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Two complexity results for the vertex coloring problem ☆

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ABSTRACT

We show that the chromatic number of $\{P_5, K_p - e\}$-free graphs can be computed in polynomial time for each fixed p. Additionally, we prove polynomial-time solvability of the weighted vertex coloring problem for $\{P_5, \overline{P_3 + P_2}\}$-free graphs.

1. Introduction

In this paper, we consider only *simple graphs*, i.e. finite undirected graphs without loops and multiple edges. A *coloring of a graph G* is an arbitrary mapping $c : V(G) \longrightarrow \mathbb{N}$, such that $c(u) \neq c(v)$ for any two adjacent vertices u and v of G. Elements of the set $\bigcup_{v \in V(G)} \{c(v)\}$ are said to be *colors*. A coloring c^* of a graph G is a *k-coloring* if $c^* : V(G) \longrightarrow \{1, \ldots, k\}$. The *chromatic number of a graph G*, denoted by $\chi(G)$, is the minimal number k, such that G has a k-coloring. For a given graph G and a number k, the *coloring problem* is to decide whether $\chi(G) \leq k$ or not. A similar *k-colorability problem* is to check whether vertices of a given graph can be colored with at most k colors. Both problems can be naturally defined in another way via partition into independent sets. An *independent set of a graph* is an arbitrary set of its pairwise non-adjacent vertices. A graph coloring is a partition of vertex set of a given graph into independent sets, called *color classes*.

For a given graph G and a function $w : V(G) \longrightarrow \mathbb{N}$, a pair (G, w) is called a *weighted graph*. For a weighted graph (G, w), the *weighted coloring problem* is to find the smallest number k, denoted by $\chi_w(G)$, such that there is a function $c : V(G) \longrightarrow 2^{\{1,2,\ldots,k\}}$, where $|c(v)| = w(v)$ for any $v \in V(G)$ and $c(v_1) \cap c(v_2) = \emptyset$ for any edge (v_1, v_2) of G. The number $\chi_w(G)$ is called the *weighted chromatic number* of (G, w). For any graph G, $\chi_{w'}(G) = \chi(G)$, where w' maps every vertex to 1. So, the weighted coloring problem generalizes the coloring problem.

A class of simple graphs is called *hereditary* if it is closed under deletion of vertices. It is well-known that any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{S}$. We write $\mathcal{X} = Free(\mathcal{S})$, and the graphs in $\mathcal{X}$ are said to be *$\mathcal{S}$-free*. If $\mathcal{S} = \{G\}$, then we write "G-free" instead of "$\{G\}$-free".

There is a natural lower bound for the chromatic number of graphs. A *clique* in a graph is a subset of its pairwise adjacent vertices. The size of a maximum clique of a graph G, denoted by $\omega(G)$, is called the *clique number* of G. Clearly, $\chi(G) \geq \omega(G)$.

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A graph is said to be *perfect* if the clique and the chromatic numbers are equal for its every induced subgraph, not necessarily proper. The class of perfect graphs coincides with $Free(\{C_5, \overline{C_5}, C_7, \overline{C_7}, \dots\})$, by The Strong Perfect Graph Theorem [5], see notation for graphs in the next section. Sometimes, computing the clique number in polynomial time helps to determine the chromatic number also in polynomial time [15,36]. More precisely, for graphs in [15,36], including perfect graphs, determining the chromatic number can be polynomially reduced to computing the clique number and the clique number can be found in polynomial time.

The computational complexity of the coloring, the weighted coloring, and the k-colorability problems and their edge variants was intensively studied for families of the forms $\{Free(\mathcal{S})|\ \mathcal{S}$ has a small number of graphs$\}$ and $\{Free(\mathcal{S})|$ every graph in $\mathcal{S}$ is small$\}$ [1–3,7,8,13,14,17–39,41,42]. The computational complexity of the coloring problem was completely determined for all the classes of the form $Free(\{G\})$ [22]. Namely, if $\subseteq_i$ is the induced subgraph relation, then the problem is polynomial-time solvable for $Free(\{G\})$ whenever $G \subseteq_i P_4$ or $G \subseteq_i P_3 + K_1$; otherwise it is NP-complete. A study of forbidden pairs was also initiated in [22].

The following result shows some recent advances in classification of the complexity of the coloring problem for $\{G_1, G_2\}$-free graphs [12]. Note that, by symmetry, the graphs G_1 and G_2 may be swapped in each of the subcases of the theorem.

Theorem 1. *Let G_1 and G_2 be two fixed graphs. The coloring problem is* NP-*complete for $Free(\{G_1, G_2\})$ if:*

1. $C_p \subseteq_i G_1$ *for* $p \geq 3$, *and* $C_q \subseteq_i G_2$ *for* $q \geq 3$
2. $K_{1,3} \subseteq_i G_1$, *and* $K_{1,3} \subseteq_i G_2$ *or* $\overline{K_2 + O_2} \subseteq_i G_2$ *or* $C_r \subseteq_i G_2$ *for* $r \geq 4$ *or* $K_4 \subseteq_i G_2$
3. G_1 *and* G_2 *contain a spanning subgraph of a* $2K_2$ *as an induced subgraph*
4. *bull* $\subseteq_i G_1$, *and* $K_{1,4} \subseteq_i G_2$ *or* $\overline{C_4 + K_1} \subseteq_i G_2$
5. $C_3 \subseteq_i G_1$, *and* $K_{1,p} \subseteq_i G_2$ *for* $p \geq 5$
6. $C_3 \subseteq_i G_1$ *and* $P_{22} \subseteq_i G_2$
7. $C_p \subseteq_i G_1$ *for* $p \geq 5$, *and* G_2 *contains a spanning subgraph of a* $2K_2$ *as an induced subgraph*
8. $C_p + K_1 \subseteq_i G_1$ *for* $p \in \{3, 4\}$ *or* $\overline{C_q} \subseteq_i G_1$ *for* $q \geq 6$, *and* G_2 *contains a spanning subgraph of a* $2K_2$ *as an induced subgraph*
9. $K_5 \subseteq_i G_1$ *and* $P_7 \subseteq_i G_2$
10. $K_6 \subseteq_i G_1$ *and* $P_6 \subseteq_i G_2$.

