WEYL n-ALGEBRAS AND KONTSEVICH INTEGRAL OF UNKNOT

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Abstract. Given a Lie algebra with a scalar product, one may consider the latter as a symplectic structure on a *dg*-scheme, which is the spectrum of the Chevalley–Eilenberg algebra. In the first section we explicitly calculate the first order deformation of the differential on the Hochschild complex of the Chevalley–Eilenberg algebra. The answer contains the Duflo character. This calculation is used in the last section. There we sketch the definition of the Wilson loop invariant of knots, which is hopefully equal to the Kontsevich integral, and show that for unknot they coincide. As a byproduct we get a new proof of the Duflo isomorphism for a Lie algebra with a scalar product.

Introduction

In [Mar1] we built perturbative Chern–Simons invariants by means of factorization complex of Weyl n-algebras. In the present paper we continue this line and introduce the Wilson loop invariant. This invariant is supposed to be equal to the Bott–Taubes invariant and the Kontsevich integral. In fact, we are interested in the only question here: to calculate the Wilson loop invariant of unknot in S^3. This problem appears to be connected with the Duflo isomorphism.

We consider Duflo isomorphism for Lie algebras with a scalar product, which is much simpler to prove, that the general statement from [Duf]. There are (at least) two proofs of the Duflo isomorphism for Lie algebra with a scalar product. In [AM] authors use a quantization of Weil algebra. In [BNLT] the Kontsevich integral of knots and link is used. Our sketch of proof (see remark before Proposition 7) is related with the both. The work [Kri] also connects these two approaches and it would be very interesting to compare it with our arguments.

The first section is nearly not connected with the rest of the paper, but is of independent interest. Here we make a very concrete calculation of the first order deformation of the Hochschild complex of Chevalley–Eilenberg algebra of a Lie algebra, the deformation is given by the scalar product. This calculation is closely connected with [Mar2] and may be rephrased in the style of this paper, see Remark 2.

In the second section we give a very short survey of results about e_n-algebras and factorization complex we need. For fundamentals we refer the reader to [Lur] and for a much more detailed survey than ours we refer to [Gin]. At the end of the section we describe a Construction, which plays a crucial role in the next section.

In the third section we apply the Construction to quantum Chevalley–Eilenberg algebra, the role of which for perturbative Chern–Simons invariants is explained in [Mar1, Appendix]. The central result is Proposition 7. The calculation we

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make here strongly reminds the one from the first section. I would be glad to understand better reasons of this similarity. This section must be considered as an announcement, it contains no proofs.

Everything is over a field $\Bbbk$ of characteristic 0.

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1. Quantization of the Chevalley–Eilenberg complex

1.1. **Hochschild homology of Chevalley–Eilenberg complex.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The Chevalley–Eilenberg algebra $\mathrm{Ch}^\bullet(\mathfrak{g})$ is a super-commutative dg-algebra $S^*(\mathfrak{g}^\vee[1])$ generated by the dual space $\mathfrak{g}^\vee$ placed in degree 1. The differential is a derivation of this free super-commutative algebra defined on generator by the tensor $\mathfrak{g}^\vee \to \mathfrak{g}^\vee \wedge \mathfrak{g}^\vee$ dual to the bracket. The Jacoby identity guarantees that this is a differential. In terms of [ASZK] the Chevalley–Eilenberg algebra may be thought as the function ring of a Q-manifold.

With any $\mathfrak{g}$-module E one may associate module $\mathrm{Ch}^\bullet(\mathfrak{g}, E)$ over $\mathrm{Ch}^\bullet(\mathfrak{g})$ as follows. As $S^*(\mathfrak{g}^\vee[1])$-module it is freely generated by E and the differential is defined by its value on $E\otimes 1$ given by the tensor $E \to E\otimes\mathfrak{g}^\vee$ of the $\mathfrak{g}$-action. As a complex, $\mathrm{Ch}^\bullet(\mathfrak{g}, E)$ calculates cohomology of $\mathfrak{g}$ with coefficients in E.

The $\mathrm{Ch}^\bullet(\mathfrak{g})$-module $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee_{ad})$ corresponding to the adjoint $\mathfrak{g}$-module may be thought as a cotangent complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$. The de Rham differential $d_{dR}\colon \mathrm{Ch}^\bullet(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee_{ad})$, which is a derivation of $\mathrm{Ch}^\bullet(\mathfrak{g})$-modules, is tautologically defined on generators. Define the module of differential forms of $\mathrm{Ch}^\bullet(\mathfrak{g})$ as the power series of $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee_{ad})$ over $\mathrm{Ch}^\bullet(\mathfrak{g})$, that is $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$. It is a super-commutative algebra and the de Rham differential acting on it in the usual way is a derivation.

For a unital dg-algebra A define the reduced (or normalized) Hochschild complex $C_*(A)$ (see e. g. [Lod, Ch 1.1]) as the total complex of the bi-complex with the $(-i)$-th term

$$\prod_{i\geq 0}(A\otimes \underbrace{A/\Bbbk\otimes\cdots\otimes A/\Bbbk}_{i}), \tag{1}$$

the first differential coming from A and the second differential given by

$$\begin{aligned} &a_0\otimes a_1\otimes a_2\otimes\cdots\otimes a_i\mapsto\\ &a_0a_1\otimes a_2\otimes\cdots\otimes a_i - a_0\otimes a_1a_2\otimes\cdots\otimes a_i+\dots\\ &+(-1)^{i+\deg a_i(\deg a_0+\dots\deg a_{i-1})}a_ia_0\otimes a_1\otimes\cdots\otimes a_{i-1}. \end{aligned}\tag{2}$$

Here one have to choose representatives of quotients $A/\Bbbk$, then apply formula and take quotients again, the result does not depend on choices. Note, that usual (and right, because homotopy invariant) definition uses direct sum instead of product, but we need the one we gave. For a usual algebra reduced Hochschild complex calculates $\mathrm{Tor}_*^{A\otimes A^o}(A, A)$.

The following proposition is a variant of the Hochschild–Kostant–Rosenberg isomorphism.

