DOI 10.1007/s11141-015-9594-1
Radiophysics and Quantum Electronics, Vol. 58, No. 3, August, 2015
(Russian Original Vol. 58, No. 3, March, 2014)

SOLITONS IN AN EXTENDED NONLINEAR SCHRÖDINGER EQUATION WITH PSEUDO STIMULATED SCATTERING AND INHOMOGENEOUS CUBIC NONLINEARITY

N. V. Aseeva,* E. M. Gromov, and V. V. Tyutin

UDC 537.86

We consider the soliton dynamics within the framework of an extended nonlinear Schrödinger equation with pseudo stimulated scattering, which occurs from the damped low-frequency waves, and the spatially inhomogeneous cubic nonlinearity. It is shown that the pseudo stimulated scattering, which leads to a shift of the spectrum of the soliton wave numbers to the long-wavelength region, and the nonlinearity, which increases with the coordinate and shifts the soliton spectrum to the short-wavelength region, can be in balance. The soliton solution, which results from this balance, is explicitly obtained.

1. INTRODUCTION

The intense high-frequency (HF) wave packets are known to propagate in nonlinear media without changing their shapes, i.e., be solitons. The soliton solutions emerge in many topical problems in various fields of physics when simulating the intense-wave propagation in dispersive media [1–7].

The dynamics of rather long HF wave packets is described by the second-order approximation of the theory of dispersion of nonlinear waves, in which the nonlinear Schrödinger equation [8, 9] is the basic one. The soliton solutions in this equation appear as a result of the balance of the dispersive expansion and nonlinear compression of the wave packets.

To solve many applied problems, it is necessary to decrease the soliton length to several wavelengths. Such a decrease is usually accompanied by the stimulated scattering from the low-frequency (LF) medium perturbations. The stimulated scattering from the spatially homogeneous HF time modes of the medium (Raman stimulated scattering), has been studied in sufficient detail [1]. For solitons, allowance for this scattering leads to the frequency downshift of the soliton spectrum [8]. The possibility of compensating for the Raman frequency shift has by now been rather comprehensively studied [8–16].

However, for some media, the short-soliton propagation is accompanied by excitation of the damped LF waves. The model for describing the stimulated scattering of the HF waves from the damped LF waves, which is called pseudo stimulated scattering, is proposed in [17], in which it is shown that the pseudo stimulated scattering shifts the soliton wave-number spectrum to the long-wavelength region. The mechanism of compensation for the pseudo stimulated scattering due to the spatially inhomogeneous dispersion is considered in [17–19].

In this work, the soliton dynamics is considered within the framework of an extended nonlinear Schrödinger equation with pseudo stimulated scattering and the spatially inhomogeneous cubic nonlinearity. It is shown that the nonlinearity, which increases with the coordinate, shifts the soliton wave-number

* naseeva@hse.ru

National Research University "Higher School of Economics," Nizhny Novgorod, Russia. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 58, No. 3, pp. 227–234, March 2014. Original article submitted May 16, 2014; accepted March 27, 2015.

spectrum to the short-wavelength region and can compensate for the pseudo stimulated scattering, which shifts the soliton wave-number spectrum to the long-wavelength region. The soliton class whose existence is caused by the balance of the pseudo stimulated scattering and the inhomogeneous cubic nonlinearity is found. This work is the continuation of [19].

2. BASIC EQUATION AND INTEGRAL RELATIONSHIPS

Let us consider the dynamics of the envelope function $U(\xi,t)$ of the intense high-frequency field $U(\xi,t)\exp(i\omega t - i\kappa\xi)$ in a medium with allowance for the nonlinear interaction with the damped LF wave perturbations $n(\xi,t)$ propagating with the spatially inhomogeneous velocity $V(\xi)$ and causing the HF-radiation loss. In the approximation of unidirectional propagation along the coordinate ξ, the initial system of the wave-field dynamics corresponds to the Zakharov-type system of equations [20, 21]

$$2i\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial \xi^2} - nU = 0, \tag{1}$$

$$\frac{\partial n}{\partial t} + V(\xi)\frac{\partial n}{\partial \xi} - \nu\frac{\partial^2 n}{\partial \xi^2} = -\frac{\partial(|U|^2)}{\partial \xi}, \tag{2}$$

