *loc. cit.* by an application of Chevalley involution). Note also that U' is generated by $t_{ii}, \bar{t}_{ii},\ 1 \le i \le n;\ t_{i+1,i}, \bar{t}_{i,i+1},\ 1 \le i \le n-1$. We denote by $U'_{\le 0}$ the subalgebra of U' generated by $t_{ii}, \bar{t}_{ii}, \bar{t}_{i,i+1}$. It acts on the field $\mathbb{C}(\widetilde{T} \times \mathbb{C}^*)$ as follows: $\bar{t}_{i,i+1}$ acts trivially for any $1 \le i \le n-1$, and $\bar{t}_{ii} = t_{ii}^{-1}$ acts by multiplication by $t_i^{-1}v^{1-i}$. We define the *universal Verma module* $\mathfrak{M}$ over U' as $\mathfrak{M} := U' \otimes_{U'_{\le 0}} \mathbb{C}(\widetilde{T} \times \mathbb{C}^*)$.

Recall that $M = \oplus_{\underline{d}} M_{\underline{d}},\ M_{\underline{d}} = K^{\widetilde{T} \times \mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{\mathbb{C}[\widetilde{T} \times \mathbb{C}^*]} \mathbb{C}(\widetilde{T} \times \mathbb{C}^*)$ (cf. [2], 2.7).

We define the following operators on M (well defined since the correspondences $\mathfrak{E}_{\underline{d},i}$ are smooth, and the $\widetilde{T} \times \mathbb{C}^*$-fixed point sets are finite):

$$\begin{aligned}
&t_{ii} = t_i v^{d_{i-1}-d_i+i-1} :\ M_{\underline{d}} \to M_{\underline{d}};\ \bar{t}_{ii} = t_{ii}^{-1};\\
&\bar{t}_{i,i+1} = (v^{-1} - v)t_{i+1}^{i} t_i^{-i-1} v^{(2i+1)d_i-(i+1)d_{i-1}-id_{i+1}-2i+1}\mathbf{p}_*\mathbf{q}^* :\ M_{\underline{d}} \to M_{\underline{d}-i};\\
&t_{i+1,i} = (v^{-1} - v)t_{i+1}^{-i-1} t_i^{i} v^{id_{i-1}+(i+1)d_{i+1}-(2i+1)d_i-1}\mathbf{q}_*(\mathfrak{L}_i \otimes \mathbf{p}^*) :\ M_{\underline{d}} \to M_{\underline{d}+i}.
\end{aligned}$$

According to Theorem 2.12 of [2], these operators satisfy the relations in U', i.e. they give rise to the action of U' on M. Moreover, there is a unique isomorphism $\psi :\ M \to \mathfrak{M}$ carrying $[\mathcal{O}_{\mathfrak{Q}_0}] \in M$ to the lowest weight vector $1 \in \mathbb{C}(\widetilde{T} \times \mathbb{C}^*) \subset \mathfrak{M}$.

6.2 Quantum Gelfand–Tsetlin basis

The construction of Gelfand–Tsetlin basis for the representations of quantum $\mathfrak{gl}_n$ goes back to M. Jimbo [11]. We will follow the approach of [19]. Given $\underline{\widetilde{d}}$ and the corresponding Gelfand–Tsetlin pattern $\Lambda = \Lambda(\underline{\widetilde{d}})$ (see Subsection 2.10), we define $\xi_{\underline{\widetilde{d}}} = \xi_\Lambda \in \mathfrak{M}$ by the formula (5.12) of [19]. According to Proposition 5.1 of *loc. cit.*, the set $\{\xi_{\underline{\widetilde{d}}}\}$ (over all collections $\underline{\widetilde{d}}$) forms a basis of $\mathfrak{M}$.

Recall the basis $\{[\underline{\widetilde{d}}]\}$ of M introduced in 2.16 of [2]: the direct images of the structure sheaves of the torus-fixed points. The following theorem is proved absolutely similarly to Theorem 2.11, using Proposition 5.1 of [19].

