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On the extension and generation of set-valued mappings of bounded variation

by

V. V. Chistyakov (Nizhny Novgorod) and A. Rychlewicz (Łódź)

Abstract. We study set-valued mappings of bounded variation of one real variable. First we prove the existence of an extension of a metric space valued mapping from a subset of the reals to the whole set of reals with preservation of properties of the initial mapping: total variation, Lipschitz constant or absolute continuity. Then we show that a set-valued mapping of bounded variation defined on an arbitrary subset of the reals admits a regular selection of bounded variation. We introduce a notion of generated set-valued mappings and show that, under suitable assumptions, set-valued mappings (with arbitrary domains) which are Lipschitzian, of bounded variation or absolutely continuous are generated by certain families of mappings with nice properties. Finally, we prove a Castaing type representation theorem for set-valued mappings of bounded variation.

1. Introduction. This paper is devoted to single- and set-valued mappings of bounded variation of one real variable. Our aim is to extend certain selection theorems, obtained recently by the first author ([2], [5]–[10]) under the assumption that the domain of mappings under consideration is an interval, to the case when the domain of set-valued mappings is an arbitrary subset of the reals $\mathbb{R}$. It is natural to consider mappings of bounded variation $f : E \to X$, where X is a metric space, on an arbitrary nonempty set $E \subset \mathbb{R}$, since the notion of (Jordan) variation of f depends only on the order relation on E and the distance function d in the target space X. Single-valued functions and mappings $f : E \to X$ of bounded variation with arbitrary $\emptyset \neq E \subset \mathbb{R}$ have already been treated in different contexts: [1], [27] (if $X = \mathbb{R}$) and [4]–[6], [13] (if X is a metric or normed space). We also extend selection results for Lipschitzian and absolutely continuous set-valued mappings from [17], [18], [26], [28] and [29] to the case of an arbitrary domain $E \subset \mathbb{R}$.

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First, we study extensions of a metric space valued mapping from a subset of $\mathbb{R}$ to the whole $\mathbb{R}$ with preservation of properties of the initial mapping: total variation, Lipschitz constant or absolute continuity (Theorem 1). In Section 4 we prove an existence theorem for regular selections of a given set-valued mapping (Theorem 2), introduce a notion of generated set-valued mappings and show that, under suitable assumptions, set-valued mappings (with arbitrary domains) which are Lipschitzian, of bounded variation or absolutely continuous are generated by certain families of mappings (Theorem 3). Finally, in Section 5 we prove a Castaing type representation theorem for set-valued mappings of bounded variation (Theorem 4).

2. Preliminaries and main results. In this section we recall some definitions and facts needed for our results.

Let (X, d) be a metric space and $E \subset \mathbb{R}$ be a nonempty set. The (*total*) *Jordan variation* of a mapping $f : E \to X$ is defined by

$$V(f, E) = \sup_T \sum_{i=1}^{m} d\big(f(t_i), f(t_{i-1})\big) \tag{1}$$

where the supremum is over all partitions $T = \{t_i\}_{i=0}^m \subset E$ of E, i.e., $m \in \mathbb{N}$ and $t_{i-1} < t_i$, $i = 1, \ldots, m$. Denote by $\mathrm{BV}(E; X)$ the set of all mappings $f : E \to X$ for which $V(f, E) < \infty$; these mappings are called *of bounded Jordan variation* on E. A mapping $g : E \to X$ is said to be *Lipschitzian* on E if its (minimal) *Lipschitz constant*, defined by

$$L(g, E) = \sup\{d(g(t), g(s))/|t - s| \mid t, s \in E,\ t \neq s\}, \tag{2}$$

is finite. We set $\mathrm{Lip}(E; X) = \{g : E \to X \mid L(g, E) < \infty\}$. A map $h : E \to X$ is said to be *absolutely continuous* on E (in symbols, $h \in \mathrm{AC}(E; X)$) if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any $n \in \mathbb{N}$ and any finite collection $\{\alpha_i, \beta_i\}_{i=1}^n \subset E$ with $\alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \ldots \le \alpha_n < \beta_n$ the condition $\sum_{i=1}^n (\beta_i - \alpha_i) \le \delta(\varepsilon)$ implies $\sum_{i=1}^n d(h(\beta_i), h(\alpha_i)) \le \varepsilon$.

Note that the following embeddings hold: $\mathrm{Lip}(E; X) \subset \mathrm{AC}(E; X)$ for any $E \subset \mathbb{R}$, $\mathrm{Lip}(E; X) \subset \mathrm{BV}(E; X)$ if E is bounded, and $\mathrm{AC}(E; X) \subset \mathrm{BV}(E; X)$ if E is compact (see, e.g., [6]).

Let us recall the main properties of the variation $V(\cdot, \cdot)$ (see [4]–[6]). If $f \in \mathrm{BV}(E; X)$, we set $f(E) = \{f(t) \in X \mid t \in E\}$ and $\omega(f, E) = \sup\{d(f(t), f(s)) \mid t, s \in E\}$. We have: (i) $\omega(f, E) \le V(f, E)$; (ii) $V(f, E_1) \le V(f, E_2)$ for $E_1 \subset E_2 \subset E$; (iii) $V(f, E) = V(f, E_t^-) + V(f, E_t^+)$ for $t \in E$ where $E_t^- = E \cap (-\infty, t]$ and $E_t^+ = E \cap [t, \infty)$; (iv) if $J \subset \mathbb{R}$ and $\psi : J \to E$ is nondecreasing, then $V(f, \psi(J)) = V(f \circ \psi, J)$ where $(f \circ \psi)(t) = f(\psi(t))$, $t \in J$; (v) $V(f, E) = \sup\{V(f, E \cap [a, b]) \mid a, b \in E,\ a \le b\}$; (vi) if $s = \sup E \in (\mathbb{R} \setminus E) \cup \{\infty\}$, then $V(f, E_t^-) \to V(f, E)$ as $E \ni t \to s$; (vii) if $i = \inf E \in (\mathbb{R} \setminus E) \cup \{-\infty\}$, then $V(f, E_t^+) \to V(f, E)$ as $E \ni t \to i$;