where ν is the coefficient of the HF-radiation loss due to LF perturbations. In particular, this system describes the dynamics of the intense electromagnetic or Langmuir waves in an isotropic plasma with allowance for their interaction (via the ponderomotive nonlinearity) with the damped ion-acoustic waves, which propagate with velocity that is inhomogeneous along the coordinate and are responsible for the HF-radiation loss. Assuming that the spatial scale of inhomogeneity of the velocity $V(\xi)$ of the LF waves is much greater than that of the envelope function of the HF wave packet, i.e., $D \gg D_{|U|}$, in the third-order approximation of the dispersion theory, the nonlinear response n of the medium includes the term with a small antisymmetric spatial nonlocality with respect to the coordinate ξ: $n = -|U|^2/V(\xi) - \nu\,[\partial(|U|^2)/\partial\xi]/V(0)$. The term that is antisymmetric with respect to the coordinate ξ is caused by the HF-radiation loss due to the LF waves. In this case, Eqs. (1) and (2) are reduced to the extended nonlinear Schrödinger equation with a weak antisymmetric Kerr (ponderomotive) response of the medium

$$2i\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial \xi^2} + 2\alpha(\xi)U\,|U|^2 + \mu U\frac{\partial(|U|^2)}{\partial \xi} = 0, \tag{3}$$

where $\alpha(\xi) = 1/[2V(\xi)]$ is the cubic-nonlinearity coefficient, whose spatial nonuniformity is stipulated by the spatial nonuniformity of the LF-wave velocity, and the coefficient $\mu = \nu/V(0)$. The last term in Eq. (3) describes the stimulated scattering of the HF field from the LF waves, which is responsible for the HF-wave loss and is a spatial analog of the stimulated Raman scattering in optics (pseudo stimulated scattering).

For the zero boundary conditions $U_{\xi\to\pm\infty} \to 0$ at infinity, Eq. (3) yields the following relationships for the integral moments of the wave field:

$$\frac{\mathrm{d}N}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-\infty}^{+\infty} |U|^2\,\mathrm{d}\xi = 0, \tag{4}$$

$$2\frac{\mathrm{d}P}{\mathrm{d}t} \equiv 2\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-\infty}^{+\infty} K\,|U|^2\,\mathrm{d}\xi = -\mu L + \int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}\alpha}{\mathrm{d}\xi}|U|^4\,\mathrm{d}\xi, \tag{5}$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-\infty}^{+\infty}\left[\frac{\partial(|U|^2)}{\partial\xi}\right]^2\mathrm{d}\xi = 2\int\limits_{-\infty}^{+\infty}\frac{\partial^2(|U|^2)}{\partial\xi^2}\,\frac{\partial(K\,|U|^2)}{\partial\xi}\,\mathrm{d}\xi, \tag{6}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-\infty}^{+\infty}|U|^4\,\mathrm{d}\xi = \int\limits_{-\infty}^{+\infty}K\frac{\partial(|U|^4)}{\partial\xi}\,\mathrm{d}\xi, \tag{7}$$

$$N\frac{\mathrm{d}\bar{\xi}}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-\infty}^{+\infty} \xi\,|U|^2\,\mathrm{d}\xi = P, \tag{8}$$

where the complex amplitude of the HF field is $U = |U|\,\exp(i\varphi)$, and $K = \partial\varphi/\partial\xi$ is the additional wave number of the packet.

3. ANALYTICAL RESULTS

3.1. Evolution of integral moments

To analyze the system of Eqs. (4)–(8), we assume that the spatial scales of the nonuniformity of the cubic-nonlinearity coefficient α and the additional wave number K are much longer than the nonuniformity scale of the envelope function U of the wave packet, i.e., $\{D_\alpha, D_K\} \gg D_{|U|}$, and approximate the spatial distribution of the wave number in the vicinity of the wave-packet barycenter $\bar{\xi} = N^{-1}\int_{-\infty}^{+\infty} \xi\,|U|^2\,\mathrm{d}\xi$ by the linear dependence $K(\xi,t) \approx K(\bar{\xi}) + (\partial K/\partial\xi)_{\bar{\xi}}\,(\xi-\bar{\xi})$. In this case, under the condition $(\partial|U|/\partial\xi)_{\bar{\xi}} = 0$, we obtain the equality $(\partial K/\partial\xi)_{\bar{\xi}} = -2\,(|U|^{-1}\,\partial|U|/\partial\xi)_{\bar{\xi}}$ from the imaginary part of Eq. (3). For the wave packets whose amplitude and length are related by the soliton-like relationship, with allowance for Eqs. (4) and (8), we obtain the following formula for the spatial distribution of the wave number:

