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SURVEY ARTICLES

Closing Lemmas

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Abstract—Numerous papers deal with the closing lemma and variations thereof. The survey considers various types of closing lemmas. The ideas of proofs of the main results are presented, and schemes for constructing important examples are given. We illustrate the application of closing lemmas to the description of residual sets in spaces of dynamical systems.

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INTRODUCTION

The C^r closing lemma is one well-known problem in the theory of dynamical systems. Usually, the lemma is meant to refer to a bifurcation problem in which there is a nonclosed orbit with some return property (for example, nontrivial recurrence or nonwandering). The problem is to perturb the original dynamical system so as to obtain a C^r-close system that has a periodic orbit passing through a given point. In fact, the problems obtained vary dramatically in the solution methods as well as the difficulty level depending on the type of recurrence (chain recurrence, prolongational recurrence, etc.), the topology on the space of considered dynamical systems, and the constraints imposed on the admissible perturbations. There are other subtleties as well, and so various closing lemmas should be considered at present.

Along with obviously being of interest in themselves, almost all versions of closing lemmas have inevitably arisen when studying structural stability and also when attempting to describe residual sets in spaces of dynamical systems. Recently the interest in these lemmas has been renewed in connection with the Palis program [107–109], where new candidates for generic dynamical systems are presented. The interest in closing lemmas has never faded since the beginning of the so-called hyperbolic revolution (the term is due to Anosov [7]) at the end of the 1950s. In the last two decades, some progress has been made in the solution of numerous problems posed at the dawn of that revolution. (It suffices to note the solution of the Palis–Smale problem on necessary and sufficient conditions for structural stability.) In this survey, within our competence, we present the results obtained so far concerning closing lemmas and their application to the description of generic dynamical systems.

Essentially, the problem of closing a nonperiodic trajectory arose from the following excerpt from Poincaré's famous paper "New Methods in Celestial Mechanics" [119, vol. 1, p. 82] :

...here is a fact that I could not prove rigorously but which nevertheless seems very plausible.

Given two equations of the form indicated in Section 13[1] *and a particular solution of these equations, there always exists a periodic solution (albeit possibly with a very large period) such that the difference between the two solutions is as small as desired on a time interval as large as desired.*

Needless to say, Poincaré (who knew about linear vector fields specifying minimal flows on the torus) meant not an arbitrary Hamiltonian system but a generic one. Recall that, on a symplectic manifold M, Hamiltonian vector C^r fields equipped with the Whitney C^r topology form a Baire space $\mathcal{H}^r\ (M)$, i.e., a space in which an arbitrary G_δ-subset is everywhere dense. (A G_δ-subset is a countable intersection of open everywhere dense subsets.) Any G_δ-subset of a Baire space is said

[1] The equations in question are an autonomous system of analytic Hamiltonian equations. [*Author's remark.*]

to be *residual*, and any element of a residual subset is said to be *generic*. It is clear from the context that Poincaré considered trajectories on an arbitrary compact regular energy surface (i.e., a level surface of the Hamiltonian function). Therefore, omitting the inaccuracies and the smoothness restriction (Poincaré considered analytic Hamiltonian systems, $r = \omega$), the above excerpt from Poincaré can be restated as follows.

For a generic Hamiltonian system in $\mathcal{H}^r(M)$, periodic trajectories are everywhere dense on any compact regular energy surface. Moreover, let the relation $H_0 = c$ define a regular compact energy surface M_0 of a generic Hamiltonian H_0. Then, for any point $x_0 \in M_0$ and for arbitrary numbers $\varepsilon > 0$ and $T > 0$, there exists a point $y_0 \in M_0$ lying on a periodic trajectory such that

$$d(\phi_t(x_0), \phi_t(y_0)) < \varepsilon \quad \text{for all} \quad 0 \le t \le T.$$

Here ϕ_t is the time t shift along the trajectories of the system, and d is the metric on M.

The following claim, which can be naturally called the *Poincaré problem on the denseness of periodic trajectories of a generic Hamiltonian system* (this problem will be discussed in Section 3), is an intermediate step towards the proof of the above-represented assertion (which could be called the *Poincaré conjecture on the denseness of periodic trajectories of a generic Hamiltonian system*). For brevity, to stick to the style and spirit of the exposition, we refer to this claim as the C^r *Poincaré closing lemma*. Thus, suppose that a point $x_0 \in M$ lies on a regular compact energy surface M_0 of a Hamiltonian H_0 that specifies a Hamiltonian vector field $\vec{v} \in \mathcal{H}^r(M)$, and assume that the trajectory of $\vec{v}$ passing through x_0 is nonperiodic. Then

in the space $\mathcal{H}^r(M)$, there exists a field $\vec{w} \in \mathcal{H}^r(M)$ that is arbitrarily close to $\vec{v}$ and whose trajectory passing through x_0 is periodic.

Hermann [73] showed that this Poincaré lemma fails for sufficiently large r (see also [74]). However, this lemma holds for $r = 1$ [128].

More than 50 years after Poincaré's paper [119], René Thom modified the above-represented excerpt from Poincaré citation in one of his preprints [149] and stated it as a separate assertion in bifurcation theory. Thom thought that this assertion was easy to prove and stated it in the form of a lemma. (He even presented a half-page proof [149, pp. 5–6] but later conceded that it was unsatisfactory.) The original assertion dealt with an arbitrarily small perturbation making a nontrivially recurrent[2] or nonwandering[3] nonperiodic point periodic. Then the cases of nontrivial recurrence and nonwandering were separated, and the closing lemma, which is referred to as the classical closing lemma in what follows (in our opinion, it could be called the *Thom closing lemma*), and the improved closing lemma emerged.

We present the statement of these lemmas only for the case of diffeomorphisms. (For the case of vector fields, the statement is similar.) Let $\mathit{Diff}^{\,r}(M)$ be the space of C^r diffeomorphisms of a manifold M equipped with the C^r topology, and let $f \in \mathit{Diff}^{\,r}(M)$ have a nontrivially recurrent point $x_0 \in M$. The following assertion is called the *classical C^r closing lemma.*

For every neighborhood $U(f)$ of the diffeomorphism f in the space $\mathit{Diff}^{\,r}(M)$, there exists a diffeomorphism $g \in U(f)$ with periodic point x_0.

For a nonwandering nonperiodic point x_0, the corresponding assertion is called the *improved C^r closing lemma* [122]. Clearly, the improved closing lemma implies the classical one, because a nontrivially recurrent point is nonwandering.

In the C^0 topology, both the classical and the improved closing lemma can readily be proved, because either the orbit of x_0 or orbits close to x_0 pass arbitrarily close to x_0. Let us indicate the fundamental difficulties encountered in the proof of the classical C^r closing lemma for $r \ge 1$. Since x_0 is a nontrivially recurrent point, it follows that there exists an $\varepsilon > 0$ small enough that the points $f(x_0), \ldots, f^{k-1}(x_0)$ do not lie in the ε-neighborhood $U_\varepsilon(x_0)$ of x_0 and $f^k(x_0)$ lies in $U_\varepsilon(x_0)$ for some $k \ge 3$ (see Fig. 1). In other words, $f^k(x_0)$ is the first point of the positive semiorbit of

[2] A nontrivially recurrent point is a point that is nonperiodic and belongs to the ω- or α-limit set of itself.

[3] A nonwandering point is a point such that each of its neighborhoods meets arbitrarily large iterations of itself.

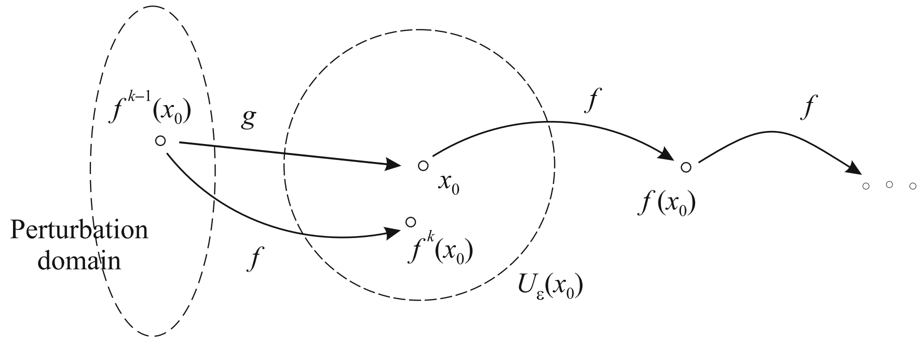

Fig. 1. Perturbation of a diffeomorphism f near $f^{k-1}(x_0)$.

the point x_0 that returns into $U_\varepsilon(x_0)$. Naturally, one can try to perturb f in a neighborhood of $f^{k-1}(x_0)$ so as to obtain a diffeomorphism $g \in U(f)$ with $g(f^{k-1})(x_0) = x_0$, which, at first glance, has the periodic point x_0. However, in general, it is not guaranteed that the intermediate points $f(x_0), \ldots, f^{k-2}(x_0)$ lie outside the perturbation domain. Therefore, one cannot ensure that all the relations

$$f(x_0) = g(x_0), \quad \ldots, \quad f^{k-1}(x_0) = g^{k-1}(x_0)$$

are true. Therefore, one cannot yet claim that the diffeomorphism g has the periodic point x_0. If $r = 0$, then there exists a desired perturbation domain. [One can take the domain that is the preimage with respect to f of a sufficiently thin tube lying in $U_\varepsilon(x_0)$ and connecting the points x_0 and $f^k(x_0)$; this preimage does not contain the points $f^i(x_0)$, $1 \le i \le k-2$.] If, for $r \ge 1$, one tries to diminish the perturbation domain for a fixed point $f^k(x_0)$ so as to exclude intermediate points, then one can get a "large derivative" and the exit of the perturbing diffeomorphism g from the given neighborhood $U(f)$. On the other hand, if one tries to diminish the perturbation domain by choosing a point $f^k(x_0)$ closer to x_0, then one should take larger k, and hence the number of intermediate points is larger as well. Similar difficulties arise if we try to perturb f near the points $f(x_0), \ldots, f^{k-2}(x_0)$ so as to hit $f^{-1}(x_0)$. This "tug-of-war" makes the classical (and so much the more the improved) C^r closing lemma a challenge for $r \ge 1$. Peixoto [112, 113] was the first to indicate that the proof of the C^r lemma in the general case for $r \ge 1$ is nontrivial.

Note one earlier paper that had been published before the closing lemmas were stated. In 1939, when describing structurally stable diffeomorphisms of the circle S^1, Maier [17] essentially proved the classical C^r closing lemma for $f \in \mathit{Diff}^{\,r}(S^1)$ with any $r \ge 1$ (see Theorem 5 in [17]). (Note that, following Poincaré, Maier referred to nontrivially recurrent points as Poisson-stable points.) Later, Pliss [19] and Peixoto [112] reproved Maier's result and used it in the proof of the classical C^r closing lemma for vector fields without singular points on a torus.

The main contribution to the topic is due to Charles Pugh. In 1964, he announced the classical C^1 closing lemma for diffeomorphisms, flows, and vector fields on two- and three-dimensional manifolds [120]. (These results formed his doctoral thesis.) In 1967, Pugh [121, 122] published the proof of the classical and improved C^1 closing lemmas for manifolds of an arbitrary dimension. As to the classical and improved C^r closing lemmas for $r \ge 2$, there are few results so far. Smale [147] stated the improved C^r closing lemma for $r \ge 2$ as one of the problems of the XXI century (Problem 10).

The improved C^1 closing lemma is essentially used when proving that the set of Kupka–Smale diffeomorphisms whose nonwandering set is the closure of the set of hyperbolic periodic points [122] is residual in the space $\mathit{Diff}^{\,1}(M)$. Kupka–Smale diffeomorphisms may have very complicated dynamics that does not admit a finite description. At the end of the 1950s, the work of Andronov, Pontryagin [2], and Peixoto [112] raised hope that structurally stable systems are residual in the space $\mathit{Diff}^{\,1}(M)$. Therefore, after this conjecture had be shown to be false [145], attempts started to find other sets that would be residual and consist of systems with comprehensible dynamics. One direction of these attempts was related to generalizations of the notion of nonwandering.

As a consequence, it became necessary to obtain closing lemmas with weakened requirements on the recurrence of nonperiodic points (like, for example, prolongational recurrence in the sense of Auslender, Pugh, etc.).[4]

The most general type of recurrence is the chain recurrence. Recall that an *ε-chain of length n* from a point p to a point q is defined as a sequence $\{p_0 = p, p_1, \ldots, p_n = q\}$ such that $d(f(p_{j-1}), p_j) < \varepsilon$ for $1 \le j \le n$, where d is a metric on M. A point p is said to be *chain recurrent* if, for each $\varepsilon > 0$, there exists an ε-chain of arbitrarily large length from p to p. The assertion on the bifurcation of a periodic point from a chain recurrent point could naturally be referred to as a *strengthened C^r closing lemma*. Under additional constraints on the dynamical systems, such lemmas were proved in [13, 30, 37, 46, 53].

In lemmas with weakened recurrence requirements, Pugh singled out a part whose investigation could advance the proof of these lemmas. Namely, if we take two points p, $q \in M$ with $\omega(p) \cap \alpha(q) \neq \varnothing$ [i.e., there exists a point that is an accumulation point of the positive semiorbit of p and the negative semiorbit of q under a diffeomorphism $f \in Diff^{\,r}(M)$], then it is natural to try to perturb the diffeomorphism in the C^r topology so as to ensure that the points p and q lie on the same orbit. The following assertion (sometimes, for brevity, we refer to it as the *connecting lemma*) is called the *C^r orbit connecting lemma*.

In any neighborhood $U(f)$ of the diffeomorphism f in the space $Diff^{\,r}(M)$, there exists a diffeomorphism $g \in U(f)$ for which the points p and q lie on one orbit.

The C^r orbit connecting lemma is harder to prove than the improved (and so much the more the classical) C^r lemma, because one deals with two orbits instead of one.

Mane suggested a more general version of connecting lemmas for orbits and stated the so-called "make or break lemma." We call it the *C^r Mane dichotomy*. In the notation of the connecting lemma, it can be stated as follows.

In any neighborhood $U(f)$ of the diffeomorphism f in the space $Diff^{\,r}(M)$, there exists a diffeomorphism $g \in U(f)$ such that either the points p and q lie on one orbit or $\omega(p) \cap \alpha(q) = \varnothing$.

There are important modifications of the connecting lemma that lead to connecting lemmas for stable and unstable manifolds of invariant sets of diffeomorphisms. (One such lemma is known as the lemma on the bifurcation of homoclinic or heteroclinic points.) Let us present the statement of one version of the C^r connecting lemma for invariant manifolds (which, in a sense, is an assertion on the bifurcation of a homoclinic point from an almost homoclinic one) for the classical case in which $W^u_f(p)$ and $W^s_f(p)$ are the unstable and stable manifolds, respectively, of a hyperbolic periodic point p of a diffeomorphism $f \in Diff^{\,r}(M)$. Suppose that there exist almost homoclinic points associated with p,

$$\big(clos\ W^u_f(p) \cap W^s_f(q)\big) \bigcup \big(W^u_f(p) \cap clos\ W^s_f(q)\big) - \{p\} \neq \varnothing.$$

Then for each neighborhood $U(f)$ of f in the space $Diff^{\,r}(M)$, there exists a diffeomorphism $g \in U(f)$ coinciding with f in some neighborhood of p such that $W^u_g(p) \cap W^s_g(p) - \{p\} \neq \varnothing$.

Substantial progress in the analysis of the C^1 connecting lemma for invariant manifolds is due to Hayashi [70]. In particular, his result implies the C^1 lemma on the bifurcation of a homoclinic point associated with an isolated hyperbolic set. The Hayashi result gave an impetus for a series of papers dealing with the connecting lemma both for invariant manifolds and for orbits.

In the present survey, we consider various types of the closing lemma. The main ideas of the proofs of the classical and improved C^1 closing lemmas are given in Section 1. C^1 connecting lemmas for invariant manifolds and orbits as well as the strengthened variant of the C^1 Mane dichotomy for vector fields are considered in Section 2. The Poincaré problem on the denseness of periodic trajectories and close results (including the Hermann counterexample) are presented in Section 3. The Anosov closing lemma, which is considered in Section 4, is an analog of the Poincaré problem on the denseness of periodic trajectories of a generic Hamiltonian system (in the sense that

[4] Note that the idea of these notions goes back to Bendixson's notion [40] of extension of a saddle separatrix.

nothing has to be perturbed) for diffeomorphisms with uniform hyperbolic structure. Various C^r lemmas for $r \geq 2$ are presented in Section 5. In this section, we also describe the construction of the most interesting and important (from our viewpoint) counterexamples. Closing lemmas for special dynamical systems and foliations are gathered in Section 6. Finally, in Section 7, we demonstrate applications of closing lemmas to the description of residual sets in spaces of dynamical systems.

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1. CLASSICAL AND IMPROVED C^1 LEMMAS

Pugh proved [121, 122] the classical and improved C^1 lemmas for manifolds of arbitrary dimension. As Pugh recollects [127], he was told about the closing problem by his teacher P. Hartman in 1963 after a seminar at which Hartman had listened to M. Peixoto's talk. However, note that his proofs contain several difficult points (see [123] and Remark 11.1.4 in [134]). Although these proofs were recognized by the mathematical community, the search for shorter and clearer proofs was continued [81, 128, 84], and now there is a clear exposition. Let us present the main ideas of the proof of the classical C^1 lemma. Let a C^1 diffeomorphism $f : M \to M$ of a smooth manifold M have a nontrivially recurrent point $x_0 \in \omega(x_0)$. Take an arbitrary neighborhood $U(f)$ of f in the space $\mathit{Diff}^{\,1}(M)$. First, Pugh (Theorem 5.1 in [121]) noted that it suffices to obtain a periodic orbit that does not necessarily pass through x_0 but passes arbitrarily closely to x_0. This remark pertains to any $r \geq 1$ and is called the *local C^r closing lemma*. Let us state this assertion rigorously.

For any neighborhood U_0 of f in $\mathit{Diff}^{\,r}(M)$ and any neighborhood $V(x_0)$ of x_0, there exists a $g \in U_0$ with a periodic orbit passing through $V(x_0)$.

If $g \in U_0$ has a periodic point $y_0 \in V(x_0)$, then one can construct a C^r diffeomorphism h arbitrarily close to the identity and taking y_0 to x_0. Then the diffeomorphism $h \circ g \circ h^{-1}$ is C^r-close to f and has periodic point x_0. Note that this assertion was proved in Theorem 5.1 in [121] for C^1 smoothness, but the proof can be reproduced word for word for C^r, $r \geq 1$. A vast majority of the proofs of closing lemmas are based on the local lemma.

On the basis of the local closing lemma, Pugh singled out the main direction in the proof of the classical lemma.

For given neighborhoods $U_0 \subset \mathit{Diff}^{\,1}(M)$ and $V(x_0) \subset M$, it is unnecessary to close the orbit segment x_0, $f(x_0)$, $\ldots$, $f^k(x_0)$ with extreme points x_0, $f^k(x_0) \in V(x_0)$, because "between" these extreme points there may be intermediate points $f(x_0)$, $\ldots$, $f^{k-1}(x_0)$. One should close an orbit segment $f^m(x_0)$, $\ldots$, $f^n(x_0)$ such that the extreme points $f^m(x_0)$ and $f^n(x_0)$ lie in $V(x_0)$ but "between" them there are no intermediate points $f^j(x_0)$, where $0 \leq m \leq j \leq n \leq k$; see Fig. 2. In a sense, the word "between" implies that there exists a desired perturbation domain that covers the points $x_m = f^m(x_0)$ and $x_n = f^n(x_0)$. A clear proof of the existence of a desired orbit segment can be found in [84, 128]. At the beginning of Section 2, we present the idea of how to single out such an orbit segment.

After finding such an orbit segment, one constructs an ε-chain from the terminal point x_n to the initial point x_m, $n > m$ (see Fig. 2) or to some point x_k, where $m \leq k < n$. The jumps in the ε-chain are concentrated in disjoint balls. One can find perturbations of the original diffeomorphism implementing these jumps by the following standard technique.

Let $B_r \subset \mathbb{R}^m$ be a closed ball of radius r. For an arbitrary number $0 < \delta < 1$, by δB_r we denote the ball with the same center and with radius δr. It is referred to as the *δ-core* of B_r. For arbitrary given points P and Q in the interior of B_r, there exists a C^∞ diffeomorphism $h : \mathbb{R}^m \to \mathbb{R}^m$ that is the identity mapping outside B_r and takes P to Q. If P and Q lie in δB_r, then such a diffeomorphism h is called a *motion of the point P into the point Q in the δ-core* of

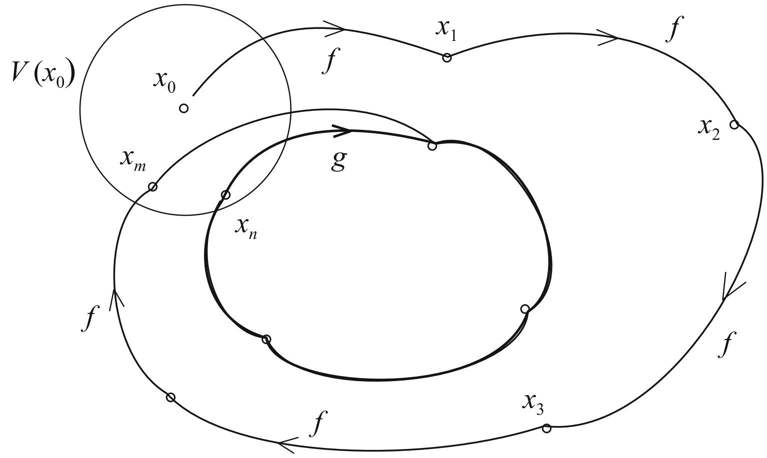


Fig. 2. Closing the orbit $x_i = f^i(x_0)$.

the ball B_r. By using the standard bump function, one can prove the following assertion ([156, Lemma 2.1]).

Lemma 1.1. *For any $\varepsilon > 0$, there exists $0 < \delta < 1$ such that, for any closed ball $B_r \subset \mathbb{R}^m$ and for arbitrary points P and Q in δB_r, there exists a motion $h : \mathbb{R}^m \to \mathbb{R}^m$ of the point P to the point Q in the δ-kernel of B_r such that all first partial derivatives of the mapping $h - id$ are less than ε.*

Indeed, take a C^∞ bump function $\alpha : \mathbb{R}^m \to \mathbb{R}$ such that $\alpha = 1$ in $B_{\frac{1}{3}}$ and $\alpha = 0$ outside $B_{\frac{2}{3}}$; moreover, the partial derivatives of the function α do not exceed 6. One can assume that the center B_r coincides with the origin of $\mathbb{R}^m$ and identify the points with the corresponding vectors. Then

$$h(\vec{x}) = \vec{x} + \alpha\left(\frac{\vec{x} - \vec{P}}{r}\right)(\vec{Q} - \vec{P})$$

is the desired motion, since

$$\|\vec{Q} - \vec{P}\| < 2r\delta, \qquad \left\|\frac{\partial}{\partial x_i}(h - id)\right\| \le 6\frac{1}{r}2\delta r = \varepsilon.$$

Note that the radius of B_r plays no role in the statement of the lemma. Therefore, the motion in the δ-kernel of the ball B_r is close in the C^1 topology to the identical one if ε and r are sufficiently small. Then the composition $h \circ f$ is C^1-close to f. By using the mapping $\exp_z : TM_z \to M$, which locally identifies the tangent space with a neighborhood of a point on the manifold, one can prove the analog of Lemma 1.1 for the manifold M. (For a rigorous proof, see [128, Th. 6.1].)