(viii) if $f_n : E \to X$ for $n \in \mathbb{N}$ and $d(f_n(t), f(t)) \to 0$ as $n \to \infty$ for all $t \in E$, then $V(f, E) \le \liminf_{n\to\infty} V(f_n, E)$; (ix) the image $f(E)$ is a totally bounded and separable subset of X; moreover, the closure $\overline{f(E)}$ of $f(E)$ in X is compact if X is complete.

In what follows we shall need the concept of metric convexity (due to Menger [24]): a metric space (X, d) is said to be *metrically convex* if for any $x, y \in X$ with $x \ne y$ there exists $z \in X$, $x \ne z \ne y$, such that $d(x, z) + d(z, y) = d(x, y)$. Clearly, any normed linear space is metrically convex. Another example of a metrically convex space is given below.

Given a mapping $\widetilde{f} : \mathbb{R} \to X$, we denote its restriction to $E \subset \mathbb{R}$ by $\widetilde{f}|_E$.

The first main result (an extension theorem) will be proved in Section 3:

THEOREM 1. *Let $\emptyset \ne E \subset \mathbb{R}$ and (X, d) be a complete metric space.*

(a) *If $f \in \mathrm{BV}(E; X)$, then there exists $\widetilde{f} \in \mathrm{BV}(\mathbb{R}; X)$ such that $\widetilde{f}|_E = f$ and $V(\widetilde{f}, \mathbb{R}) = V(f, E)$.*

(b) *If (X, d) is metrically convex and $g \in \mathrm{Lip}(E; X)$, then there exists $\widetilde{g} \in \mathrm{Lip}(\mathbb{R}; X)$ such that $\widetilde{g}|_E = g$ and $L(\widetilde{g}, \mathbb{R}) = L(g, E)$.*

(c) *If (X, d) is metrically convex, $\mathbb{R} \setminus \overline{E}$ (the complement of the closure $\overline{E}$ of E in $\mathbb{R}$) is a finite union of disjoint open intervals and $h \in \mathrm{AC}(E; X)$, then there exists $\widetilde{h} \in \mathrm{AC}(\mathbb{R}; X)$ such that $\widetilde{h}|_E = h$.*

To treat set-valued mappings (s.v.m., for short) of bounded variation, we introduce some notation and terminology.

Let (X, d) be a metric space. Denote by $\dot{2}^X$, $\dot{2}^X_{\mathrm{cl}}$, $\dot{2}^X_{\mathrm{cb}}$ and $\dot{2}^X_{\mathrm{c}}$ the families of all nonempty subsets of X, all nonempty closed subsets of X, all nonempty closed bounded subsets of X, and all nonempty compact subsets of X, respectively. The *Hausdorff distance* $D = D_d$ is defined by the formula ([21])

$$D(A, B) = \max\{\mathrm{e}(A, B), \mathrm{e}(B, A)\}, \qquad A, B \in \dot{2}^X,$$

where

$$\mathrm{e}(A, B) = \sup_{x\in A} \mathrm{dist}(x, B) \quad \text{and} \quad \mathrm{dist}(x, B) = \inf_{y\in B} d(x, y).$$

It is known that D is a *metric* on $\dot{2}^X_{\mathrm{cb}}$ and $\dot{2}^X_{\mathrm{c}}$ and a *pseudometric* (i.e., a metric with possibly infinite values) on $\dot{2}^X_{\mathrm{cl}}$. A general example of a metrically convex metric space (which is not a normed linear space) is given in [11, Theorem 4.1]: if (X, d) is a continuum (i.e. connected and compact), then $(\dot{2}^X_{\mathrm{cl}}, D_d)$ is metrically convex if and only if (X, d) is metrically convex.

Given two nonempty sets E and X, any mapping $F : E \to \dot{2}^X$ is called a *s.v.m.* (or a *multifunction*) from E into X. The set $F(t) \subset X$ is called the *value* of F at $t \in E$. A (single-valued) mapping $f : E \to X$ is said to be a *selection* of F if $f(t) \in F(t)$ for all $t \in E$. If (X, d) is a metric space and

$\emptyset \neq E \subset \mathbb{R}$, the properties of $F : E \to \dot{2}^X$ of being *of bounded* (*Jordan*) *variation*, *Lipschitzian* or *absolutely continuous* are introduced along the same lines as above (cf. (1) and (2)) with the metric d there replaced by the Hausdorff distance $D = D_d$; the respective spaces of s.v.m. will be denoted by $\mathrm{BV}(E; \dot{2}^X)$, $\mathrm{Lip}(E; \dot{2}^X)$ and $\mathrm{AC}(E; \dot{2}^X)$.

Of particular interest are those selections of $F : E \to \dot{2}^X$ that preserve certain regularity properties of F. The second main result is the existence of regular selections, which will be proved in Section 4:

THEOREM 2. *Let $\emptyset \neq E \subset \mathbb{R}$ be an arbitrary set, (X, d) a complete metric space, $F : E \to \dot{2}^X_{\mathrm{c}}$ a s.v.m. with compact values, $t_0 \in E$ and $x_0 \in X$.*

(a) *If $F \in \mathrm{BV}(E; \dot{2}^X_{\mathrm{c}})$, then there exists a selection $f \in \mathrm{BV}(E; X)$ of F with*

$$d(x_0, f(t_0)) = \mathrm{dist}(x_0, F(t_0)) \quad \textit{and} \quad V(f, E) \leq V(F, E). \tag{3}$$

Now suppose also that (X, d) is metrically convex.