Proposition 1. *Formula*

$$a_0\otimes a_1\otimes\cdots\otimes a_i\mapsto a_0\, d_{dR}a_1\cdots d_{dR}a_i \tag{3}$$

defines a morphism from the reduced Hochschild complex $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ *of Chevalley–Eilenberg algebra to its differential forms* $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$*, which is a quasi-isomorphism.*

Proof. Direct calculation shows, that this is a morphism. The proof of Proposition 2 implies that this is a quasi-isomorphism. □

Equip $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ with a descending filtration F: the subcomplex $F_k C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ consists of chains $a_0 \otimes a_1 \otimes \cdots \otimes a_i$ such that $\deg a_0 \geq k$.

Proposition 2. *The spectral sequence associated with filtration* F *on* $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ *degenerates at the second sheet, complex* $E_1^{p,0}$ *is isomorphic to* $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ *and* $E_1^{p,>0} = 0$.

Proof. The associated graded object to the filtration in hand is the tensor product of $S^*(\mathfrak{g}^\vee[1])$ and the normalized standard complex, which calculates homology of algebra with coefficients in the augmentation module $\mathrm{Ch}^\bullet(\mathfrak{g}) \to \Bbbk$. More precisely, the latter complex is the total complex of the bicomplex, which is the direct product $\prod_i (\mathrm{Ch}^\bullet(\mathfrak{g})/\Bbbk)^{\otimes i}$ and the second differential of $a_1 \otimes \cdots \otimes a_i$ is

$$a_1 \cdot a_2 \otimes \cdots \otimes a_i - a_1 \otimes a_2 \cdot a_3 \otimes \cdots \otimes a_i + \cdots \pm a_1 \otimes \cdots \otimes a_{i-1} \cdot a_i, \tag{4}$$

where a_i are elements of the augmentation ideal, which is identified with $\mathrm{Ch}^\bullet(\mathfrak{g})/\Bbbk$. The cohomology of this complex is easy to find by means of the spectral sequence associated with the bicomplex with the first differential (4). It degenerates at the first sheet for trivial reasons and equals to $\Bbbk[[\mathfrak{g}^\vee]]$ sitting in degree 0.

Equip $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ with the stupid filtration and consider map (3) of filtered complexes. In the light of the above, the associated map of spectral sequences gives an isomorphism on the first sheet. It follows that the first differentials also coincide. Thus the first differential of our spectral sequence is such as stated and higher differentials vanish for dimensional reasons. □

Note, that $F_i C_i(\mathrm{Ch}^\bullet(\mathfrak{g}))$ consists of chains $a_0 \otimes a_1 \otimes \cdots \otimes a_i$ such that $\deg a_{>0} = 1$. Taking into account the proposition we get the following.

Corollary 1. *Every cycle in* $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ *may be presented by a sum of chains* $a_0 \otimes a_1 \otimes \cdots \otimes a_i$ *with* $\deg a_{>0} = 1$.

It seems to be an interesting question to find an explicit formula for these cycles.

1.2. **Invariant vector fields.** Along with the Hochschild complex as above one may consider the Hochschild complex $C_*(A, M)$ of a dg-algebra A with coefficients in a A-bimodule M (see e. g. [Lod, Ch 1.1]). It is given by the same formulas (1) and (2), but a_0 now is an element of M. For a usual algebra reduced Hochschild complex calculates $\mathrm{Tor}_*^{A \otimes A^o}(A, M)$.

The $\mathrm{Ch}^\bullet(\mathfrak{g})$-module of 1-forms $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee)$ is a bimodule as well, because the algebra is supercommutative. Introduce the Hochschild complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$ with coefficients in this bimodule $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}), \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee))$.

Proposition 3. *Formulas*

$$\begin{aligned} a_0 \otimes a_1 \otimes \cdots \otimes a_i &\mapsto a_0\, d_{dR} a_1 \otimes a_2 \otimes \cdots \otimes a_i \\ a_0 \otimes a_1 \otimes \cdots \otimes a_i &\mapsto \pm a_0\, d_{dR} a_i \otimes a_1 \otimes \cdots \otimes a_{i-1}, \end{aligned} \tag{5}$$

where the sign is defined by the Koszul rule, define morphisms from the Hochschild complex $C_*(\mathfrak{g})$ *to the Hochschild complex with coefficients* $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}), \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee))$ *of degree 1.*

Proof. This is a direct calculation. □

The following proposition describes these morphisms in terms of quasi-isomorphism (1).

Recall some basic facts from Lie group theory. For a finite-dimensional Lie algebra $\mathfrak{g}$ denote by $U_\mathfrak{g}$ its enveloping algebra. This is a Hopf algebra dual to the Hopf algebra of formal functions $F(G)$ on the formal group associated with $\mathfrak{g}$. The Poincaré–Birkhoff–Witt map from the symmetric power of $\mathfrak{g}$ to its universal enveloping $i_{PBW}\colon S^*\mathfrak{g} \to U_\mathfrak{g}$ provides an isomorphism between them as adjoint $\mathfrak{g}$-modules. It is dual to the exponential coordinate map $exp^*\colon F(G) \to \Bbbk[[\mathfrak{g}^\vee]]$. Maps

$$\mathcal{L}_L\colon F(G) \to F(G) \otimes \mathfrak{g}^\vee \quad \text{and} \quad \mathcal{L}_R\colon F(G) \to F(G) \otimes \mathfrak{g}^\vee \tag{6}$$

dual to the multiplications

$$U_\mathfrak{g} \otimes \mathfrak{g} \to U_\mathfrak{g} \quad \text{and} \quad \mathfrak{g} \otimes U_\mathfrak{g} \to U_\mathfrak{g}$$

correspondingly being contracted with a Lie algebra element is derivation along the left and right invariant vector fields given by this element. After identification of G and $\mathfrak{g}$ by the exponential mapping, maps (6) are given by elements of $Vect(\mathfrak{g}) \otimes \mathfrak{g}^\vee$. Applying constant trivialization of the tangent bundle to $\mathfrak{g}$ one may identify such a tensor with a section of trivial vector bundle with fiber $\mathrm{End}(\mathfrak{g})$ over $\mathfrak{g}$. In other words, this section is the transformation matrix between the constant basis of the tangent bundle and the one given by left (right) invariant vector fields. By e. g. [Reu, Ch. 3.4] they are given by formulas

$$\mathrm{id} \pm \frac{1}{2}\mathrm{Ad} + \sum_{n\geq 1} \frac{B_{2n}}{(2n)!}\mathrm{Ad}^{2n} \tag{7}$$

(”$+$” for the first and ”$-$” for the second tensor), where Ad the structure tensor of the $\mathfrak{g}$, being considered as linear function on $\mathfrak{g}$ taking values in $\mathrm{End}(\mathfrak{g})$ and B_n are Bernoulli numbers:

$$\sum_{n\geq 0} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}. \tag{8}$$

Recall that Proposition 1 identifies $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ with complex $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$. In the same way one can build a quasi-isomorphism between $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}), \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee))$ and $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee \otimes \Bbbk[[\mathfrak{g}^\vee]]^{ad})$.