It is polynomial-time solvable for $Free(\{G_1, G_2\})$ if:

1. G_1 *is an induced subgraph of a* P_4 *or a* $P_3 + K_1$
2. $G_1 \subseteq_i K_{1,3}$, *and* $G_2 \subseteq_i$ *hammer or* $G_2 \subseteq_i$ *bull or* $G_2 \subseteq_i P_5$
3. $G_1 \neq K_{1,5}$ *is a forest on at most six vertices or* $G_1 = K_{1,3} + 3K_1$, *and* $G_2 \subseteq_i$ *paw*
4. $G_1 \subseteq_i sK_2$ *or* $G_1 \subseteq_i P_5 + O_s$ *for* $s > 0$, *and* G_2 *is a complete graph or* $G_2 \subseteq_i$ *hammer*
5. $G_1 \subseteq_i P_4 + K_1$ *or* $G_1 \subseteq_i P_5$, *and* $G_2 \subseteq_i \overline{P_4 + K_1}$ *or* $G_2 \subseteq_i \overline{P_5}$
6. $G_1 \subseteq_i K_2 + O_2$, *and* $G_2 \subseteq_i \overline{2K_2 + K_1}$ *or* $G_2 \subseteq_i \overline{P_3 + O_2}$ *or* $G_2 \subseteq_i \overline{P_3 + K_2}$
7. $G_1 \subseteq_i \overline{K_2 + O_2}$, *and* $G_2 \subseteq_i 2K_2 + K_1$ *or* $G_2 \subseteq_i P_3 + O_2$ *or* $G_2 \subseteq_i P_3 + P_2$
8. $G_1 \subseteq_i K_2 + O_s$ *for* $s > 0$ *or* $G_1 = P_5$, *and* $G_2 \subseteq_i \overline{K_2 + O_t}$ *for* $t > 0$
9. $G_1 \subseteq O_4$ *and* $G_2 \subseteq \overline{P_3 + O_2}$
10. $G_1 \subseteq P_5$, *and* $G_2 \subseteq_i C_4$ *or* $G_2 \subseteq \overline{P_3 + O_2}$.

A complete complexity dichotomy for the coloring problem is hard to obtain even in the following cases: (a) two forbidden induced subgraphs, each on at most four vertices [24]; (b) two connected forbidden induced subgraphs, each on at most five vertices [32]. For all but three cases either NP-completeness or polynomial-time solvability was shown in the family of all the hereditary classes, defined by four-vertex forbidden induced structures [24]. The remaining three classes $Free(\{O_4, C_4\})$, $Free(\{K_{1,3}, O_4\})$, $Free(\{K_{1,3}, O_4, \overline{K_2+O_2}\})$ are stubborn. A similar result was obtained in [32] for two connected five-vertex forbidden induced fragments, where the number of open cases was 13. A list of the open cases is presented below (the numbers in parentheses show the quantities of such kind sets).

1. $\{K_{1,3}, G\}$, where $G \in \{bull, butterfly\}$ (2)
2. $\{fork, bull\}$ (1)
3. $\{P_5, G\}$, where G is an arbitrary connected five-vertex complement graph of the line graph of a forest with 3 leaves in each connected component and $G \notin \{K_5, gem\}$ (10).

Recently, the number of the open cases was reduced to 10 [18,36] by proving that the coloring problem can be solved in polynomial time for $Free(\{P_5, \overline{P_5}\})$, $Free(\{K_{1,3}, bull\})$, $Free(\{P_5, \overline{P_3 + O_2}\})$. In this paper, we reduce the number to eight by showing that the coloring problem can be solved for $\{P_5, \overline{P_3 + P_2}\}$-free or $\{P_5, K_p - e\}$-free graphs in polynomial time. More generally, we prove polynomial-time solvability of the weighted coloring problem for $\{P_5, \overline{P_3 + P_2}\}$-free graphs.

2. Notation

As usual, P_n, C_n, O_n, K_n stand for a simple path, chordless cycle, empty graph, complete graph on n vertices, respectively. A graph $K_{p,q}$ is a complete bipartite graph with p vertices in the first part and q vertices in the second. A graph $K_p - e$ is obtained from a K_p by deleting an arbitrary edge. The graph *paw* is obtained from a $K_{1,3}$ by adding a new edge, incident to degree two vertices. The graphs *fork*, *gem*, *hammer*, *bull*, *butterfly* have vertex set $\{v_1, v_2, v_3, v_4, v_5\}$. Edge set for a *fork* is $\{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_4, v_5)\}$, for a *gem* is $\{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$, for a *hammer* is $\{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_1, v_4), (v_4, v_5)\}$, for a *bull* is $\{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_1, v_4), (v_2, v_5)\}$, for a *butterfly* is $\{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_1, v_4), (v_1, v_5), (v_4, v_5)\}$.

A formula $N(x)$ means the neighborhood of a vertex x of some graph. For a graph G and a set $V' \subseteq V(G)$, $G(V')$ denotes the subgraph of G, induced by V'. A graph $G_1 + G_2$ is the disjoint union of graphs G_1 and G_2, having non-intersected sets of vertices. A graph kG is the disjoint union of k copies of a graph G. A graph $\overline{G}$ is the complement graph of G.

3. Auxiliary results

3.1. Decomposition by clique separators and its applications to the weighted coloring problem

A *clique separator* in a graph is a clique whose removal increases the number of connected components. For example, a graph $K_p - e$ has a clique separator with $p-2$ vertices. If a graph G has a clique separator Q, then $V(G) \setminus Q$ can be partitioned into non-empty subsets A and B, such that any element of A is not adjacent to any element of B. Let $G_1 \triangleq G(A \cup Q)$ and $G_2 \triangleq G(B \cup Q)$, where the symbol $\triangleq$ means the equality by definition. We repeat a similar decomposition until no further decomposition is possible. The whole process can be represented by a binary decomposition tree whose leaves correspond to some induced subgraphs of G without clique separators. There is an algorithm, having the computational complexity $O(mn)$, for constructing some binary decomposition tree for any graph with n vertices and m edges [40].

Lemma 1. *For any weighted graph* (G, w), $\chi_w(G) = \max(\chi_w(G_1), \chi_w(G_2))$.