$$K(\xi,t) = k(t)\left[1 - \frac{\alpha'(\bar{\xi})\,(\xi-\bar{\xi})}{\alpha(\bar{\xi})}\right], \tag{9}$$

where $k(t) = K(\bar{\xi})$ is the wave number at the wave-packet barycenter and $\alpha'(\bar{\xi}) \equiv (\partial\alpha/\partial\xi)_{\bar{\xi}}$ is the cubic-nonlinearity gradient at the wave-packet barycenter. With allowance for Eq. (9), the system of Eqs. (4)–(8) takes the form

$$2N\frac{\mathrm{d}k}{\mathrm{d}t} = -\mu L + \alpha'(\bar{\xi})M, \qquad \frac{\mathrm{d}L}{\mathrm{d}t} = 3\frac{\alpha'(\bar{\xi})}{\alpha(\bar{\xi})}kL, \tag{10}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\alpha'(\bar{\xi})}{\alpha(\bar{\xi})}kM, \qquad \frac{\mathrm{d}\bar{\xi}}{\mathrm{d}t} = k. \tag{11}$$

The equilibrium state of Eqs. (10) and (11) satisfies the conditions

$$k = 0, \qquad \mu L_0 = \alpha'(\bar{\xi}_0)M_0, \tag{12}$$

where M_0 and L_0 are the integral moments of the initial wave packet. It follows from Eq. (12) that the pseudo stimulated scattering is compensated by the cubic nonlinearity, which increases with the coordinate ($\alpha'(\bar{\xi}_0) > 0$). In the equilibrium regime, the wave packet propagates with a zero wave number and the integral parameters corresponding to their initial values N, L_0, and M_0.

For the further analysis of the systems of Eqs. (10) and (11), the spatial inhomogeneity of the cubic nonlinearity is simulated by the function

$$\alpha(\xi) = \alpha_0\,[1 + \tanh(\xi/D)], \tag{13}$$

where D is the characteristic scale of the cubic-nonlinearity inhomogeneity, which coincides with the above-mentioned inhomogeneity scale of the LF-wave velocity $V(\xi)$.[1] The system of Eqs. (10) and (11) has the first integral

$$2\,(y^2 - y_0^2) + \lambda\,[8\rho + 8\ln(\cosh\rho) - 6\tanh\rho - \tanh^2\rho] - 2\tanh\rho - \tanh^2\rho = 0 \tag{14}$$

[1] The possibility of fabricating optical fibers with cubic nonlinearity that is inhomogeneous along the longitudinal coordinate is experimentally demonstrated in [22]

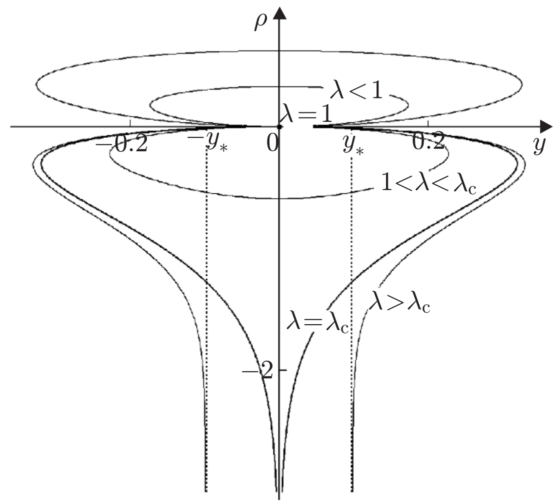

Fig. 1. The curves corresponding to the first integral (14) for $y_0 = 0$ and various λ.

with the function $y = k\sqrt{N/(\alpha_0 M_0)}$, $\rho = \bar{\xi}/D$, and the coefficients $y_0 = y(0)$ and $\lambda = \mu L_0 D/(\alpha_0 M_0)$. Figure 1 shows the curves corresponding to the first integral (14) for the initial condition $y_0 = 0$ and various values of λ.