Proposition 4. *Under quasi-isomorphism as above, maps (5)*

$$\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee \otimes \Bbbk[[\mathfrak{g}^\vee]]^{ad})$$

are induced by (6), where $\Bbbk[[\mathfrak{g}^\vee]]$ *is identified with* $F(G)$ *by the exponential mapping That is (5) are given by formulas (7).*

Proof. Recall, that in the proof of Proposition 2 we considered the direct product of terms of standard complex calculating $\mathrm{Tor}_*^{\mathrm{Ch}^\bullet(\mathfrak{g})}(\Bbbk, \Bbbk)$ and identified it with $\Bbbk[[\mathfrak{g}^\vee]]$. Consider also the complex calculating $\mathrm{Ext}^*_{\mathrm{Ch}^\bullet(\mathfrak{g})}(\Bbbk, \Bbbk)$, here we take direct sum rather that direct product. The former complex is dual to the latter one. As in the proof of Proposition 2, spectral sequence argument shows, that cohomology of the latter complex is isomorphic to $S^*(\mathfrak{g})$. The Yoneda product equips it with multiplication that turns it to the universal enveloping algebra of $\mathfrak{g}$, this is easy to check. As the unrestricted version of $\mathrm{Tor}_*^{\mathrm{Ch}^\bullet(\mathfrak{g})}(\Bbbk,\Bbbk)$ is dual to it, this is formal functions on the group. The quasi-isomorphism (3) is dual to the PBW isomorphism, that is it is given by exponential coordinates. Formulas (5) define left and right action of the Lie algebra on the functions on the group. It proves the statement. □

Remark 1. Maps (5) may be thought as the Atiyah class of the diagonal of the *dg*-manifold which is a spectrum of $\mathrm{Ch}^\bullet(\mathfrak{g})$. Analogous maps and formulas for a usual complex manifold play crucial role in [Mar2].

1.3. **Quantization.** Let now $\mathfrak{g}$ be an finite-dimensional Lie algebra with a non-degenerate invariant scalar product $\langle\cdot\,,\cdot\rangle$. The scalar product may be thought as a constant symplectic structure of degree -2 on a *dg*-manifold (or Q-manifold), which is a spectrum of $\mathrm{Ch}^\bullet(\mathfrak{g})$. That is we define a Poisson bracket on $\mathrm{Ch}^\bullet(\mathfrak{g})$ on generators by $\{x,y\} = \langle x,y\rangle$ and extend it on the whole algebra by the Leibnitz rule. In terms of [ASZK] we get a *QP*-manifold.

A symplectic structure gives a first order deformation of the product of functions on a manifold and thus deforms the Hochschild complex. Our aim is to calculate it in our case.

More precisely, consider a ring $\Bbbk[\varepsilon]$, where $\deg\varepsilon = 2$ and $\varepsilon^2 = 0$ and the Chevalley–Eilenberg complex $\mathrm{Ch}^\bullet(\mathfrak{g})\otimes\Bbbk[\varepsilon]$ over $\Bbbk[\varepsilon]$ with the differential as before and the product given by $x\cdot y = x\wedge y + \frac{1}{2}\varepsilon\langle x,y\rangle$. Take Hochschild complex of $\Bbbk[\varepsilon]$-algebra $\mathrm{Ch}^\bullet(\mathfrak{g})\otimes\Bbbk[\varepsilon]$, that is all tensor product are taken over $\Bbbk[\varepsilon]$. It is a module over $\Bbbk[\varepsilon]$. Multiplication by ε defines 2-step filtration on it. Consider the spectral sequence associated with this filtration. The 0-th sheet is $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))\otimes\Bbbk[\varepsilon]$. The following proposition describes d_0 of this spectral sequence, which is the first order deformation of the differential in the Hochschild complex.

Proposition 5. *Contract tensors (6) from $Vect(\mathfrak{g})\otimes\mathfrak{g}^\vee$ with the pairing $\langle\cdot\,,\cdot\rangle$ and consider the resulting element of $Vect(\mathfrak{g})\otimes\mathfrak{g}$ as a differential operator on $\mathrm{Ch}^\bullet(\mathfrak{g},\Bbbk[[\mathfrak{g}^\vee]]^{ad})$ of the second order, where term $\cdot\otimes\mathfrak{g}$ differentiates $\mathrm{Ch}^\bullet(\mathfrak{g})$ and term $Vect(\mathfrak{g})\otimes\cdot$ differentiates $\Bbbk[[\mathfrak{g}^\vee]]$. Under quasi-isomorphism (3) differential d_0 of the above-mentioned spectral sequence is given by half-sum of these operators on the complex $\mathrm{Ch}^\bullet(\mathfrak{g},\Bbbk[[\mathfrak{g}^\vee]]^{ad})$. By (7), the matrix of this differential operator is given by*

$$\mathrm{id} + \sum_{n\geq 1}\frac{B_{2n}}{(2n)!}\mathrm{Ad}^{2n}, \tag{9}$$

B_n are Bernoulli numbers, Ad *is the structure tensor of the $\mathfrak{g}$, being considered as linear function on $\mathfrak{g}$ taking values in* $\mathrm{End}(\mathfrak{g})$.

Proof. By the very definition, the differential presenting the first order deformation of the Hochschild complex in the direction given by a symplectic form is given by the formula

$$\begin{aligned} d_0(a_0\otimes a_1\otimes\cdots\otimes a_n) = \frac{1}{2}\{a_0,a_1\}\otimes a_2\otimes\cdots\otimes a_n -\\ -\frac{1}{2}a_0\otimes\{a_1,a_2\}\otimes\cdots\otimes a_n + \cdots \pm \frac{1}{2}\{a_n,a_0\}\otimes a_1\otimes\cdots\otimes a_{n-1}, \end{aligned} \tag{10}$$

where $\{\ ,\ \}$ is the Poisson bracket, associated with the symplectic form. Apply it to the Chevalley–Eilenberg complex. By Corollary 1, any class in $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ may be presented by a cycle with degree one elements on non zero places. As the Hochschild complex is reduced, it follows, that in (10) only the first and the last term do not vanish. These terms are given by maps (5). Applying Proposition 4 we complete the proof. $\square$

Thus the proposition defines on the algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ a differential operator of order 2 and of cohomological degree -1. On this algebra another differential operator of the same order and degree is defined, call it the Brylinski differential after [Bry] and denote it by d_{Br}; in terms of the above proposition it is given by the unit matrix. They are not chain homotopic, but by the following proposition they become such after conjugation by an automorphism of complex $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$. This automorphism equals to multiplication by the Duflo character.