Theorem 6.3 *The isomorphism $\psi : M \to \mathfrak{M}$ of Subsection 6.1 takes $[\underline{\widetilde{d}}]$ to $c_{\underline{\widetilde{d}}}\xi_{\underline{\widetilde{d}}}$ where*

$$c_{\underline{\widetilde{d}}} = (v^2-1)^{-|\underline{d}|} v^{|\underline{d}|+\sum_i i d_{i-1}d_i - \sum_i \frac{2i+1}{2} d_i^2 - \frac{1}{2}\sum_{i,j} d_{i,j}^2} \prod_i t_i^{i(d_i-d_{i-1})} \prod_j t_j^{\sum_{k\geq j} d_{k,j}}.$$

6.4 Quantum Casimirs

Let Cas_k^v be the quantum Casimir element of the completion of the quantum universal enveloping algebra $U_v(\mathfrak{gl}_k)$. The quantum Casimir element is defined in section 6.1. of [16] in terms of the universal R-matrix lying in the completion of $U_v(\mathfrak{gl}_k) \otimes U_v(\mathfrak{gl}_k)$. According to [16], Proposition 6.1.7, the eigenvalue of Cas_k^v on the Verma module over $U_v(\mathfrak{gl}_k)$ with the highest weight λ_k is $v^{-(\lambda_k,\lambda_k+2\rho_k)}$. This means that the operator Cas_k^v is diagonal in the Gelfand–Tsetlin basis, and the eigenvalue of Cas_k^v on the basis vector $\xi_{\underline{\widetilde{d}}} = \xi_\Lambda$ is $v^{-\sum_{j\leq k}\lambda_{kj}(\lambda_{kj}+k-2j+1)}$ (with $t_i v^{j-1-d_{ki}} = v^{\lambda_{ki}}$). Consider the following "corrected" Casimir operators

$$\widetilde{Cas_k^v} := Cas_k^v \cdot \prod_{j=1}^k t_{jj}^{k-2} v^{\sum_{j=1}^k (\lambda_{nj}-j)(\lambda_{nj}-j+1) - \frac{k(k-1)(k-2)}{3}}$$

Lemma 2.14 admits the following

Corollary 6.5 a) *The operator of tensor multiplication by the class $[\mathcal{D}_k]$ in M is diagonal in the basis $\{[\underline{\widetilde{d}}]\}$, and the eigenvalue corresponding to $\underline{\widetilde{d}} = (d_{ij})$ equals $\prod_{j\leq k} t_j^{2-2d_{kj}} v^{d_{kj}(d_{kj}-1)}$.*

b) *The isomorphism $\psi : M \to \mathfrak{M}$ carries the operator of tensor multiplication by $[\mathcal{D}_k]$ to the operator $\widetilde{Cas_k^v}^{-\frac{1}{2}}$.*

Proof a) is immediate.

b) straightforward from a) and the formula for eigenvalues of Cas_k^v. □

6.6 Quantum Gelfand–Tsetlin algebra and K-rings

Recall the Whittaker vector $\mathfrak{k} = \sum_{\underline{d}} \mathfrak{k}_{\underline{d}}$ of [2] 2.30. According to Proposition 2.31 of *loc. cit.*, $\psi[\mathcal{O}_{\underline{d}}] = \mathfrak{k}_{\underline{d}}$.

Consider the "quantum Gelfand–Tsetlin algebra" $\mathcal{G} \subset \mathrm{End}(\mathfrak{M})$ generated by all the $\widetilde{Cas_k^v}^{-\frac{1}{2}}$ over the field $\mathbb{C}(\widetilde{T} \times \mathbb{C}^*)$. We denote by $\mathcal{I}_{\underline{d}} \subset \mathcal{G}$ the annihilator ideal of the vector $\mathfrak{k}_{\underline{d}} \in \mathfrak{V}$, and we denote by $\mathcal{G}_{\underline{d}}$ the quotient algebra of $\mathfrak{G}$ by $\mathcal{I}_{\underline{d}}$. The action of $\mathcal{G}$ on $\mathfrak{k}_{\underline{d}}$ gives rise to an embedding $\mathcal{G}_{\underline{d}} \hookrightarrow \mathfrak{M}_{\underline{d}}$.

Proposition 6.7 a) $\mathcal{G}_{\underline{d}} \xrightarrow{\sim} \mathfrak{M}_{\underline{d}}$.

b) *The composite morphism* ψ : $K^{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{\mathbb{C}[\widetilde{T}\times\mathbb{C}^*]} \mathbb{C}(\widetilde{T} \times \mathbb{C}^*) = M_{\underline{d}} \xrightarrow{\sim} \mathfrak{M}_{\underline{d}} \xrightarrow{\sim} \mathcal{G}_{\underline{d}}$ *is an algebra isomorphism.*

c) *The algebra* $K^{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{\mathbb{C}[\widetilde{T}\times\mathbb{C}^*]} \mathbb{C}(\widetilde{T} \times \mathbb{C}^*)$ *is generated by* $\{[\mathcal{D}_k] : k \geq 2,\ d_k \neq 0 \neq d_{k-1}\}$.

Proof The proof is the same as for Proposition 2.17. □

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