Proof. Let c_1 and c_2 be optimal weighted colorings of (G_1, w) and (G_2, w), respectively. Let $\bigcup_{v \in V(G_1)} c_1(v) \triangleq \{col_1, \ldots, col_p\}$ and $\bigcup_{u \in V(G_2)} c_2(u) \triangleq \{col'_1, \ldots, col'_q\}$. Without loss of generality, $q \geq p, \forall v \in Q\ c_1(v) = \{col_{i_1^{(v)}}, \ldots, col_{i_k^{(v)}}\}$ and $c_2(v) = \{col'_{i_1^{(v)}}, \ldots, col'_{i_k^{(v)}}\}$. Let us define a weighted coloring c of (G, w) as follows. For any $x \in V(G_2)$, $c(x) \triangleq c_2(x)$. For any $y \in V(G_1) \setminus V(G_2)$ and $i \in \{1, \ldots, p\}$, $col'_i \in c(y)$ if and only if $col_i \in c_1(y)$. Hence, G can be colored in $\chi_w(G_2)$ colors. So, $\chi_w(G) = \chi_w(G_2)$. ■

For a given graph, any maximal its induced subgraph without proper clique separators will be called a *C-block* of the graph. Leaves of a decomposition tree of any graph correspond to its C-blocks. Let $\mathcal{X}$ be a class of graphs. The set of all graphs whose every C-block belongs to $\mathcal{X}$, denoted by $[\mathcal{X}]_C$, will be called the *C-closure of* $\mathcal{X}$.

Theorem 2. *If the (weighted) coloring problem can be solved in polynomial time for a hereditary class* $\mathcal{X}$*, then it is so for* $[\mathcal{X}]_C$.

Proof. Clearly, $[\mathcal{X}]_C$ is hereditary. Every C-block of any graph $G \in [\mathcal{X}]_C$ belongs to $\mathcal{X}$. A decomposition tree for G can be constructed in $O(|V(G)| \cdot |E(G)|)$ time. Hence, by the previous lemma, the (weighted) coloring problem can be polynomially solved for $[\mathcal{X}]_C$. ■

3.2. Modular decomposition and its applications to the weighted coloring problem

Let G be a graph. A set $M \subseteq V(G)$ is a *module* in G if either x is adjacent to all the elements of M or to none of them for each $x \in V(G) \setminus M$. A module in a graph is *trivial* if it contains only one vertex or all vertices of the graph; otherwise it is *non-trivial*. A graph containing no non-trivial modules is said to be *prime*. For instance, a P_4 is prime and a C_4 is not prime.

Modular decomposition of graphs is an algorithmic technique, based on the following decomposition theorem due to T. Gallai.

Theorem 3 ([11]). *Let G be a graph with at least two vertices. Then, exactly one of the following conditions holds:*

(1) *G is not connected*
(2) *$\overline{G}$ is not connected*
(3) *G and $\overline{G}$ are connected, and there is a set V' with at least four elements and a unique partition $P(G)$ of $V(G)$, such that*
(a) *$G(V')$ is a maximal prime induced subgraph of G*
(b) *for each $V'' \in P(G)$, V'' is a module (perhaps, trivial) in G and $|V'' \cap V'| = 1$.*

By Theorem 3, there are decomposition operations of three types. First, if G is not connected, then disconnect it into its connected components $G_1, \ldots, G_p$. Second, if $\overline{G}$ has connected components $\overline{G_1}, \ldots, \overline{G_q}$, then decompose G into $G_1, \ldots, G_q$. At length, if G and $\overline{G}$ are connected, then its maximal modules are pairwise disjoint and they form the partition $P(G)$. The graph G is decomposed into the subgraphs in $\{G(V'') \mid V'' \in P(G)\}$. Additionally, every element of $P(G)$ is contracted to obtain a graph, which is isomorphic to $G(V')$. In other words, $G(V')$ is the induced subgraph of G, obtained by taking one element in each of the elements of $P(G)$.

The decomposition process above can be represented by a uniquely determined tree, called the *modular decomposition tree* $T(G)$ of G. Its vertices are some induced subgraphs of G. For the first two decomposition operations, the vertex of $T(G)$ corresponding to G has the children corresponding to all the connected components of G or $\overline{G}$, respectively. For the third decomposition operation, the children correspond to all the graphs in $\{G(V'') \mid V'' \in P(G)\}$. Moreover, we associate the graph $G(V')$ with the vertex of $T(G)$ corresponding to G. A modular decomposition tree can be constructed in $O(n+m)$-time for any graph with n vertices and m edges [6].

Clearly, for any function w, we have $\chi_w(G) = \max_i(\chi_w(G_i))$, where $G_1, \ldots, G_p$ are the connected components of G. Similarly, if $\overline{G_1}, \ldots, \overline{G_q}$ are the connected components of $\overline{G}$, then $\chi_w(G) = \sum_{i=1}^{q} \chi_w(G_i)$.

Lemma 2. *Let (G, w) be a weighted graph and $P(G)$ be its modular decomposition. Then $\chi_w(G) = \chi_{w^*}(G(V'))$, where $w^*(v) = \chi_w(G(V''))$ for each $v \in V'$, $V'' \in P(G)$, $\{v\} = V' \cap V''$.*

Proof. Contraction of any $V'' \in P(G)$ to v and assignment $w(v) = \chi_w(G(V''))$ produce a weighted subgraph of G whose weighted chromatic number is at most $\chi_w(G)$. For the subgraph, every element of $N(v)$ cannot have a color coinciding with one of the $\chi_w(G(V''))$ colors of v. Hence, the chromatic number of the subgraph is at least $\chi_w(G)$, i.e. it is equal to $\chi_w(G)$. Therefore, $\chi_w(G) = \chi_{w^*}(G(V'))$. ■

Let $[\mathcal{X}]_P$ be the set of all graphs whose every prime induced subgraph belongs to $\mathcal{X}$. Clearly, $[\mathcal{X}]_P$ is hereditary whenever $\mathcal{X}$ is hereditary. The sums $\sum_{v \in V(G)} w(v)$ and $\sum_{v \in V'} w^*(v)$ are equal. The theorem below follows from the previous lemma and the possibility for constructing modular decomposition tree in linear time [6].

Theorem 4. *If the weighted coloring problem can be solved for a hereditary class $\mathcal{X}$ in polynomial time, then it is so for $[\mathcal{X}]_P$.*

3.3. Bipartite Ramsey Theorem

The well-known Ramsey Theorem states that any graph has a sufficiently large independent set or a sufficiently large clique. There is its analogue for bipartite graphs. A *matching* in a graph is a subset of pairwise non-adjacent edges. The following result is a corollary of Theorem 2 from [10] for $H = K_{s,s}$.