In the proof of the C^1 lemma, **the closure of the original diffeomorphism f is constructed as the composition of f with motions in δ-kernels of pairwise disjoint sufficiently small balls**.

In what follows, we use a theorem dealing with a sequence of isomorphisms of Euclidean spaces. Let us explain how such a sequence arises in the closing lemma. Take a neighborhood U_0 of a nontrivially recurrent point $x_0 \overset{\text{def}}{=} x$. Then there exists a strictly increasing sequence of positive integers $j_1, \ldots, j_k, \ldots$ such that $f^{j_k}(x) \in U_0$ and $f^j(x) \notin U_0$ for $j \neq j_k$. In other words, this is a sequence of iterations for which x returns into U_0. Without loss of generality, one can assume that the neighborhood U_0 is a map of the atlas, and it is identified with the Euclidean space $\mathbb{R}^{\dim M}$, whose origin is identified with the point x. Then the tangent space T_xM is canonically isomorphic to $\mathbb{R}^{\dim M}$, and in what follows T_xM is identified with $\mathbb{R}^{\dim M}$. Set $f^{j_k}(x) \overset{\text{def}}{=} y_k$. The points y_k belong to a nontrivially recurrent orbit. Therefore, each point of the sequence $\{y_k\}_{k=1}^{\infty}$ is a condensation point. The mapping f^{j_k} is a C^1 diffeomorphism that, in the linear approximation at the point x, is equal to the derivative $Df^{j_k}(x): T_xM \to T_{y_k}M$, which is an isomorphism of linear spaces. Set

$$Df^{j_k}(x) \stackrel{\text{def}}{=} F_k, \qquad T_{y_k}M \stackrel{\text{def}}{=} \mathbb{R}_k^{\dim M}.$$

Thus, we have a sequence of points $y_k \in \mathbb{R}^{\dim M}$, which has accumulation points, and a sequence of isomorphisms $F_k : \mathbb{R}^{\dim M} \to \mathbb{R}_k^{\dim M}$. Under these conditions, we have the following assertion, which was proved by Jiehua Mai [84]. (A clear exposition of the proof of Theorem 1.1 can be found in [32].)

Theorem 1.1. *For each $\delta > 0$, there exist points y_s and y_l $(s > l)$ and a point set $\{x_1, \dots, x_\mu\}$, where $x_1 = y_s$ and $x_\mu = y_l$, such that the following conditions are satisfied.*

1. $j_l + j_\mu < j_s$ *[i.e., the number μ should be chosen so as to ensure that j_μ does not exceed $j_s - j_l$, which, in turn, implies that the point $f^{j_\mu}(y_l) = f^{j_\mu + j_l}(x)$ belongs to the negative semiorbit of the point $y_s = f^{j_s}(x)$].*

2. *For any $i = 1, \dots, \mu - 1$, the point $F_i(x_{i+1})$ lies in the δ-core of the ball $B(F_i(x_i))$; moreover, the following assertions hold.*

(a) *The balls $B(F_i(x_i))$ are pairwise disjoint.*

(b) *Each ball $B(F_i(x_i))$ contains none of the points $y_l, y_{l+1}, \dots, y_s$; see Fig.* 3.

It was shown that **the number μ depends only on the sequence of isomorphisms F_i and the number δ and is independent of the sequence of points** (the only requirement being that it should have accumulation points). After defining μ (and, therefore, after fixing a finite set of isomorphisms $F_1, \dots, F_\mu$), for this sequence of points y_k, we define a finite segment y_l whose extreme points are sufficiently close, and the indices l and y_s differ by a sufficiently large number.

Now the proof of the classical lemma can be carried out as follows. By g_i we denote the motion of the point $F_i(x_i)$ to the point $F_i(x_{i+1})$ in the δ-core of the ball $B(F_i(x_i))$ and form the composition $g = f \circ g_1 \circ \dots \circ g_{\mu-1}$. The diffeomorphism g coincides with f outside all balls $B(F_i(x_i))$ and with $f \circ g_i$ inside $B(F_i(x_i))$. Note that the definition of the sequence $j_1, j_2, \dots$ implies that, for any $i = 1, \dots,$ the points

$$f(f^{j_i}x_i), \quad f^2(f^{j_i}x_i), \quad \dots, \quad f^{j_{i+1}-j_i-1}(f^{j_i}x_i)$$

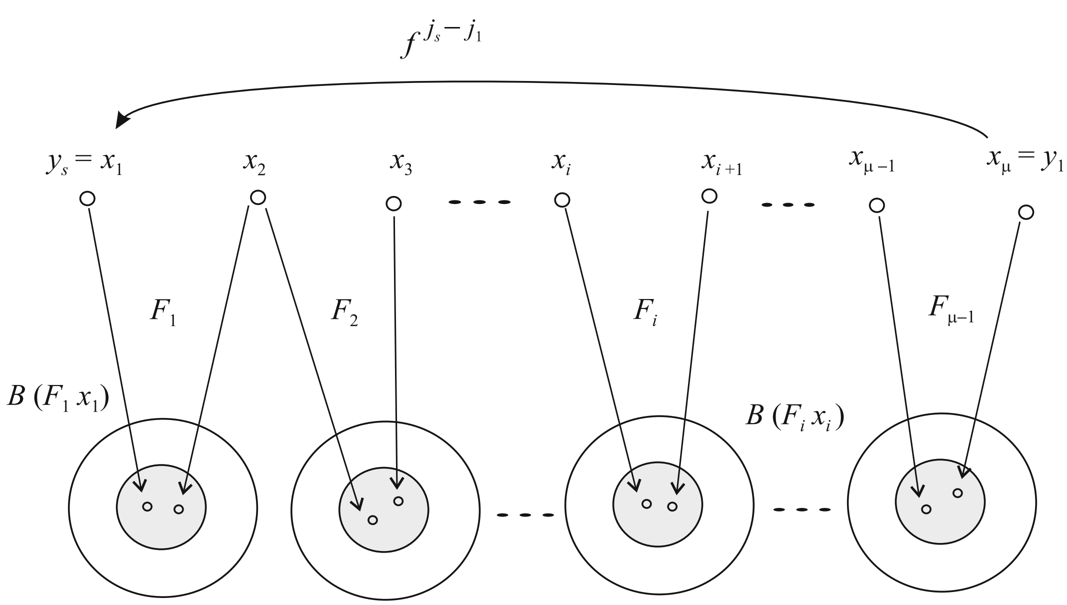


Fig. 3.

do not lie in the neighborhood U_0. Therefore, under the action of g, the point $f^{j_s}x = y_s = x_1$ takes the following path:

$$y_s = x_1 \xrightarrow{f^{j_1}} f^{j_1}(x_1) \xrightarrow{g_1} f^{j_1}(x_2) \xrightarrow{f^{j_2-j_1}} f^{j_2}(x_2) \xrightarrow{g_2} f^{j_2}(x_3) \xrightarrow{f^{j_3-j_2}} \ldots$$
$$\ldots \xrightarrow{f^{j_\mu - j_{\mu-1}}} f^{j_\mu}(x_\mu) = f^{j_\mu + j_l}(x) \xrightarrow{f^{j_s-(j_\mu+j_l)}} f^{j_s}(x) = x_1 = y_s.$$

Therefore, the point $y_s = f^{j_s}(x)$ is a periodic point of the diffeomorphism g, which can be made arbitrarily close to f in the C^1 topology, since this depends completely on the diameters of the balls $B(F_i(x_i))$ and the motions in their δ-cores (which can be made arbitrarily small).

The improved C^1 closing lemma can be proved in a similar way (for a brief proof, see [85]). On the basis of the main idea of the paper [84], the improved C^1 closing lemma was proved in [151] for a nonsingular (Df is injective on the tangent spaces of all points) endomorphism of a compact closed manifold in [151].

Note Pliss's paper [20], in which he considered the nonautonomous two-dimensional system $\dot{\vec{x}} = P(t)\vec{x} + X(t,\vec{x})$ with diagonal matrix $P(t) = \operatorname{diag}(p(t), q(t))$ formed by integrally separated functions $p(t)$ and $q(t)$; i.e., $(\operatorname{sgn} t)\int_0^t [p(\tau) - q(\tau)]\,d\tau > c|t| - d$ for some $c > 0$ and $d > 0$. The nonlinearity $X(t,\vec{x})$ is small and has a small Lipschitz constant. If $\vec{x}(t, t_0, \vec{x}_0)$ is a solution of the original system with the initial conditions $(t_0, \vec{x}_0)$, then, for each C^1-neighborhood $U(X)$ of the nonlinearity X, there exists a nonlinearity $Y \in U(X)$ such that the corresponding solution satisfies the condition $\vec{y}(\theta, -\theta, \vec{x}(-\theta, 0, \vec{x}_0)) = 0$ for sufficiently small $\vec{x}_0$. The proof is of interest from the viewpoint of the closing technique, because the perturbation is in a sense written out in closed form. Note that the conditions imposed on the system can readily be verified.

To finish our discussion of the classical lemma, we note three papers by Lin Zhensheng, C^r Closing Lemma I, II, III, *Ann. of Diff. Equat.*, 1990, vol. 6, pp. 59–67; 1991, vol. 7, pp. 68–77; 1992, vol. 8, pp. 307–322, where the proof of the classical C^r lemma is presented for $r \geq 2$. (The case of $\dim M \geq 3$ was considered in the first paper, the case of $\dim M = 2$ in the second one, and in the third paper, the case of $r = \infty$ was considered separately. In the last two papers, the author corrected mistakes found in the previous ones.) From our viewpoint, which is shared by Carlos Gutierrez [67] and Jehua Mai [86], the proof presented there is incorrect, because the C^r perturbations with $r \geq 2$ are not arbitrarily small in these works, even though, the C^1 perturbations are.

2. C^1 LEMMAS ON CONNECTING ORBITS AND MANIFOLDS

As was mentioned in the introduction, connecting lemmas split into two classes, connecting lemmas for orbits and connecting lemmas for invariant manifolds of points. This classification proves to be convenient in spite of the fact that connecting lemmas for invariant manifolds can be formally considered as connecting lemmas for orbits. The extraction of connecting lemmas for invariant manifolds is related to specific properties of problems to be solved in which a small perturbation of a dynamical system should generate a heteroclinic or homoclinic orbit rather than connect particular points by a single orbit.

The first results pertaining to the C^r lemma on the bifurcation of homoclinic points were obtained by Takens [148], who proved the C^1 lemma on the bifurcation of a homoclinic point in the class of conservative (symplectic) dynamical systems, and by Robinson [131], who proved the C^r ($r \geq 1$) lemma on the bifurcation of a homoclinic point for a diffeomorphism of a sphere (in the case of a fixed hyperbolic point). Then these results were developed by Newhouse [99] and Pixton [116]. Mané [89] proved the C^1 and C^2 lemmas on the bifurcation of a homoclinic point for a periodic hyperbolic point p of a diffeomorphism of an arbitrary closed manifold under the assumption that the measure of the point p is positive with respect to some invariant probability measure.

Recall that, in the original situation, which is considered in the connecting lemma for orbits, there is an accumulation point (say, $z \in M$) of the positive semiorbit $O^+(p)$ of the point p and the negative semiorbit $O^-(q)$ of the point q of a diffeomorphism $f \in \mathit{Diff}^{\,r}(M)$, $z \in \omega(p) \cap \alpha(q)$. For a given neighborhood of f in the space $\mathit{Diff}^{\,r}(M)$, it is natural to take points $x \in O^+(p)$ and $y \in O^-(q)$ near z and perturb f so as to ensure that the resulting diffeomorphism g brings x to y. That strategy can only be successful if the perturbation domain contains no intermediate point $f(p), \ldots, f^i(p), \ldots,$

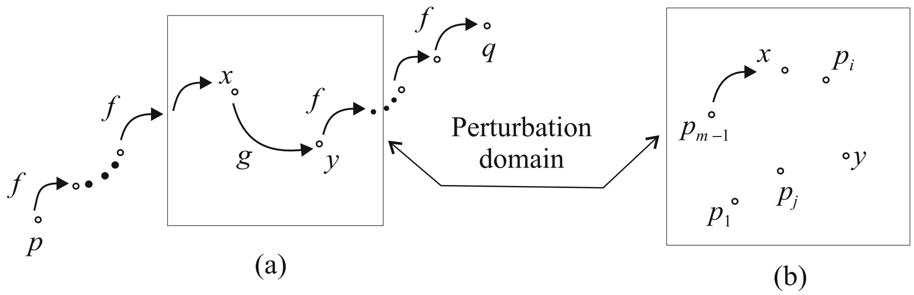


Fig. 4.

$f^{m-1}(p) = f^{-1}(x)$, $f(y), \ldots, f^k(y) = q$, because otherwise, one cannot guarantee that the orbits of the points p and q of the perturbed diffeomorphism g pass through the points x and y, respectively; see Fig. 4 (a). In 1997, the Japanese mathematician Hayashi [70] suggested a procedure for avoiding this difficulty. Let us outline the main ideas of this procedure in a particular situation. Suppose temporarily that an intermediate point $p_j = f^j(p)$, $1 \le j \le m-2$, lies near $p_{m-1} = f^{m-1}(p) = f^{-1}(x)$. If we simultaneously perturb f so as to ensure that g brings $f^j(p)$ to $f^{m-1}(p)$, then the presence of intermediate points $f^{j+1}(p), \ldots, f^{m-2}(p)$ in the perturbation domain plays no role. The pair $f^j(p)$, $f^{m-1}(p)$ is referred to as a *cutting pair*. In a similar way, one can introduce the notion of a cutting pair for intermediate points in the semiorbit $O^-(q)$. In the general case, cutting pairs have the form $f^j(p)$, $f^l(p)$, $1 \le j \le l-1 \le m-1$, or $f^{-j}(q)$, $f^{-l}(q)$, where $1 \le l \le j-1 \le k-1$, Fig. 4 (b). One of Hayashi's ideas is to choose not only a *connecting pair* of points x and y but also cutting pairs in an optimal way. If points in cutting pairs are sufficiently close to each other, then one can construct the desired perturbation in pairwise disjoint balls containing cutting pairs and one connecting pair (x, y). Then the perturbation of the semiorbits $O^+(p)$ and $O^-(q)$ at points lying in the ball with the connecting pair (x, y) does not violate the desired condition: both semiorbits still pass through the corresponding points x and y of the connecting pair. The perturbations inside the above-mentioned balls are similar to the perturbation constructed in the proof of the classical or improved closing lemma; namely, the corresponding ε-chain is constructed. Since these last lemmas have been proved in the smoothness class C^1, Hayashi [70] proved the C^1 connecting lemma. However, note that, in the classical and improved closing lemmas, the considered point is originally nonperiodic. Accordingly, Hayashi additionally assumed that z is nonperiodic. Because of this assumption, the C^1 connecting lemma for orbits has not been proved in full generality yet.

Connecting Orbits

First, consider results concerning the connecting lemma for orbits. For the first time, such a lemma was stated as Problem 23 in the famous list of 50 problems by Palis and Pugh [110]. Let us present its statement.

Let U_1 and U_2 be two open domains such that the topological closure of the positive f-orbit of U_1 meets the topological closure of the negative f-orbit of U_2. Does there exist a diffeomorphism g C^r-close to f such that the positive g-orbit of U_1 meets the negative g-orbit of U_2?

We return to the discussion of this problem later. The C^0 connecting lemma holds for orbits on arbitrary manifolds, although the proof is not as easy as that of the classical or improved C^0 closing lemma [140, 141]. Let us outline the Pugh example [126], which illustrates that the C^1 connecting lemma does not hold for flows on the plane in general. Note that there are two nonequivalent topologies, weak and strong, for noncompact manifolds. Unless otherwise specified, we assume the Whitney strong topology, which permits one to control the perturbation "at infinity."

Consider a smooth flow f^t whose phase portrait is shown in Fig. 5 on the plane or an open disk. The flow f^t has two saddles, two sources, and two sinks. The trajectories of f^t in the square $ABCD$ are vertical lines. The points p and q lie on separatrices l_p and l_q, respectively, such that $\omega(l_p) \cap \alpha(l_q) = l_0$, where l_0 is the unbounded trajectory coming from infinity, going to infinity, and such that $B, D \in l_0$.

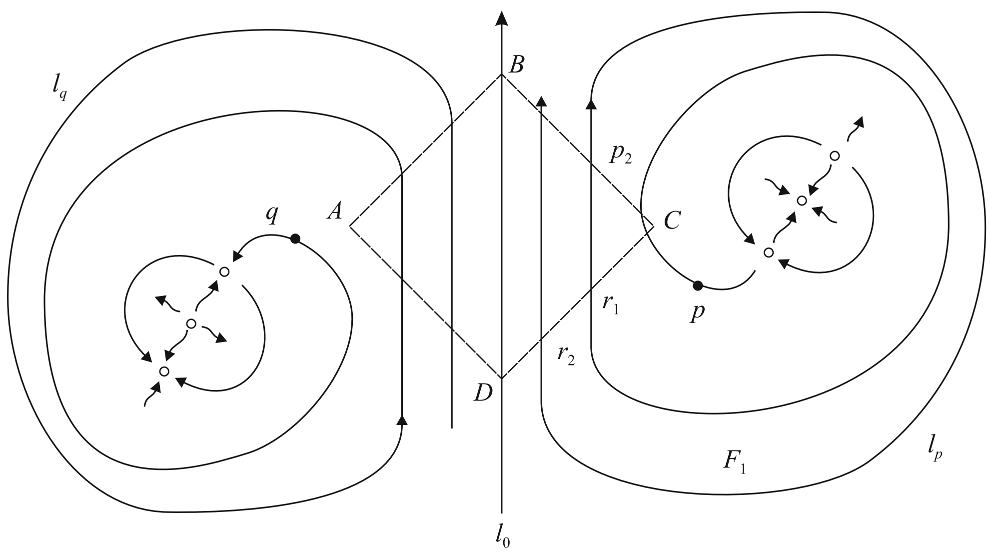


Fig. 5.

By p_1, p_2 ... (respectively, r_0, r_1, ...) we denote the time-successive points of intersection of the separatrix l_p with the segment BC (respectively, CD). By F_n we denote the flow box bounded by the segments $[p_n; p_{n+1}] \subset BC$ and $[r_n; r_{n+1}] \subset CD$ and the arcs $[p_n; r_n] \subset l_p$ and $[p_{n+1}; r_{n+1}] \subset l_p$. Figure 5 represents F_1. We place the plug shown in Fig. 11 into each F_n and denote the resulting flow by $\overline{f}^t$.

Then on each $[p_n; p_{n+1}]$ there exists a segment W_n^s such that any positive semitrajectory entering F_n through W_n^s tends either to a sink or to a saddle. Likewise, on each $[r_n; r_{n+1}]$ there exists a segment W_n^u such that any negative semitrajectory entering F_n through W_n^u tends (in the negative direction) either to a sink or to a saddle.

By construction, the segment AC (which is not shown in Fig. 5 but can readily be imagined) is a contact-free segment for f^t. The segments ABC and ADC are also contact-free in the topological sense. By π^u we denote the first return map $ADC \to AC$ along arcs lying in $ABCD$. In other words, each point $x \in ADC$ is taken by this map to the first point of intersection $\pi^u(x)$ of AC with the positive semitrajectory passing through $x \in ADC$. Likewise, by π^s we denote the first return map $ABC \to AC$ along arcs lying in $ABCD$ but in the negative direction. Set $I_n = \pi^u(W_n^u)$. By I'_n we denote the open interval between the segments I_n and I_{n+1}. In what follows, the length of an interval I is denoted by $|I|$. Take numbers a and λ such that

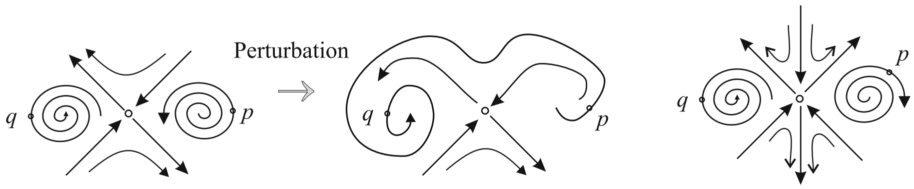
$$0 < a < 1 < \lambda a, \qquad \sum_1^\infty \left(1 + \frac{1}{\lambda}\right) a^n \le \frac{1}{\sqrt{2}} |AB|.$$

Find a flow f^t such that $|I_n| = a^n$ and $|I'_n| = a^n/\lambda$. The above inequalities guarantee that the intervals I_n and I'_n of this length can be placed on AC.

Consider a perturbed flow $\overline{f}^t$ that is ε-close to f^t. To denote similar objects related to $\overline{f}^t$, we use the bar above a letter. Set $\overline{I}_n = \overline{\pi}^u(\overline{W}_n^u)$ and $\overline{J}_n = \overline{\pi}^s(\overline{W}_n^s)$. The open interval between $\overline{J}_n$ and $\overline{J}_{n+1}$ will be denoted by $\overline{J}'_n$. Recall that the closeness of dynamical systems on noncompact manifolds is defined in terms of the Whitney strong C^1 topology. This permits one to control the perturbation of the flow f^t at the "points at infinity" corresponding to the points B and D in the obvious sense. Therefore, the following estimates hold for all sufficiently small $\varepsilon > 0$:

$$(1-\varepsilon)a^n \le |\overline{I}_n| \le (1+\varepsilon)a^n, \qquad (1-\varepsilon)a^n \le |\overline{J}_n| \le (1+\varepsilon)a^n, \tag{1}$$

$$(1-\varepsilon)\frac{a^n}{\lambda} \le |\overline{I}'_n| \le (1+\varepsilon)\frac{a^n}{\lambda}, \qquad (1-\varepsilon)\frac{a^n}{\lambda} \le |\overline{J}'_n| \le (1+\varepsilon)\frac{a^n}{\lambda}. \tag{2}$$



(a) (b)

Fig. 6.

We assume that they are true for the flow $\overline{f}^t$. The key technical assertion (it is stated as the "Gap-lemma" in [126]) is the following: there exists an $\varepsilon = \varepsilon(a, \lambda) > 0$ such that if relations (1) and (2) hold and if the interval $\overline{I}'_n$ meets $\overline{J}'_n$ for some n, then

$$\overline{I}'_{n+1} \subset \overline{J}_{n+1} \cup \overline{J}'_{n+1} \cup \overline{J}_n.$$

It follows from this inclusion that the separatrix $\overline{l}_{\overline{p}}$ of the flow $\overline{f}^t$ either successively meets all intervals $\overline{I}'_n$ and hence cannot get even in a neighborhood of the point $\overline{q}$ or gets into the plug for some n. In either case, $\overline{l}_{\overline{p}}$ does not merge with $\overline{l}_{\overline{q}}$. Thus, the C^1 connecting lemma does not hold in general for noncompact manifolds.

As follows from Arnaud's theorem [33] given below, the existence of a trajectory l_0 with empty limit set in the above-represented Pugh example proves to be a necessary condition for the possibility to connect orbits.