For $\lambda = 1$, the curve corresponding to the first integral given by Eq. (14) degenerates to the initial point with $y = 0$ and $\rho = 0$, which corresponds to equilibrium of the pseudo stimulated scattering and the increasing cubic nonlinearity. In this case, the integral moments and the packet wave number do not vary in time. For $0 < \lambda < \lambda_c = 1/(8\ln 2 - 5) \approx 1.83$, the curves corresponding to the first integral in Eq. (14) are localized, which corresponds to the dynamic equilibrium of the pseudo stimulated scattering and the increasing nonlinearity, i.e., the parameters of the wave packet periodically vary in time. For $\lambda > \lambda_c$, the soliton wave number tends to $-y_* = -\sqrt{(\lambda - \lambda_c)/(2\lambda_c)}$ with time.

3.2. Soliton solution

Let us consider the solution to Eq. (3) in the form of the stationary wave $U(\xi, t) = \psi(\xi)\exp(i\Omega t)$. For the nonlinearity profile in the form of Eq. (13), Eq. (3) yields

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} + 2\alpha_0\left[1 + \tanh(\xi/D)\right]\psi^3 - 2\Omega\psi + 2\mu\psi^2\frac{\mathrm{d}\psi}{\mathrm{d}\xi} = 0. \tag{15}$$

Let us assume that the spatial scale D of the inhomogeneous nonlinearity is much longer than the scale of the inhomogeneity of the envelope function of the wave packet, i.e., $D \gg D_\psi \equiv D_{|U|}$. With allowance for the estimate $\varepsilon \sim \xi/D \sim D_\psi/D \sim \mu \ll \alpha_0$ and the approximation $\tanh(\xi/D) \approx \xi/D$, the solution to Eq. (15) is sought in the form $\psi = \psi_0 + \psi_1$, where the term $\psi_1 \sim \varepsilon\psi_0 \ll \psi_0$. In the zero- and the first-order approximations with respect to the parameter ε, we obtain

$$\frac{\mathrm{d}^2\psi_0}{\mathrm{d}\xi^2} + 2\alpha_0\psi_0^3 - 2\Omega\psi_0 = 0, \tag{16}$$

$$\frac{\mathrm{d}^2\psi_1}{\mathrm{d}\xi^2} + \left(6\alpha_0\psi_0^2 - 2\Omega\right)\psi_1 = \frac{2}{D}\psi_0^3\xi - 2\mu\psi_0^2\frac{\mathrm{d}\psi_0}{\mathrm{d}\xi}. \tag{17}$$

Equation (16) has a solution in the form of the classical soliton $\psi_0 = A_0/\cosh(\xi/\Delta)$, where $\Omega = \alpha_0 A_0^2/2$ and $\Delta = 1/(A_0\sqrt{\alpha_0})$. With allowance for Eq. (16), after the change of the coordinate $\eta = \xi/\Delta$ and the function $\Psi = \psi_1 D/(2A_0)$, Eq. (17) takes the form

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\eta^2} + \left(\frac{6}{\cosh^2\eta} - 1\right)\Psi = -\frac{\eta}{\cosh^3\eta} + \frac{5}{4}\frac{\mu}{\mu_*}\frac{\sinh\eta}{\cosh^4\eta}, \tag{18}$$

where $\mu_* = 5\alpha_0/(4DA_0^2)$ is the equilibrium value of the parameter of the pseudo stimulated scattering. For the "initial" condition $\Psi(0) = 0$, Eq. (18) has the exact solution

$$\Psi(\eta) = \frac{1}{4\cosh\eta}\left[4\Psi'(0)\tanh\eta + \tanh\eta - \eta + \frac{\mu}{\mu_*}(\tanh\eta)\ln(\cosh\eta)\right] + \frac{1}{12}\left(\frac{\mu}{\mu_*} - 1\right)(\tanh^2\eta)\sinh\eta, \tag{19}$$