Lemma 3. *Any bipartite graph G having parts A and B, each on $n > s^{s+1}$ vertices, contains subsets $A' \subseteq A$, $B' \subseteq B$, $|A'| = |B'| = \lfloor (\frac{n}{s})^{\frac{1}{s}} \rfloor$, such that $A' \cup B'$ induces a matching or $G(A' \cup B')$ is complete bipartite.*

3.4. Connected $\{P_5, K_p - e\}$-free graphs without clique separators

Lemma 4. *Let G be a connected $\{P_5, K_p - e\}$-free graph $(p \geq 3)$ without clique separators, and let Q be its maximum clique. Then the graph G is O_3-free or $|Q| \leq (p+1)^{p+2}(p-2)$.*

Proof. Assume that $|Q| > (p+1)^{p+2}(p-2)$. Let $N(Q) \triangleq \{y \notin Q \mid \exists x \in Q, (y, x) \in E(G)\}$. Let us consider the bipartite subgraph H of G, induced by all the edges between Q and $N(Q)$. Every element of $N(Q)$ is adjacent to at most $p-3$ elements of Q, as G is $K_p - e$-free and Q is maximum. Every element of Q has a neighbor in $N(Q)$, as G has no clique separators. Hence, H has a matching with at least $\lfloor \frac{|Q|}{p-2} \rfloor$ edges. Let G' be a subgraph of H, induced by all vertices of some maximum matching of H. Clearly, the graph G' is $K_{p-2,p-2}$-free and each of its parts has at least $\lfloor \frac{|Q|}{p-2} \rfloor$ vertices. Clearly, $\lfloor \frac{|Q|}{p-2} \rfloor > (p+1)^{p+2}$. Let $N_1 \triangleq \{u_1, u_2, \ldots, u_k\}$ be a maximum subset of $Q \cap V(G')$, such that $N(Q)$ has vertices $v_1, v_2, \ldots, v_k$, where $v_i \in N(u_i) \setminus \bigcup_{j \neq i} N(u_j)$ for each i. By the previous lemma for $s = p+1$, $k \geq \lfloor (\frac{1}{p+1} \lfloor \frac{|Q|}{p-2} \rfloor)^{\frac{1}{p+1}} \rfloor \geq p+1 \geq 4$. The set $N_2 \triangleq \{v_1, v_2, \ldots, v_k\}$ must be independent or a clique. Indeed, $G'(N_2)$ must be P_3-free; otherwise some vertices $v_{i_1}, v_{i_2}, v_{i_3}$, the vertex u_{i_1}, and an arbitrary element of $N_1 \setminus \{u_{i_1}, u_{i_2}, u_{i_3}\}$ induce a P_5 in G. In other words, $G'(N_2)$ is the disjoint union of complete graphs. If $G'(N_2)$ is not complete and not empty, simultaneously, then there are vertices $v_{j_1}, v_{j_2}, v_{j_3}$, such that $(v_{j_1}, v_{j_2}) \in E(G)$, $(v_{j_1}, v_{j_3}) \notin E(G)$, $(v_{j_2}, v_{j_3}) \notin E(G)$. The vertices $v_{j_1}, v_{j_2}, u_{j_2}, u_{j_3}, v_{j_3}$ induce a P_5 in G.

Suppose that N_2 is independent. Then, there is no vertex v_i having a neighbor $w \notin Q \cup N(Q)$. Otherwise, to avoid an induced P_5, w must be adjacent to all the vertices of N_2. Hence, v_1, w, v_2, u_1, u_3 induce a P_5 in G. Hence, for each i, each neighbor of v_i that is outside Q must belong to $N(Q)$. Let $w_i \in N(Q)$ be a neighbor of v_i. There are three non-neighbors $u_{k_1}, u_{k_2}, u_{k_3}$ of w_i, as G' is $K_{p-2,p-2}$-free. Let $u' \in Q \setminus \{u_i\}$ be a neighbor of w_i. Then, (w_i, v_{k_1}) and (w_i, v_{k_2}) are edges of G; otherwise $v_i, w_i, u', u_{k_1}, v_{k_1}$ or $v_i, w_i, u', u_{k_2}, v_{k_2}$ induce a P_5. But, the vertices $v_{k_2}, w_i, v_{k_1}, u_{k_1}, u_{k_3}$ induce a P_5. Hence, $N(w_i) \cap Q = \{u_i\}$. Therefore, any neighbor of v_i that lies outside Q must be adjacent to u_i and non-adjacent to $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k$, simultaneously. Moreover, it is also true for any vertex of the connected component of $G(N(u_i) \setminus Q)$ containing v_i. Assume that w_i has a neighbor $v^* \in N(Q)$, non-adjacent to u_i. Clearly, $(v_i, v^*) \notin E(G)$. As N_1 is maximum, there is a number i^*, such that $i^* \neq i$ and $(v^*, u_{i^*}) \in E(G)$. As G' is $K_{p-2,p-2}$-free and $k \geq p+1$, there is a vertex $u_{i^{**}}$, such that $i^{**} \notin \{i, i^*\}$ and $(u_{i^{**}}, v^*) \notin E(G)$. The vertices $v_i, w_i, v^*, u_{i^*}, u_{i^{**}}$ induce a P_5 in G. Therefore, none of the vertices of the connected component of $G(N(u_i) \setminus Q)$ containing v_i has a neighbor in $N(Q)$, non-adjacent to u_i. Hence, Q is a clique separator.