Theorem 2.1. *On a surface M (not necessarily compact), let there be given a vector field $X \in \mathfrak{X}^r(M)$, $r \geq 1$, that has points p, $q \in M$ such that $\omega(p) \cap \alpha(q) \neq \varnothing$. Suppose that the intersection $\omega(p) \cap \alpha(q)$ contains at least one one-dimensional nonperiodic trajectory with nonempty ω- or α-limit set. Then, for each neighborhood $U(X)$ of the field X in $\mathfrak{X}^1(M)$, there exists a field $Y \in U(X) \cap \mathfrak{X}^r(M)$ such that Y has a trajectory passing through p and q.*

If the intersection $\omega(p) \cap \alpha(q)$ may contain equilibria (along with one-dimensional trajectories with empty limit sets), then, as the following theorem in [33] shows, the C^1 connecting lemma takes place provided that all equilibria are hyperbolic. [Note that, in this case, the intersection $\omega(p) \cap \alpha(q)$ can contain only hyperbolic saddles.]

Theorem 2.2. *On a surface M (not necessarily compact), let there be given a vector field $X \in \mathfrak{X}^r(M)$, $r \geq 1$, that has points p, $q \in M$ such that $\omega(p) \cap \alpha(q) \neq \varnothing$. Let all equilibria of X be hyperbolic. Suppose that the intersection $\omega(p) \cap \alpha(q)$ contains at least one trajectory (not necessarily one-dimensional) with nonempty ω- or α-limit set. Then, for each neighborhood $U(X)$ of the field X in $\mathfrak{X}^1(M)$, there exists a field $Y \in U(X) \cap \mathfrak{X}^r(M)$ such that Y has a trajectory passing through p and q.*

A possible scenario for the connecting lemma is shown in Fig. 6 (a). Since every trajectory on a compact surface has nonempty ω- or α-limit set, we see that Theorem 2.2 has the following corollary.

Corollary 2.1. *On a compact surface M, let there be given a vector field $X \in \mathfrak{X}^r(M)$, $r \geq 1$, that has points p, $q \in M$ such that $\omega(p) \cap \alpha(q) \neq \varnothing$. Let all equilibria of the field X be hyperbolic. Then, for each neighborhood $U(X)$ of the field X in $\mathfrak{X}^1(M)$, there exists a vector field $Y \in U(X) \cap \mathfrak{X}^r(M)$ such that Y has a trajectory passing through p and q.*

If the intersection $\omega(p) \cap \alpha(q)$ contains nonhyperbolic fixed points as in Fig. 6 (b), then the problem remains open.

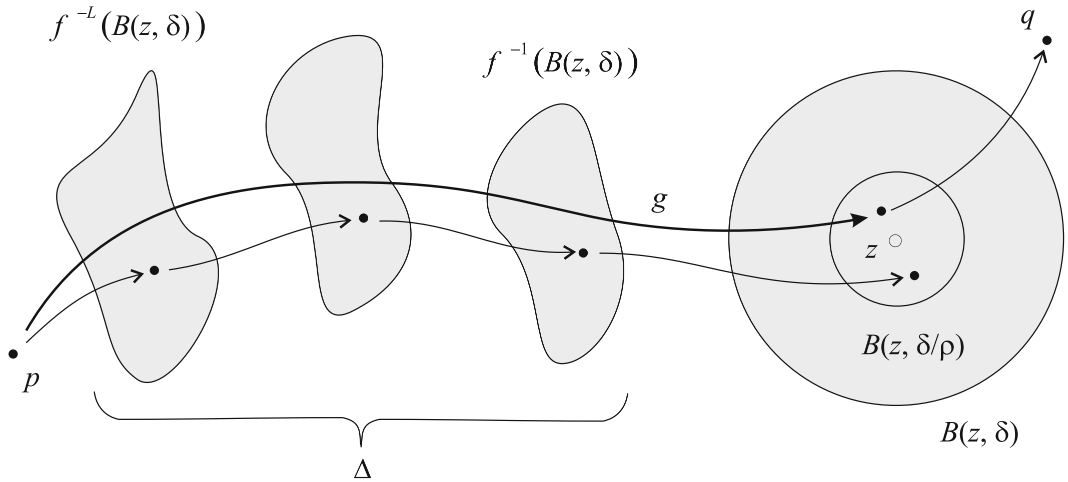


Fig. 7.

Let us return to the general case. By unifying the methods in [128, 84, 70], Lan Wen and Zhihong Xia [156] obtained so far the most general version of the C^1 connecting lemma for orbits. By $B(z, \delta)$ we denote the (metric) ball with center z and radius δ.

Theorem 2.3. *Let $f : M \to M$ be a diffeomorphism of a closed manifold M, and let $z \in M$ be a nonperiodic point of f. Then, for each neighborhood $U(f)$ of f in the space $\mathit{Diff}^{\,1}(M)$, there exist numbers $\varrho > 1$, $L \in \mathbb{N}$, and $\delta_0 > 0$ with the following property: if points p, $q \in M$ lie outside the set*

$$\Delta \stackrel{\text{def}}{=} \bigcup_{n=1}^{L} f^{-n}(B(z, \delta))$$

and the strictly positive f-orbit and the negative f-orbit of q meet the ball $B(z, \frac{\delta}{\varrho})$ for $0 < \delta \le \delta_0$, then there exists a diffeomorphism $g \in U(f)$ such that $g = f$ outside Δ and the points p and q lie on a common g-orbit (*see Fig.* 7).

This theorem is also true for a noncompact manifold M provided that the trajectory of the point $z \in M$ has at least one accumulation point. Theorem 2.3 readily implies the following version of the C^1 connecting lemma (Theorem A in [156]) for orbits.

Theorem 2.4. *Let $f : M \to M$ be a diffeomorphism of a closed manifold M, and let p, $q \in M$ be points such that $\omega(p) \cap \alpha(q) \neq \varnothing$. Suppose that the intersection $\omega(p) \cap \alpha(q)$ contains a nonperiodic point $z \in M$ of f. Then, for each neighborhood $U(f)$ of f in $\mathit{Diff}^{\,1}(M)$, there exists a diffeomorphism $g \in U(f)$ such that q belongs to the g-orbit of p. Moreover, there exists an $L \in \mathbb{N}$ such that, for each sufficiently small $\delta > 0$, the diffeomorphism $g \in U(f)$ can be chosen to satisfy $g = f$ outside $\bigcup_{n=1}^{L} f^{-n}(B(z, \delta))$.*

Thus, to study the C^1 connecting lemma completely, it remains to consider the case in which the intersection $\omega(p) \cap \alpha(q)$ consists of periodic points.

The proof of Theorem 2.3 is based on a systematic application of the so-called ε-kernel transition of fixed length from one point to another. Before providing a rigorous definition, recall that, just as in the case of the classical closing lemma, a neighborhood of a point is identified with the use of the standard mapping $\exp^{-1}$ with the tangent space T_zM, and in the linear approximation, a diffeomorphism f can be considered as Df. This explains, in a sense, why the main technical assertions are stated for isomorphisms of linear spaces.

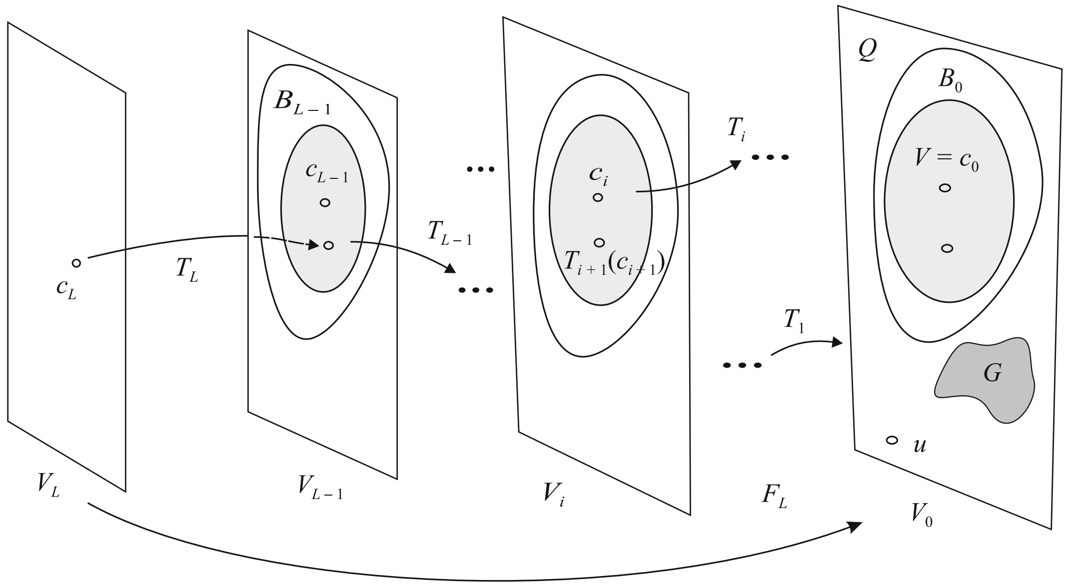


Fig. 8. An ε-kernel transition from $u = F_L(c_L)$ to v.

Let us present the definition of an ε-kernel transition. Let there be given a finite sequence of linear spaces $V_0, \ldots, V_L$ (where the number $L \in \mathbb{N}$ is fixed) each of which is isomorphic to $\mathbb{R}^{\dim M}$ and a sequence of linear transformations $T_i : V_i \to V_{i-1}$, $1 \le i \le L$. Let $F_j = T_1 \circ \cdots \circ T_j$ and $F_0 = id$. Next, let there be given points u, $v \in V_0$, sets Q, $G \subset V_0$, and a ball $B_0 \subset V_0$ such that $v \in B_0 \subset Q$ and $B_0 \cap G = \varnothing$. Then an *ε-kernel transition from u to v contained in Q and avoiding G* is a sequence of points c_i $(0 \le i \le L)$ and balls $B_i \subset V_i$ $(0 \le i \le L-1)$ such that $c_0 = v$, $c_L = F_L^{-1}(u)$, and

$$c_i \in \varepsilon B_i, \quad T_{i+1}(c_{i+1}) \in \varepsilon B_i, \quad B_i \subset F_i^{-1}(Q), \quad B_i \cap F_i^{-1}(G) = \varnothing, \quad 0 \le i \le L-1.$$

In other words, an ε-kernel transition is an ε-chain from c_L to $c_0 = v$ with some constraints on the position of the intermediate points; see Fig. 8.

Two ε-kernel transitions $\{c_i, B_i\}$ and $\{c_i', B_i'\}$ are said to be *disjoint* if $B_i \cap B_i' = \varnothing$ for all $0 \le i \le L$. Disjoint ε-kernel transitions are used in the proof of Theorem 2.3 for points that form cutting pairs and one connecting pair. Note that if u belongs to the positive semiorbit of v, $u = f^k(v)$, and moreover, $k \ge L+1$, then an application of the ε-kernel transition from u to v gives the classical closing lemma. If u and v form a cutting pair, then v belongs to the positive semiorbit of u, $v = f^k(u)$. After an application of the ε-kernel transition to such a cutting pair, one can neglect intermediate points $f(u), \ldots, f^{k-1}(u)$ that get into the perturbation domain.

The following theorem, which can be viewed as a generalization of Theorem 1.1, essentially claims the existence of an optimal set of cutting pairs and one connecting pair and the existence of ε-kernel transitions between the points of the chosen pairs. In what follows, it is convenient to treat the sequence $\{x\}_1^s$ as the numbered sequence of positive iterations of p that get into some small neighborhood of z and the sequence $\{y\}_1^t$ as the numbered sequence of negative iterations of q that get into the same neighborhood of z. The points p_i and q_i are prototypes of cutting pairs, and the points x and y form a prototype of a connecting pair.

Theorem 2.5. *Let there be given an infinite sequence of linear spaces $V_0, \ldots, V_i, \ldots$ each of which is isomorphic to $\mathbb{R}^{\dim M}$ and a sequence of linear transformations $T_i : V_i \to V_{i-1}$. Then for any $\varepsilon > 0$, there exist numbers $\sigma > 1$ and $L \in \mathbb{N}$ satisfying the following property. For arbitrary finite sequences $\{x\}_1^s$ and $\{y\}_1^t$ of points in V_0 with the order $>$ of the form*

$$x_1 < x_2 < \cdots < x_s < y_t < y_{t-1} < \cdots < y_1$$

introduced on their union $N \stackrel{\text{def}}{=} \{x\}_1^s \cup \{y\}_1^t$, *there exist points* $x \in \{x\}_1^s \cap B(x_s, \sigma|x_s - y_t|)$ *and* $y \in \{y\}_1^t \cap B(x_s, \sigma|x_s - y_t|)$ *and* k *ordered pairs* $\{p_i, q_i\} \subset N \cap B(x_s, \sigma|x_s - y_t|)$ *such that*

1. $x_1 \le p_1 \le q_1 < p_2 \le q_2 < \dots < p_{k'} \le q_{k'} < x < y < p_{k'+1} \le q_{k'+1} < \dots < p_k \le q_k \le y_1$;
2. *there exists an* ε*-kernel transition of length* L *from* x *to* y *contained in* $B(x_s, \sigma|x_s - y_t|)$;
3. *for each* $i = 1, \dots, k$, *there exists an* ε*-kernel transition of length* L *from* p_i *to* q_i *contained in* $B(x_s, \sigma|x_s - y_t|)$;
4. *all* ε*-kernel transitions avoid the set* $N - [x, y] - [p_1, q_1] - \dots - [p_k, q_k]$, *where* $[a, b]$ *stands for the set* $\{c \in N | a \le c \le b\}$;
5. *all* $k + 1$ ε*-kernel transitions are pairwise disjoint.*

Now if we use ε-kernel transitions for all pairs p_i, q_i (from p_i to q_i) and an ε-kernel passage for the pair x, y (from x to y), then we obtain an orbit connecting x_1 with the point y_1. The proof of Theorem 2.5 is quite cumbersome and uses a special filling of the space V_0 in a neighborhood of the point x_s by $(\dim M)$-dimensional parallelepipeds. Then one chooses a maximal and a minimal point (in the sense of the order introduced in the theorem) in each parallelepiped and takes pairwise disjoint intervals $[x, y]$, $[p_1, q_1]$, ..., $[p_k, q_k]$ that contain all points of the sequence N lying in all parallelepipeds. Note that actually there are some constraints on the ε-kernel transitions, but we omit them to avoid cumbersome technical details. Just as the classical C^1 closing lemma follows from Theorem 1.1 (see Section 1), one can derive Theorem 2.3 from Theorem 2.5.

By Theorem 2.3, Problem 23 in [110], mentioned at the beginning of this section, has the following partial solution: if the intersection of the topological closure of the positive f-orbit of U_1 with the topological closure of the negative f-orbit of U_2 contains at least one nonperiodic point whose orbit has accumulation points, then there exists a diffeomorphism g C^1-close to f such that the positive g-orbit of U_1 meets the negative g-orbit of U_2.

A "uniform" analog of Theorem 2.3 was obtained in [152]. Namely, the existence of a neighborhood $U_1(f) \subset U(f)$ such that the conclusion of Theorem 2.3 holds for f as well as for any diffeomorphism f_1 in $U_1(f)$ is proved in addition to the existence of numbers $\varrho > 1$, $L \in \mathbb{N}$, and $\delta_0 > 0$. Developing the method of proof of Theorem 2.3, Bonatti and Crovisier [46] proved the following assertion.

Theorem 2.6. *Let* $f : M \to M$ *be a diffeomorphism of a compact manifold* M *such that all periodic orbits of* f *are hyperbolic. Suppose that there exist points* p, $q \in M$ *such that there exists an* ε*-chain from* p *to* q *for each* $\varepsilon > 0$. *Then for every neighborhood* U *of* f *in the space* $\mathit{Diff}^{\,1}(M)$, *there exists a diffeomorphism* $g \in U$ *for which* p *and* q *lie on the same orbit.*

By the Kupka–Smale theorem, a generic C^1 diffeomorphism satisfies the assumptions of Theorem 2.6. That is why Theorem 2.6 is substantially used in the description of residual sets (see Section 7) in the space $\mathit{Diff}^{\,1}(M)$. Note that, in particular, Theorem 2.6 implies that, for a generic diffeomorphism, any chain-recurrent point can be made periodic by an arbitrarily small C^1 perturbation.

By using the main method of his paper [70], Hayashi proved the strengthened variant of the Mañé C^1 dichotomy for vector fields [71]. Before stating the precise result, let us give necessary definitions (as usual, only for the case of a diffeomorphism). Following [39], we refer to the set

$$J_1^+(p) = \{q \in M : \quad \exists\, z_k \to p, \quad \exists\, n_k \to +\infty \\ \text{such that} \;\; f^{n_k}(z_k) \to q \text{ as } k \to +\infty\}$$

as the *first-order prolongational* ω*-limit set of the point* p *in the sense of Auslander*. In a similar way, one can define $J_1^+(N)$ for an arbitrary set $N \subset M$. Set $J_m^+(p) = J_1^+(J_{m-1}^+(p))$ for $m \ge 2$. By replacing $n_k \to +\infty$ by $n_k \to -\infty$, one can define the *first-order prolongational* α*-limit set* $J_1^-(p)$ *of the point* p *in the sense of Auslander* and, by induction, the set $J_m^-(p)$ for $m \ge 2$. A point p is said to be *prolongationally recurrent in the sense of Auslander* if $p \in J_m^+(p) \cup J_m^-(p)$ for some $m \ge 1$. Since $p \in J_1^+(p)$ if and only if $p \in J_1^-(p)$, it follows that each nonwandering point is prolongationally recurrent in the sense of Auslander. If we replace the condition of nonwandering in the improved closing lemma by prolongational recurrence, then we obtain a more general assertion, which could be called the *Auslander C^r closing lemma*. Such a lemma was considered for vector fields on the plane and the sphere in [114, 115] for $r \geq 1$; see Section 5.

The prolongational ω-limit set of a point p is defined as the set

$$\tilde{\omega}(p) = \{q \in M : \exists\, z_k \to p,\ \exists\, n_k \to +\infty\ \exists\, f_k \to f \\ \text{such that } f_k^{n_k}(z_k) \to q \text{ as } k \to +\infty\}.$$

In a completely similar way, one defines the prolongational α-limit set $\tilde{\alpha}(p)$ and the corresponding notions for flows and vector fields. Obviously, $J_1^+(p) \subset \tilde{\omega}(p)$ and $J_1^-(p) \subset \tilde{\alpha}(p)$. The following assertion was proved in [71].

Theorem 2.7. *Let there be a C^1 vector field $\vec{X} \in \chi^1(M)$ given on a closed manifold M, and assume that there exist points p, $q \in M$ such that $\tilde{\omega}_{\vec{X}}(p) \cap \tilde{\alpha}_{\vec{X}}(q) \neq \varnothing$. Then for each neighborhood $U(\vec{X}) \subset \chi^1(M)$ of $\vec{X}$, there exists a $\vec{Y} \in U(\vec{X})$ such that either the points p and q lie on the same trajectory of the vector field $\vec{Y}$ or $\tilde{\omega}_{\vec{Y}}(p) \cap \tilde{\alpha}_{\vec{Y}}(q) = \varnothing$.*

Connecting Invariant Manifolds

First, we present some definitions. Let f be a C^1-smooth diffeomorphism of a closed d-dimensional ($d \geq 2$) Riemannian manifold M. A set $\Lambda \subset M$ invariant with respect to f is said to be *hyperbolic* if the restriction $T_\Lambda M$ of the tangent bundle TM of M to Λ can be represented as the Whitney sum $E^s_\Lambda \oplus E^u_\Lambda$ of df-invariant subbundles E^s_Λ and E^u_Λ ($\dim E^s_x + \dim E^u_x = \dim M$, $x \in \Lambda$) and there exist constants $C_s > 0$, $C_u > 0$, and $0 < \lambda < 1$ such that

$$\|df^n(v)\| \leq C_s \lambda^n \|v\|, \qquad v \in E^s_\Lambda, \qquad n > 0,$$
$$\|df^{-n}(v)\| \leq C_u \lambda^n \|v\|, \qquad v \in E^u_\Lambda, \qquad n > 0.$$

The hyperbolic structure results in the existence of so-called stable and unstable manifolds, which comprise points with the same asymptotic behavior under positive and negative iterations, respectively. More exactly, for each $x \in \Lambda$, there exists an injective immersion $I^s_x : \mathbb{R}^s \to M$, whose image $W^s(x) = I^s_x(\mathbb{R}^s)$ is referred to as the *stable manifold of the point* x, such that the following properties hold.

1. The tangent space of $W^s(x)$ at the point x coincides with E^s_x, $T_x W^s(x) = E^s_x$.
2. A point $y \in M$ belongs to $W^s(x)$ if and only if $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$.
3. $f(W^s(x)) = W^s(f(x))$.
4. If $x, y \in \Lambda$, then either $W^s(x) = W^s(y)$ or $W^s(x) \cap W^s(y) = \varnothing$.

The *unstable manifold* $W^u(x)$, $x \in \Lambda$, is defined as the stable manifold with respect to the diffeomorphism f^{-1}. The unstable manifolds have similar properties. By virtue of property 3, the stable and unstable manifolds are said to be *invariant*.

A hyperbolic set $\Lambda \subset M$ of a diffeomorphism $f : M \to M$ is said to be *isolated* if it has a compact neighborhood U (the so-called *isolating neighborhood*) such that $\bigcup_{n \in \mathbb{Z}} f^n(U) = \Lambda$. It is well known that an isolated maximal hyperbolic set can be uniquely represented as a union of pairwise disjoint isolated transitive sets, which are called basic sets. By $W^s(\Lambda)$ and $W^u(\Lambda)$, we denote the stable and unstable manifolds, respectively, of the set Λ. A point z is called a *homoclinic point associated with* Λ if $z \in W^s(\Lambda) \cap W^u(\Lambda) - \Lambda$. A point is called an *almost homoclinic point associated with* Λ if it lies in

$$(clos\, W^s(\Lambda) \cap W^u(\Lambda)) \cup (W^s(\Lambda) \cap clos\, W^u(\Lambda)) - \Lambda.$$

Let a C^r diffeomorphism $f : M \to M$ have an almost homoclinic point associated with an isolated hyperbolic set Λ. The following assertion will be referred to as the *C^r lemma on the generation of a homoclinic point.*

For any neighborhood U of the diffeomorphism f in the space $Diff^r(M)$, there exists a diffeomorphism $g \in U$ coinciding with f in some neighborhood of Λ and such that g has a homoclinic point associated with Λ.

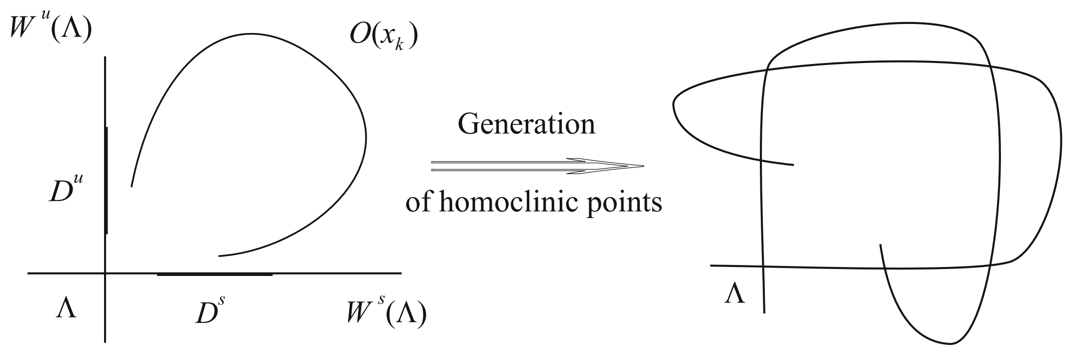


Fig. 9.