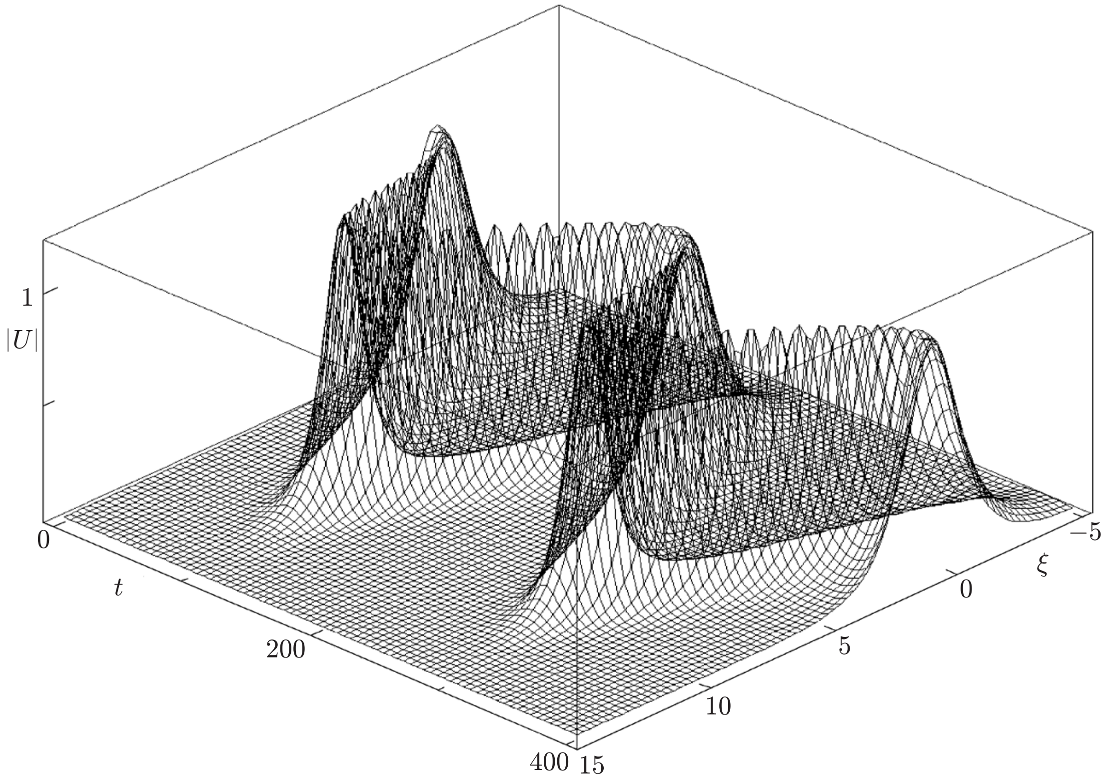

Fig. 2. Numerical simulation of the spatiotemporal distribution of the absolute value of the wave-packet envelope function for $\mu = 3/64 = 3\mu_*/4$.

which is similar to the solutions obtained in [19, 23]. For $\mu = \mu_*$, the solution to Eq. (19) satisfies the zero boundary conditions at infinity $\Psi(\eta \to \pm\infty) \to 0$. This solution exists as a result of balance of the pseudo stimulated scattering and the increasing nonlinearity.

4. NUMERICAL RESULTS

Let us consider the numerical solution to the initial problem of the dynamics of the wave packet $U(\xi, t = 0) = 1/\cosh \xi$ within the framework of Eq. (3) for the approximation of the spatial inhomogeneity of the cubic nonlinearity in the form $\alpha(\xi) = 1 + \tanh(\xi/10)$ and various parameters μ. The state of the balance of the pseudo stimulated scattering and the spatially increasing cubic nonlinearity is achieved in the case where $\mu_* = 1/16$. For $\mu = 1/16$, the initial pulse numerically tends to the spatially localized distribution with the zero wave number, which is close to the above-obtained analytical solution given by Eq. (19) for $\alpha_0 = A_0 = 1$ and $\mu = \mu_*$. If the parameter μ of the pseudo stimulated scattering deviates from the equilibrium value μ_*, the numerical simulation shows the nonstationary solitons with the amplitude $|U|$ and the wave number k, which are variable in time t, e.g., for $\mu = 3/64 = 3\mu_*/4$ (see Figs. 2 and 3).