Given a Lie group G, equip it with the left invariant volume form (which is the right invariant as well, due to the invariant scalar product). Equip its Lie algebra $\mathfrak{g}$ with the constant volume form and denote by $j \in \Bbbk[[\mathfrak{g}^\vee]]$ the Jacobian of the exponential mapping. The Duflo character is the power series on $\mathfrak{g}$ which is the square root of the Jacobian and is given by

$$j^{\frac{1}{2}} = \exp \sum_{n=1}^{\infty} \frac{B_{2n}}{4n(2n)!}\ \mathrm{Tr}(\mathrm{Ad}^{2n}), \tag{11}$$

where B_n are Bernoulli numbers from (8) and Ad is the linear function on $\mathfrak{g}$ taking values in $\mathrm{End}(\mathfrak{g})$ as above.

Proposition 6. *Under quasi-isomorphism (3) differential d_0 on $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ is chain homotopic to $j^{-\frac{1}{2}} \circ d_{Br} \circ j^{\frac{1}{2}}$, where $j^{\frac{1}{2}}$ is the operator of multiplication of $\Bbbk[[\mathfrak{g}^\vee]]$ on the Duflo character and $j^{-\frac{1}{2}}$ is the inverse operator.*

Proof. We will use differential operator notation for endomorphisms of complex $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ and will use Einstein summation convention. For example, $d_{Br} = g_{ij}\partial/\partial x^i\ \partial/\partial\, d_{dR}x^j$, where g_{ij} is the scalar product x_i is a basis in $\mathfrak{g}^\vee$ and d_{dR} is the de Rham differential (we think of $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ as of differential forms on $\mathrm{Ch}^\bullet(\mathfrak{g})$ as in the first section). By Proposition 1.3,

$$d_0 - d_{Br} = \sum_{n\geq 1} \frac{B_{2n}}{(2n)!} (\mathrm{Ad}^{2n})^i_j g_{ik}\partial/\partial x^k\ \partial/\partial\, d_{dR}x^j, \tag{12}$$

where g_{ij} is the scalar product and Ad^* is the element of $\Bbbk[[\mathfrak{g}^\vee]] \otimes \mathrm{End}(\mathfrak{g})$. Consider differential operators of order 2 given by

$$H_{2n-1} = (\mathrm{Ad}^{2n-1})^i_j g_{ik}\partial/\partial x^k \wedge \partial/\partial x^j.$$

We leave to the reader to check that

$$[d_{CE}, H_{2n-1}] = 2(\mathrm{Ad}^{2n})^i_j g_{ik}\partial/\partial x^k\ \partial/\partial\, d_{dR}x^j - \frac{1}{2n}[d_{Br}, \mathrm{Tr}(\mathrm{Ad}^{2n})],$$

where d_{CE} is the differential in the Chevalley–Eilenberg complex; all other terms vanish due to the Jacoby rule. Comparing it with (12) we see, that $d_0 - d_{Br}$ is chain homotopic to $[d_{Br}, \ln j^{\frac{1}{2}}]$. This implies the statement. $\square$

Remark 2. The above proposition can be formulated and proved in a coordinate-free manner for any QP-manifold in terms of [ASZK]. In the setting of [Mar2] (see Remark 1) it describes the differential on differential forms on a complex symplectic manifold, which are Hochschild homology of the structure sheaf, coming from the first order deformation of the structure sheaf along the symplectic structure. It seems that being applied to the cotangent bundle of a complex manifold, it gives an alternative way of calculating the Todd class of this manifold.

Remark 3. Proposition 6 was inspired by the proof of the Duflo isomorphism for a Lie algebra with an invariant scalar product from [AM]. As we will see below, the calculation above is connected with another proof of the Duflo isomorphism, the one from [BNLT].

2. e_n-ALGEBRAS

2.1. **e_n-algebras.** The main character of what follows is a unital algebra over the operad e_n, the operad of rational chains of the little discs operad. Recall that this dg-operad and its cohomology for $n > 1$ is the shifted Poisson operad, which is generated by an associative commutative product $\cdot$ of degree 0 and a Lie bracket $\{\,,\,\}$ of degree $1-n$, they constrained by the Leibnitz rule. It is known , that for $n > 1$ operad e_n is formal, that is quasi-isomorphic ot its cohomology, but we will not need it. A e_∞-algebra is a unital homotopy commutative algebra and e_0-algebra be a complex with a marked cocycle.

The embedding of spaces induces the map of operads $e_k \to e_n$ for $k < n$. It induces a functor from e_n-algebras to e_k-algebras which we denote by obl_k^n. In particular, functor obl_n^∞ makes e_n-algebra from any commutative (that is e_∞-) algebra.

For our purpose it will be more convenient to consider the operad of rational chains of Fulton–MacPherson operad, see [Mar1] and references therein for details. The latter operad is homotopy equivalent to e_n and below we will make no difference between them, that is when saying e_n-algebra we shall mostly mean an algebra over this operad.

Operations of operad of little discs are spaces of n-balls embedded in a radius one n-ball. Group $SO(n)$ acts by rotations on the big ball. To take into account this action one may consider $SO(n)$ as an operad with only 1-ary operations and take the semi-direct product of this operad and the little discs operad. The result is called the framed little discs operad, see [SW]. The dg-operad of chains of this operad we denote by fe_n.

An alternative and better way to take into account the $SO(n)$-action is to consider equivariant chains. It gives us a dg-operad colored by $BSO(n)$, see e. g. [Mar1]. Modules over this operad are $SO(n)$-equivariant complexes. Call these modules by equivariant e_n-algebras. In general, the category of such algebras is not the same as the one of fe_n-algebras. But for $n = 2$ commutativity of the group simplifies things and these categories are essentially the same.