If we require g to have a transversal homoclinic point, then we obtain the C^r *lemma on the bifurcation of a transversal homoclinic point.* This lemma plays an important role, because little information on the dynamics can be derived in practice from the existence of almost homoclinic points. On the other hand, Smale [144] proved that a transversal homoclinic point is an accumulation point of invariant compact sets on which the diffeomorphism acts like a finite Markov chain.

The existence of an almost homoclinic point associated with an isolated hyperbolic set Λ implies the existence of an almost homoclinic sequence. Let us give a precise definition. Let U be an isolating neighborhood for Λ, and let $D^s \subset W^s(\Lambda) \cap U$ and $D^u \subset W^u(\Lambda) \cap U$ be the fundamental domains of the restrictions $f|_{W^s(\Lambda)-\Lambda}$ and $f|_{W^u(\Lambda)-\Lambda}$, respectively; i.e.,

$$D^s = clos\,(W^s_\varepsilon(\Lambda) - f(W^s_\varepsilon(\Lambda)))\,, \qquad D^u = clos\left(W^u_\varepsilon(\Lambda) - f^{-1}(W^u_\varepsilon(\Lambda))\right)$$

for some $\varepsilon > 0$. A sequence of finite orbits $O(x_k) = \{f^i(x_k) : m_k \le i \le n_k,\ x_k \in M\}$ is called an *almost homoclinic sequence associated with* Λ if the initial points $f^{m_k}(x_k)$ tend to D^s, the terminal points $f^{n_k}(x_k)$ tend to D^u, and at least one point in $O(x_k)$ lies outside U; see Fig. 9.

The following assertion was proved in [70].

Theorem 2.8. *Suppose that a diffeomorphism $f : M \to M$ of a closed Riemannian manifold M has an isolated hyperbolic set Λ and an almost homoclinic sequence associated with Λ. Then for each neighborhood U of f in $Diff^1(M)$, there exists a diffeomorphism $g \in U$ coinciding with f in some neighborhood of Λ and such that g has a homoclinic point associated with Λ.*

The idea of the choice of a connecting pair and cutting pairs was used in the proof of this theorem. As a consequence, we obtain the following assertion.

Theorem 2.9. *Suppose that a diffeomorphism $f : M \to M$ of a closed Riemannian manifold M has an isolated hyperbolic set Λ and an almost homoclinic point associated with Λ. Then for each neighborhood U of f in $Diff^1(M)$, there exists a diffeomorphism $g \in U$ coinciding with f in some neighborhood of Λ and such that g has a homoclinic point associated with Λ.*

Let us present two theorems which can be derived from Theorem 2.3. (For the proof, see [156].)

Theorem 2.10. *Suppose that a diffeomorphism $f : M \to M$ of a closed Riemannian manifold M has an isolated hyperbolic set Λ, and let the intersection $clos\,W^s(\Lambda) \cap clos\,W^u(\Lambda) - \Lambda$ contain nonperiodic points. Then for each neighborhood U of f in $Diff^1(M)$, there exists a diffeomorphism $g \in U$ coinciding with f in some neighborhood of Λ and such that g has a homoclinic point associated with Λ.*

Theorem 2.11. *Suppose that a diffeomorphism $f : M \to M$ of a closed Riemannian manifold M has an isolated hyperbolic set Λ and there exists a family of periodic orbits not lying in Λ*

and accumulating to Λ. *Then for each neighborhood* U *of* f *in* $Diff^1(M)$, *there exists a diffeomorphism* $g \in U$ *coinciding with* f *in some neighborhood of* Λ *and such that* g *has a homoclinic point associated with* Λ.

Theorem 2.9 does not say anything about the character of the homoclinic point obtained by the perturbation and the invariant manifolds whose intersection forms the homoclinic point. (It is only clear that a transversal homoclinic point can be obtained.) The following refinement of Theorem 2.9 was obtained in [91] for a two-dimensional manifold.

Theorem 2.12. *Suppose that a diffeomorphism* $f : M^2 \to M^2$ *of a closed two-dimensional Riemannian manifold* M^2 *has a basic set* Λ *and an almost homoclinic point associated with* Λ. *Then one of the following conditions is satisfied for each periodic point* $p \in \Lambda$.

(i) *Outside* Λ, *there exists a transversal homoclinic point associated with* p.

(ii) *For each neighborhood* U *of* f *in* $Diff^1(M^2)$, *there exists a diffeomorphism* $g \in U$ *coinciding with* f *in some neighborhood of* Λ *and such that* g *has homoclinic tangency associated with* p.

Therefore, either the unstable and stable manifolds of p themselves meet transversally outside the set Λ, and then no perturbation is required for obtaining homoclinic points, or by an arbitrarily small (in the C^1 topology) perturbation one can ensure that the unstable and stable manifolds have a point of tangency. In the most interesting case where Λ has an almost homoclinic point but the unstable and stable manifolds of the set Λ are disjoint, for each periodic point $p \in \Lambda$ there exists an arbitrarily small perturbation under which the unstable and stable manifolds of p have a point of tangency. Note that even in the case of the coarsest (quadratic) tangency, there exist arbitrarily small perturbations leading to complicated dynamics [14, 15, 101].

3. POINCARÉ PROBLEM ON THE DENSENESS OF PERIODIC ORBITS

First, recall the main notions of the theory of Hamiltonian systems. A skew-symmetric 2-form ω on a manifold is said to be *symplectic* if it is closed ($d\omega = 0$) and nondegenerate; the latter means that if $\omega(\vec{X}, \vec{Y}) = 0$ for all $\vec{Y}$, then $\vec{X} = 0$. An even-dimensional manifold $M = M^{2m}$ equipped with a symplectic form is said to be *symplectic*. On the tangent space T_zM at an arbitrary point $z \in M^{2m}$, the form ω defines the nondegenerate skew-symmetric inner product

$$\omega(\vec{v}, \vec{u}) = \sum_{i,j=1}^{2m} a_{ij} v_i u_j = \vec{v}^T A \vec{u}, \tag{3}$$

where $A = (a_{ij})$ is a skew-symmetric matrix ($A^T = -A$). By virtue of nondegeneracy, ω naturally defines an isomorphism of the tangent space T_zM onto the cotangent space T_z^*M by the formula $\vec{u} \mapsto \omega(\cdot, \vec{u})$. By J_A we denote the inverse isomorphism $T_z^*M \to T_zM$. Let a C^r function $H : M \to \mathbb{R}$, $r \geq 1$, be given on the symplectic manifold M. Then $dH \in T^*M$. The C^{r-1} vector field $\vec{X}_H = J_A(dH)$ defined in accordance with (3) by the formula $\omega(\vec{X}_H, \vec{Y}) = dH(\vec{Y})$ is called the *Hamiltonian field* associated with (the Hamiltonian) H. Sometimes the vector field $\vec{X}_H$ is referred to as the skew-symmetric gradient of the function H. The flow ϕ_H generated by the field $\vec{X}_H$ is called the *Hamiltonian flow*. The space of Hamiltonian C^r smooth vector fields will be denoted by $\mathcal{H}^r(M)$. If c is a regular value of a Hamiltonian H, then $H^{-1}(c)$ is called an *energy manifold*.

The space $\mathbb{R}^{2m}$ equipped with the symplectic form (3), where A is a constant matrix, is a standard example of a symplectic manifold. The natural projection $\pi : \mathbb{R}^{2m} \to \mathbb{R}^{2m}/\mathbb{Z}^{2m} \cong T^{2m}$ induces a symplectic structure on the torus T^{2m}. In this case, we have

$$\vec{X}_H = J_A \nabla H, \qquad J_A = -A^{-1}, \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2m}}\right). \tag{4}$$

As was mentioned above, the C^1 Poincaré closing lemma was proved by Pugh and Robinson [128]. (See the statement of the Poincaré lemma in Introduction.) Actually, they proved a stronger result by showing that there exists a desired perturbation that preserves an energy manifold. Let us present their theorem.

Theorem 3.1. *Let a point $x_0 \in M$ lie on a regular compact energy manifold M_0 of a C^2-Hamiltonian H_0 that defines a Hamiltonian C^1 vector field $\vec{v}_0 \in \mathcal{H}^1(M)$. Then in the space $\mathcal{H}^1$ (M), there exists a field $\vec{w} \in \mathcal{H}^1$ (M) arbitrarily close to $\vec{v}_0$ such that M_0 is its energy manifold and x_0 lies on a periodic trajectory of $\vec{w}$.*

The proof follows the scheme of proof of the improved C^1 closing lemma, but one additionally shows that the perturbation in the direction perpendicular to the family of energy manifolds can be eliminated. (See [128, Sec. 9] for details.) As a consequence, we obtain the weakened Poincaré conjecture on the denseness of periodic trajectories of a generic Hamiltonian system in the class C^1.

Theorem 3.2. *For a generic Hamiltonian vector field in the space $\mathcal{H}^1$ (M), the periodic trajectories are everywhere dense on compact energy manifolds.*

Let us present Hermann's example [73], which shows that the Poincaré lemma fails for sufficiently large r. The example is constructed on the even-dimensional torus $\mathbb{T}^{2n+2}$, $n \geq 1$, with the coordinates $\theta = (\theta_1, \dots, \theta_{2n+2})$. We denote the last coordinate by $\theta_{2n+2} = r$.

First, recall some facts from "multidimensional" arithmetics. We say that a vector

$$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$$

satisfies the Diophantine condition with exponent $\tau \geq 0$ if there exists a $C > 0$ such that

$$\left| \sum_{i=1}^{d} k_i \alpha_i \right| \geq C \left(\sum_{i=1}^{d} |k_i| \right)^{-\tau}$$

for each $k = (k_1, \dots, k_d) \in \mathbb{Z}^d - \{0\}$. The set of such vectors is denoted by $DC_\tau(\mathbb{R}^d)$. If $\tau \geq d-1 \geq 1$, then the complement $\mathbb{R}^d - DC_\tau(\mathbb{R}^d)$ of the set $DC_\tau(\mathbb{R}^d)$ has zero Lebesgue measure. Moreover, the vector $(1, s\alpha) \in \mathbb{R}^{d+1}$ satisfies the Diophantine condition with exponent $\tau_1 = \sup\{\tau, d+0\}$ for Lebesgue almost all $s \in \mathbb{R} - \{0\}$.

Take $\alpha = (\alpha_1, \dots, \alpha_{2n+1}) \in \mathbb{R}^{2n+1} \in DC_\tau(\mathbb{R}^{2n+1})$, where $\tau \geq 2n+1 \geq 3$ and the numbers $\{\alpha_1, \dots, \alpha_{2n+1}\}$ are rationally independent. On $\mathbb{T}^{2n+2}$ we introduce a symplectic structure of the form (3) such that the isomorphism J_A is given by the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & \alpha_1 \\ -1 & 0 & 0 & 0 & \dots & 0 & \alpha_2 \\ 0 & 0 & 0 & 1 & \dots & 0 & \alpha_3 \\ 0 & 0 & -1 & 0 & \dots & 0 & \alpha_4 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \alpha_{2n+1} \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & \dots & -\alpha_{2n+1} & 0 \end{bmatrix}$$

Since $\det J_A = \alpha_{2n+1}^2 \neq 0$, it follows that there exists a matrix A that defines the desired symplectic structure. Take the Hamiltonian $H_0(\theta_1, \dots, \theta_{2n+1}, r) = \sin(2\pi r)$. One can readily see that $\nabla H_0 = (0, \dots, 0, \cos(2\pi r))$ and system (4) acquires the form

$$\begin{cases} \dot{\theta}_1 & = & 2\alpha_1 \pi \cos(2\pi r) \\ \dots & \dots & \dots \\ \dot{\theta}_{2n+1} & = & 2\alpha_{2n+1} \pi \cos(2\pi r) \\ \dot{r} & = & 0. \end{cases}$$

Obviously, each value $c \in (-1, +1)$ is regular for the Hamiltonian H_0, and the energy manifold $H_0^{-1}(c)$ consists of two tori T_{1c}^{2n} and T_{2c}^{2n} on each of which the corresponding Hamiltonian flow is minimal, because the numbers $\{\alpha_1, \dots, \alpha_{2n+1}\}$ are rationally independent. To show that the above-represented system provides the desired example, Hermann [73] proved that the above-mentioned properties of the Hamiltonian H_0 are preserved under small C^r perturbations; namely, for each $r > \sup\{2\tau_1; \tau + \tau_1 + 1\}$, there exists a neighborhood $U(H_0)$ of H_0 in the C^r topology such that all values of an arbitrary $H \in U(H_0)$ in the interval $[-1/2, 1/2]$ are regular, and, for any $c \in [-1/2, 1/2]$, the energy manifold $H^{-1}(c)$ consists of two tori on each of which the corresponding Hamiltonian flow is conjugate to a flow on the torus T^{2n}_{1c} and T^{2n}_{2c} (and hence is minimal and has no periodic orbit). Indeed, for the Hamiltonians sufficiently C^2-close to H_0, each of the two tori $T_{c,j} \in H^{-1}(c)$ $(j = 1, 2)$ is the graph of some C^2 function $\psi_{c,j} : T^{2n+1} \to \mathbb{R}$: $H(\theta, \psi_{c,j}(\theta)) = c$ for all $\theta \in T^{2n+1}$. Under the action of the projection $T^{2n+2} \to T^{2n+1}$ of the form $(\theta, r) \to \theta$, the vector field $\vec{X}_H|_{T_{c,j}} = J_A \nabla H|_{T_{c,j}}$ is projected to the vector field $\vec{Y}_{c,j}$. Consider its "normalization,"

$$\vec{Z}_{c,j} = \frac{1}{\phi_{c,j}} \vec{Y}_{c,j}, \quad \text{where} \quad \phi_{c,j} = \frac{\partial H}{\partial r}(\cdot, \psi_{c,j}).$$

One can readily see that the kth component $(1 \le k \le 2n+1)$ of this field is equal to

$$-\sum_{i=1}^{2n+1} c_{ki} \frac{\partial \psi_{c,j}}{\partial \theta_i} + \alpha_k,$$

where $J_A = (c_{ij})$. Since $c_{ki} = -c_{ik}$, we have

$$\int\limits_{T^{2n+1}} \vec{Z}_{c,j} dm = \alpha_A, \tag{5}$$

where dm is the Haar measure. Somewhat exaggerating, one can say that the field $\vec{Z}_{c,j}$ has rotation number α_A. On the two-dimensional torus, one could make the conclusion that the field $\vec{Z}_{c,j}$ is conjugate to a linear field that, by virtue of the rational independence of the coordinates of the vector α_A, is a minimal vector field. However, in the multidimensional case (the torus T^{2n+1} is at least three-dimensional), a field is not conjugate to a linear field in general. Therefore, we cannot claim yet that the vector field $\vec{X}_H|_{T_{c,j}}$ does not have periodic trajectories. The proof of the conjugacy with a linear field is carried out with the use of a fine result due to Hermann [72], in which it is required that the vector field be close to a linear field in the C^r topology for a sufficiently large r. Let us present this result.

First, recall that if a diffeomorphism $f : T^d \to T^d$ of the torus T^d preserves the measure dm and can be represented in the form $f = Id + \zeta$, where ζ is a 1-periodic function of each of its arguments, then, by analogy with the circle $T^1 = S^1$, one can define the rotation number

$$\varrho_m(f) = \int\limits_{T^d} \zeta dm.$$

(More precisely, this is one way to introduce a characteristics similar to the Poincaré classical rotation number.) By g_t we denote the time t shift along the trajectories of the vector field $\vec{Z}_{c,j}$. By virtue of (5), $\varrho(g_t) = t\alpha_A$. Take an $s \in \mathbb{R} - \{0\}$ such that the vector $(1, s\alpha_A) \in \mathbb{R}^{2n+2}$ satisfies the Diophantine condition with exponent $\tau_1 = \sup\{\tau, 2n + 1 + 0\}$. Now we use the following Proposition 2.6.1 in [72, Chap. XIII].

Let $(1, s\alpha) \in \mathbb{R}^{d+1}$ satisfy the Diophantine condition with exponent τ_1, and let $r > 2\tau_1$. Then in the space $\mathit{Diff}^{\,r}_m(T^d)$, there exists a neighborhood V of the translation $R_\alpha : T^d \to T^d$ by the vector α such that every diffeomorphism $g \in V \cap \mathit{Diff}^{\,r}_m(T^d)$ with $\varrho(g) = \alpha$ is $C^{r-\tau_1-0}$ conjugate to R_α.

Here $\mathit{Diff}^{\,r}_m(T^d)$ stands for the space of C^r diffeomorphisms $T^d \to T^d$ preserving the measure dm. Thus, if the Hamiltonian H is C^{r+1}-close to H_0, then the vector field $\vec{Z}_{c,j}$ is conjugate to the constant vector field α_A on T^{2n+2}. This implies the desired result; for details, see [73].

It was shown in [74] that the C^r smoothness requirement cannot be weakened in general. The paper [74] also provides other examples on compact symplectic manifolds that are not homological to a torus.

Let a symplectic 2-form ω define a symplectic structure on an even-dimensional compact manifold M^{2m}. By $Diff^{\,r}_{\omega}(M^{2m})$ we denote the space of symplectic C^r diffeomorphisms of M^{2m}, that is, diffeomorphisms preserving the symplectic (or volume) form ω. The following C^1 connecting lemma for pseudo-orbits (an analog of Theorem 2.6) holds for symplectic C^1 diffeomorphisms [37, 46, 53].

Theorem 3.3. *Let $f \in Diff^{\,1}_{\omega}(M^{2m})$ be a symplectic diffeomorphism of a compact manifold M^{2m} such that, for any number $s \in \mathbb{N}$, the set of periodic points of f of period s is finite. Suppose that there exist points p, $q \in M$ such that, for each $\varepsilon > 0$, there exists an ε-chain from p to q. Then, for each neighborhood U of f in $Diff^{\,1}_{\omega}(M^{2m})$, there exists a diffeomorphism $g \in U$ such that p and q lie on the same orbit.*

The proof is carried out by the same scheme as Theorem 2.6 with regard of the symplecticity of the diffeomorphism.

Closing lemmas play an important role in the analysis of various classes of holomorphic mappings of the multidimensional complex space $\mathbb{C}^k$; see the survey [160] and Appendix B in [93], where the role of the classical closing lemma in the proof of the denseness of hyperbolic mappings in families of polynomials of degree 2 is discussed. By $\mathcal{E}$ we denote the space of holomorphic mappings $\mathbb{C}^k \to \mathbb{C}^k$ equipped with the topology of uniform convergence on compact subsets. In [56], Fornaess and Sibony proved the classical closing lemma (in the above-mentioned topology of the space $\mathcal{E}$) in classes of holomorphic endomorphisms, biholomorphic mappings, biholomorphic conservative (i.e., volume-preserving) mappings, biholomorphic symplectomorphisms $\mathbb{C}^{2k} \to \mathbb{C}^{2k}$, and holomorphic Hamiltonians on $\mathbb{C}^2$. The proofs are very different from the ones considered here and are specific to complex analysis and complex dynamics. (These specific features essentially single out complex dynamics in the theory of dynamical systems.) Note also the papers [57, 130], where close issues were considered.

4. ANOSOV LEMMA

The assertion that is now referred to as the Anosov closing lemma historically arose from Lemma 13.1 in [5]. (See also [3, 4], where fundamental results, based on Lemma 13.1, on the countability of periodic motions in Anosov systems and the denseness of periodic motions in Anosov systems with an integral invariant were announced.) This lemma deals with an Anosov flow[5] f^t on a closed manifold M. For a point $z \in M$, by $l(z, f_t z)$ we denote the trajectory arc of time length t with endpoints z and $f_t z$. Let us present Lemma 13.1 in [5].

Lemma 4.1. *For each $\epsilon > 0$, there exists a $\delta > 0$ with the following property. Let a point $z \in M$ and a number $\tau > \epsilon$ satisfy the inequality $d(z, f_\tau z) < \delta$. Then there exists a periodic trajectory l_0 of the flow f^t such that the Hausdorff distance between the arc $l(z, f_\tau z)$ and the curve l_0 is less than ϵ.*

This assertion illustrates how the local hyperbolicity condition and the nonlocal recurrence condition imply the existence of a periodic motion in a system. To demonstrate the idea of how to use hyperbolicity, considered a weakened version of Lemma 4.1, which is due to Franks [58, Proposition 1.7]. (Note that the assumption of Lemma 4.1 holds in an arbitrary neighborhood of a nonwandering point.)

Lemma 4.2. *Let $f : M \to M$ be an Anosov diffeomorphism of a manifold M. Then periodic points are dense in the nonwandering set $NW(f)$.*

Scheme of proof of Lemma 4.2. Let U be a neighborhood of a point $z \in NW(f)$. Without loss of generality, one can assume that U lies in a neighborhood with a product structure. In what

[5] This means that the entire M is a uniform hyperbolic set of the flow f^t.

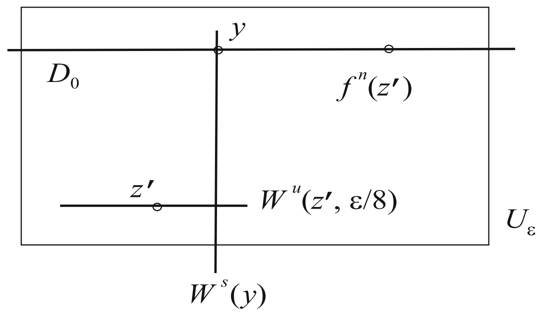


Fig. 10.

follows, it is convenient to denote $W^{u(s)}_\varepsilon(p)$ by $W^{u(s)}(p,\varepsilon)$. Take an $\varepsilon > 0$ small enough that

$$U_\varepsilon \stackrel{\text{def}}{=} W^u(z,\varepsilon) \times W^s(z,\varepsilon) \subset U.$$

Since z is a nonwandering point, it follows that there exists a $z' \in U_{\varepsilon/4}$ such that $W^u(z',\varepsilon/8) \subset U_{\varepsilon/4}$ and $f^n(z') \in U_{\varepsilon/4}$. The number n can be assumed to be large enough that $C_u\lambda^{-n} > 8$ and $C_s\lambda^n < 1/8$, where $C_s > 0$, $C_u > 0$, and $0 < \lambda < 1$ are the constants occurring in the definition of hyperbolicity; see Section 2. Then

$$W^u\left(f^n(z'), \frac{\varepsilon}{2}\right) \subset f^n\left(W^u(z'), \frac{\varepsilon}{8}\right) \stackrel{\text{def}}{=} D.$$

By D_0 we denote the component of D containing the point $f^n(z')$; see Fig. 10. Define a mapping $h : D_0 \to D_0$ as follows: first, we map D_0 into $W^u(z',\varepsilon/8)$ by f^{-n} and then project $W^u(z',\varepsilon/8)$ into D_0 along the stable manifolds. Since f^{-n} acts as a contraction on unstable manifolds, it follows that h has a fixed point $y \in D_0$. Obviously, $f^n(W^s(y)) = W^s(y)$. Set $D' = W^s(y, 3\varepsilon/2)$. Then

$$f^n(D') \subset W^s\left(f^n(y), \frac{3\varepsilon}{16}\right) \subset D'.$$

Therefore, f^n has a fixed point $x \in U_\varepsilon$. The proof of the lemma is complete.