Suppose that N_2 is a clique. Let Q' be a maximal clique of G that includes N_2. Suppose $v \in N(Q) \setminus Q'$. Since N_1 is maximum, v has neighbors in N_1, say, $u_1, \ldots, u_q$. As G' is $K_{p-2,p-2}$-free, $q \leq p-3$. To avoid a P_5, induced by v, u_1, a vertex in $\{u_{q+1}, \ldots, u_k\}$ and some two vertices in $\{v_{q+1}, \ldots, v_k\}$, non-adjacent to v, v must be adjacent to at least $k-q-1$ vertices among $v_{q+1}, \ldots, v_k$. Suppose that $(v, v_{k-1}) \in E(G)$. The vertex v must be adjacent to at least $q-1$ elements of $\{v_1, \ldots, v_q\}$; otherwise there are two vertices $v_{i'}, v_{i''}$ in $\{v_1, \ldots, v_q\} \setminus N(v)$, such that $v_{i'}, v_{k-1}, v, u_{i''}, u_k$ induce a P_5 in G. Hence, v is adjacent to at least $k-2$ vertices of N_2. Hence, to avoid an induced $K_p - e$, $v \in Q'$. As Q' is maximal, v cannot exist, i.e. $Q' = N(Q)$. Moreover, $V(G) = Q \cup N(Q)$, since $N(Q)$ is a clique separator otherwise. Hence, G is O_3-free, as Q and $N(Q)$ are cliques. ■

3.5. Irreducible $\{P_5, \overline{P_3 + P_2}\}$-free graphs

A connected prime graph without clique separators is said to be *irreducible*.

Clearly, any $\{P_5, \overline{P_3 + P_2}, C_5\}$-free graph is perfect, by The Strong Perfect Graph Theorem. Let G be an irreducible $\{P_5, \overline{P_3 + P_2}\}$-free graph containing an induced $C_5 = (v_1, v_2, v_3, v_4, v_5)$. We associate the following notation with G, taking the indices modulo 5 throughout this section:

- $V_i \triangleq \{x \notin V(C_5) | N(x) \cap V(C_5) = \{v_{i-1}, v_{i+1}\}\}$,
- $V'_i \triangleq \{x \notin V(C_5) | N(x) \cap V(C_5) = \{v_{i-1}, v_i, v_{i+1}\}\}$,
- $V''_i \triangleq \{x \notin V(C_5) | N(x) \cap V(C_5) = V(C_5) \setminus \{v_i\}\}$,
- $V'''_i \triangleq \{x \notin V(C_5) | N(x) \cap V(C_5) = \{v_{i-2}, v_i, v_{i+2}\}\}$,
- V'''' be the set of all the vertices, adjacent to all the vertices of the 5-cycle.

The following statement is true, as G is P_5-free.

Lemma 5. *Every element of $V(G) \setminus V(C_5)$, having a neighbor on the 5-cycle, belongs to $\bigcup_{i=1}^{5}(V_i \cup V'_i \cup V''_i \cup V'''_i) \cup V''''$.*

Lemma 6. *The following statements are true:*

(1) *Every element of V'''' is adjacent to every element of $\bigcup_{i=1}^{5}(V_i \cup V''_i \cup V'''_i)$.*

(2) *The set V'''' is a clique. For each i, V_i is independent and V''_i is a clique.*

(3) (a) *If $V_i \neq \emptyset$, then every element of V_i is adjacent to every element of $V_{i-1} \cup V_{i+1} \cup V'_i \cup V''_{i-2} \cup V''_{i+2}$, not adjacent to any element of $V'_{i-2} \cup V'_{i+2} \cup V''_i$, and $V'_{i-1} \cup V'_{i+1} \cup V''_{i-1} \cup V''_{i+1} = \emptyset$.*

(b) *For each i, every element of V'_i is adjacent to every element of $V'_{i-1} \cup V'_{i+1} \cup V''_{i-2} \cup V''_i \cup V''_{i+2}$, every element of V''_i is adjacent to every element of $V''_{i-2} \cup V''_{i+2}$.*

(c) *For each i, any two non-adjacent elements of V'_i have the same sets of neighbors in $V'_i \cup V'_{i-2} \cup V'_{i+2} \cup V''_{i-1} \cup V''_{i+1}$.*

(4) (a) *For each i, every element of V_i is adjacent to at most one element of $V_{i+2} \cup V_{i-2}$. Moreover, for any i and $j \in \{i-2, i+2\}$, there are no two elements of V_i having neighbors in V_j.*

(b) *If an element of V_i and an element of V_j are adjacent, where $j \in \{i-2, i+2\}$, then $V_{\frac{i+j}{2}} \cup \bigcup_{s=1}^{5}(V'_s \cup V''_s) = \emptyset$.*

(5) *For each i, none of the elements of $V_i \cup V'_i$ has a neighbor outside $\bigcup_{i=1}^{5} N(v_i)$.*

(6) *For each i, every element of V''_i that has a neighbor outside $\bigcup_{i=1}^{5} N(v_i)$ is adjacent to every element of $V''_{i-1} \cup V''_{i+1}$.*

Proof. (1) Let $a \in V''''$ and $b \in \bigcup_{i=1}^{5}(V_i \cup V''_i \cup V'''_i)$ be non-adjacent vertices. If $b \in V_i$, then $a, b, v_{i-1}, v_i, v_{i+1}$ induce a $\overline{P_3 + P_2}$. If $b \in V''_i$, then $a, b, v_{i-1}, v_{i-2}, v_{i+1}$ induce a $\overline{P_3 + P_2}$. If $b \in V'''_i$, then $a, b, v_i, v_{i+1}, v_{i+2}$ induce a $\overline{P_3 + P_2}$.

(2) The set V'''' is a clique; otherwise any two non-adjacent its vertices, v_1, v_2, v_4 induce a $\overline{P_3 + P_2}$. For each i, the set V_i is independent; otherwise any two adjacent its vertices, v_{i-1}, v_i, v_{i+1} induce a $\overline{P_3 + P_2}$. For each i, the set V''_i is independent; otherwise any two adjacent its vertices, v_{i-1}, v_i, v_{i+1} induce a $\overline{P_3 + P_2}$.