Consider the latter case in some detail. The cohomology of fe_2 is known as the Batalin–Vilkovisky (BV) operad, see e. g. [SW]. It is generated by the product $\cdot$ and the bracket $\{\,,\,\}$ obeying the same relations as ones for e_2 and an additional 1-ary operation Δ of degree -1 obeying relations

$$\Delta^2 = 0, \qquad \{a, b\} = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}\Delta(a)b - a\Delta(b).$$

2.2. **Factorization complex.** Given a framed n-manifold (that is the one with the tangent bundle trivialized) M and a e_n-algebra the factorization complex $\int_M A$ is defined, see e. g. [Mar1] and references therein. The idea of the definition is straightforward: discs embedded in M define a right module over e_n and the factorization complex is the tensor product over e_n of this right module with the left module given by A.

To extend the above definition to unframed manifold one need the algebra A to be equivariant. Locally one may choose framing on M and apply the definition and the use the equivariance to identify results for different framings.

The important property of the factorization complex is its behavior with respect to gluing, see e. g. [Gin] and references therein. Let M_1 and M_2 be two manifolds with isomorphic boundaries B, then for a e_n-algebra A there is a map of complexes

$$\int_{M_1} A \otimes \int_{M_2} A \to \int_{M_1 \cup_B M_2} A.$$

It follows that for $k < n$, a k-manifold M^k and a e_n-algebra A, the complex $\int_{M^k \times I^{n-k}} A$ is a e_k-algebra, and it is equivariant, if A is such. In particular, for a n-manifold M with boundary B the complex $\int_{B \times I} A$ is a (homotopy) algebra, and the map above equip $\int_M A$ with a module structure over it. In terms of this action the gluing rule may be written as

$$\int_{M_1 \cup_B M_2} A = \int_{M_1} A \underset{\int_{B \times I} A}{\otimes} \int_{M_2} A. \tag{13}$$

Another important property of the factorization complex is a kind of homotopy invariance:

$$\int_{M^k \times I^{n-k}} A = \int_{M^k} \mathrm{obl}_k^n A.$$

Below we will make no difference between sides of this equality and will denote simply by $\int_{M^k} A$. In particular, the factorization complex on the disk is quasi-isomorphic as a complex to the algebra itself.

Example 1. Let A be an equivariant e_2-algebra. Then its factorization complex on the disc $\int_{D^2} A$, which is A itself, is a module over $\int_{S^1 \times I^1} A = \int_{S^1} \mathrm{obl}_1^2 A$, which is the Hochschild homology complex of $\mathrm{obl}_1^2 A$. The equivariance of A is substantial here, without it the Hochschild complex of e_2-algebra A does not act on A, and if the equivariance structure is chosen, the action depends on this choice. To see it, note that $S^1 \times I^1$ is a framed manifold, that is why we do not need equivariance to take its factorization complex for any, not only equivariant algebra. But this framing, which comes from the constant framing on the square after gluing together two opposite edges, can not be continued on the whole disc from the annulus. Thus to construct the desired action by gluing annulus with the disc one need to identify factorization complex with different framings, and here one needs equivariance.

This may be generalized for any n by replacing the circle with $(n-1)$-sphere.

2.3. **Weyl n-algebras.** The species of equivariant e_n-algebras we need is the Weyl n-algebras, we refer to [Mar1] and [CPT$^+$] for the definition. To built such an algebra one need a super-vector space V with a super-skew-symmetric non-degenerate bilinear form on it. The e_n-algebra associated with such data is denoted by $\mathcal{W}^n(V)$. In analogy with the usual Weyl algebra, it is deformation of the polynomial algebra generated by V in the direction given by pairing. In fact, this is an algebra over the field of Laurent formal series of the quantization parameter h, but this must be ignored, assuming loosely speaking, that $h = 1$.

There are some important properties we need. Firstly, being considered as e_k-algebra, where $k < n$, it is commutative. In other words, $\mathrm{obl}_k^n \mathcal{W}^n(V) = \mathrm{obl}_k^\infty \Bbbk[V]$ for any $k < n$, where $\Bbbk[V]$ is the polynomial algebra.

The following property crucial for the construction of the perturbative invariants in [Mar1]: for any n-manifold M complex $\int_M \mathcal{W}^n(V)$ has one dimensional cohomology ([Mar1, Proposition 11]). I conjecture, that for any $k < n$ the factorization complex $\int_{N^k \times I^{n-k}} \mathcal{W}^n(V)$ for any k-dimensional manifold N^k is again a Weyl algebra.

Example 2. Let V be a vector space. Equip $V \oplus V^\vee[-1]$ with the standard form of degree -1. Then $\mathcal{W}^2(V \oplus V^\vee[-1])$ is the space of polyvector fields on $V^\vee$ and standard operations on it — Gerstenhaber bracket and cup product — are operations of cohomology of e_2.

As any Weyl algebra, $\mathcal{W}^2(V \oplus V^\vee[-1])$ is equivariant. Thus it is acted by operad fe_2 and by its cohomology, which is the BV operad. The operation Δ is equal to the de Rham differential, being polyvector fields identified with differential forms by mean of the constant volume form. Another choice of the volume form leads to another fe_2-structure with the same underlying e_2-structure.

2.4. **Action.** For an associative (or e_1-) algebras the notion of modules plays the central role. The higher generalization of this notion is a e_n-algebra acting on a e_{n-1}-algebra, for definition and discussion see e. g [Gin] and references therein. Constructively it may be defined by means of the Swiss cheese operad, what is especially convenient for algebras over the operad of chains of the Fulton–MacPherson operad. In the same way as operations of the little discs operad are given by configurations space of $\mathbb{R}^n$, operations of the Swiss cheese operad are given by spaces of distinct points on $\mathbb{R}^{\geq 0} \times \mathbb{R}^{n-1}$. There are points of two types: on the boundary and in the interior. Correspondingly, it gives a colored operad with two colors. If a e_n-algebra B acts on a e_{n-1}-algebra A, then elements of B sit on the interior points and elements of A — on boundary ones. For further details we refer the reader to [Vor].

Note, that the action of the Swiss cheese operad may be formulated in terms of factorization sheaves, for the definition of the latter see e. g. [Gin] and references therein. Namely, such an action is equivalent to a factorization sheaf on the semi-space such that a restriction on the boundary and the interior are constant factorization sheaves, corresponding to e_{n-1}-algebra A and e_n-algebra B.