Following [134], we refer to the closure of the union of all ω- and α-limit sets as the *limit set of the homeomorphism* f,

$$L(f) = clos \bigcup_{z\in M} (\omega(z) \cup \alpha(z)).$$

The chain-recurrent set of f will be denoted by $\mathfrak{R}(f)$. Obviously,

$$clos(Per(f)) \subset L(f) \subset NW(f) \subset \mathfrak{R}(f).$$

Presently, the Anosov closing lemma is understood as the following assertion [134].

Theorem 4.1. *Let $f : M \to M$ be a diffeomorphism of a compact Riemannian manifold M. The following assertions hold.*

1. *Suppose that the chain-recurrent set $\mathfrak{R}(f)$ of the diffeomorphism f has hyperbolic structure. Then periodic points are dense in $\mathfrak{R}(f)$, and*

$$clos(Per(f)) = \mathfrak{R}(f) = L(f) = NW(f).$$

2. *Suppose that the limit set $L(f)$ of the diffeomorphism f has hyperbolic structure. Then periodic points are dense in $L(f)$, and*

$$clos(Per(f)) = L(f).$$

3. *Suppose that the nonwandering set $NW(f)$ of the diffeomorphism f has hyperbolic structure. Then periodic points are dense in the nonwandering set of the restriction of f to $NW(f)$; i.e.,*

$$clos(Per(f)) = NW(f|_{NW(f)}).$$

The proof is based on the following tracing theorem, which has an independent value (for tracing see [117]). Recall that a point y is ϵ-*tracing* a δ-chain $\{x_j\}_{j_1}^{j_2}$ if $d(f^j(y), x_j) < \epsilon$ for all $j_1 \leq j \leq j_2$. Below $U_\eta(N)$ stands for the η-neighborhood of the set N.

Theorem 4.2. *Let Λ be a hyperbolic invariant compact set. For any $\varepsilon > 0$, there exist $\delta > 0$ and $\eta > 0$ such that if $\{x_j\}_{j_1}^{j_2}$ is a δ-chain lying in $U_\eta(\Lambda)$, then there exists a point y, which is ε-tracing $\{x_j\}_{j_1}^{j_2}$. If the δ-chain $\{x_j\}_{j_1}^{j_2}$ is periodic, then y is also periodic. If $j_1 = -\infty$, $j_2 = +\infty$, and Λ is isolated, then $y \in \Lambda$.*

Let us schematically show how to use the shadowing theorem in the proof of the first assertion of Theorem 4.1. Take a chain-recurrent point $z \in \mathfrak{R}(f)$. Then, for each $\delta > 0$, there exists a periodic δ-chain $\{x_j\}_{j_1}^{j_2}$ such that $d(z, x_{j_1}) < \delta$. Moreover, such a δ-chain can be chosen to consist of chain-recurrent points. By Theorem 4.2, there exists a periodic point y that ε-shadows $\{x_j\}_{j_1}^{j_2}$. Since $\delta + \varepsilon$ can be arbitrarily small, we have $z \in clos(Per(f))$. The remaining assertions of Theorem 4.1 can be proved in a similar way.

There is a version of the closing lemma for a nonuniformly hyperbolic set, which was proved by Katok [77]. (See also Theorem S.4.13 in [78].)

5. C^r LEMMAS FOR $r \geq 2$

Classical and Improved C^r Lemmas

As was mentioned above, there are numerous results on the classical and improved C^r lemmas for $r \geq 2$. Most of them deal with smooth transformations of the circle and the segment and also with flows on surfaces.

Mappings of the circle and the segment. The circle S^1 is the first manifold on which the classical C^r closing lemma was proved for all $r \geq 1$, including C^ω, even before the statement itself of the lemma appeared [17]. Let us outline the proof, because its main idea is used in most proofs of closing lemmas for $r \geq 2$.

Let $x_0 \in S^1$ be a nontrivially recurrent point of a C^r diffeomorphism $f : S^1 \to S^1$. Take the lift $\overline{f} : \mathbb{R} \to \mathbb{R}$ with respect to the universal covering $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$. By $\overline{R}_\lambda : \mathbb{R} \to \mathbb{R}$ we denote the translation $x \mapsto x + \lambda$ covering the rotation $R_\lambda : S^1 \to S^1$. Take an $\varepsilon > 0$. It suffices to show that there exists $|\lambda_0| < \varepsilon$ such that $R_{\lambda_0} \circ f$ has a periodic point x_0. Take a lift $\overline{x}_0 \in \mathbb{R}$ of that point. Without loss of generality, one can assume that $\overline{x}_0 = 0$. Since x_0 is a nontrivially recurrent point, it follows that there exists an $m \in \mathbb{Z}$ and a $k \in \mathbb{N}$ such that either $m - \varepsilon/2 < \overline{f}^k(\overline{x}_0) < m$ or $m < \overline{f}^k(\overline{x}_0) < m + \varepsilon/2$. To be definite, we assume that the first inequality holds. (In the case of the second inequality, the proof can be carried out in a similar way.) Obviously, if $\lambda > 0$, then $R_\lambda \circ f > f$ and there exists $0 < \lambda_* < \varepsilon$ such that $\overline{f}^k(\overline{x}_0) + \lambda_* > m$. We have

$$\left(\overline{R}_\lambda \circ \overline{f}\right)^k(\overline{x}_0) = \left(\overline{R}_\lambda \circ \overline{f}\right) \circ \left(\overline{R}_\lambda \circ \overline{f}\right)^{k-1}(\overline{x}_0) \geq \overline{R}_\lambda \circ \overline{f}^k(\overline{x}_0) = \overline{f}^k(\overline{x}_0) + \lambda.$$

Therefore, $\left(\overline{R}_{\lambda_*} \circ \overline{f}\right)^k(\overline{x}_0) > m$. On the other hand, $\left(\overline{R}_0 \circ \overline{f}\right)^k(\overline{x}_0) = \overline{f}^k(\overline{x}_0) < m$. Since the point $\left(\overline{R}_\lambda \circ \overline{f}\right)^k(\overline{x}_0)$ continuously depends on λ, it follows that there exists $0 < \lambda_0 < \lambda_* < \varepsilon$ such that $\left(\overline{R}_{\lambda_0} \circ \overline{f}\right)^k(\overline{x}_0) = m$. It follows that $R_{\lambda_0} \circ f$ has the periodic point x_0. The proof of the improved C^r closing lemma reproduces this proof almost word for word.

The above-represented proof demonstrates essentially the only method, which can be called the "field rotation," for proving closing lemmas in the smoothness classes with $r \geq 2$. In the C^1 lemma, the bifurcation is concentrated in small balls, and the closure is performed in the δ-kernels of these balls (which automatically makes higher-order derivatives large), while for $r \geq 2$ one uses a bifurcation, which, owing to a sufficiently large size of the deformation domain, practically does not affect higher-order derivatives. On the circle, a "rotation of the field" is simply a translation, and for vector fields it implies the addition of a constant vector field of desired direction.

The following super-strengthened closing lemma can be proved for diffeomorphisms of the circle [13].

Theorem 5.1. *Suppose that a C^r diffeomorphism $f : S^1 \to S^1$ has a chain-recurrent point, $r \geq 1$. Then, for each neighborhood U of f in the space $Diff^r(S^1)$, there exists a $g \in U$ such that all points of the diffeomorphism g are periodic.*

The most important case is one in which the rotation number $rot(f)$ is irrational. Note that, in this case, a chain-recurrent point can be wandering, but then $r = 1$. According to a remarkable result due to Hermann [72], f can be approximated by an analytic diffeomorphism g_0 that is analytically conjugate to a rotation R_α, $g_0 = h \circ R_\alpha \circ h^{-1}$. Obviously, R_α can be approximated by the rotation $R_{p/q}$, all of whose orbits are periodic. Then $g = h^{-1} \circ R_{p/q} \circ h$ is the desired diffeomorphism.

Now consider C^r smooth mappings of the interval $I = [0; 1]$ into itself. By $End^r_{pm}(I)$ we denote the set of C^r smooth (in particular, continuous) piecewise monotone mappings of I into itself equipped with the natural metric $d_r(f, g) = \max_{x\in I} \max_{i=0,\dots,r} |f^{(i)}x - g^{(i)}x|$ if r is finite and $d_\infty(f, g) = \sum_{i=0}^{\infty} \dfrac{1}{2^i} \max\limits_{x\in I}\{\min(1, |f^{(i)}x - g^{(i)}x|)\}$. For $f \in End^r_{pm}(I)$, we set

$$Rec(f) = \{x \in I : \forall\varepsilon > 0,\ \exists n > 0 : |f^n x - x| < \varepsilon\}.$$

Obviously, $Per(f) \subset Rec(f) \subset NW(f)$. The relation

$$clos(Per(f)) = clos(Rec(f)) \tag{6}$$

was proved in [161] for any continuous mapping $f \in End^0_{pm}(I)$; this relation can be treated as an analog of the Poincaré lemma (i.e., nothing should be perturbed): there exists a periodic orbit passing near any nontrivially recurrent point. Note that relation (6) is not necessarily true for an arbitrary homeomorphism of the circle S^1. As to the approximation of nonwandering points by periodic ones, the following assertion was proved in [161].

Theorem 5.2. *Suppose that $f \in End^r_{pm}(I)$, $0 \leq r \leq \infty$, has a nonwandering point $x \in NW(f)$. Then for each $\varepsilon > 0$, there exists a polynomial g such that $d_r(f, g) < \varepsilon$ and x belongs to the closure of the set of periodic points of g, $x \in clos(Per(g))$.*

Theorem 5.2 is used to prove that the relation $clos(Per(f)) = NW(f)$ holds for a generic mapping $f \in End^r_{pm}(I)$, $2 \leq r \leq \infty$. This relation was proved earlier by Jacobson [24] for a generic f in the space $End^1_{pm}(S^1)$.

Consider one-to-one mappings of the circle $S^1 \cong \mathbb{R}/\mathbb{Z}$ (which is equipped with the positive sense) with discontinuities. Let $A = \{a_i\}_{i=1}^k$ and $B = \{b_i\}_{i=1}^k$ be two families of cyclically ordered points that divide S^1 into intervals $I_i = (a_i, a_{i+1})$ and $J_i = (b_i, b_{i+1})$, respectively, where $a_{k+1} = a_1$ and $b_{k+1} = b_1$. A mapping

$$f : S^1 - \bigcup_{i=1}^{k} a_i \to S^1 - \bigcup_{i=1}^{k} b_i$$

is said to be *piecewise C^r-diffeomorphic*, $r \geq 0$, if it is a one-to-one mapping and the restriction $f|_{I_i}$ is a C^r smooth diffeomorphism[6] on $J_{\sigma(i)}$. Let us introduce the C^r-topology on the set of such mappings. For a given $\varepsilon > 0$, we say that g belongs to the ε-neighborhood $U_\varepsilon(f)$ of a mapping f if there exists an orientation-preserving C^r diffeomorphism $h : S^1 \to S^1$ ε-close to the identity mapping and such that $h(clos\, I_i(f)) = clos\, I_i(g)$, $i = 1, \dots, k$, and $g \circ h$ is ε-close to f in the Whitney C^r topology on each $clos\ (I_i(f))$. The space of piecewise diffeomorphic mappings of a circle with a fixed k, equipped with the above-mentioned topology, is denoted by $\mathcal{M}^{r+0}(k)$.

[6] A C^0 diffeomorphism is understood as a homeomorphism.

Take the symbols $J_1, \dots, J_k$ and $A_1, \dots, A_k$, and for $x \in S^1$ define the route

$$i_f(x) = (i_0(x), \dots, i_n(x), \dots)$$

of the point x, where $i_n(x) = J_i$ if $f^n(x) \in I_i$ and $i_n(x) = A_i$ if $f^n(x) = a_i$. In the latter case, we assume that $i_m(x) = A_i$ for all $m \geq n$. By $A^\infty = \bigcup_{i=0}^\infty f^{-i}(A)$ and $B^\infty = \bigcup_{i=0}^\infty f^i(B)$, we denote the preimages and "images," respectively, of the points of discontinuity. If $f^{-i}(a_s) = b_p$ for some $1 \leq p, s \leq k$, $i \geq 1$, then we set $f^{-i-1}(a_s) = f^{-1}(b_p) = \varnothing$. By $O(x) = \bigcup_{-\infty}^{+\infty} f^n(x)$ we denote the orbit of a point $x \notin A^\infty \cup B^\infty$.

Let $x \in I_\nu = [a_\nu, a_{\nu+1}]$ be a nontrivially recurrent point ($x \notin A^\infty \cup B^\infty$), and let $q_1(r)$ be the first positive iteration such that $f^{q_1(r)}(x) \in (x, a_{\nu+1})$. By induction, we define $q_n(r)$ as the first iteration such that $f^{q_n(r)}(x) \in (x, f^{q_{n-1}(r)}(x))$. In a similar way, one can define the numbers $q_n(l)$. [For example, $q_1(l)$ is the first positive iteration such that $f^{q_1(l)}(x) \in (a_\nu, x)$.] Consider the block $B_n^r = [i_0(x), \dots, i_{q_n(r)}(x)] \subset i_f(x)$ and define $r_n \in \mathbb{N} \cup \{0\}$ as the maximum number of repetitions of the block B_n^r in $i_f(x)$ after the symbol $i_{q_n(r)}(x)$. Formally, $i_k(x) = i_{k+jq_n(r)}(x)$ for $0 \leq j \leq r_n$ and $0 \leq k \leq q_n(r)$. The sequence $R(x) = \{r_1(x), \dots, r_n(x), \dots\}$ is referred to as the *right t-expansion of the point x*. By replacing the numbers $q_n(r)$ by $q_n(l)$, we obtain the definition of the *left t-expansion of x*, which we denote by $L(x)$. The following theorem was proved in [16].

Theorem 5.3. *Suppose that $f \in \mathcal{M}^{r+0}(k)$, $r \geq 1$, is increasing on each monotonicity interval. Let $x \in S^1$ be a nontrivially recurrent point, and let $L(x) = \{l_i\}_1^\infty$ and $R(x) = \{r_i\}_1^\infty$ be its left and right t-expansions, respectively. If $\varlimsup_{n\to\infty} l_n \geq 3$ and $\varlimsup_{n\to\infty} r_n \geq 3$, then for each neighborhood $U(f)$, there exists a $g \in U(f)$ that has periodic point x.*

In what follows, when describing results for flows on surfaces, we also present other results dealing with the one-dimensional dynamics (because there is a close relationship between them if one considers the mappings induced by a flow on a secant line).

Examples by Gutierrez and Carroll. As early as in 1975, Pugh [124] casted doubt on the validity of the classical and improved C^r lemmas for $r \geq 2$ for manifolds of dimensions ≥ 2. To justify his words, Pugh constructed a flow on the torus with a prolongationally recurrent point of the first order in the Pugh sense, for which the C^2 closing lemma fails (see Section 5). The idea of that construction was used in most counterexamples to various closing lemmas (except for the Poincaré lemma, where a different idea is used). In 1978, Gutierrez [64] showed that the classical (and so much the more the improved) C^r closing lemma with $r \geq 2$ is not true for noncompact surfaces in general. More precisely, he proved the following assertion.

Theorem 5.4. *Let $T^2 - \{p\}$ be a punctured torus, $p \in T^2$, and suppose that, for a given $r \geq 2$, there exists a vector field $\vec{X} \in \mathfrak{X}^\infty(T^2)$ with a nontrivially recurrent trajectory such that, in some neighborhood U of the field $\vec{X}|_{T^2-\{p\}}$ in the space $\mathfrak{X}^r(T^2 - \{p\})$, any vector field $\vec{Y} \in U$ has no periodic trajectory.*

It is assumed that the space $\mathfrak{X}^r(T^2 - \{p\})$ is equipped with the Whitney strong topology. This means that a fundamental system of neighborhoods of the field $\vec{X}|_{T^2-\{p\}}$ is defined as follows. Let $\{U_i\}$ be a locally finite cover of the set $T^2 - \{p\}$, and let $\{\varepsilon_i\}$ be a family of positive numbers. Then the fundamental system of neighborhoods is generated by the open sets

$$U(\{U_i\}, \{\varepsilon_i\}) = \{\vec{V} \in \mathfrak{X}^r(T^2 - \{p\}) : ||\vec{V}|_{U_i} - \vec{X}|_{U_i}||_r < \varepsilon_i\}.$$

The construction of the vector field $\vec{X} \in \mathfrak{X}^\infty(T^2)$ starts from a homeomorphism $f : C \to C$ of the circle. The homeomorphism f is nontransitive and has the irrational rotation number $rot\, f = \frac{1}{2}(\sqrt{5} - 1)$. There exists a continuous mapping $h : C \to S^1 \cong \mathbb{R}/\mathbb{Z}$ that half-conjugates f with $R_{rot\, f}$ and maps each pasted-in interval into the corresponding point.

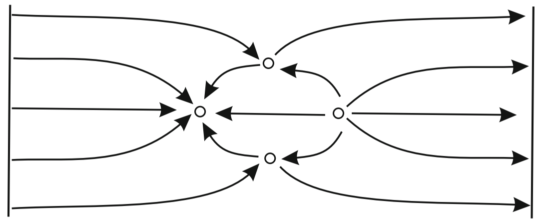


Fig. 11.

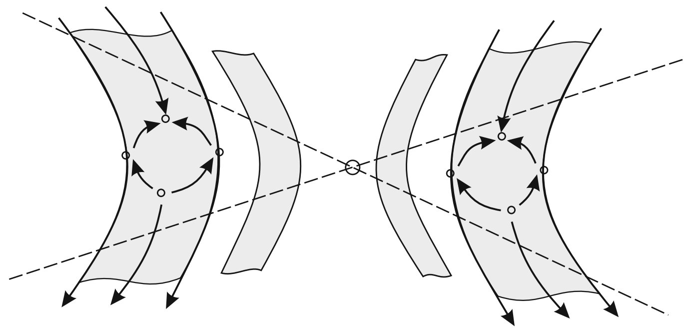


Fig. 12.

The standard suspension $sus\,f$ over f is a flow on T^2 without fixed points, which induces the first return map f on the zero meridian. One can assume that the circle C is embedded as the zero meridian in T^2, $C \subset T^2$. The trajectories of the flow $sus\,f$ passing through the interval $h^{-1}(0)$ form the so-called Denjoy cell. If false saddles (that is saddles each of which has exactly two saddle sectors) are placed to endpoints of the interval $h^{-1}(0)$, then $sus\,f$ becomes smoothable [62]. More accurately, $sus\,f$ with false saddles is topologically equivalent to a C^∞ flow; moreover, the homeomorphism implementing this equivalence can be chosen to be arbitrarily close to the identity mapping. Now, in the Denjoy cell near specially chosen intervals $h^{-1}(R^n_{rot\,f}(0))$, which accumulate to $h^{-1}(0)$, we paste the "plugs" shown in Fig. 11. Each plug contains one source, one sink, and two saddles. All fixed points are structurally stable. Next, the segment $h^{-1}(0)$ is deleted from the torus, and the resulting cut is contracted into the deleted point p. Obviously, this can be carried out so as to obtain a C^∞ flow (we denoted it by f^t) on the punctured torus $T^2 - \{p\}$. Moreover, the vector field $X|_{T^2-\{p\}}$ generating f^t can be multiplied by a smooth positive function so as to ensure that it can be extended smoothly to the deleted point p up to an equilibrium.

The arrangement of the specially chosen intervals $h^{-1}(R^n_{rot\,f}(0))$, which now accumulate to the puncture, is chosen with regard of the continued fraction expansion of the rotation number, $rot\,f = [1, \ldots, 1, \ldots]$. The corresponding plugs accumulate to the puncture as well; see Fig. 12. Since the space $\mathfrak{X}^r(T^2 - \{p\})$ is equipped with the strong Whitney topology, it follows that the perturbations of the vector field $X|_{T^2-\{p\}}$ are subjected to some constraints near p. Gutierrez [64] proved that, for any perturbation sufficiently close to $X|_{T^2-\{p\}}$, every one-dimensional trajectory either gets into one of plugs and tend to the fixed point or tends to the puncture. (This happens both in the positive and the negative direction.) Consequently, all vector fields in some neighborhood of the field $X|_{T^2-\{p\}}$ have no periodic trajectories.

A close idea was used by Carroll [52] to construct a C^∞ smooth vector field for which no sufficiently small C^2 twisting along a closed transversal leads to the appearance of periodic trajectories. Let us present a rigorous definition. Let C be a closed simple transversal of some C^∞ vector field X on the torus T^2. The curve C has a cylindrical neighborhood A holomorphic to $C \times (-1; +1)$. Without loss of generality, one can assume that the trajectories of the flow f^t cross A along segments of the form $\{x\} \times (-1; +1)$ and are directed from $C \times \{-1\}$ to $C \times \{+1\}$. Let us specify an orientation on $C \times \{+1\}$. The C^r *twisting of the field* X *along* C is the addition to X of a C^r vector field Y whose support lies in A and which "shifts" all points of $C \times \{+1\}$ only in one direction, either positive or negative.

Recall that the topological closure of a nontrivially recurrent semitrajectory is called a *quasi-minimal set*. Every flow on the torus has at most one quasi-minimal set. The following theorem was proved in [52].

Theorem 5.5. *On the torus T^2, there exists a C^∞ vector field X with finitely many equilibria and a nowhere dense quasi-minimal set $\Omega(X)$ such that the following conditions are satisfied.*

1. *X has a simple closed transversal C that meets all nontrivially recurrent semitrajectories of the field X.*

2. *Each sufficiently small C^2 twisting of the field X along C results in a vector field that has either a quasi-minimal set coinciding with $\Omega(X)$ or a nonwandering set consisting of finitely many equilibria.* (*In particular*, *in the latter case*, *there are neither nontrivially recurrent semitrajectories nor periodic trajectories.*)

Like the above-represented Gutierrez construction, the construction of the field X starts from a Denjoy flow with a rotation number, whose continued fraction forms a bounded sequence. Along with the cell position, the cell width decay rate is taken into account in [52]. This permits constructing a field with finitely many equilibria.

An even more exotic example was considered in [68]. More precisely, on a nonorientable surface of genus 4, there exists a supertransitive Kupka–Smale C^∞ flow[7] f^t and a family of flows $\{f^t_\mu\}_{\mu\geq 0}$, $f^t_0 = f^t$, obtained from f^t by a nontrivial C^∞ twisting such that all flows $\{f^t_\mu\}_{\mu\geq 0}$ are topologically equivalent to f^t. The construction is based on a special example of a minimal rearrangement of five intervals on the circle; moreover, the transformation is orientation-reversing on two of the intervals. (Such intervals are referred to as flips.)

Flows on surfaces: sufficient conditions. First results concerning C^r closing lemmas for flows were obtained by Peixoto [112, 113] when proving that structurally stable flows on an oriented closed surface M^2 coincide with Morse–Smale flows. Peixoto (see Lemma 4 in [112]) showed that if a nontrivially recurrent trajectory in a minimal set passes through a point $p \in M^2$ of a C^r-smooth flow and p does not lie in the limit set of any separatrix, then there exists a flow that is arbitrarily close in the C^r topology and has a periodic trajectory passing through p; here $r \geq 1$. The desired perturbation was constructed by a rotation of a field along a transversal (twisting). For the case in which the point p lies in the limit set of a separatrix, Peixoto proved the following analog of the C^r connecting lemma (see Lemma 5 in [112]). Let p belong to the intersection of the α-limit set of some ω-separatrix with the ω-limit set of some α-separatrix. Then there exists a flow that is arbitrarily close in the C^r topology and has one more separatrix connection than the original flow. An analysis of the proof shows that both results hold for a point of a nontrivially recurrent trajectory that does not necessarily belong to a minimal set. The refined proofs can be found in [18, 61]. These results were generalized by Peixoto in [69] to foliations defined by direction fields specified by principal curvatures. (The so-called umbilical points, at which the curvature is the same in all directions, are singularities.) Similar bifurcations were considered by Aranson [10] when studying flows of the first degree of structural instability.