(b) *If $F \in \mathrm{Lip}(E; \dot{2}^X_{\mathrm{c}})$, then there exists a selection $f \in \mathrm{Lip}(E; X)$ of F satisfying conditions* (3) *and $L(f, E) \leq L(F, E)$.*

(c) *If $\mathbb{R} \setminus \overline{E}$ is a finite union of disjoint open intervals and $F \in \mathrm{AC}(E; \dot{2}^X_{\mathrm{c}})$, then there exists a selection $f \in \mathrm{AC}(E; X)$ of F satisfying conditions* (3).

When E is a (closed) interval, particular cases of this theorem are contained in [17] (part (a), $X = \mathbb{R}^n$), [18] ((b) and (c), $X = \mathbb{R}^n$, F convex and nonconvex valued), [29] ((c), $X = \mathbb{R}^n$, F nonconvex valued), [26, Suppl.] ((b), X a Banach space, the graph $\mathrm{Gr}(F) = \{(t, x) \in E \times X \mid x \in F(t)\}$ compact), [28] ((b), X a metric space), [5] ((a), X a Banach space, F continuous and $\mathrm{Gr}(F)$ compact), [6] ((a)–(c), X a Banach space and $\mathrm{Gr}(F)$ compact), [7] ((a)–(c), X a Banach space) and [2], [10] ((a)–(c), X a metric space).

Let X and E be nonempty sets and X^E be the set of all mappings from E into X. Any nonempty family $\mathcal{F} \subset X^E$ of mappings generates a s.v.m. $\mathcal{F} : E \to \dot{2}^X$ according to the rule $\mathcal{F}(t) = \{f(t) \mid f \in \mathcal{F}\}$, $t \in E$. We say that a s.v.m. $F : E \to \dot{2}^X$ is *generated* by a (nonempty) family $\mathcal{F} \subset X^E$ (or is *$\mathcal{F}$-generated*) if $F(t) = \mathcal{F}(t)$ for all $t \in E$; note that the family $\mathcal{F}$ here is not uniquely determined in general and that any mapping $f \in \mathcal{F}$ is a selection of F. On the other hand, given an arbitrary s.v.m. $F : E \to \dot{2}^X$, one cannot infer easily that F is $\mathcal{F}$-generated for some $\mathcal{F} \subset X^E$: at least this is the problem of existence of (a large enough family of) selections of F.

If $\emptyset \neq E \subset \mathbb{R}$ and (X, d) is a metric space, then a family $\mathcal{G} \subset X^E$ is called: *equi-Lipschitzian* if $\mathcal{L}(\mathcal{G}, E) = \sup_{g \in \mathcal{G}} L(g, E)$ is finite; *pointwise compact* if $\mathcal{G}(t)$ is a compact subset of X for all $t \in E$.

The third main result is on the generated s.v.m. of bounded variation:

THEOREM 3. *Let $\emptyset \neq E \subset \mathbb{R}$ and (X,d) be a metric space.*

1. *Let $F \in \mathrm{BV}(E; \dot{2}^X_{\mathrm{c}})$, $\nu(t) = V(F, E \cap (-\infty, t])$, $t \in E$, and $J = \nu(E)$. Suppose that either* (a) *E is an interval and F is continuous*, *or* (b) *(X,d) is metrically convex. Then there exists a pointwise compact equi-Lipschitzian family $\mathcal{G} \subset \mathrm{Lip}(J; X)$ such that*

$$F(t) = \{g(\nu(t)) \mid g \in \mathcal{G}\} \quad \text{for all } t \in E. \tag{4}$$

2. *Conversely*, *if $\nu \in \mathrm{BV}(E; \mathbb{R})$, $J = \nu(E)$, $\mathcal{G} \subset \mathrm{Lip}(J; X)$ is an equi-Lipschitzian family and $F : E \to \dot{2}^X$ is given by* (4), *then $F \in \mathrm{BV}(E; \dot{2}^X)$ and $V(F,E) \leq \mathcal{L}(\mathcal{G}, J) V(\nu, E)$.*

Similar results hold for Lipschitzian s.v.m. (cf. Lemma 1 in Section 4) and absolutely continuous s.v.m. (see remarks at the end of Section 4).

Finally, in Section 5 we prove a Castaing type representation theorem for s.v.m. of bounded variation (Theorem 4).

3. Extensions of metric space valued mappings. Most of this section is devoted to the proof of the extension Theorem 1. In what follows we need a structural lemma (Lemma A) for mappings of bounded variation and the fundamental result in the theory of metric convexity (Lemma B).

LEMMA A ([6, Lemma 3.3]). *Let (X,d) be a metric space*, *$\emptyset \neq E \subset \mathbb{R}$ and $f : E \to X$. Then $f \in \mathrm{BV}(E; X)$* (*respectively*, *$f \in \mathrm{AC}(E; X)$ with E compact*) *if and only if $f = g \circ \nu$ on E*, *where $\nu : E \to \mathbb{R}$ is bounded and nondecreasing* (*respectively*, *bounded*, *nondecreasing and absolutely continuous*) *with image $J = \nu(E)$*, *$g \in \mathrm{Lip}(J; X)$ and $L(g, J) \leq 1$. In the necessity part one can set $\nu(t) = V(f, E \cap (-\infty, t])$, $t \in E$.*

LEMMA B ([24], see also [15, Sec. 2]). *If (X,d) is a complete and metrically convex metric space*, *then for any $x, y \in X$ there exists an isometrical embedding $\varphi : [0, d(x,y)] \to X$ such that $\varphi(0) = x$ and $\varphi(d(x,y)) = y$.*