(3) Let a_1 be an element of V_i. It is adjacent to every element of $V_{i-1} \cup V_{i+1} \cup V'_{i-1} \cup V'_{i+1}$; otherwise G contains a P_5, induced by a_1, some element of the set, and $v_{i-1}, v_{i-2}, v_{i+2}$ or $v_{i+1}, v_{i+2}, v_{i-2}$. Hence, $V'_{i-1} \cup V'_{i+1}$ must be empty; otherwise some its element, $a_1, v_{i-1}, v_i, v_{i+1}$ induce a $\overline{P_3 + P_2}$. The vertex a_1 is adjacent to every element of $V'_i \cup V''_{i-2} \cup V''_{i+2}$; otherwise some its element, $a_1, v_{i-1}, v_i, v_{i+1}$ induce a $\overline{P_3 + P_2}$. If $V''_{i-1} \cup V''_{i+1}$ has an element b_1, then $(a_1, b_1) \in E(G)$; otherwise $a_1, v_{i+1}, v_i, b_1, v_{i-2}$ or $a_1, v_{i-1}, v_i, b_1, v_{i+2}$ induce a P_5. Hence, $a_1, b_1, v_{i-1}, v_i, v_{i+1}$ induce a $\overline{P_3 + P_2}$. If a_1 has a neighbor $b_2 \in V''_i$, then $a_1, b_2, v_{i-1}, v_i, v_{i+1}$ induce a $\overline{P_3 + P_2}$. If a_1 has a neighbor $b_3 \in V'_{i-2} \cup V'_{i+2}$, then $v_{i+2}, b_3, a_1, v_{i-1}, v_i$ or $v_{i-2}, b_3, a_1, v_{i+1}, v_i$ induce a P_5.

Let a_2 be an element of V'_i. It is adjacent to every element of $V'_{i-1} \cup V'_{i+1}$; otherwise G contains a P_5, induced by a_2, some element of the set, and $v_{i-1}, v_{i-2}, v_{i+2}$ or $v_{i-2}, v_{i+1}, v_{i+2}$. The vertex a_2 is adjacent to every element of V''_i; otherwise some element of V''_i, $a_2, v_{i-1}, v_i, v_{i+1}$ induce a $\overline{P_3 + P_2}$. The vertex a_2 is adjacent to every element of $V''_{i-2} \cup V''_{i+2}$; otherwise some its element, $a_2, v_i, v_{i-2}, v_{i+2}$ induce a P_5.

Every element of V''_i is adjacent to every element of $V''_{i-2} \cup V''_{i+2}$; otherwise an element of V''_i, and an element of $V''_{i-2} \cup V''_{i+2}$, and $v_{i-1}, v_{i+1}, v_{i+2}$ or $v_{i-1}, v_{i-2}, v_{i+1}$ induce a $\overline{P_3 + P_2}$.

Let a' and a'' be arbitrary non-adjacent vertices of V'_i, b' be an element of $V'_i \cup V'_{i-2} \cup V'_{i+2} \cup V''_{i-1} \cup V''_{i+1}$, adjacent to a' and not adjacent to a''. If $b' \in V'_i$, then $a', a'', b', v_{i-1}, v_{i+1}$ induce a $\overline{P_3 + P_2}$. If $b' \in V'_{i-2} \cup V'_{i+2} \cup V''_{i-1} \cup V''_{i+1}$, then $a'', v_{i+1}, a', b', v_{i-2}$ or $a'', v_{i-1}, a', b', v_{i+2}$ induce a P_5.

(4) Let v' be an arbitrary vertex of V_i. Without loss of generality, let it be adjacent to $u' \in V_{i-2}$ and $u'' \in V_{i-2} \cup V_{i+2}$. If $u'' \in V_{i-2}$, then u' and u'' are not adjacent, by the second part of this lemma. Hence, $v', u', u'', v_{i-1}, v_{i+2}$ induce a $\overline{P_3 + P_2}$. If $u'' \in V_{i+2}$, then $(u', u'') \in E(G)$, by this lemma (part 3-a), and $v', u', u'', v_{i-1}, v_{i-2}$ induce a $\overline{P_3 + P_2}$.

If elements v^* and v^{**} of V_i have neighbors u^*, u^{**} in V_j, respectively, then (v^*, u^{**}) and (v^{**}, u^*) are not edges of G, by the previous sentences. By Lemma 6 (part 2), (v^*, v^{**}) and (u^*, u^{**}) are not edges of G. Therefore, v^*, v^{**}, u^*, u^{**}, and v_{i-1} or v_{i+1} induce a P_5 in G.

Let w_1 and w_2 be arbitrary adjacent elements of V_i and V_j, respectively, and $w_3 \in V_{\frac{i+j}{2}} \cup \bigcup_{s=1}^{5} (V'_s \cup V''_s)$. If $w_3 \in V_{\frac{i+j}{2}}$, then, by Lemma 6 (part 3-a), $(w_3, w_2) \in E(G)$ and $(w_3, w_1) \in E(G)$. Then $v_i, v_{\frac{i+j}{2}}, w_1, w_2, w_3$ induce a $\overline{P_3 + P_2}$. If $w_3 \in \bigcup_{i=1}^{5} V'_i$, then $w_3 \in V'_i$ or $w_3 \in V'_j$, by Lemma 6 (part 3-a). By Lemma 6 (part 3-a), w_3 is adjacent to w_1 and not adjacent to w_2 in the first case, and it is adjacent to w_2 and not adjacent to w_1 in the second. Then $v_i, w_3, w_1, w_2, v_{\frac{i+j}{2}+2}$ or $v_j, w_3, w_2, w_1, v_{j+2}$ induce a P_5. If $w_3 \in \bigcup_{i=1}^{5} V''_i$, then $w_3 \in V''_i$ or $w_3 \in V''_j$, by Lemma 6 (part 3-a). By Lemma 6 (part 3-a), w_3 is adjacent to w_2 and not adjacent to w_1 in the first case, and it is adjacent to w_1 and not adjacent to w_2 in the second. Then $\{w_1, w_2, w_3\} \cup N(w_1) \cap N(w_3) \cap V(C_5)$ or $\{w_1, w_2, w_3\} \cup N(w_2) \cap N(w_3) \cap V(C_5)$ induce a $\overline{P_3 + P_2}$.

(5) For each i, any element of $V_i \cup V'_i$ has no neighbor outside $\bigcup_{i=1}^{5} N(v_i)$, as G contains an induced P_5 otherwise.

(6) Let x be a vertex in V''_i that has a neighbor $y \notin \bigcup_{i=1}^{5} N(v_i)$, let z be an arbitrary element of $V''_{i-1} \cup V''_{i+1}$. If $(x, z) \notin E(G)$, $(y, z) \notin E(G)$, then y, x, v_{i-2}, z, v_i or y, x, v_{i+2}, z, v_i induce a P_5. If $(x, z) \notin E(G)$, $(y, z) \in E(G)$, then $x, y, z, v_{i-2}, v_{i+2}$ induce a $\overline{P_3 + P_2}$. ■

Lemma 7. *If $\bigcup_{j=1}^{5} V'''_j = \emptyset$, then $|V(G)| \le 15$ or G is O_3-free.*

Proof. Let $\hat{V}$ be the subset of all the elements of $\bigcup_{i=1}^{5} N(v_i)$ having at least one neighbor outside $\bigcup_{i=1}^{5} N(v_i)$. By Lemma 6 (parts 2,3-b,6), $\hat{V} \cap \bigcup_{i=1}^{5} V''_i$ is a clique. This fact and Lemma 6 (parts 1,2,5) imply that $\hat{V}$ is a clique. This set must be empty; otherwise it is a clique separator of G.