It is known that for any e_n-algebra there exist a universal e_{n+1} algebra $\mathrm{End}(A)$ acting on it ([Lur]). In other words, to define an action of e_{n+1} algebra B on A is the same as to produce an morphism of e_{n+1}-algebras $B \to \mathrm{End}(A)$. For an associative (or e_1-) algebra the End-object is its Hochschild cohomology complex.

Let V be a vector space. Equip $V \oplus V^\vee[1-n]$ with the standard form of degree $(1-n)$. Then $\mathcal{W}^n(V \oplus V^\vee[1-n])$ is $\mathrm{End}(\Bbbk[V])$, where $\Bbbk[V]$ is the polynomial algebra. To see it one may construct an action of $\mathcal{W}^n(V \oplus V^\vee[1-n])$ on $\Bbbk[V]$ directly by using the Swiss cheese operad and the Fulton–MacPherson compactification. Then one need to check, that the resulting map $\mathcal{W}^n(V \oplus V^\vee[1-n]) \to \mathrm{End}(\Bbbk[V])$ is a quasi-isomorphism.

The action commutes with taking the factorization complex. That is if an equivariant e_{n+1}-algebra B acts on an equivariant e_n-algebra A, then for a k-manifold

N the e_{n-k+1}-algebra $\int_{N^k\times I^{n-k+1}} B$ acts on e_{n-k}-algebra $\int_{N^k\times I^{n-k}} A$. It follows immediately from definitions of the Swiss cheese operad and the factorization complex. It seems plausible, that under appropriate conditions $\int_{N^k\times I^{n-k+1}} \mathrm{End}(A) = \mathrm{End}(\int_{N^k\times I^{n-k}} A)$.

Example 3. Consider the polynomial algebra $A = \Bbbk[V]$ as an associative algebra. Its Hochschild cohomology complex $C^*(A, A)$ (which, as it was mentioned above, is $\mathcal{W}^2(V \oplus V^\vee[-1])$) acts on it. It follows, that $\int_{S^1} C^*(A, A)$, which is a e_1-algebra, acts on $\int_{S^1} A$. The latter complex is the Hochschild homology complex of A, which is known to be quasi-isomorphic to the direct sum of shifted differential forms (see e. g. [Lod]). It is shown in [NT], that the first complex is quasi-isomorphic to the differential operators on differential forms, what is in good agreement with the speculation preceding the present example.

In the next section the following Construction, which generalizes Example 1, plays a crucial role.

Construction. Let A be an equivariant e_n-algebra. Then for any $k < n$ the e_{n-k}-algebra $\int_{S^k} A$ naturally acts on $\mathrm{obl}^n_{n-k-1} A$. The corresponding action of the Swiss cheese operad is defined as follows. Embed $\mathbb{R}^{\geq 0} \times \mathbb{R}^{n-k-1}$ linearly into $\mathbb{R}^n$. Put at any point of this semi-space the factorization complex of A on the k-sphere lying into the $k+1$ space perpendicular to the semi-space, with its center on $0 \times \mathbb{R}^{n-k-1}$ and passing through this point. In particular, for points on $0 \times \mathbb{R}^{n-k-1}$ we get the sphere of zero diameter, that is a point and the factorization complex is A itself.

In other words, consider a map $\mathbb{R}^n \to \mathbb{R}^{\geq 0} \times \mathbb{R}^{n-k-1}$ which sends a point to the pair consists of radius k-sphere lying into the $k+1$ space perpendicular to the semi-space passing through this point and the orthogonal projection on $\mathbb{R}^{n-k-1}$. Then the direct image of the factorization sheaf on $\mathbb{R}^n$ is the desired factorization sheaf on $\mathbb{R}^{\geq 0} \times \mathbb{R}^{n-k-1}$.

3. Wilson loop

3.1. **Quantum Chevalley–Eilenberg algebra.** Given a Lie algebra $\mathfrak{g}$ with an invariant scalar product, in [Mar1, Appendix] (see also [CPT$^+$, 3.6.2]) a e_3-*dg*-algebra $\mathrm{Ch}^\bullet_h(\mathfrak{g})$ is defined as follows. Take the Weyl 3-algebra given by the space $\mathfrak{g}^\vee[1]$ with the scalar product and equip it with a differential $\frac{1}{h}\{\cdot, q\}$, where $\{\ ,\ \}$ is the image of the Lie bracket under the map $L_\infty \to e_3$ (see e. g. [Mar1, Proposition 2]) and q is the degree 3 element, which is composition of the Lie bracket on g and the scalar product. Call this e_3-algebra the quantum Chevalley–Eilenberg algebra.

Consider the Hochschild complex $C_*(\mathrm{Ch}^\bullet_h(\mathfrak{g}))$. Here and in what follows we will consider unrestricted Hochschild chains, that is the Hochschild complex is the direct product of its terms.

The Hochschild complex is the factorization complex $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$. As $\mathrm{Ch}^\bullet_h(\mathfrak{g})$ is e_3-algebra, the Hochschild complex is e_2-algebra. Consider it as e_1-algebra, that is take $\mathrm{obl}^2_1 \int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$. By the very definition it is equal to $\int_{S^1} \mathrm{obl}^3_2 \mathrm{Ch}^\bullet_h(\mathfrak{g})$. We mentioned above an important property of Weyl algebras: $\mathrm{obl}^n_k \mathcal{W}^n(V) = \mathrm{obl}^\infty_k \Bbbk[V]$ for any $k < n$. It follows, that $\mathrm{obl}^3_2 \mathrm{Ch}^\bullet_h(\mathfrak{g}) = \mathrm{obl}^\infty_2 \mathrm{Ch}^\bullet(\mathfrak{g})$. Thus $\mathrm{obl}^3_2 \mathrm{Ch}^\bullet_h(\mathfrak{g})$ is just the super-commutative Chevalley–Eilenberg algebra. Its Hochschild complex is again super-commutative algebra quasi-isomorphic to $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ by Proposition 1.To recap, $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ as e_1-algebra, that is $\mathrm{obl}^2_1 \int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ is isomorphic to $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$.