Proof of Theorem 1. (a) Set $E_t^- = E \cap (-\infty, t]$ for $t \in \mathbb{R}$, and $\nu(t) = V(f, E_t^-)$ for $t \in E$. Then $\nu : E \to \mathbb{R}$ is bounded and nondecreasing and

$$\omega(\nu, E) = V(f, E) \quad \text{where} \quad \omega(\nu, E) = \sup_{t \in E} \nu(t) - \inf_{t \in E} \nu(t).$$

By Lemma A, there exists $g \in \mathrm{Lip}(J; X)$ with $J = \nu(E)$ such that $L(g, J) \leq 1$ and $f = g \circ \nu$ on E. We extend ν onto $\mathbb{R}$ as follows: given $t \in \mathbb{R}$, we set $\widetilde{\nu}(t) = \sup\{\nu(s) \mid s \in E_t^-\}$ if $E_t^- \neq \emptyset$, and $\widetilde{\nu}(t) = \inf\{\nu(s) \mid s \in E\}$ if $E_t^- = \emptyset$. It is clear that $\widetilde{\nu} : \mathbb{R} \to \mathbb{R}$ is bounded, nondecreasing, $\widetilde{\nu}|_E = \nu$ and $\omega(\widetilde{\nu}, \mathbb{R}) = \omega(\nu, E)$. Moreover, $\widetilde{\nu}(\mathbb{R}) \subset \overline{\nu(E)} = \overline{J}$. Note that g extends by continuity to a unique $\widetilde{g} \in \mathrm{Lip}(\overline{J}; X)$ and $L(\widetilde{g}, \overline{J}) = L(g, J) \leq 1$. Defining $\widetilde{f} = \widetilde{g} \circ \widetilde{\nu}$ on $\mathbb{R}$, we have: $\widetilde{f}|_E = f$ and

$$V(\widetilde{f},\mathbb{R})\le L(\widetilde{g},\overline{J})\omega(\widetilde{\nu},\mathbb{R})\le\omega(\widetilde{\nu},\mathbb{R})=\omega(\nu,E)=V(f,E).$$

Since $\widetilde{f}|_E=f$, we also have $V(\widetilde{f},\mathbb{R})\ge V(f,E)$.

(b) Again, there exists a unique $\overline{g}\in\mathrm{Lip}(\overline{E};X)$ such that $\overline{g}|_E=g$ and $L(\overline{g},\overline{E})=L(g,E)$. Define $\widetilde{g}$ to be $\overline{g}$ on $\overline{E}$. The difference $\mathbb{R}\setminus\overline{E}$ is at most a countable union of disjoint open intervals (a_k,b_k). If b_k-a_k is finite, set $x_k=\overline{g}(a_k)$ and $y_k=\overline{g}(b_k)$, denote by $\varphi_k:[0,d(x_k,y_k)]\to X$ the isometrical embedding from Menger's theorem (Lemma B) such that $\varphi_k(0)=x_k$ and $\varphi_k(d(x_k,y_k))=y_k$, define $\widetilde{g}$ on the interval (a_k,b_k) by

$$\widetilde{g}(t)=\varphi_k((t-a_k)d(x_k,y_k)/(b_k-a_k)),\qquad t\in(a_k,b_k),$$

and note that $d(x_k,y_k)/(b_k-a_k)\le L(\overline{g},\overline{E})=L(g,E)$. Since φ_k is an isometry, for all $t,s\in[a_k,b_k]$ we have

$$(5)\qquad d(\widetilde{g}(t),\widetilde{g}(s))=(d(x_k,y_k)/(b_k-a_k))|t-s|\le L(g,E)|t-s|.$$

If $a_k=-\infty$ and $b_k<\infty$, we set $\widetilde{g}(t)=\overline{g}(b_k)$, $t\in(-\infty,b_k]$, and if $a_k>-\infty$ and $b_k=\infty$, we set $\widetilde{g}(t)=\overline{g}(a_k)$, $t\in[a_k,\infty)$.

Clearly, $\widetilde{g}|_E=g$. It remains to verify that

$$d(\widetilde{g}(t),\widetilde{g}(s))\le L(g,E)|t-s|\quad\text{for all } t,s\in\mathbb{R}.$$

This inequality is clear if $t,s\in\overline{E}$. If $t\in\overline{E}$ and $s\notin\overline{E}$, we may suppose that $s\in(a_k,b_k)$ and $b_k\le t$, so that the triangle inequality and (5) yield

$$d(\widetilde{g}(t),\widetilde{g}(s))\le d(\overline{g}(t),\overline{g}(b_k))+d(\widetilde{g}(b_k),\widetilde{g}(s))\le L(g,E)|t-s|.$$

Finally, if $t,s\notin\overline{E}$, we may assume that $t\in(a_m,b_m)$, $s\in(a_k,b_k)$ and $b_k\le a_m$, and so

$$\begin{aligned}d(\widetilde{g}(t),\widetilde{g}(s))&\le d(\widetilde{g}(t),\widetilde{g}(a_m))+d(\overline{g}(a_m),\overline{g}(b_k))+d(\widetilde{g}(b_k),\widetilde{g}(s))\\&\le L(g,E)((t-a_m)+(a_m-b_k)+(b_k-s))=L(g,E)|t-s|.\end{aligned}$$

It follows that $L(\widetilde{g},\mathbb{R})\le L(g,E)$, and so $L(\widetilde{g},\mathbb{R})=L(g,E)$ as $\widetilde{g}|_E=g$.