The examples by Gutierrez [64] and Carroll [52] make Pugh's conjecture that the classical C^r closing lemma is not true in general for $r \geq 2$ even for vector fields on compact surfaces plausible. This increases the role of assertions with sufficient conditions that imply such a lemma. First, note that the C^r closing lemma for diffeomorphisms of the circle implies the C^r lemma for vector fields without equilibria on the torus for any $r \geq 1$. As to vector fields with equilibria, the problem in full generality remains open (even if one restricts oneself to the class of fields with structurally stable equilibria).

Let $\alpha = [a_0, \ldots, a_n, \ldots]$ be the continued fraction expansion of an irrational number α. (For continued fractions, see [23, 72].) We say that α has an *unbounded type* if $\limsup_{n\to\infty} a_n = \infty$. The following remarkable theorem was proved in [63].

Theorem 5.6. *Let a vector field $X \in \mathfrak{X}^r(T^2)$, $r \geq 1$, have a nontrivially recurrent semitrajectory l passing through a point p and finitely many equilibria. If the rotation number $rot\, X = \alpha$ of the*

[7] Recall that a flow is called a Kupka–Smale flow if it has no separatrix connections and all of its fixed points are hyperbolic.

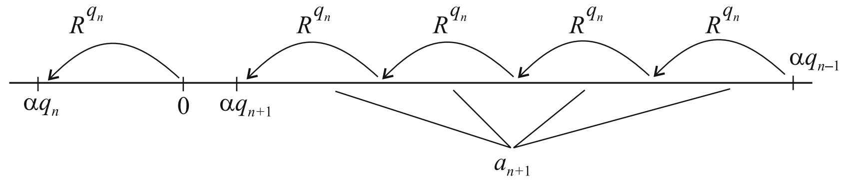


Fig. 13.

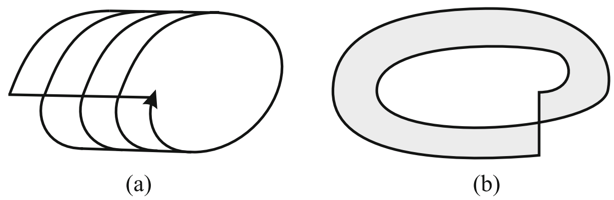


Fig. 14. The annulus A (a) and a figure eight-shaped flow box (b).

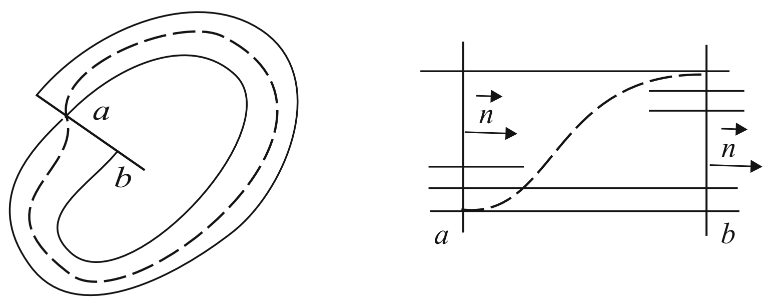


Fig. 15.

vector field X has unbounded type, then, for each neighborhood $U(X) \subset \mathfrak{X}^r(T^2)$ of X in the space $\mathfrak{X}^r(T^2)$, there exists a $Y \in U(X)$ such that there exists a periodic trajectory of the vector field Y passing through p.

Let us outline the idea of the proof. Let p_n/q_n be the convergents. It is well known that $p_{n+1} = a_{n+1}p_n + p_{n-1}$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$. Since X has a nontrivially recurrent semitrajectory, it follows that X has a simple closed transversal T meeting all nontrivially recurrent semitrajectories. Then, on some open domain, X induces the first return map $T \to T$ with rotation number α. This first return map is semiconjugate to the rotation $R_\alpha \overset{\text{def}}{=} R$ of the unit circle S^1 by α. Figure 13 illustrates the position of the first q_{n+1} points of the orbit $R^n(0) = \alpha n \pmod 1$ near the point 0. One can show (with the help of Fig. 13) that there exists an interval I_n whose first q_{n+1} iterations under R are adjacent to each other but their interiors are disjoint.

Since the mutual arrangement of points is preserved under semiconjugation, it follows that there exists an annulus A bounded by two segments of a transversal T and two arcs of a nontrivially recurrent semitrajectory l; see Fig. 14. The annulus A is cut by arcs of the nontrivially recurrent semitrajectory l into a_{n+1} so-called figure eight-shaped flow boxes.

Since $\limsup_{n\to\infty} a_n = \infty$, it follows that there exists a flow box without fixed points. Now, in that box, one can perform a local rotation of the vector field so as to ensure that the nontrivially recurrent semitrajectory passing through the corner point becomes a periodic trajectory; see Fig. 15.

It is well known [23] that irrational numbers of unbounded type form a set of full Lebesgue measure. Therefore, Theorem 5.6 implies that the classical C^r closing lemma, $r \geq 1$, holds for "most" C^r smooth vector fields with nontrivially recurrent semitrajectories on the torus.

Lloyd [83] considered the case of flows on the torus with irrational rotation numbers of bounded type and showed that the C^r closing lemma is true for all $r \geq 1$ in this case as well if all fixed points are hyperbolic and the divergence is zero at each saddle. The last condition can be replaced by the conservativeness of the flow in a neighborhood of the saddle together with the possibility of C^2 linearization. The proof is based on the following Denjoy type theorem, which is of interest in itself.

Theorem 5.7. *Let $f : S^1 \to S^1$ be an orientation-preserving mapping of degree 1 with irrational rotation number $\alpha = [a_0, \dots, a_n, \dots]$ and with C^1 smoothness outside finitely many points. Suppose that the function $\log Df$ is of bounded variation on the set $L \subset S^1$ where f has positive derivative Df. If f and f^{-1} have wandering intervals, then $\lim_{n\to\infty} a_n = \infty$.*

Theorem 5.6 was generalized in [52] as follows. Let a vector field $X \in \mathfrak{X}^r(T^2)$, $r \geq 1$, have an irrational rotation number $\alpha = [a_0, a_1, \dots, a_n, \dots]$ with $\limsup_{n\to\infty} a_n \geq k$. If X has at most $k+3$ equilibria, then there exist C^r twistings of the field X along some simple closed transversal C that are arbitrarily close to the identity mapping and result in the appearance of periodic trajectories.

The main idea of [63] was used in [12] for vector fields on a closed orientable hyperbolic surface. By that time, Aranson and Grines [11] introduced an analog of the Poincaré rotation number, namely, the homotopic rotation class. Let us describe this invariant in more detail.

Let Δ be the Poincaré model of the Lobachevskii plane; i.e., Δ is the unit disk on the complex z-plane equipped with a metric of constant negative curvature. The circle $S_\infty = \partial\Delta = (|z| = 1)$ is referred to as the *absolute*. It is known that, for any orientable closed surface M of genus ≥ 2, there exists an isometry group Γ of the plane Δ such that $\Delta/\Gamma \cong M$. By $\pi : \Delta \to \Delta/\Gamma \cong M$ we denote the natural projection, which is a universal covering. Being equipped with the metric induced by the mapping π, M is an orientable hyperbolic surface.

Let $l = \{m(t) \in M : t \geq 0\}$ be a semi-infinite continuous curve without self-intersections on M, and let $\overline{l} = \{\overline{m}(t) \in \Delta : t \geq 0\}$ be a lift of l to Δ. Suppose that $\overline{l}$ tends to a single point $\sigma \in S_\infty$ in the Euclidean metrics on the closed disk $\Delta \cup S_\infty$ as $t \to +\infty$. In this case, we say that the *curve $\overline{l}$ has an asymptotic direction.*

Aranson and Grines [11] proved that if l is a nontrivially recurrent semitrajectory on M, then its arbitrary lift $\overline{l}$ has an irrational asymptotic direction. In [11], the set $\mu(l)$ of asymptotic directions of all possible lifts of the semitrajectory l was called the *homotopic rotation class* of the semitrajectory l. The notion of a chain (or continued) fraction of the homotopic rotation class $\mu(l)$ and the notion of a chain fraction of nonconstant type were introduced in [12]. The following assertion is the main result of [12].

Theorem 5.8. *Suppose that a vector field $X \in \mathfrak{X}^r(M^2)$, $r \geq 1$, defined on a closed orientable hyperbolic surface M^2 has a nontrivially recurrent semitrajectory l passing through a point p and finitely many equilibria. If the homotopy rotation class $\mu(l)$ has a nonconstant type, then, for any neighborhood $U(X) \subset \mathfrak{X}^r(M^2)$ of the field X in the space $\mathfrak{X}^r(M^2)$, there exists a $Y \in U(X)$ such that the vector field Y has a periodic trajectory passing through p.*

In [31], the condition on $\mu(l)$ was stated in the form of a Koebe–Morse coding of geodesics [79, 95], and the following assertion was proved.

Theorem 5.9. *Let a vector field $X \in \mathfrak{X}^r(M^2)$, $r \geq 1$, defined on a closed orientable hyperbolic surface M^2 have a nontrivially recurrent semitrajectory l passing through a point p and finitely many equilibria. Suppose that the Koebe–Morse code of the coasymptotic geodesic $g(l)$ admits c-expansions of unbounded type. Then for each neighborhood $U(X)$ of the vector field X in the space $\mathfrak{X}^r(M^2)$, there exists a $Y \in U(X)$ such that the vector field Y has a periodic trajectory passing through p.*

The notion of an expansion for a point of the absolute was introduced in [38], and a similar theorem (including hyperbolic surfaces with boundary) was proved.

Gutierrez [65] suggested his own development of his Theorem 5.6 for a compact orientable hyperbolic surface. Let $E : [a, b) \to [a, b)$ be a rearrangement of intervals defined on the half-open interval $[a, b) \subset \mathbb{R}$. An interval $[s, t] \subset [a, b)$ is called a *virtual orthogonal edge* of the rearrangement E if the restriction of E to $[s, t]$ is continuous and $s < E(s) < E^2(s) = t$. For a given $k \in \mathbb{N}$, by $\mathfrak{B}_k$ we denote the set of rearrangements $E : [a, b) \to [a, b)$ such that, for some sequence of points $b_n \to a$, $a < b_n$, and for each $n \in \mathbb{N}$, the rearrangement $E_n : [a, b_n) \to [a, b_n)$ induced by the rearrangement E has at least $\chi(M^2) + k + 3$ pairwise disjoint virtual orthogonal edges, where $\chi(M^2)$ is the Euler characteristic of the surface M^2. Set $\mathfrak{B} = \bigcap_{k \geq 1} \mathfrak{B}_k$. The following theorem was proved in [65].

Theorem 5.10. *Let a vector field $X \in \mathfrak{X}^r(M^2)$, $r \geq 1$, defined on a compact orientable surface with negative Euler characteristic $\chi(M^2)$ have a nontrivially recurrent semitrajectory l passing through a point p and finitely many equilibria. Let the first return map induced by the field X on a contact-free segment Σ passing through p be semiconjugate to a rearrangement of intervals from the set $\mathfrak{B}$. Then, for any neighborhood $U(X) \subset \mathfrak{X}^r(M^2)$ of the field X in the space $\mathfrak{X}^r(M^2)$, there exists a $Y \in U(X)$ such that the vector field Y has a periodic trajectory passing through p.*

Note that the fact that a rearrangement belongs to the set $\mathfrak{B}$ is independent of the choice of Σ and is exclusively a characteristics of asymptotic properties of the semitrajectory l.

The idea of the proof of Theorems 5.8, 5.9, and 5.10 is the same: for a given neighborhood $U(X) \subset \mathfrak{X}^r(M^2)$, one constructs a figure eight-shaped equilibrium-free flow box arbitrarily close to the point p and such that a local rotation of the vector field X inside the box does not lead outside $U(X)$ and results in the appearance of a periodic trajectory.

Note the paper [66], where many results for flows on surfaces were stated in terms of generalized rearrangements of intervals on a circle or a segment.

Closure of Prolongationally Recurrent Points

Following [124], we say that a point p is *prolongationally recurrent of the first-order in the sense of Pugh* if $\omega(p) \cap \alpha(p) \neq \varnothing$. A point p is said to be *prolongationally recurrent of the nth order in the sense of Pugh* if there exist points $p_0 = p$, $p_1, \ldots, p_n = p$ such that

$$\alpha(p_{i+1}) \cap \omega(p_i) \neq \varnothing, \quad 0 \leq i \leq n-1. \tag{7}$$

Theorem 2.3 readily implies the following assertion (Theorem B in [156]).

Theorem 5.11. *Suppose that $f : M \to M$ is a diffeomorphism of a closed manifold M and a point p is a prolongationally recurrent point of the nth order in the sense of Pugh*; *i.e., there exist points $p_0 = p$, $p_1, \ldots, p_n = p$ such that relation* (7) *holds. Suppose that each intersection $\alpha(p_{i+1}) \cap \omega(p_i)$ contains a nonperiodic point. Then, for each neighborhood $U(f)$ of the diffeomorphism f in the space $Diff^1(M)$, there exists a diffeomorphism $g \in U(f)$ such that p is a periodic point of g.*

In the general case, for $r = 1$, the closing lemma for a prolongationally recurrent point in the sense of Pugh remains an open problem.

Let us present Pugh's example [124] showing that this lemma is not true in general for $r \geq 2$. Since the prolongational recurrence in the sense of Pugh implies the prolongational recurrence in the sense of Auslander, it follows that the same example implies that the C^2 Auslander closing lemma is not true in the general case as well. The main idea is to block a prolongationally recurrent trajectory by domains that are the attraction or repelling domains of trivial sinks or sources, respectively. Then the prolongationally recurrent trajectory of the perturbed system gets into one of these domains and hence cannot be "closed."

On a torus T^2, consider a flow without rest points and with exactly one periodic trajectory l_0 onto which the remaining trajectories wind both in the positive and negative directions. The trajectory l_0 is a structurally unstable double-asymptotic trajectory. Take a strip bounded by two trajectories and place a plug with eight rest points (two sinks, two sources, and four saddles) into it; see Fig. 16 (a). The resulting flow is denoted by f^t. One can assume that f^t is C^r smooth, where $r \geq 2$. The properties of f^t will be refined below.

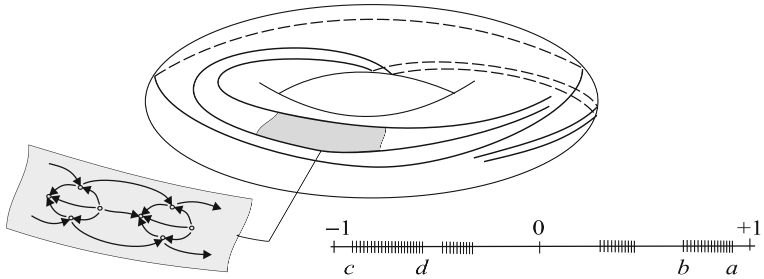


Fig. 16. The plug and the arrangement of the points a, b, c, and d on Σ.

Through an arbitrary point x_0 of the trajectory l_0, we draw a contact-free segment Σ with a parametrization $y : \Sigma \to [-1; +1]$ such that $y(x_0) = 0$. The flow f^t induces the first return map ψ on Σ. To be definite, we assume that ψ topologically stretches from x_0 for $y \geq 0$ and contracts to x_0 for $y \leq 0$. In other words, the germ of the mapping ψ at $y = 0$ has the form $y \mapsto y - y^2$. The following assertion is a key technical point. For arbitrary points $0 \leq \psi(a) \leq b \leq a \leq 1$, there exists a limit

$$\varrho(a, b) = \lim_{n\to\infty} \varrho_n(a, b), \quad \text{where} \quad \varrho_n(a, b) = \frac{\psi^n(a) - \psi^n(b)}{\psi^n(a) - \psi^{n+1}(a)},$$

which continuously depends on the points a and b. Moreover, if two first return maps ψ and ψ' are C^2 close, then the corresponding expressions ϱ_n and ϱ'_n are close as well for certain n (note that this is not the case if ψ and ψ' are only C^1 close).

By l_a and l_b we denote the α-separatrices that wind on l_0 in the positive direction and enter the boundary of the repelling domain W^u of the extreme source. The points a and b are the first points of intersection of l_a and l_b with Σ. Let us refine the flow f^t by requiring that $\varrho(a, b) = 1/2$. Therefore, the length of the interval between two successive intersections of W^u and Σ is approximately equal to the length of the previous intersection of W^u and Σ.

By c and d we denote the first points of intersection of the ω-separatrices l_c and l_d with Σ, which wind on l_0 in the negative direction, where $0 \geq \psi^{-1}(c) \geq d \geq c \geq -1$; see Fig. 16 (b). The separatrices l_c and l_d enter the boundary of the attraction domain W^s of the extreme sink that belongs to the "plug." Since ψ^{-1} is attracting on the interval $[-1; 0]$, it follows that the limit

$$\varrho(d, c) = \lim_{n\to\infty} \varrho_n(d, c) = \lim_{n\to\infty} \frac{\psi^{-n}(d) - \psi^{-n}(c)}{\psi^{-n}(d) - \psi^{-n+1}(c)}$$

exists and has properties similar to those of $\varrho(a, b)$. Let us refine the flow f^t by requiring that $\varrho(d, c) = 3/4$. Therefore, the length of the interval between two successive intersections of W^s with Σ is "much more" than the length of the previous intersection of W^s and Σ.

Each point on the separatrix l_a, say $A_0 \in l_a$, is prolongationally recurrent in the sense of Pugh. Consider a C^2-perturbation of the original flow and show that there cannot appear a periodic trajectory passing through the point A_0. Indeed, obviously, such a trajectory cannot occur if there exists a periodic trajectory homotopic to l_0 near l_0 for the perturbed flow. Therefore, it suffices to consider a perturbation for which l_0 disappears. But then, by virtue of the approximate relations $\varrho'(a', b') \approx 1/2$ and $\varrho'(d', c') \approx 3/4$, the trajectory of the perturbed flow passing through A_0 should get into the attraction domain of one of the sinks.

Let us proceed to vector fields for which the Auslander C^r closing lemma can be proved for $r \geq 2$. Recall the definition of an ε-chain of vector fields of length n. Given a vector field $X \in \mathfrak{X}^r(M)$ generating a flow f^t on the manifold M, we say that the trajectory arc has time length T if one endpoint of the arc passes into the other endpoint by the time T shift along the trajectory. Let $\varepsilon = \varepsilon(x)$ and $T = T(x)$ be positive continuous functions defined on M. If M is compact, then ε and T are assumed to be constant positive quantities. An (ε, T)-*chain* from a point $p \in M$ to a point $q \in M$ is defined as a sequence of arcs $[x_1, y_1]$, ..., $[x_n, y_n]$ such that $[x_i, y_i]$ has time length $\geq T(x_i)$ and $d(x_i, y_{i+1}) < \varepsilon(y_i)$ for all $1 \leq i \leq n$, where $p = x_1$ and $q = x_{n+1}$.

We say that an (ε, T)-chain is *δ-shadowed* by a trajectory arc if the chain lies in the δ-neighborhood of the arc and vice versa. Following [114], we say that an equilibrium *s satisfies the shadowing property* if s has a neighborhood $U(s)$ such that every $(\varepsilon, 1)$-chain in $U(s)$ is δ-shadowed by a trajectory arc, and, in addition, $\delta \to 0$ uniformly as $\varepsilon \to 0$. By using the Grobman–Hartman theorem, one can readily show that hyperbolic equilibria satisfy the shadowing property.

Consider a vector field X on the plane $\mathbb{R}^2$. Let s be an equilibrium of the field X, and let $J(s)$ be its Jacobian at s. An isolated equilibrium s is said to be *semihyperbolic* if $J(s) = 0$ and the trace of the Jacobian is nonzero, $\sigma(s) \neq 0$. The topological structure of such an equilibrium can be described. One can assume that s is the origin $(0, 0)$. Then, by using a nonsingular linear change of coordinates in a neighborhood of s, one can reduce the field X to the form $\dot{x} = P_2(x, y)$, $\dot{y} = y + Q_2(x, y)$, where $P_2(x, y)$ and $Q_2(x, y)$ are functions such that their expansions at the origin start from terms of order ≥ 2. Then it is well known (see Theorem 65 in [1]) that s is locally topologically equivalent to either a saddle–node, or a hyperbolic saddle, or a hyperbolic node.

The following Auslander C^r closing lemma for vector fields on the plane was proved in [114].

Theorem 5.12. *Let p be a prolongationally recurrent point in the sense of Auslender of a vector field $X \in \mathfrak{X}^r(\mathbb{R}^2)$, $r \geq 1$. Suppose that each equilibrium of X either is semihyperbolic or satisfies the shadowing condition. Then, for each neighborhood $U(X)$ of the field X in $\mathfrak{X}^r(\mathbb{R}^2)$, there exists a field $Y \in U(X)$ such that Y has a periodic trajectory passing through p.*

Note that if a vector field on the plane does not have an equilibrium, then it also does not have nonwandering and prolongationally recurrent points. The main construction in the proof of Theorem 5.12 is that of a chain of flow boxes and trajectory arcs that connect the centers of these boxes. Then one constructs the desired perturbation (a functional rotation of the field) in the flow boxes.

In [115], the condition on the recurrence type was weakened, but the condition on the type of the equilibria was strengthened. More precisely, the following assertion was proved in [115].

Theorem 5.13. *Let p be a chain-recurrent point of a vector field $X \in \mathfrak{X}^r(\mathbb{R}^2)$, $r \geq 1$, and let X have only hyperbolic equilibria. Then, for each neighborhood $U(X)$ of the field X in $\mathfrak{X}^r(\mathbb{R}^2)$, there exists a field $Y \in U(X)$ such that Y has a periodic trajectory passing through p.*

As was mentioned in [115], the requirement of hyperbolicity of all equilibria in Theorem 5.13 can be weakened to the following condition: each equilibrium should have a neighborhood in which every ε-orbit is δ-shadowed by a trajectory arc, and in addition, $\delta \to 0$ uniformly as $\varepsilon \to 0$. An ε-orbit is defined as a C^1 smooth oriented curve at each of whose points the tangent vector is ε-close to the vector of the field X at the same point.

6. SPECIAL CASES

Consider closing lemmas for special dynamical systems and foliations whose orbits or leaves are subjected to some constraints (for example, to conditions on the recurrence type).

Almost Wandering Points

A nonwandering point $p \in NW(f)$ is said to be *almost wandering* if there exists a neighborhood $U(p)$ of that point and a positive integer N such that the orbit $O(x) = \{f^n(x) : n \in \mathbb{Z}\}$ of an arbitrary point $x \in M$ has at most N intersections with $U(p)$. Let us present an example of a diffeomorphism with such a point. Figure 17 presents the stable and unstable manifolds of saddle fixed points of a diffeomorphism of a two-dimensional sphere. In addition to two saddle points, there are two sources and two sinks. By virtue of the presence of a heteroclinic orbit, the point p is nonwandering. On the other hand, p is almost wandering, because there exists a neighborhood $U(p)$ that has at most two points of intersection with an arbitrary orbit. The above-represented example and the following Theorems 6.1 and 6.2 are due to Pugh [125].