Let V_i be non-empty for some i, and let v be an arbitrary element of V_i. The set $\{v, v_i\}$ is not a module in G if and only if v has a neighbor in $V_{i+2} \cup V_{i-2}$, by Lemma 6 (parts 1,2,3-a,5). Hence, $|V_i| \le 2$, by Lemma 6 (parts 1,2,3-a,4-a,5); otherwise some two of its elements constitute a module in G. Additionally, $\bigcup_{s=1}^{5} (V'_s \cup V''_s) = \emptyset$, by Lemma 6 (part 4-b). By the previous lemma (part 1), V'''' must be empty; otherwise $V(G) \setminus V''''$ is a non-trivial module. Hence, $|V(G)| \le 5 + \sum_{j=1}^{5} |V_j| \le 15$.

Suppose that $\bigcup_{i=1}^{5} V_i = \emptyset$. One may show that V'_i is a clique for each i; otherwise any two non-adjacent its elements constitute a module, by Lemma 6 (parts 1,3-b,3-c,5). Suppose that G has three pairwise non-adjacent vertices. None of them belongs to $V(C_5)$, by Lemma 6 (parts 2 and 3-b) and the fact that V'_i is a clique for each i. If one of them belongs to V'''', then the second and third must belong to V'_i and to V'_{i+2} for some i, by Lemma 6 (parts 1,2,3-b) and the fact that V'_i is a clique for each i. The graph G has a P_5, induced by the three vertices and v_i, v_{i+2}. Suppose that none of the three vertices belongs to V''''. Clearly, at least one of them must belong to $\bigcup_{s=1}^{5} V'_s$, by Lemma 6 (parts 2 and 3-b). Suppose that it belongs to $V'_{i'}$. Then, the other two must belong to $V'_{i'-2} \cup V'_{i'+2} \cup V''_{i'-1} \cup V''_{i'+1}$, by Lemma 6 (part 3-b) and the fact that V'_i is a clique for each i. Hence, by Lemma 6 (parts 2 and 3-b) and the fact, one of them belongs to $V'_{i'+2}$, the second to $V''_{i'+1}$ or one belongs to $V'_{i'-2}$, the second to $V''_{i'-1}$. Hence, a vertex in V'_i, a vertex in $V'_{i'+2}$, a vertex in $V''_{i'+1}$, $v_{i'}$, and $v_{i'+2}$ or $v_{i'-2}$ induce a P_5. We have a contradiction with our assumption. ■

Lemma 8. *Let $V'''_i \neq \emptyset$. Then the following statements are true:*

1. $|V'''_i| = 1$, $V'''_{i-1} = V'''_{i+1} = \emptyset$, $\bigcup_{j=1}^{5} V'_j = \emptyset$, *and* $\bigcup_{j=1, j \neq i}^{5} V''_j = \emptyset$.
2. *The element of* V'''_i *is adjacent to every element of* V_i. *Every element of* $(\bigcup_{j=1, j\neq i} V_i) \cup V''_i \cup V'''_{i-2} \cup V'''_{i+2}$ *is not adjacent to the element of* V'''_i.
3. *If* $V''_i \neq \emptyset$, *then* $\bigcup_{j=1, j\neq i}^{5} V_j = \emptyset$, *and every element of* $V''_i \cup V'''_i$ *has no neighbor outside* $\bigcup_{i=1}^{5} N(v_i)$.

Proof. Let a be an arbitrary element of V'''_i.

(1) If there is a vertex $b_1 \in V'''_i \setminus \{a\}$, then a and b_1 must be adjacent; otherwise $v_i, v_{i+2}, v_{i-2}, a, b_1$ induce a $\overline{P_3 + P_2}$. Then, $v_i, v_{i+1}, v_{i+2}, a, b_1$ induce a $\overline{P_3 + P_2}$. If there is a vertex $b_2 \in V'''_{i-1} \cup V'''_{i+1}$, then a and b_2 are not adjacent; otherwise

$v_i, v_{i+1}, v_{i+2}, a, b_2$ or $v_i, v_{i-1}, v_{i-2}, a, b_2$ induce a $\overline{P_3+P_2}$. Then, $v_{i-2}, a, v_i, v_{i+1}, b_2$ or $v_{i+2}, a, v_i, v_{i-1}, b_2$ induce a P_5. Assume that there is a vertex $b_3 \in \bigcup_{j=1}^{5} V'_j \cup \bigcup_{j=1, j\neq i}^{5} V''_j$. If $b_3 \in V'_{i-2} \cup V'_{i+2}$ and $(a, b_3) \notin E(G)$, then $b_3, v_{i-2}, a, v_i, v_{i+1}$ or $b_3, v_{i+2}, a, v_i, v_{i-1}$ induce a P_5. If $b_3 \in V''_{i-2} \cup V''_{i+2}$ and $(a, b_3) \notin E(G)$, then $a, b_3, v_i, v_{i+1}, v_{i+2}$ or $a, b_3, v_i, v_{i-1}, v_{i-2}$ induce a $\overline{P_3+P_2}$. If $b_3 \in V'_{i-2} \cup V'_{i+2} \cup V''_{i-2} \cup V''_{i+2}$ and $(a, b_3) \in E(G)$, then $b_3, a, v_i, v_{i+1}, v_{i+2}$ or $b_3, a, v_i, v_{i-1}, v_{i-2}$ induce a $\overline{P_3+P_2}$. If $b_3 \in V'_{i-1} \cup V'_{i+1} \cup V''_{i-1} \cup V''_{i+1}$ and $(a, b_3) \notin E(G)$, then $a, b_3, v_i, v_{i-1}, v_{i-2}$ or $a, b_3, v_i, v_{i+1}, v_{i+2}$ induce a $\overline{P_3+P_2}$. If $b_3 \in V'_{i-1} \cup V'_{i+1}$ and $(a, b_3) \in E(G)$, then $v_{i-1}, b_3, a, v_{i+2}, v_{i+1}$ or $v_{i+1}, b_3, a, v_i, v_{i-2}, v_{i-1}$ induce a P_5. If $b_3 \in V''_{i-1} \cup V''_{i+1}$ and $(a, b_3) \in E(G)$, then $a, b_3, v_i, v_{i+1}, v_{i+2}$ or $a, b_3, v_i, v_{i+1}, v_{i+2}$ induce a $\overline{P_3+P_2}$. If $b_3 \in V'_i$ and $(a, b_3) \notin E(G)$, then $v_{i-1}, b_3, v_{i+1}, v_{i+2}, a$ induce a P_5. If $b_3 \in V'_i$ and $(a, b_3) \in E(G)$, then $a, b_3, v_i, v_{i+1}, v_{i+2}$ induce a $\overline{P_3+P_2}$.