(c) We extend h by continuity to $\overline{h}:\overline{E}\to X$. Since $h\in\mathrm{AC}(E;X)$, it is uniformly continuous on E, and so $\overline{h}$ is (uniformly) continuous on $\overline{E}$. Hence (cf. [13, Lemma IV.5.6]), $\overline{h}$ is absolutely continuous on $\overline{E}$ with respect to the the same function $\delta(\cdot)$ as for h. Writing $\mathbb{R}\setminus\overline{E}=\bigcup_{k=1}^N(a_k,b_k)$ (disjoint union), we define the desired extension $\widetilde{h}:\mathbb{R}\to X$ in the same manner as in (b) above. Denote by L the largest number of $d(x_k,y_k)/(b_k-a_k)$ for b_k-a_k finite, $k=1,\dots,N$. By (5) (with $\widetilde{g}$ and g replaced by $\widetilde{h}$ and h, respectively), $\widetilde{h}$ is absolutely continuous on $\mathbb{R}$ with respect to the function $\widetilde{\delta}(\varepsilon)=\min\{\delta(\varepsilon/2),\varepsilon/(2\max\{1,L\})\}$, $\varepsilon>0$. ∎

Remarks. (a) In the case $X=\mathbb{R}$ Theorem 1(a) is due to Saks [27, Ch. 7, Sec. 4, Lemma (4.1)].

(b) Theorem 1(b) is related to the classical extension theorems ([19], [23], [25]). If $X=\mathbb{R}$, it gives a particular case of the Kirszbraun–McShane theorem (cf. [12, 2.10.43]). Also, it extends the results of [12, 2.5.16] and [6, step 3 in the proof of Theorem 5.1] in the case of a Banach space $(X, \|\cdot\|)$; in this case one may take

$$\varphi_k(\varrho) = x_k + \varrho(y_k - x_k)/\|x_k - y_k\|, \quad \varrho \in [0, \|x_k - y_k\|], \quad \text{if} \quad x_k \neq y_k.$$

However, as is shown in the following two examples, the assumption that X is metrically convex is essential for Theorem 1(b).

EXAMPLES. 1. Let $E = X = \{0, 1\}$, $\chi(0) = 0$ and $\chi(1) = 1$. Clearly, $\chi \in \mathrm{Lip}(E; X)$ and $L(\chi, E) = 1$. If $\widetilde{\chi} : \mathbb{R} \to X$ is any extension of χ, then it is the characteristic function of the set $\{t \in \mathbb{R} \mid \widetilde{\chi}(t) = 1\} \neq \mathbb{R}$ and, hence, discontinuous.

2. The pathwise connectedness of X is not sufficient either. In fact, let $\lambda : \mathbb{R} \to \mathbb{R}$ be a continuous nowhere differentiable function (cf. [14, Example 3.8]), $X = \{(t, \lambda(t)) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$ be its graph and $t_0, t_1 \in \mathbb{R}$ be such that $\lambda(t_0) \neq \lambda(t_1)$. Define $g : E = \{t_0, t_1\} \to X$ by $g(t_0) = (t_0, \lambda(t_0))$ and $g(t_1) = (t_1, \lambda(t_1))$. It is clear that $g \in \mathrm{Lip}(E; X)$. The unique continuous extension $\widetilde{g} : \mathbb{R} \to X$ of g is given by $\widetilde{g}(t) = (t, \lambda(t))$ for all $t \in \mathbb{R}$; however, $\widetilde{g}$ is not Lipschitzian.

4. Generated set-valued mappings. Theorem 2 is based on and is a generalization of the following result on the existence of regular selections for compact-valued s.v.m. defined on intervals:

LEMMA C ([2, Theorems 2, 3]). *Let $I \subset \mathbb{R}$ be an arbitrary interval, (X, d) a metric space, $F : I \to \dot{2}^X_{\mathrm{c}}$ a s.v.m., $t_0 \in I$ and $x_0 \in F(t_0)$, and let $\mathfrak{F}$ denote either* BV, Lip *or* AC. *If $F \in \mathfrak{F}(I; \dot{2}^X_{\mathrm{c}})$, then it admits a selection $f \in \mathfrak{F}(I; X)$ such that $f(t_0) = x_0$ and $V(f, I) \leq V(F, I)$. In the case $\mathfrak{F} = \mathrm{Lip}$ the selection f can be additionally chosen such that $L(f, I) \leq L(F, I)$.*

The conditions of Lemma C are sharp in the sense that if $\dot{2}^X_{\mathrm{c}}$ is replaced by $\dot{2}^X_{\mathrm{cb}}$, then the inequality $V(f, I) \leq V(F, I)$ does not hold in general (cf. [2, Example 2]). On the other hand, examples from [16] show that a continuous s.v.m. $F : [a, b] \to \dot{2}^{\mathbb{R}^2}_{\mathrm{c}}$ need not admit continuous selections.

We also need a result on the metric convexity of the hyperspace $\dot{2}^X_{\mathrm{c}}$:

LEMMA D. *If (X, d) is a metrically convex metric space, then so is $(\dot{2}^X_{\mathrm{c}}, D_d)$.*

Proof. We follow the proof outlined in [20]. Let $A, B \in \dot{2}^X_{\mathrm{c}}$ and $A \neq B$. Then $r = D(A, B) > 0$. For $\varepsilon > 0$ we set $A_\varepsilon = \{x \in X \mid \mathrm{dist}(x, A) \leq \varepsilon\}$. Since $D(A, B) = \inf\{\varepsilon > 0 \mid A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\}$, we have $A \subset B_r$ and $B \subset A_r$. Setting $\Gamma = \{(\alpha, \beta) \in A \times B \mid d(\alpha, \beta) \leq r\}$ and making use of the metric convexity of X, for each $(\alpha, \beta) \in \Gamma$ choose $c_{\alpha,\beta} \in X$ such that

$$d(\alpha, c_{\alpha,\beta}) = d(\beta, c_{\alpha,\beta}) = d(\alpha, \beta)/2.$$