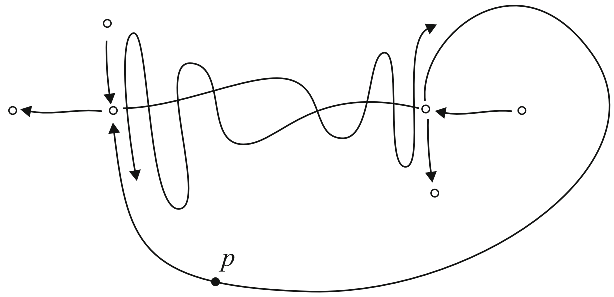


Fig. 17.

Theorem 6.1. *If* $p \in NW(f)$ *is an almost wandering point of a* C^r *diffeomorphism* $f : M \to M$, $r \geq 1$, *then, for any neighborhood* U *of the diffeomorphism* f *in the space* $\mathit{Diff}^{\,r} M$, *there exists a* $g \in U$ *with a periodic point* p.

The proof is based on the fact that, by virtue of the almost wandering property, for p there exists a neighborhood U_0 and a sequence $p_k \to p$ of points that, under positive iterations, first leave U_0 and then get in U_0 so as to ensure that $f^{n_k}(p_k) \to p$, $f^n(p_k) \notin U_0$, $0 < n < n_k$. Obviously, there exist C^r diffeomorphisms $g_k, h_k : M \to M$ such that $g_k = h_k = id$ outside U_0, $g_k(p) = p_k$, and $h_k(f^{n_k}(p_k)) = p$. For any neighborhood U, there exists a k such that $h_k \circ f \circ g_k \in U$. One can readily see that $g = h_k \circ f \circ g_k$ is the desired diffeomorphism.

The following theorem deals with the two-dimensional torus T^2. The tangent space at each point is canonically identified with the plane $\mathbb{R}^2$ since the torus is a flat homogeneous Riemannian manifold with a transitive symmetry group. A diffeomorphism $f : T^2 \to T^2$ is said to be *nonoverturning* if, for each nonzero vector $\vec{v} \in T_x(T^2) \cong \mathbb{R}^2$, there exists a half-plane $H_v \subset \mathbb{R}^2$ such that (1) $df^n(\vec{v}\,) \in H_v$ for all $n \geq 0$ and for all $x \in T^2$; (2) $\angle(\vec{v}, \partial H_v) \geq \beta > 0$, where β is independent of $\vec{v}$. An example of a nonoverturning diffeomorphism is given by a rigid displacement of the torus.

Theorem 6.2. *Let* $f : T^2 \to T^2$ *be a nonoverturning* C^r *diffeomorphism*, $r \geq 1$, *and let* p *be a nontrivially recurrent point of* f. *Then, for any neighborhood* U *of the diffeomorphism* f *in the space* $\mathit{Diff}^{\,r} M$, *there exists a* $g \in U$ *with a periodic point* p.

In [125] the notion of a nonoverturning diffeomorphism was generalized to an arbitrary manifold, and an assertion similar to Theorem 6.2 was proved.

Note Hirsch's paper [75] in which so-called cooperative and competing vector fields in the space $\mathbb{R}^3$ were considered. Recall that a vector field $\vec{F}$ is said to be *cooperative* (respectively, *competing*) if all off-diagonal elements of the Jacobian $D\vec{F}(\vec{x})$ are nonnegative (respectively, nonpositive) at each point $\vec{x}$. A vector field $\vec{F}$ is said to be *irreducible* if all matrices $D\vec{F}(\vec{x})$ are irreducible, where $\vec{x}$ belongs to the domain $X \subset \mathbb{R}^3$. A domain X is said to be *p-convex* if, together with two arbitrary points $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ such that $u_i \leq v_i$, the entire segment connecting $\vec{u}$ and $\vec{v}$ lies in X. For a cooperative irreducible system whose solutions are defined for all $t \geq 0$ in some p-convex open domain X, the improved C^r closing lemma was proved in [75] for all $r \geq 1$. One basic step of the proof is to show that a nonwandering point $p \in X$ is strictly nonwandering (i.e., that one can construct a transversal surface S through p such that the successive intersections of the positive semitrajectory passing through p form a sequence of points converging to p). Next, an argument resembling the proof of Theorem 6.1 is used. The improved C^r closing lemma for a competing system satisfying some constraints was also proved in [75].

Mappings of the Annulus

Recall that, for orientation-preserving diffeomorphisms, the closing lemma holds even in the analytic class, because the "closing" is carried out with the use of a rigid displacement. The following generalization of this result was obtained in [49].

Theorem 6.3. *Let $f : A \to A$ be a twisting diffeomorphism of the annulus $A = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$. If f is a nonwandering point, then for each $\varepsilon > 0$, there exists an $\alpha \in (-\varepsilon, +\varepsilon)$ such that the mapping*

$$(x, y) \mapsto f(x, y) + (0, \alpha)$$

has a periodic orbit.

A similar theorem was proved in [49] for a twisting diffeomorphism of the two-dimensional torus homotopic to the mapping $(x, y) \mapsto (x + y, y)$.

Foliations and Actions of Groups

By $\mathcal{FOL}^{l,k}(M)$, $1 \le k \le l$, we denote the space of C^l foliations of codimension 1 equipped with the C^k topology on a manifold M of dimension $d = \dim M$. The C^k topology on this space is introduced via the C^{k-1} topology on the set of fields of tangent $(d-1)$-planes as follows. The set of tangent $(d-1)$-planes forms a Grassmann bundle on M of a smoothness class C^{l-1}. Each foliation $\mathcal{F}$ corresponds to a field of tangent $(d-1)$-planes $P\mathcal{F}$, which is a cross-section in the Grassmann bundle. The closeness of foliations $\mathcal{F}$, $\mathcal{G} \in \mathcal{FOL}^{l,k}(M)$ implies the closeness of the fields $P\mathcal{F}$ and $P\mathcal{G}$ in the C^{k-1} Whitney topology.

Let a foliation $\mathcal{F}$ be given by an action of the group $\mathbb{R}^{d-1}$ on M. If this action is nondegenerate and orientable, then each leaf is homeomorphic to some manifold of the form $\mathbb{T}^i \times \mathbb{R}^{d-i-1}$, $0 \le i \le d-1$, where $\mathbb{T}^i$ is the i-dimensional torus. The following assertion was proved in [138].

Theorem 6.4. *Let a foliation $\mathcal{F} \in \mathcal{FOL}^{l,2}(M)$, $l \ge 2$, defined by a nondegenerate orientable action of the group $\mathbb{R}^2$ be defined on a closed 3-manifold M. If $\mathcal{F}$ has no leaf homeomorphic to $\mathbb{R}^2$, then, for any neighborhood $U(\mathcal{F})$ of the foliation $\mathcal{F}$ in $\mathcal{FOL}^{l,2}(M)$, there exists a foliation $\mathcal{G} \in U(\mathcal{F})$ defined by a nondegenerate orientable action of the group $\mathbb{R}^2$ such that all leaves of $\mathcal{G}$ are compact.* (*They are two-dimensional tori.*)

If we suppose that, under the assumptions of Theorem 6.4, $\mathcal{F}$ has no compact leaves, then each leaf is homeomorphic to a cylinder and is everywhere dense on M (i.e., $\mathcal{F}$ is a minimal foliation). As was shown in [138], in this case, any leaf can be "closed" into a compact 2-torus.

An invariant of the topological equivalence similar to the Poincaré rotation number was introduced in [137] for a foliation of codimension 1 with trivial holonomy group on $\mathbb{T}^3$ (see also [136]). The invariant, called the rotation functional, is a pair (λ, μ) of numbers specifying the asymptotic behavior of the leaves. If λ and μ are rational numbers, then all leaves are compact (tori). If λ and μ are rationally independent irrational numbers, then the leaves are planes. If at least one of the numbers λ and μ is irrational and these numbers are rationally dependent, then the leaves are cylinders. One can show that if a foliation on $\mathbb{T}^3$ has no compact leaves, then it has a trivial holonomy group; consequently, at least one of the numbers in the rotation functional is irrational.

The following assertion supplementing Theorem 6.4 was proved in [30].

Theorem 6.5. *Let a foliation $\mathcal{F} \in \mathcal{FOL}^{l,r}(\mathbb{T}^3)$, $1 \le k \le l \le \infty$, without compact leaves be defined on $\mathbb{T}^3$. Suppose that one of numbers of the rotation functional of the foliation $\mathcal{F}$ satisfies the Diophantine condition with some positive exponent. Then for any neighborhood $U(\mathcal{F})$ of the foliation $\mathcal{F}$ in $\mathcal{FOL}^{l,r}(\mathbb{T}^3)$, there exists a foliation $\mathcal{G} \in U(\mathcal{F})$ such that all leaves of $\mathcal{G}$ are compact* (*2-tori*).

Recall that a C^r-action of the group $\mathbb{Z}^k$ on S^1 is defined as a homeomorphism $\varrho : \mathbb{Z}^k \to \mathit{Diff}^{\,\mathrm{r}}(\mathrm{S}^1)$ such that the mapping $(\gamma, x) \to \varrho(\gamma)(x)$, $x \in S^1$, is C^r-smooth for any $\gamma \in \mathbb{Z}^k$. The space $G^r(\mathbb{Z}^k, S^1)$, $0 \le r \le \infty$, of such actions is equipped with the natural C^r-topology and admits a complete metric for finite r. The following strengthened closing lemma was proved in [30].

Theorem 6.6. *If the rotation number of one of the mappings $\varrho(\gamma)$ of an action $\varrho \in G^\infty(\mathbb{Z}^k, S^1)$ is irrational and satisfies the Diophantine condition with some positive exponent, then, for any $\epsilon > 0$ and any finite number $r \in \mathbb{N}$, there exists a $\varrho_c \in G^\infty(\mathbb{Z}^k, S^1)$ ϵ-close to ϱ in the space $G^r(\mathbb{Z}^k, S^1)$ and such that all orbits ϱ_c are compact.*

Ergodic Closing Lemma

This lemma was introduced by R. Mané [88] in the investigation of necessary conditions for the structural stability of dynamical systems. Mané considered a diffeomorphism in some neighborhood of which any diffeomorphism has only hyperbolic periodic points. The problem was to prove that all nonwandering points of the original diffeomorphism are hyperbolic. The assumption of the contradiction implies the existence of a nonwandering, nonperiodic, and nonhyperbolic point. It is natural to try to perturb a diffeomorphism so as to obtain a nonhyperbolic periodic point. The desired small C^1 perturbation could be provided by the improved C^1 closing lemma, but, unfortunately, this lemma provides no information on the closeness of the periodic and original orbits. Therefore, Mané suggested the following strengthening of the improved closing lemma, which is referred to as the *Mané C^r closing lemma*.

Let a diffeomorphism $f \in \mathit{Diff}^{\,r} M$ have a nonwandering nonperiodic point $x_0 \in M$. Then, for any ε-neighborhood $U_\varepsilon(x_0)$ of the point x_0 and for any neighborhood U of the diffeomorphism f in the space $\mathit{Diff}^{\,r} M$, there exists a $g \in U$ such that g has a periodic point y in $U_\varepsilon(x_0)$ and the inequalities $d(f^i(x_0), g^i(y)) < \varepsilon$ hold for all $0 \leq i \leq m$, where m is the minimum period of the point y.

One can see that the Mané C^r closing lemma is not related to notions of ergodic theory. However, the corresponding result obtained by Mané in the investigation of this problem was based on an invariant measure; therefore, Mané called it the ergodic closing lemma. Recall that a subset $N \subset M$ (M is considered to be compact) is referred to as a *set of full measure* (with respect to μ) if $\mu(N) = 1$ for the normalized measure μ. Note that since each f-invariant measure "lives" on a nonwandering set, one can assume that a set of full measure consists of nonwandering points (but does not necessarily coincide with the nonwandering set). Therefore, it is natural to consider the problem how "large," from the viewpoint of an invariant measure, is the set of nonwandering points for which the Mané C^r closing lemma holds.

For a fixed $\varepsilon > 0$ and a point $x \in M$, by $O_\varepsilon(x)$ we denote the ε-neighborhood of the orbit of x. By $\Sigma_r(f)$ we denote the set of points $x \in M$ for which the above-represented Mané C^r closing lemma holds under the additional assumption that the equality $f = g$ holds on $M - O_\varepsilon(x)$ for all $\varepsilon > 0$. The following assertion was proved in [88].

Theorem 6.7. *The set $\Sigma_1(f)$ is a set of full measure.*

Obviously, Theorem 6.7 can be restated as follows: $\mu\,(M - \Sigma_1(f)) = 0$, where μ is an arbitrary f-invariant measure. Theorem 6.7 was generalized in [32] to a noncompact manifold M and was stated as $\mu\,(\mathit{Rec}(f) - \Sigma_1(f)) = 0$, where $\mathit{Rec}(f)$ stands for the set of recurrent points.

7. APPLICATION TO DESCRIPTION OF GENERIC SYSTEMS

Closing lemmas were mainly used in two directions: for the investigation of structural stability (e.g., see [5, 6, 17, 58, 70, 88, 90, 112, 120]) and for proving the largeness of some class of dynamical systems in the space of dynamical systems. In this section, we focus on the latter aspect and restrict the exposition to dynamical systems on compact manifolds.

The following so-called *general denseness problem* obtained by Pugh in [122] was the first important result that substantially used the improved C^1 closing lemma.

Theorem 7.1. *Let M be a compact manifold. Then diffeomorphisms for which the nonwandering set coincides with the topological closure of the set of hyperbolic periodic points form a residual set in the space $\mathit{Diff}^{\,1}(M)$.*

The method of proof became the main method for such assertions. By $\mathcal{C}_M$ we denote the family of compact subsets of the manifold M. Consider a mapping $\Gamma : \mathit{Diff}^{\,1}(M) \to \mathcal{C}_M$ that takes each diffeomorphism $f \in \mathit{Diff}^{\,1}(M)$ to the set $\mathit{clos}\,\mathit{Per}_h(f)$, where $\mathit{Per}_h(f)$ is the set of hyperbolic periodic points of the diffeomorphism f. By virtue of the stability of a hyperbolic periodic point under a perturbation, the mapping Γ is lower semicontinuous; i.e.,

$$clos\,Per_h(f) \subset \liminf_{k\to\infty} clos\,Per_h(f_k)$$

for any sequence $f_k \to f$.[8] The Hausdorff metric equips $\mathcal{C}_M$ with the structure of a topological space. Therefore, one can speak of points of continuity of the mapping Γ. It is a well-known fact from general topology that *the points of continuity of a lower-semicontinuous mapping form a residual set*. By $\mathcal{G}$ we denote the set of points of continuity of the mapping Γ. Then $\mathcal{G}$ satisfies Theorem 7.1. Indeed, should $f \in \mathcal{G}$ have a nonwandering point x_0 near which there is no periodic orbit, this would contradict the continuity of Γ at f, because, by the improved closing lemma, the diffeomorphism f can be approximated by diffeomorphisms with a hyperbolic periodic point x_0.

Note that in [122] Theorem 7.1 was stated for vector fields in a more general form with regard of the second part of the Kupka–Smale theorem implying that the stable and unstable manifolds of periodic trajectories meet transversally. We present the main statements for diffeomorphisms and note essential differences in the case of vector fields.

Since the $\omega(\alpha)$-limit set of an arbitrary point lies in a nonwandering set, we see that the following result in [36] can be treated as a supplement of Theorem 7.1: for a generic diffeomorphism in the space $Diff^{\,1}(M)$, any $\omega(\alpha)$-limit set of an arbitrary point is approximated by periodic orbits in the Hausdorff metric.

Recall that, before Theorem 7.1 arose, the set of Kupka–Smale diffeomorphisms in the space $Diff^{\,r}(M)$ had been proved to be residual for any $1 \le r \le \infty$ in [80, 143]. Therefore, by taking into account Theorem 7.1, one can single out the following residual set, which consists of diffeomorphisms f satisfying the following conditions: (1) all periodic points of f are hyperbolic; (2) the invariant manifolds of periodic points meet transversally; (3) the periodic points are dense in the nonwandering set $NW(f)$. In connection with the proof of the connecting lemma and its generalization (see Theorem 2.6, which implies that any chain-recurrent point can be brought into a periodic point by an arbitrarily small C^1 perturbation), the last condition can be strengthened. In complete analogy with the proof of Theorem 7.1, one can show that, for a generic diffeomorphism f, any chain-recurrent point can be approximated by periodic points. Summarizing the preceding, we find that there exists a residual set $KSP(M) \subset Diff^{\,1}(M)$[9] of diffeomorphisms $f \in KSP(M)$ with the following properties:

- all periodic points of f are hyperbolic;
- the invariant manifolds of periodic points meet transversally;
- the periodic points are dense in the chain-recurrent set $\mathfrak{R}(f)$ of f; i.e.,

$$clos(Per(f)) = L(f) = NW(f) = \mathfrak{R}(f), \tag{8}$$

where $L(f) = clos\bigcup_{z\in M}\omega(\alpha)(z)$ is the closure of the union of all $\omega(\alpha)$-limit sets of f.

Condition (8) has been essentially refined for the space $Diff^{\,1}(S^1)$ of diffeomorphisms of the circle S^1 and for the space $\chi^1(M^2)$ of vector fields on an orientable compact surface M^2: a chain recurrent set consists of finitely many periodic orbits. Therefore, Morse–Smale systems are generic in $Diff^{\,1}(S^1)$ and $\chi^1(M^2)$. (Moreover, they form an open dense set.) It is known that Morse–Smale systems are structurally stable [105, 111], and in the spaces $Diff^{\,1}(S^1)$ and $\chi^1(M^2)$, they can be classified with the use of clear finite invariants [17, 112]. However, starting from the spaces $Diff^{\,1}(M^n)$ for $n \ge 2$ and $\chi^1(M^n)$ for $n \ge 3$, the situation becomes much more complicated. In fact, these spaces do not contain residual sets with a similar clear dynamics admitting classification with the use of constructive and finite invariants. By way of illustration, we present three results.

Let T^n be the n-dimensional torus, and let $T^2\sharp T^2$ be the connected sum of two tori (that is, an orientable closed surface of genus 2). On T^2, there exist DA-diffeomorphisms with a one-dimensional expanding attractor and a contracting repeller [146]. By using these diffeomorphisms, Robinson

[8] The lower limit $\liminf A_k$ of a sequence of sets $A_k \in \mathcal{C}_M$ is defined as the family of points $x \in M$ such that $x = \lim_{k\to\infty} x_k$ for some $x_k \in A_k$.

[9] The abbreviation KSP is formed by the first letters of the names Kupka, Smale, and Pugh.

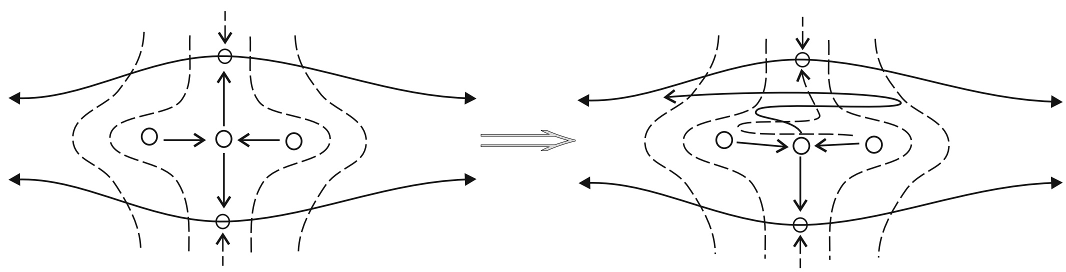


Fig. 18. A modified DA-diffeomorphism (a) and its perturbation (b).

and Williams [135] constructed a diffeomorphism $f_0 \in \mathit{Diff}^{\,r}(T^2\sharp T^2)$, $r \geq 1$, whose nonwandering set consists of exactly one expanding attractor and contracting repeller with the following property: for any finite-parameter family $\mathcal{S} \subset \mathit{Diff}^{\,r}(T^2\sharp T^2)$ of diffeomorphisms that contains f_0, there exists a diffeomorphism that is arbitrarily close to f_0 in the space $\mathit{Diff}^{\,r}(T^2\sharp T^2)$ and has the same nonwandering set but is not conjugate to any diffeomorphism in the family $\mathcal{S}$. The idea of the example is the presence of irremovable tangencies of stable and unstable manifolds of points from the expanding attractor and the contracting repeller, respectively. The second result was obtained by Simon [142], who showed that, in any residual set in the space $\mathit{Diff}^{\,r}(T^3)$, $r \geq 1$, there is an uncountable family of conjugacy (and even Ω-conjugacy) classes. The idea of all constructions is well illustrated by the following "least-dimensional" Williams example [157]; see Fig. 18. On T^2, take a modified DA-diffeomorphism with a one-dimensional expanding attractor Ω, which differs from the classical DA-diffeomorphism in the sense that there are two sources and a saddle s instead of one source in the domain $T^2 - \Omega$; see Fig. 18 (a). Both unstable separatrices of the saddle s coincide with the separatrices of two saddles, say s_1 and s_2, that belong to Ω. A standard perturbation (in addition to the original paper [157], such a perturbation was described in the monograph [134], see Theorem 5.1) of the diffeomorphism results in the generation of heteroclinic transverse intersections of separatrices of the saddle s with the separatrices of the saddles s_1 and s_2, see Fig. 18 (b). One can readily see that the separatrix of the saddle s has heteroclinic tangencies with the stable manifolds of points of the set Ω.

Let us represent an in a sense positive result due to Mané [88], which follows from his ergodic closing lemma (Theorem 6.7).

Theorem 7.2. *In the space* $\mathit{Diff}^{\,1}(M^2)$, *where* M^2 *is a compact two-dimensional manifold*, *there exists a residual set* $\mathfrak{G}$ *such that one of the following possibilities holds for* $f \in \mathfrak{G}$.

1. f *has either infinitely many sinks or infinitely many sources.*
2. f *is* Ω*-stable.*

This theorem cannot be generalized to manifolds of dimension ≥ 3, because there are robustly transitive diffeomorphisms that do not have hyperbolic structure on the entire manifold (robust transitivity implies that all diffeomorphisms in some neighborhood of a given diffeomorphism are transitive). Consequently, these diffeomorphisms are not Ω-stable because, by [106], an Ω-stable diffeomorphism necessarily has a hyperbolic nonwandering set.

The first example of a robustly transitive diffeomorphism that does not admit hyperbolic structure on the entire manifold, was constructed by Shub [139] on the four-dimensional torus $T^4 = T^2 \times T^2$. Somewhat exaggerating, one can say that the Shub example is a skew product of an Anosov diffeomorphism $T^2 \times \{\cdot\} \to T^2 \times \{\cdot\}$ by a DA-diffeomorphism $\{\cdot\} \times T^2 \to \{\cdot\} \times T^2$. Mané [87] proved the existence of an open domain $U \subset \mathit{Diff}^{\,1}(T^3)$ such that any $f \in U$ is mixing (and hence robustly transitive) but is not an Anosov diffeomorphism. The constructions start from an Anosov diffeomorphism $f_0 : T^3 \to T^3$ with a two-dimensional expanding bundle and a one-dimensional contracting bundle. Next, an analog of the Smale surgery [146] is applied to f_0 for the construction of a DA-diffeomorphism (the Smale surgery can be represented as the double bifurcation "saddle $\to$

saddle + saddle–node → saddle + node + saddle”), but the surgery is performed not in the standard direction of the one-dimensional contracting bundle (then one would obtain a two-dimensional expanding Williams attractor [158]) but in the direction of the “weakest” one-dimensional expanding foliation. As a result, one obtained a diffeomorphism that is not an Anosov diffeomorphism but preserves the mixing property specific to f_0. There are different examples; see [42, 45].