Now let us apply the Construction from the previous section to $A = \mathrm{Ch}^\bullet_h(\mathfrak{g})$, $n = 3$ and $k = 1$. It gives an action of e_2-algebra $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ on $\mathrm{obl}^3_1 \mathrm{Ch}^\bullet_h(\mathfrak{g})$, which is $\mathrm{obl}^\infty_1 \mathrm{Ch}^\bullet(\mathfrak{g})$. That is we get a map from e_2-algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ to the Hochschild cohomology complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$ by the universal property, which is easily seen to be a quasi-isomorphism. The Hochschild cohomology complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$ is known to be equal to $\mathrm{Ch}^\bullet(\mathfrak{g}, U^{ad}_\mathfrak{g})$, where $U_\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$.

To be more precise, in this way we get a map to $\mathrm{Ch}^\bullet(\mathfrak{g}, U^{ad}_\mathfrak{g}) \otimes \Bbbk[[h]]$. The e_1-structure on this complex comes from the one on the universal enveloping algebra. On the other hand, as it is shown in the previous paragraph, $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ as e_1-algebra isomorphic to $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$. Thus an explicit form of this map, which is supplied by the proposition below, implies the Duflo isomorphism.

Proposition 7. *The map of complexes*

$$\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad}) = \int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, U^{ad}_\mathfrak{g}) \otimes \Bbbk[[h]] \tag{14}$$

as above is chain homotopic to the one induced by the composition

$$\Bbbk[[\mathfrak{g}^\vee]] \xrightarrow{\exp(h(\cdot,\cdot))} S^*\mathfrak{g} \otimes \Bbbk[[h]] \xrightarrow{j^{\frac{1}{2}}} S^*\mathfrak{g} \otimes \Bbbk[[h]] \xrightarrow{PBW} U_\mathfrak{g} \otimes \Bbbk[[h]], \tag{15}$$

where the first arrow is given by the scalar product multiplied by h, the second is contraction with the Duflo character (11) and the third one is the PBW map.

Sketch of proof. As it was mentioned above, $\mathrm{Ch}^\bullet_h(\mathfrak{g})$ as e_2-algebra is isomorphic to commutative algebra $\mathrm{Ch}^\bullet(\mathfrak{g})$. It follows, that the map induced by the unit embedding $\mathrm{Ch}^\bullet(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ is the morphism of e_2-algebras and in composition with (14) it gives the standard map $\mathrm{Ch}^\bullet(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, U^{ad}_\mathfrak{g})$. Thus we know the image of a subalgebra $\mathrm{Ch}^\bullet(\mathfrak{g})$ under (14). One may see, that the whole map (14) may be uniquely determined from it as the one compatible with the Lie bracket coming from e_2-structure. To see it one may use the faithful action of $\int_{S^1 \times S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ on $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ as in the sketch of proof of Proposition 8.

So our closest purpose is to calculate the bracket on $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]])$, which is $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$. As we will see below it is enough to calculate the bracket with an element which is image of $a \in \mathrm{Ch}^\bullet(\mathfrak{g})$ under the embedding map as above. Given an element $b \in \int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$, the bracket $\{a, b\}$ may be presented graphically as follows. Consider the solid torus $D^2 \times S^1$ and consider two circles in it: $C = (0, S^1)$, call it the big one and $c = (\{x \in D^2 \mid |x| = 1/2\}, *)$, call it the small one. The cycle in the factorization complex of the solid torus, which is $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$, presenting $\{a, b\}$ equals to $C_b \otimes ([c] \otimes a)$, where by C_b we denote the image in $\int_{D^2 \times S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ under the embedding $C \hookrightarrow D^2 \times S^1$. One may see, that cycle $[c] \otimes a$ is equal to $c_{d_{dR}a}$, where d_{dR} is the de Rham differential. If $a = x_1 \wedge \cdots \wedge x_i$, then $d_{dR}x = \sum \pm d_{dR}x_i \, x_1 \wedge \ldots \hat{x_i} \cdots \wedge x_n$.

Let us now start pull the small circle to unlink it from the big one. That is consider a family of cycles c^t where c^t is a family of circles in the solid torus such that c^0 is the small circle, c^1 is a circle unlinked with the big circle and only one circle in the family intersects the big one. Until circles do not intersect, nothing happens, the cycle $C_b \otimes c^t_a$ remains in the same class. But as soon as they intersect each other, class changed by the one which is a derivation of b. The calculation shows, that for $b = d_{dR}x_0 \, x_1 \wedge \cdots \wedge x_n$ it is given by sum of maps (5) contracted with x_0 and multiplied by $x_1 \wedge \cdots \wedge x_n$. The reasoning is analogous to Proposition 5: unlinking influences only on points close to the intersection point. When the small circle is unlinked from the big one, $C_b \otimes c^1_a$ vanishes, because $c^1_a = [c^1] \otimes a$ is a boundary.

Note, that e_2-algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ is in fact a fe_2-algebra. Thus instead of the Lie bracket one may calculate the operator Δ corresponding to the rotation. Given an element $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad}) \ni x = \sum a_i b_i$, where a_i are in the odd part and b_i in the even part, one may show, that

$$\Delta x = \sum \{a_i, b_i\}.$$

Apply to it calculations from the previous paragraph. Comparing it with Proposition 5 we see, that operator Δ on $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ coincides with operator d_0 from there. Proposition 6 implies that the Duflo character supplies an isomorphism between this operator and d_{Br}. To complete the proof one have to verify that d_{Br} is operator Δ for fe_2-algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, U^{ad}_{\mathfrak{g}})$. $\square$

While proving the proposition we found, that operator Δ on fe_2-algebra $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ equals to the first order deformation of the Hochschild differential of $\mathrm{Ch}^\bullet(\mathfrak{g})$ that we discussed in the first section. I have no explanation of this coincidence.

3.2. **Invariants of knots.** In [Mar1] we constructed invariants of manifolds using Weyl n-algebras. Below we develop this ideas for manifolds with embedded links. Let us restrict ourselves with a 3-sphere with a knot in it.

As it was observed in [Mar1], cohomology of the factorization complex of a Weyl n-algebra $\mathcal{W}^n(V)$ on a closed n-manifold is one-dimensional. If V lies in degree 1 and the manifold is 3-sphere (or homology sphere), then the generator of this cohomology is given by the class $[p] \otimes S^{\mathrm{top}}V$, where p is a point in the manifold. As it was explained in [Mar1, Appendix], factorization complex $\int_{S^3} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ is isomorphic to the one of the underlying Weyl 3-algebra, because the Chevalley–Eilenberg differential is inner, one need to consider here unrestricted chains, that is take direct product rather than the direct sum. It is easy to see that the generator in the cohomology of $\int_{S^3} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ is given by $[p] \otimes S^{\mathrm{top}}\mathfrak{g}^\vee$. Call it the standard cycle. The idea of invariants we construct is to produce another cycle and compare it with the standard one.