Defining the compact set $C=\{c_{\alpha,\beta}\mid(\alpha,\beta)\in\Gamma\}$, let us show that $A\neq C\neq B$ and $D(A,B)=D(A,C)+D(C,B)$. In fact, if $c\in C$, then $c=c_{\alpha,\beta}$ for some $(\alpha,\beta)\in\Gamma$, and so $d(c,\alpha)=d(c_{\alpha,\beta},\alpha)\le r/2$. Hence, $C\subset A_{r/2}$. Given $\alpha\in A$, there exists $\beta\in B$ with $(\alpha,\beta)\in\Gamma$, whence $d(\alpha,c_{\alpha,\beta})\le r/2$. Therefore, $A\subset C_{r/2}$. It follows that $D(A,C)\le r/2$. Similarly, $D(B,C)\le r/2$. Since always $D(A,B)\le D(A,C)+D(B,C)$, the proof is complete. ∎

Proof of Theorem 2. (a) Since (X,d) is complete, so is $(\dot{2}^X_{\rm c},D)$ (see [3, Theorem II-5]), and hence, by Theorem 1(a), there exists $\widetilde{F}\in\mathrm{BV}(\mathbb{R};\dot{2}^X_{\rm c})$ such that $\widetilde{F}|_E=F$ and $V(\widetilde{F},\mathbb{R})=V(F,E)$. Choose $y_0\in F(t_0)$ such that $d(x_0,y_0)=\mathrm{dist}(x_0,F(t_0))$. Applying Lemma C with $\mathfrak{F}=\mathrm{BV}$ and $I=\mathbb{R}$, we find a selection $\widetilde{f}\in\mathrm{BV}(\mathbb{R};X)$ of $\widetilde{F}$ such that $\widetilde{f}(t_0)=y_0$ and $V(\widetilde{f},\mathbb{R})\le V(\widetilde{F},\mathbb{R})$. Then $f=\widetilde{f}|_E\in\mathrm{BV}(E;X)$ is a selection of F, the first condition in (3) holds and

$$V(f,E)\le V(\widetilde{f},\mathbb{R})\le V(\widetilde{F},\mathbb{R})=V(F,E).$$

(b) By the assumptions on X and Lemma D, $(\dot{2}^X_{\rm c},D)$ is complete and metrically convex, and so Theorem 1(b) yields $\widetilde{F}\in\mathrm{Lip}(\mathbb{R};\dot{2}^X_{\rm c})$ such that $\widetilde{F}|_E=F$ and $L(\widetilde{F},\mathbb{R})=L(F,E)$. The rest of the proof is similar to that of (a). In the same way one can prove (c). ∎

In the rest of this section we study $\mathcal{F}$-generated s.v.m. of bounded variation. We start with Lipschitzian s.v.m. and Lipschitzian families of mappings.

LEMMA 1. *Suppose that either* (a) $E\subset\mathbb{R}$ *is an interval and* (X,d) *is an arbitrary metric space*, *or* (b) $\emptyset\neq E\subset\mathbb{R}$ *is arbitrary and* (X,d) *is metrically convex. Given* $G:E\to\dot{2}^X_{\rm c}$, *we have*: $G\in\mathrm{Lip}(E;\dot{2}^X_{\rm c})$ *if and only if* G *is* $\mathcal{G}$*-generated by a pointwise compact equi-Lipschitzian family of mappings* $\mathcal{G}\subset\mathrm{Lip}(E;X)$.

Proof. Necessity. Denote by $\mathcal{G}$ the family of selections $g\in\mathrm{Lip}(E;X)$ of G for which $L(g,E)\le L(G,E)$. By the assumptions, Lemma C in case (a) and Theorem 2(b) in case (b), the family $\mathcal{G}$ is nonempty. Given $t\in E$, we have to show that $G(t)=\mathcal{G}(t)$. If $x\in\mathcal{G}(t)$, then $x=g(t)$ for some $g\in\mathcal{G}$, and since $g(t)\in G(t)$, it follows that $x\in G(t)$. Conversely, if $x\in G(t)$, then again according to Lemma C or Theorem 2(b) there exists $g\in\mathcal{G}$ such that $g(t)=x$, whence $x\in\mathcal{G}(t)$, and so $G(t)\subset\mathcal{G}(t)$. Clearly, $\mathcal{G}$ is pointwise compact and equi-Lipschitzian.

Sufficiency. A more general assertion holds: if $\mathcal{G}\subset X^E$ is a nonempty equi-Lipschitzian family and $G:E\to\dot{2}^X$ is $\mathcal{G}$-generated, then $G\in\mathrm{Lip}(E;\dot{2}^X)$ and $L(G,E)\le\mathcal{L}(\mathcal{G},E)$. The proof is a straightforward verification: If $t,s\in$

E and $x \in G(t)$, then there exists $g \in \mathcal{G}$ such that $x = g(t)$, and so

$$\inf_{y \in G(s)} d(x,y) \le d(g(t), g(s)) \le L(g,E)|t-s| \le \mathcal{L}(\mathcal{G},E)|t-s|.$$

It follows that $\mathrm{e}(G(t),G(s)) \le \mathcal{L}(\mathcal{G},E)|t-s|$. By the definition of the Hausdorff metric D, we have $D(G(t),G(s)) \le \mathcal{L}(\mathcal{G},E)|t-s|$, and so $G \in \mathrm{Lip}(E;\dot{2}^X)$.