In Fig. 3, the results of numerical calculation of the wave number at the point of the maximum absolute value of the wave-packet enveloping function are compared with the analytical results obtained from the systems of Eqs. (10) and (11) for various parameters μ as functions of time t. Close coincidence of the analytical and numerical results is achieved for both $\mu = \mu_*$ (the wave number is zero) and $\mu \neq \mu_*$ for the nonstationary wave packets.

For $\mu < \mu_c = \mu_*/(8 \ln 2 - 5) \approx 1.8\mu_*$, one can observe a dynamic equilibrium between the pseudo stimulated scattering and the increasing cubic nonlinearity, for which the soliton parameters periodically vary in time. For $\mu \geq \mu_c$, the wave number k tends to $-\sqrt{(\mu - \mu_c)/(3\mu_c)}$ for large times.

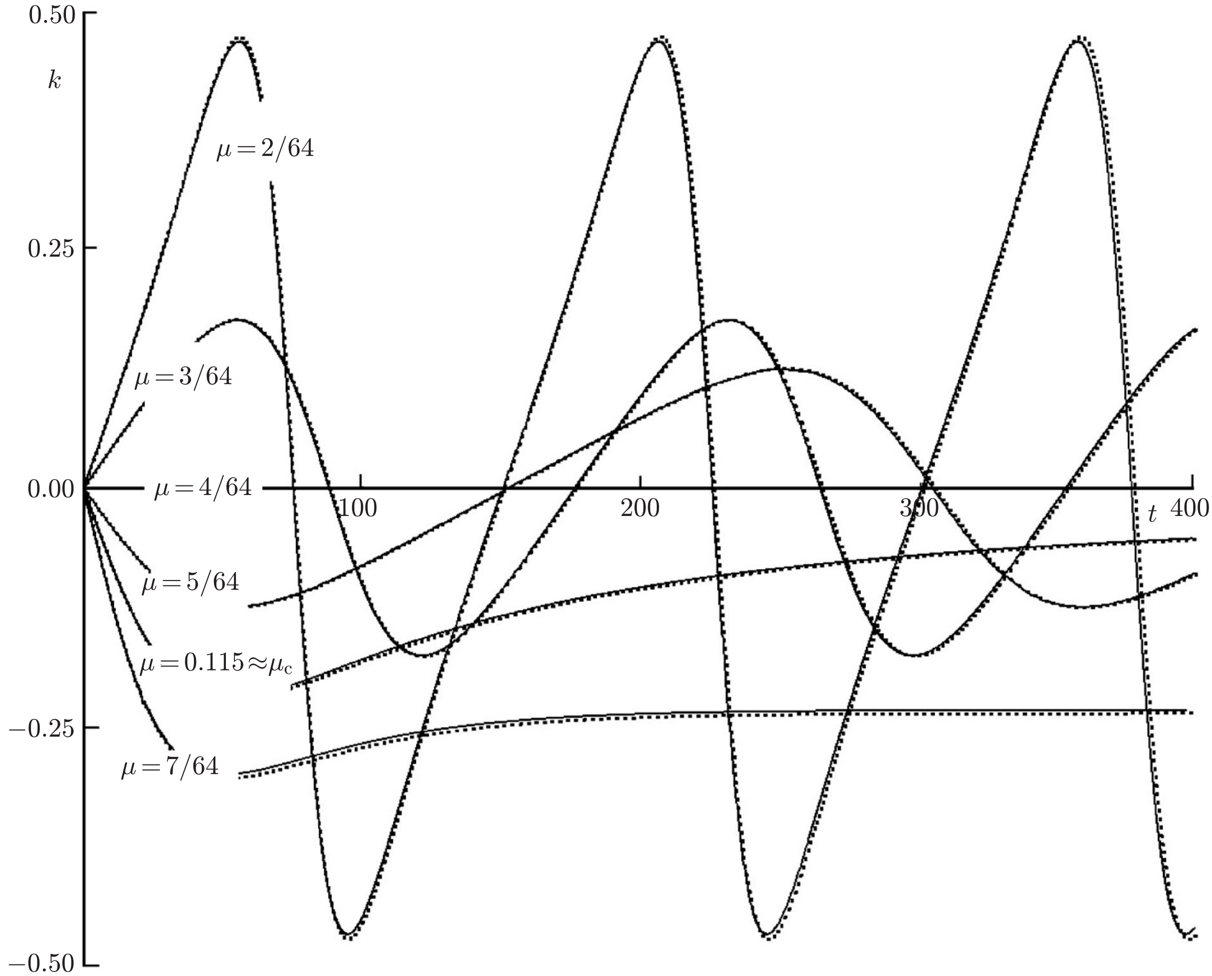

Fig. 3. Numerical (solid curves) and analytical (dotted curves) results for the local wave number k at the point of the maximum absolute value of the wave-packet envelope function for various parameters μ.