Theorem 7.2 cannot be generalized to flows on three-dimensional manifolds. Morales [94] proved that, for any closed orientable 3-manifold M^3, in the space $\chi^1(M^3)$, there exists a domain consisting of vector fields that have neither hyperbolic attractors nor hyperbolic repellers. The main element of the proof is an example of a vector field on a ball with handles (the number of handles is arbitrary but ≥ 2) that is a generalized geometric model of the Lorenz attractor. (The classical model is realized on a ball with two handles.) The vector field has a so-called singularly hyperbolic attractor and is transversal to the boundary. By reverting the time, we obtain a vector field with a singularly hyperbolic repeller. By using the Heegaard decomposition, one can construct such a flow on any closed orientable 3-manifold M^3. Note that both the attractor and the repeller in the Morales example are partially hyperbolic. Since, as Hayashi showed [70], an Ω-stable flow has a hyperbolic nonwandering set, it follows that the flow in the Morales example is not Ω-stable.

In view of the above-performed considerations and Newhouse’s results [98, 101], it becomes clear why the hope for the existence of a residual set in the spaces $\mathit{Diff}^{\,1}(M^n)$ for $n \geq 2$ and $\chi^1(M^n)$ for $n \geq 3$ with dynamics described by a finite set of constructive invariants faded in the beginning of the 1970s. Nevertheless, in connection with the achievements in the investigation of closing lemmas, there was some progress in the understanding of what classes of dynamical systems with a sufficiently clear description can form residual sets. Obviously, the impetus to describing and finding residual sets was given by the Palis strategic paper [107] (and the following papers [108, 109]), where he stated several conjectures pertaining to everywhere dense sets in spaces of dynamical systems. (Recall that a residual set is dense in a Baire space.)

Consider the Palis program in detail. The main conjecture implies the existence of a dense set $\mathbb{D}$ in the space of dynamical systems (this space is treated as the set of vector fields, diffeomorphisms, or transformations of a closed smooth manifold equipped with the uniform C^r topology, $r \geq 1$) such that each element of $\mathbb{D}$ has only finitely many attractors, the attraction domain of these attractors has full measure, and, in addition, there exists a Sinai–Ruelle–Bowen measure on each attractor. Moreover, it is assumed that the situation is preserved under generic finite-parameter and sufficiently small bifurcations. One can see that the main conjecture in a sense has ergodic character. However, by taking into account the positive results due to Sinai, Bowen, and Bowen–Ruelle [22, 47, 48] for dynamical systems with a uniform hyperbolic structure on the nonwandering set, Palis suggested auxiliary conjectures in the direction of the proof of the main conjecture, which already have “dynamic” character. We note two of them.

Strong conjecture. Each dynamical system can be C^r approximated by a hyperbolic system or a system with a homoclinic tangency or a heterodimensional cycle.

Weak conjecture. Each diffeomorphism can be C^r approximated by a Morse–Smale diffeomorphism or a diffeomorphism with a transversal homoclinic orbit.

Recall that a diffeomorphism f has a *heterodimensional cycle*, see Fig. 19, if there exist periodic saddle points p and q with stable (and hence with unstable) manifolds of distinct dimensions,

$$W^s(p) \cap W^u(q) \neq \varnothing, \qquad W^u(p) \cap W^s(q) \neq \varnothing, \qquad \dim W^u(p) \neq \dim W^u(q).$$

Note that Palis knowingly presented no rigorous definition of the hyperbolicity of a diffeomorphism (see [108, pp. 8–9]). It means sometimes the uniform hyperbolic structure on the limit set (then, by the Anosov lemma, the limit set is the closure of periodic points), sometimes the validity of the axiom A,[10] sometimes the Ω-stability, and finally, sometimes the uniform hyperbolic structure on the chain-recurrent set. (Then, by the Anosov closing lemma, the chain-recurrent set is the closure of the set of periodic points and coincides with the limit and nonwandering sets.) However, by virtue of (8), such an ambiguity is not important for the residual set $KSP(M)$.

[10] Recall that the Smale axiom A claims that the nonwandering set is hyperbolic and the periodic orbits are dense in it.

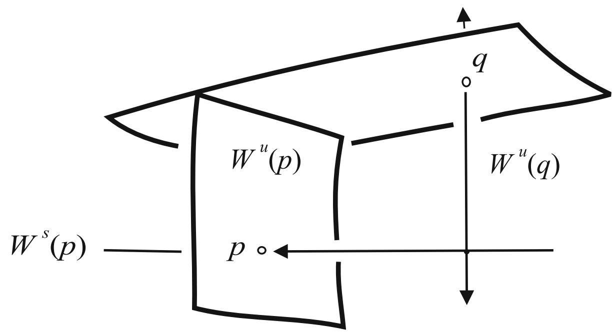


Fig. 19. A heterodimensional cycle. [The manifolds $W^s(p)$ and $W^u(q)$ meet.]

For all definitions of hyperbolicity, the Palis strong conjecture implies that the nonhyperbolicity of almost every system is always a consequence of irremovable nontransversal intersections of invariant manifolds of the points of the system. This is the case in the examples suggested by Abraham and Smale [28], Newhouse [96, 101], Williams [157], Simon [142], Robinson and Williams [135], and Diaz [55]. One can show that the strong conjecture implies the weak one.

The Palis C^1 strong conjecture for two-dimensional compact manifolds was proved in the following theorem in [129]. (Note that there are no heterodimensional cycles on two-dimensional manifolds.)

Theorem 7.3. *Let M^2 be a two-dimensional compact manifold, and let $f \in \mathit{Diff}^{\,1}(M^2)$. Then f can be C^1 approximated by A-diffeomorphisms or diffeomorphisms with homoclinic tangencies.*

First, one shows that if a diffeomorphism $f : M^2 \to M^2$ has no homoclinic tangency, then the angle between the stable and unstable manifolds of hyperbolic periodic points at their points of intersection is bounded away from zero by a positive constant. This implies the existence of a so-called dominating splitting, which is a generalization of the hyperbolic structure, on the closure of the set of periodic points. Let us give a definition (probably the definition of the dominating splitting was formally given for the first time by Mané in [90]; however, close constructions were considered even in [21, 82]) for arbitrary dimension $n = \dim M$.

Let f be a diffeomorphism of a closed n-dimensional manifold M^n equipped with some Riemannian metric. A set $\Lambda \subset M^n$ invariant under f admits *dominating splitting* if the restriction $T_\Lambda M^n$ of the tangent bundle TM^n of M^n to Λ can be represented as the Whitney sum $E \oplus F$ of Df-invariant subbundles E and F such that (1) $\dim E(x) + \dim F(x) = n$ for all $x \in \Lambda$; (2) the dimensions $\dim E(x)$ and $\dim F(x)$ are independent of $x \in \Lambda$; (3) there exist $C > 0$ and $0 < \lambda < 1$ such that

$$\frac{\|Df^k(v)\|}{\|Df^k(u)\|} \le C\lambda^k \frac{\|v\|}{\|u\|},$$

where $v \in E(x)$, $u \in F(x)$, $x \in \Lambda$, and $k \ge 1$. Geometrically, a dominating splitting means that the angle between the direction specified by the vector $Df^k(v, u)$ and F tends exponentially to zero as $k \to +\infty$, see Fig. 20. Roughly speaking, the dilation in the direction F is stronger than in the direction E, or the contraction in the direction E is stronger than in the direction F.

The key point in the proof of Theorem 7.3 is the proof of the fact that if a compact invariant set that contains neither sources no sinks but admits dominating splitting, is nonhyperbolic, then it is a finite family of normally hyperbolic circles. By analogy with considerations of the Denjoy type, it is shown that the restriction of some power of a diffeomorphism to any circle of this kind is conjugate to a rotation with an irrational rotation number. Now one can "eliminate" the nonhyperbolic part with the use of an arbitrarily small perturbation and obtain a hyperbolic diffeomorphism arbitrarily close to the original diffeomorphism f.

On a surface, an A-diffeomorphism that is not a Morse–Smale diffeomorphism necessarily has homoclinic orbits or heteroclinic tangencies. Therefore, the following result was obtained in [129]

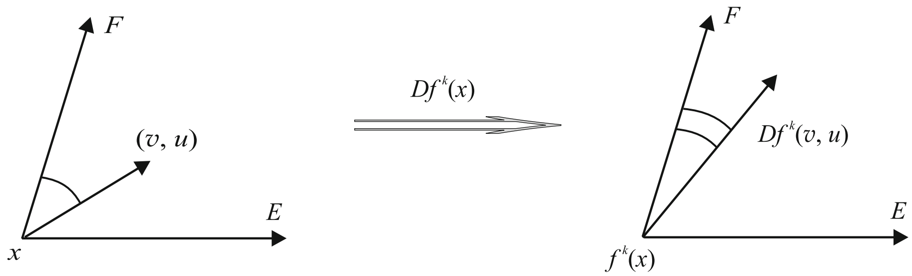


Fig. 20. Dominating splitting of $E \oplus F$.

as a corollary of Theorem 7.3. By $MS(M^2)$ we denote the set of Morse–Smale diffeomorphisms on a compact surface M^2.

Theorem 7.4. *In the set* $\mathcal{U} = \mathit{Diff}^{\,1}(M^2) - \mathit{clos}\, MS(M^2)$, *there exists an open set* $\mathcal{R}$ *dense in* $\mathcal{U}$ *and such that any diffeomorphism* $f \in \mathcal{R}$ *has a transversal homoclinic orbit.*

By taking into account the Katok result [77], we find that the closure of the interior of the set of diffeomorphisms with zero entropy is equal to the closure of the set $\mathit{clos}\, MS(M^2)$ of Morse–Smale diffeomorphisms. This fact was refined in [59].

Already manifolds of dimensions ≥ 3 can contain heterodimensional cycles, and on such manifolds, there exist diffeomorphisms that cannot be approximated by either hyperbolic diffeomorphisms or diffeomorphisms with homoclinic tangencies. For example, such diffeomorphisms include the above-mentioned robustly transitive Shub and Mané diffeomorphisms [139, 87]. In particular, the Palis strong conjecture implies that these diffeomorphisms are approximated by diffeomorphisms with heterodimensional cycles. By using the presence of a dominating splitting [153], Lan Wen obtained [154] some description of nonhyperbolic invariant sets for a C^1 generic diffeomorphism that is not hyperbolic and is not approximated by either diffeomorphisms with homoclinic tangencies or diffeomorphisms with heterodimensional cycles. By using these results and the very fine technique developed in [76] concerning the existence of a so-called central field of tangent planes, Crovisier [54] proved the Palis weak C^1 conjecture for compact n manifolds ($n \geq 3$) in the following form.

Theorem 7.5. *Every diffeomorphism of a compact manifold can be* C^1 *approximated by a Morse–Smale diffeomorphism or a diffeomorphism with a transversal homoclinic point.*

In the direction of the proof of the Palis strong conjecture, the following theorem was proved in [25] on the basis of the C^1 connecting lemma for manifolds.

Theorem 7.6. *In the space* $\mathit{Diff}^{\,1}(M)$, *where* M *is a compact manifold, there exists a residual set* $\mathfrak{R}$ *such that if* Λ *is a transitive isolated set of a diffeomorphism* $f \in \mathfrak{R}$ *in* $NW(f)$, *then one of the following possibilities is realized.*

1. Λ *is a hyperbolic set.*

2. Λ *is a nonhyperbolic set, and* f *is* C^1 *approximated by diffeomorphisms that have a heterodimensional cycle.*

In what follows, we present one more result dealing with the Palis conjecture.

A many-dimensional analog of Theorem 7.2 was obtained in [26]. Obviously, to this end one should weaken the requirement of Ω-stability, which is equivalent to axiom A (the presence of the hyperbolic structure and the denseness of periodic orbits in the nonwandering set) and the absence of cycles in the family of basic sets. Recall that the nonwandering set of an Ω-stable diffeomorphism splits into pairwise disjoint closed invariant and transitive subsets, which are said to be *basic* [146]. This representation of the nonwandering set is called the *spectral expansion.* For an arbitrary diffeomorphism, this result is used as a definition; i.e., the *spectral expansion of an* *arbitrary diffeomorphism* is understood as a representation of the nonwandering set as the union of pairwise disjoint closed invariant transitive subsets. These subsets can be introduced independently in various ways. One fruitful approach going back to Smale [146] and Newhouse [102] is the straightforward repetition of the definition of a basic set in the Smale theory as a homoclinic class. (For a detailed rigorous exposition, see [134, pp. 240–244].) Before stating subsequent results, we introduce related definitions.

Let p be a saddle periodic point of the diffeomorphism f. The *saddle homoclinic class* associated with p is defined as the set

$$H(p,f) = clos\,(W^s(p) \pitchfork W^u(p))\,,$$

where $\pitchfork$ stands for the transversal intersection. If p is a nodal periodic point (an isolated attracting or repelling periodic point), then the *homoclinic class* $H(p,f)$ is defined as its orbit $O(p)$. The homoclinic class of an arbitrary diffeomorphism is a closed invariant and transitive set. In hyperbolic theory, the homoclinic classes are basic sets. In the general case, the homoclinic classes are not isolated and may not be disjoint. However, for a generic diffeomorphism, homoclinic classes either coincide or are disjoint [26, 51]. It was proved in [26] that the presence of a spectral expansion of a generic diffeomorphism is equivalent to the finiteness of the number of homoclinic classes; in this case, there exists a dominating splitting on the nonwandering set.

A dominating splitting $NW(f) = H(p_1,f) \cup \dots \cup H(p_m,f)$ on the nonwandering set of a diffeomorphism f is said to be *R-robust* if there exists a residual set $R \subset Diff^{\,1}(M)$ containing f and such that any diffeomorphism $g \in R$ sufficiently close to f has a dominating splitting $NW(g) = H(\bar{p}_1,g) \cup \dots \cup H(\bar{p}_m,g)$ of the same type. In addition, if $g_i \to f$ $(g_i \in R)$ in the C^1 topology as $i \to \infty$, then $H(\bar{p}_j,g_i) \to H(p_j,f)$ in the Hausdorff metric on compact sets for all $j = 1,\dots,m$. The following assertion proved in [26] is the multidimensional analog of Theorem 7.2.

Theorem 7.7. *In the space* $Diff^{\,1}(M^n)$, $n \ge 3$, *there exists a residual set* $\mathfrak{G}$ *such that one of the following possibilities is realized for* $f \in \mathfrak{G}$.

1. f *has infinitely many sinks or sources.*

2. f *has finitely many sources and sinks but infinitely many homoclinic saddle classes.*

3. f *has finitely many homoclinic classes, and the nonwandering set* $NW(f)$ *admits a* $\mathfrak{G}$*-robust dominating splitting.*

In the first two cases, the nonwandering set $NW(f)$ does not admit dominating splitting. It was suggested in [44] to say that the dynamics of the diffeomorphism f is *wild*. In the last, third case, the dynamics is said to be *tame*. Note that all three cases can be realized. Examples realizing the first two cases were constructed in [43, 50]. The last case corresponds to the Ω-stability and can obviously can be realized as well.

The following assertion was proved in [26].

Theorem 7.8. *In the space* $Diff^{\,1}(M)$, *where* M *is a compact manifold, there exists a residual set* $\mathfrak{R}$ *such that if* $f \in \mathfrak{R}$ *has finitely many homoclinic classes, then exactly one of the following possibilities is realized.*

1. f *is* Ω*-stable.*

2. f *is not an A-diffeomorphism and is* C^1 *approximated by diffeomorphisms that have a heterodimensional cycle.*

Thus the Palis strong conjecture has been proved for tame diffeomorphisms in the multidimensional case. Note that the claim on the approximation by A-diffeomorphisms is replaced by the claim on the Ω-stability, and it is well known that an Ω-stable diffeomorphism is an A-diffeomorphism.

Needless to say, the connecting lemma for invariant manifolds was used for obtaining homoclinic points from almost homoclinic ones. Theorem 2.9 implies the following assertion (in [70], it was stated as Corollary 2).

Theorem 7.9. *For a* C^1 *generic diffeomorphism* $f : M \to M$ *of a closed manifold* M, *the set of transversal homoclinic points associated with a fixed hyperbolic point is everywhere dense in the set of almost homoclinic points* (*associated with the same hyperbolic point*).

In [71], this result was generalized: the everywhere denseness in the set of prolongationally homoclinic points (associated with the same hyperbolic set) was proved.

Let us briefly consider the problem on generic symplectic systems. (A fascinating survey dealing mainly with diffeomorphisms of two-dimensional manifolds can be found in [53]. For manifolds of arbitrary dimension, see [37].) Recall that $\mathit{Diff}^{\,r}_{\omega}(M^d)$ stands for the space of C^r diffeomorphisms of a compact d-manifold M^d, $d \geq 2$, preserving a symplectic or volume form ω on M^d. In addition, recall (see Section 3) that Theorem 3.1 implies the Poincaré conjecture on the denseness of periodic trajectories of a generic Hamiltonian system in the class C^1; see Theorem 3.2.

First, consider symplectic diffeomorphisms on two-dimensional compact orientable manifolds M^2. On such manifolds, the notions of symplecticity and area conservation coincide. In accordance with [148], such diffeomorphisms are said to be *conservative.* Recall that a periodic point p of a diffeomorphism $f : M^2 \to M^2$ of a two-dimensional manifold M^2 is said to be *elliptic* if the eigenvalues of $Df^s(p)$ are not real numbers,[11] where s is the period of the point p. The following assertion holds for generic conservative diffeomorphisms $M^2 \to M^2$.

Theorem 7.10. *A generic conservative diffeomorphism $f \in \mathit{Diff}^{\,1}_{\omega}(M^2)$ has the following properties.*

1. *f is transitive (i.e., there exists an orbit everywhere dense in M^2).*
2. *The saddle hyperbolic periodic points of f are everywhere dense in M^2.*
3. *If f is not conjugate to an Anosov diffeomorphism, then the elliptic periodic points are everywhere dense in M^2.*
4. *For each $\varepsilon > 0$, there exists a periodic ε-dense point.*

The genericity of properties 1 and 4 follows from Theorem 3.3 and the Poincaré recurrence theorem. The genericity of property 2 follows from [132, 162]. The genericity of property 3 is one of the main results in [100]. Now the assertions of the theorem follow from the simple remark that the intersection of finitely (and even countably) many residual sets is a residual set. Property 1 was proved in [37] for manifolds of arbitrary dimension.

The following assertion was proved in [35] for a four-dimensional symplectic manifold.

Theorem 7.11. *In the space $\mathit{Diff}^{\,1}_{\omega}(M^4)$, there exist three open sets U_1, U_2, and U_3 such that the following assertions hold.*

1. *The union $U_1 \cup U_2 \cup U_3$ is dense in $\mathit{Diff}^{\,1}_{\omega}(M^4)$.*
2. *$f \in U_1$ if and only if f is an Anosov transitive diffeomorphism.*
3. *$f \in U_2$ if and only if f is partially hyperbolic.*
4. *$f \in U_3$ if and only if f has a stable completely elliptic periodic point.*

Recall that a periodic point p of period k is said to be *completely elliptic* if the eigenvalues $Df^k(p)$ have unit absolute values and all of them are distinct.

The homoclinic points discovered by Poincaré [119] when studying the three-body problem in celestial mechanics play an important role in symplectic dynamics. Poincaré suggested that homoclinic points of a generic Hamiltonian system are dense on stable and unstable manifolds. The following assertion generalizing the corresponding Takens result [148] for two-dimensional manifolds was proved in [159] on the basis of the method of proof of Theorem 2.3.

Theorem 7.12. *Let $p \in M^d$ be a hyperbolic periodic point of a diffeomorphism $f \in \mathit{Diff}^{\,1}_{\omega}(M^d)$. Then for each point $q \in W^u(p) \cup W^s(p)$, for its arbitrary neighborhood $U(q)$, and for an arbitrary neighborhood $V(f)$ of f in $\mathit{Diff}^{\,1}_{\omega}(M^d)$, there exists a $g \in V(f)$ coinciding with f in some neighborhood of p and such that g has a homoclinic point in $U(q)$ associated with p. Moreover, there exists a residual subset $\mathfrak{B} \subset \mathit{Diff}^{\,1}_{\omega}(M^d)$ such that if $f \in \mathfrak{B}$ and p is a hyperbolic periodic point of f, then the intersection $W^u(p) \cap W^s(p)$ is everywhere dense both in $W^u(p)$ and in $W^s(p)$.*

[11] In this case, the eigenvalues are complex conjugate and have unit absolute values.

Let us return to two-dimensional closed orientable manifolds. In conclusion, we consider the application of connecting lemmas for invariant manifolds to conservative diffeomorphisms. A hyperbolic periodic point of a conservative diffeomorphism on a surface M^2 has one-dimensional invariant manifolds (i.e., is a periodic saddle point). Each of the invariant manifolds is divided by the periodic point in two components, which are referred to as separatrices. An injective immersion of the half-line $[0; +\infty)$ in a separatrix equips the latter with a parametrization such that 0 corresponds to a periodic point. Therefore, in a standard way, one can define the limit set of the separatrix L, which is denoted by $\omega(L)$. It is known [103] that if L and K are arbitrary separatrices (possibly, $L = K$), then either $K \cap \omega(L) = \varnothing$ or $K \subset \omega(L)$. The last property is generic in the smoothness class C^r for any $1 \le r \le \infty$. In particular, all separatrices of periodic saddle points of a generic conservative diffeomorphism of a closed orientable surface are nontrivially recurrent. By using this fact, Pixton [116] and Oliveira [103] proved for a generic conservative diffeomorphism of the sphere and the torus, respectively, that each hyperbolic periodic point has a transversal homoclinic point in the smoothness class C^r, $1 \le r \le \infty$. (For an arbitrary compact surface, this fact was proved by Takens [148] in the smoothness class C^1.)

Let $f : M^2 \to M^2$ be a conservative diffeomorphism of an orientable closed surface M^2 of genus ≥ 2. By $f_* : H_1(M^2) \to H_1(M^2)$ we denote the automorphism induced by f in the homology group $H_1(M^2)$. The automorphism f_* is said to be *irreducible* if the characteristic polynomial of f_* is irreducible on $\mathbb{Z}$. By $\mathfrak{I}$ we denote the set of diffeomorphisms $f \in \mathit{Diff}^{\,r}_{\omega}(M^2)$, $1 \le r \le \infty$, that induce irreducible automorphisms in the group $H_1(M^2)$. It was shown in [104] that in $\mathfrak{I}$ there exists a residual set $\mathfrak{R}$ such that any diffeomorphism $f \in \mathfrak{R}$ has transversal homoclinic points.

Mather proved [92] that, for a generic symplectic C^r $(r \ge 4)$ diffeomorphism of a compact surface, the topological closures of the separatrices of a hyperbolic periodic point coincide. This result was generalized in [150] to any $r \ge 1$ and any surface (which is not necessarily compact and orientable).

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