It remains to note that if $\mathcal{G}$ is pointwise compact, then G is compact-valued. (Note also that E here can be replaced by an arbitrary metric space.) ∎

We say that a nonempty family $\mathcal{F} \subset \mathrm{BV}(E;X)$ is *of equi-bounded variation* if $\mathcal{V}(\mathcal{F},E) = \sup_T \sum_{i=1}^m \sup_{f\in\mathcal{F}} d(f(t_i), f(t_{i-1})) < \infty$ where the first supremum is over all partitions $T = \{t_i\}_{i=1}^m$ of E. Clearly, if $\mathcal{F}$ is of equi-bounded variation, then $\mathcal{F}$ is of uniformly bounded variation and $\sup_{f\in\mathcal{F}} V(f,E) \le \mathcal{V}(\mathcal{F},E)$. Also, if $\mathcal{G} \subset \mathrm{Lip}(E;X)$ is equi-Lipschitzian, then $\mathcal{G}$ is of equi-bounded variation.

LEMMA 2. *Suppose that either* (a) *$E \subset \mathbb{R}$ is an interval and (X,d) is an arbitrary metric space*, *or* (b) *$\emptyset \ne E \subset \mathbb{R}$ is arbitrary and (X,d) is a complete metric space. If $F \in \mathrm{BV}(E;\dot{2}^X_{\mathrm{c}})$, then F is $\mathcal{F}$-generated by a pointwise compact family $\mathcal{F} \subset \mathrm{BV}(E;X)$ of mappings of uniformly bounded variation. Conversely*, *if $\mathcal{F} \subset \mathrm{BV}(E;X)$ is a family of equi-bounded variation and $F : E \to \dot{2}^X$ is $\mathcal{F}$-generated*, *then $F \in \mathrm{BV}(E;\dot{2}^X)$ and $V(F,E) \le \mathcal{V}(\mathcal{F},E)$.*

Proof. Necessity. Denote by $\mathcal{F}$ the family of all selections $f \in \mathrm{BV}(E;X)$ of F such that $V(f,E) \le V(F,E)$ and note that, by Lemma C or Theorem 2(a), $\mathcal{F}$ is nonempty. The rest of the proof is identical with the necessity part of the proof of Lemma 1.

Sufficiency. Let $t, s \in E$. Given $x \in F(t)$, there exists $f \in \mathcal{F}$ such that $x = f(t)$, and so

$$\inf_{y \in F(s)} d(x,y) \le d(f(t), f(s)) \le \sup_{f\in\mathcal{F}} d(f(t), f(s)) \equiv \alpha(t,s).$$

It follows that $\mathrm{e}(F(t),F(s)) \le \alpha(t,s)$ and $D(F(t),F(s)) \le \alpha(t,s)$, and so $V(F,E) \le \mathcal{V}(\mathcal{F},E)$. ∎

Proof of Theorem 3. 1. By Lemma A, there exists $G \in \mathrm{Lip}(J;\dot{2}^X_{\mathrm{c}})$ such that $L(G,J) \le 1$ and $F = G \circ \nu$ on E. In case (a), $\nu : E \to \mathbb{R}$ is continuous, and so G is defined on the interval $J = \nu(E)$. In case (b), G takes its values in $\dot{2}^X_{\mathrm{c}}$ where X is metrically convex. According to Lemma 1 there exists a pointwise compact equi-Lipschitzian family $\mathcal{G} \subset \mathrm{Lip}(J;X)$ such that G is $\mathcal{G}$-generated. It follows that

$$F(t) = G(\nu(t)) = \mathcal{G}(\nu(t)) = \{g(\nu(t)) \mid g \in \mathcal{G}\}, \quad t \in E.$$

2. (One can check that the family $\mathcal{F} = \{g \circ \nu \mid g \in \mathcal{G}\}$ is of equi-bounded variation.) Let $t, s \in E$. Given $x \in F(t)$, there exists a mapping $g \in \mathcal{G}$ such that $x = g(\nu(t))$. We have

$$\inf_{y \in F(s)} d(x, y) \le d(g(\nu(t)), g(\nu(s))) \le L(g, J)|\nu(t) - \nu(s)| \\ \le \mathcal{L}(\mathcal{G}, J)|\nu(t) - \nu(s)|,$$

and hence $\mathrm{e}(F(t), F(s)) \le \mathcal{L}(\mathcal{G}, J)|\nu(t) - \nu(s)|$. It follows that

$$D(F(t), F(s)) \le \mathcal{L}(\mathcal{G}, J)|\nu(t) - \nu(s)| \quad \forall t, s \in E.$$

Thus, $V(F, E) \le \mathcal{L}(\mathcal{G}, J)V(\nu, E)$. ∎

We finish this section with remarks on absolutely continuous mappings. Since Lemma A holds for $f \in \mathrm{AC}(E; X)$ with $E \subset \mathbb{R}$ compact, a variant of Lemma 2 for AC can be obtained from Lemma C and Theorem 2(c). A family $\mathcal{F} \subset \mathrm{AC}(E; X)$ is called *equi-absolutely continuous* if for each $\varepsilon > 0$ the number $\delta(\varepsilon) > 0$ from the definition of the absolute continuity of $f \in \mathcal{F}$ can be chosen independently of $f \in \mathcal{F}$. Theorem 3 holds as well if we replace "E" by "compact E" and "BV" by "AC".