Given a knot $K\colon S^1 \hookrightarrow S^3$ and a class $f \in \int_{S^1} \mathrm{Ch}^\bullet(\mathfrak{g}) = \mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ denote by K_f the direct image of this class under K. The class we are interested in is $([p] \otimes S^{\mathrm{top}}\mathfrak{g}^\vee) \otimes K_f$. By dimensional reasons, only f of degree 0 are interesting, that is in fact $f \in \Bbbk[[\mathfrak{g}^\vee]]^{inv}$. Thus we get the following definition.

Definition 1. For a knot K in $\mathbb{R}^3$ the Wilson loop invariant is the function on $\Bbbk[[\mathfrak{g}^\vee]]^{inv}$ given by

$$f \mapsto ([\infty] \otimes S^{\mathrm{top}}(\mathfrak{g}^\vee[1])) \otimes K_f \in \int_{S^3} \mathrm{Ch}^\bullet_h(\mathfrak{g}),$$

where we identify $\int_{S^3} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ with $\Bbbk[[h]]$ using the standard cycle as a generator.

In [Mar1] we showed that invariants built there are described by formulas similar to Axelrod–Singer ones. Following the same line we see, that Wilson loop invariants are connected with Bott-Taubes invariants, for a survey of the latter see e. g. [Vol]. There is another invariant of knot — the Kontsevich integral, see [CDM, Part 3]. Morally it should coincide with the Bott–Taubes invariants, see [Kon]. As far as

I know, this point is not clear, for discussion see [Les]. One may hope, that the definition above will help to elaborate this.

Our construction of the Wilson loop invariant depends on a choice of the Lie algebra with scalar product. One may give a more complicated, but universal definition of this invariants with values in the graph complex, which is the Chevalley–Eilenberg complex of Hamiltonian vector fields, in the same way as it is outlined in [Mar1, Appendix].

An interesting property of the Kontsevich integral is its value on unknot: it is equal to the Duflo character and this allows to prove the Duflo isomorphism, see [BNLT] and [CDM, Ch. 11]. The following proposition states that the Wilson loop invariant shares this property.

Proposition 8. *The Wilson loop invariant of unknot is equal to the composition*

$$\Bbbk[[\mathfrak{g}^\vee]]^{inv} \hookrightarrow \Bbbk[[\mathfrak{g}^\vee]] \to U_\mathfrak{g} \otimes \Bbbk[[h]] \to \Bbbk[[h]],$$

where the second arrow is given by (15) and the third one is the standard augmentation.

Sketch of proof. As it was discussed in Subsection 3.1, e_2-algebra $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ acts on e_1-algebra $\mathrm{Ch}^\bullet_h(\mathfrak{g})$. In Proposition 7 it is shown, that this action is not "naive", the morphism to the End-object is the composition of pairing and the Duflo character. As it was mentioned above, action is compatible with taking factorization complex: as $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ acts $\mathrm{Ch}^\bullet_h(\mathfrak{g})$ so $\int_{S^1\times S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ acts on $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$. The e_1-algebra $\int_{S^1\times S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ is the algebra of differential operators on $\mathrm{Ch}^\bullet_h(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$, see also Example 3. The complex $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ is a kind of holonomic module over these differential operators. But again it is not "naive", this action is twisted by the Duflo character.

Now apply (13) to calculate the Wilson loop invariant of knot. Cut S^3 in two solid tori un the standard way, being the infinity point inside one of them and the unknot is the middle circle of the other. Now apply (13) to calculate the Wilson loop invariant of knot. As it was mentioned, $\int_{S^1\times S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ is the algebra of differential operators and factorization complexes of solid tori are kind of holonomic modules with transversal characteristic varieties. Now the calculation of the Wilson invariant is reduced to taking the derived tensor product of these modules and comparing cycles in the result given by different $f \in \Bbbk[[\mathfrak{g}^\vee]]^{inv}$. Taking into account the Duflo twisting we get the result. □

There is another natural approach to knot invariants mentioned in [AFT]. Given a knot in a closed manifold, one may cut out a small solid torus around it to get a manifold with a boundary. Then factorization complex for a fe_3-algebra of this manifold is a module over the factorization complex of the boundary torus, which is an invariant of the knot.

The proof of the above proposition make clear what is happened when the fe_3-algebra is $\mathrm{Ch}^\bullet_h(\mathfrak{g})$. In this case the pair of algebra and module itself does not depend on the knot, this is essentially differential operators and a standard holonomic module over it. But this module contains a marked element (it is a e_0-algebra!), which is the image of the unit. And module together with this element is the invariant of the knot. Reasonings analogous to the proof of Proposition 8 show that this invariant is essentially equivalent to the Wilson loop invariant.

3.3. **Skein algebra.** It [Tur] for a Riemann surface S a skein algebra was introduced. It is generated by non self-intersecting loops on S. We claim, that there is a map from this algebra to $\int_{S\times I} \mathrm{Ch}^\bullet_h(\mathfrak{g})$. An element corresponding to a loop L maps to L_η, where $\eta \in \int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ is the canonical element, which is the preimage of $\exp(hc)$ under (15), where c is the Casimir element given by the scalar product.

The reason to propose it is the following. The skein algebra is the quantization of a Poisson algebra. The latter appears in [Gol, Wol] as a subalgebra of the Poisson algebra of functions on the moduli space of G-local systems on S. But $\int_{S\times I} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ must be thought as the quantization of the latter Poisson algebra, see [CPT$^+$].

Elements L_η plays important role because they are generating functions of Dehn twists. In other words, the cobordism corresponding to the Dehn twist gives a bimodule over $\int_{S\times I} \mathrm{Ch}^\bullet_h(\mathfrak{g})$ according to speculations in the end of the previous subsection. Then element L_η corresponding to the Dehn twist equips us the characteristic function of this module. This allows to reduce calculation of quantum invariants of manifolds to Wilson loop invariant of links similarly as it was done e. g. in [RT].

We hope to elaborate all of this elsewhere.

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