5. CONCLUSIONS

In this work, we have considered the soliton dynamics within the framework of an extended nonlinear Schrödinger equation with allowance for the pseudo stimulated scattering and the inhomogeneous cubic nonlinearity. Consideration have been performed both numerically and analytically. The analytical soliton solution of the extended nonlinear Schrödinger equation has been obtained explicitly. It results from the balance of the pseudo stimulated scattering and the spatially increasing nonlinearity.

The proposed model disregards the nonlinear dispersion and the third-order linear dispersion. The possibility of compensating for the pseudo stimulated scattering due to the spatial inhomogeneity of the dispersion characteristics of the medium with allowance for the above-mentioned higher-order effects will be considered in the forthcoming works.

This work was supported within the 2014–2015 program of the Scientific Foundation of the National Research University “Higher School of Economics” (project No. 14–01–0023).

REFERENCES

1. E. Infeld and G. Rowlands, *Nonlinear Waves, Solitons, and Chaos*, Cambridge Univ. Press, Cambridge (2000).

2. G. P. Agrawal, *Nonlinear Fiber Optics*, Academic Press, San Diego (2001).
3. Y. Yang, *Solitons in Field Theory and Nonlinear Analysis*, Springer, New York (2001).
4. Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals*, Academic Press, San Diego (2003).
5. L. A. Dickey, *Soliton Equation and Hamiltonian Systems*, World Scientific, New York (2005).
6. B. A. Malomed, *Soliton Management in Periodic Systems*, Springer, New York (2006).
7. T. Dauxois and M. Peyrard, *Physics of Solitons*, Cambridge Univ. Press, Cambridge (2006).
8. J. P. Gordon, *Opt. Lett.*, **11**, 662 (1986).
9. F. M. Mitschke and L. F. Mollenauer, *Opt. Lett.*, **11**, 659 (1986).
10. Y. J. Kodama, *Stat. Phys.*, **39**, 597 (1985).
11. B. A. Malomed and R. S. Tasgal, *J. Opt. Soc. Am. B*, **15**, 162 (1998).
12. F. Biancalama, D. V. Skrybin, and A. V. Yulin, *Phys. Rev. E*, **70**, 011615 (2004).
13. R.-J. Essiambre and G. P. Agrawal, *J. Opt. Soc. Am. B*, **14**, 314 (1997).
14. R.-J. Essiambre and G. P. Agrawal, *J. Opt. Soc. Am. B*, **14**, 323 (1997).
15. A. Andrianov, S. Muraviev, A. Kim, and A. Sysoliatin, *Laser Phys.*, **17**, 1296 (2007).
16. S. Chernikov, E. Dianov, D. Richardson, and D. Payne, *Opt. Lett.*, **18**, 476 (1993).
17. E. M. Gromov and B. A. Malomed, *J. Plasma Phys.*, **79**, 1057 (2013).
18. E. M. Gromov and B. A. Malomed, *Opt. Commun.*, **320**, 88 (2014).
19. N. V. Aseeva, E. M. Gromov, and V. V. Tyutin, *Radiophys. Quantum Electron.*, **56**, No. 3, 157 (2013).
20. V. E. Zakharov and V. E. Kuznetsov, *Physics — Uspekhi*, **40**, No. 11, 1087 (1997).
21. V. E. Zakharov, *Radiophys. Quantum Electron.*, **17**, No. 4, 326 (1974).
22. V. A. Bogatyrev, M. M. Bubnov, et al., *J. Lightwave Technol.*, **9**, 561 (1991).
23. R. Blit and B. A. Malomed, *Phys. Rev. A*, **86**, 043841 (2012).