5. A Castaing type representation. Let $(T, \mathcal{M})$ be a measurable space and (X, d) be a metric space. Recall ([3, Ch. III]) that a mapping $f : T \to X$ is *measurable* if for any open set $\mathcal{O} \subset X$ the preimage $f^{-1}(\mathcal{O}) = \{t \in T \mid f(t) \in \mathcal{O}\} \in \mathcal{M}$. A s.v.m. $F : T \to \dot{2}^X$ is *measurable* if $F^{-1}(\mathcal{O}) = \{t \in T \mid F(t) \cap \mathcal{O} \ne \emptyset\} \in \mathcal{M}$ for any open set $\mathcal{O} \subset X$. It is known (see [3, Ch. III]) that if X is a complete separable metric space, then $F : T \to \dot{2}^X_{\mathrm{c}}$ is measurable if and only if it is measurable as a mapping to the metric space $(\dot{2}^X_{\mathrm{c}}, D)$.

A mapping $f : T \to X$ is called a *measurable selection* of $F : T \to \dot{2}^X$ if it is measurable and $f(t) \in F(t)$ for all $t \in T$. It is known (see [22]) that a measurable $F : T \to \dot{2}^X$ with closed values in a complete separable metric space X admits a measurable selection. Moreover, for a s.v.m. F with nonempty closed values in a complete separable metric space X, one can choose a sequence $\{f_n\}_{n=1}^\infty$ of measurable selections for which the following representation holds (cf. [3, Ch. III]):

$$F(t) = \overline{\{f_n(t)\}_{n=1}^\infty} \quad \text{for all } t \in T, \tag{6}$$

where the bar means the closure in X. The sequence $\{f_n\}_{n=1}^\infty$ satisfying (6) is called a *Castaing representation* for F. Also, the existence of such a representation for $F : T \to \dot{2}^X_{\mathrm{cl}}$ with X a complete separable metric space is equivalent to the measurability of F ([3, Theorem III.8]).

Now, let $I = [a, b]$ and (X, d) be a complete metric space. A mapping $f \in X^I$ is said to be *proper* if it has at most a countable number of points of discontinuity on I, at each of which f has both one-sided limits. We say that $F : I \to \dot{2}^X$ *admits a Castaing* (resp. *an almost everywhere Castaing*) *representation* if there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset X^I$ of selections of F for which (6) holds (resp. holds almost everywhere).

The following theorem is based on Lemma C and the above remarks.

THEOREM 4. *Let X be a complete metric space and $F \in \mathrm{BV}(I; \dot{2}^X_{\mathrm{c}})$ be continuous. Then*: (a) *for any measurable selection f of F there exists a sequence of proper selections of F which converges to f almost everywhere on I*; (b) *F admits an almost everywhere Castaing representation by a sequence of proper selections of F.*

Proof. The idea of the proof is taken from [26, Theorem D1.9]. Since $F \in \mathrm{BV}(I; \dot{2}^X_{\mathrm{c}})$ is continuous, it is measurable and takes compact values, and so, by the remarks above, it admits a Castaing representation by its measurable selections. Hence, it suffices to prove (a).

Suppose that $f \in X^I$ is any measurable selection of F. Given $\varepsilon > 0$, using the Lusin C-property we can find a closed subset $J_\varepsilon \subset I$ such that the (Lebesgue) measure $\mathrm{meas}(I \setminus J_\varepsilon) \le \varepsilon$ and the restriction $f|_{J_\varepsilon}$ is continuous. As $I \setminus J_\varepsilon$ is open, it is the union of at most a countable number of nonintersecting intervals (α_n, β_n), $n \in \mathbb{N}$. According to Lemma C for each $n \in \mathbb{N}$ there exists a continuous selection f_n of F on $[\alpha_n, \beta_n]$ of bounded variation such that $f_n(\alpha_n) = f(\alpha_n)$. Define $\psi_\varepsilon : I \to X$ by: $\psi_\varepsilon(t) = f(t)$ if $t \in J_\varepsilon$, and $\psi_\varepsilon(t) = f_n(t)$ if $t \in (\alpha_n, \beta_n)$, $n \in \mathbb{N}$. Let us show that f_ε may be discontinuous (with left and right limits) only at points $t = \beta_n$, $n \in \mathbb{N}$. For this, it suffices to verify that $\psi_\varepsilon(t_k) \to \psi_\varepsilon(t)$ for all $\{t_k\}_{k=1}^{\infty} \subset I \setminus J_\varepsilon$ in the following two cases: $t \in J_\varepsilon \setminus \{\beta_n : n \in \mathbb{N}\}$ and $t_k \to t$ as $k \to \infty$; and $t = \beta_n$ for some $n \in \mathbb{N}$ and $t_k \to t + 0$ as $k \to \infty$.

Denote by (α_k, β_k) the interval from $I \setminus J_\varepsilon$ containing t_k. Since $\alpha_k \to t$, by the construction we have

$$\|\psi_\varepsilon(t_k) - \psi_\varepsilon(t)\| \le \|f_k(t_k) - f_k(\alpha_k)\| + \|f(\alpha_k) - f(t)\|,$$

which implies that $\psi_\varepsilon(t_k) \to \psi_\varepsilon(t)$ as $k \to \infty$ by the uniform continuity of f_k on $[\alpha_k, \beta_k]$, $k \in \mathbb{N}$, and the continuity of f on J_ε. From the definition of ψ_ε it follows that ψ_ε tends to f in measure as $\varepsilon \to +0$, i.e. $\mathrm{meas}(\{t \in I : \|\psi_\varepsilon(t) - f(t)\| > \gamma\}) \to 0$ as $\varepsilon \to +0$ for all $\gamma > 0$, from which we conclude that a suitable subsequence of ψ_ε tends to f pointwise almost everywhere on I. ∎

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Department of Mathematics
University of Nizhny Novgorod
23 Gagarin Avenue
Nizhny Novgorod 603950, Russia
E-mail: chistya@mm.unn.ru

Faculty of Mathematics
Łódź University
Stefana Banacha 22
90-238 Łódź, Poland
E-mail: anrychle@imul.math.uni.lodz.pl

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