(2) If there is an element $b' \in V_i$, non-adjacent to a, then $b', v_{i+1}, v_i, a, v_{i-2}$ induce a P_5. Let a vertex b'' belong to $(\bigcup_{j=1, j\neq i} V_i) \cup V''_i \cup V'''_{i-2} \cup V'''_{i+2}$. If $b'' \in V_{i-1} \cup V_{i+1} \cup V''_i \cup V'''_{i-2} \cup V'''_{i+2}$ and $(a, b'') \in E(G)$, then $a, b'', v_i, v_{i+1}, v_{i+2}$ or $a, b'', v_i, v_{i-1}, v_{i-2}$ induce a $\overline{P_3+P_2}$. If $b'' \in V_{i-2} \cup V_{i+2}$ and $(a, b'') \in E(G)$, then $a, b'', v_{i+1}, v_{i+2}, v_{i-2}$ or $a, b, v_{i-1}, v_{i-2}, v_{i+2}$ induce a $\overline{P_3+P_2}$.

(3) Let b''' be an arbitrary element of V''_i, and let b^* be an arbitrary element of $\bigcup_{j=1, j\neq i}^{5} V_j$. Clearly, $b^* \in V_{i-2} \cup V_{i+2}$, by Lemma 6 (part 3-a). By Lemma 8 (part 2), $(a, b^*) \notin E(G)$ and $(a, b''') \notin E(G)$. Then, $(b''', b^*) \in E(G)$, by Lemma 6 (part 3-a). Hence, $b^*, b''', v_{i-2}, a, v_i$ or $b^*, b''', v_{i+2}, a, v_i$ induce a P_5. If $c \in (N(a) \cup N(b''')) \setminus \bigcup_{i=1}^{5} N(v_i)$, then $c \in N(a) \cap N(b''')$; otherwise c, a, v_i, v_{i+1}, b''' or c, b''', v_{i+1}, v_i, a induce a P_5. Then, $a, c, b''', v_{i-2}, v_{i+2}$ induce a $\overline{P_3+P_2}$. ∎

Lemma 9. *If $\bigcup_{j=1}^{5} V'''_j \neq \emptyset$, then G has at most 23 vertices.*

Proof. Assume $V'''_i \neq \emptyset$. Suppose that $V''_i \neq \emptyset$. Hence, $\bigcup_{j=1, j\neq i}^{5} V'''_j = \bigcup_{j=1, j\neq i}^{5} V''_j = \bigcup_{j=1}^{5} V'_j = \bigcup_{j=1, j\neq i}^{5} V_j = \emptyset$, by Lemma 8 (parts 1 and 3). The set of all the vertices having a neighbor outside $\bigcup_{i=1}^{5} N(v_i)$ must be empty. Otherwise, by Lemma 6 (parts 2,5) and Lemma 8 (part 3), any vertex of this type must belong to V'''' and V'''' is a clique separator in G. The set V_i has at most one element; otherwise it is a non-trivial module, by Lemma 6 (parts 1,2,3-a) and Lemma 8 (part 2). Similarly, V''_i has at most one element; otherwise it is a non-trivial module of G. Moreover, $V'''' = \emptyset$; otherwise $V(G) \setminus V''''$ is a non-trivial module in G, by Lemma 6 (part 1). Hence, $|V(G)| \le 5 + |V_i| + |V''_i| + |V'''_i| \le 8$.

Suppose that $V''_i = \emptyset$ and $V'''_{i-2} = V'''_{i+2} = \emptyset$. Hence, $\bigcup_{j=1, j\neq i}^{5} V'''_j = \bigcup_{j=1}^{5} V''_j = \bigcup_{j=1}^{5} V'_j = \emptyset$, by Lemma 8 (part 1). The set of all the vertices having a neighbor outside $\bigcup_{i=1}^{5} N(v_i)$ is a clique separator, by Lemma 6 (parts 2,5) and Lemma 8 (part 1). Hence, this set must be empty. Clearly, $V'''' = \emptyset$; otherwise $V(G) \setminus V''''$ is a non-trivial module in G, by Lemma 6 (part 1). For each j, V_j contains at most three elements; otherwise some its two vertices constitute a non-trivial module in G, by Lemma 6 (parts 1,2,3-a,4-a,5) and Lemma 8 (part 2). Hence, $|V(G)| \le 5 + |V'''_i| + \sum_{j=1}^{5} |V_j| \le 21$, by Lemma 8 (part 1).

Suppose that $V''_i = \emptyset$ and $|V'''_{i-2}| + |V'''_{i+2}| > 0$. Hence, $|V'''_{i-2}| + |V'''_{i+2}| = 1$, by Lemma 8 (part 1). Without loss of generality, $|V'''_{i-2}| = 1$. Hence, $V'''_{i-1} = V'''_{i+1} = V'''_{i+2} = \bigcup_{j=1}^{5} V''_j = \bigcup_{j=1}^{5} V'_j = \emptyset$, by Lemma 8 (part 1). For each j, V_j contains at most three elements; otherwise some two of its vertices constitute a non-trivial module in G, by Lemma 6 (parts 1,2,3-a,4-a,5) and Lemma 8 (part 2). Let a and b be the elements of V'''_i and V'''_{i-2}, respectively. Then, $(a, b) \notin E(G)$, by Lemma 8 (part 2), and $N(a) \setminus \bigcup_{i=1}^{5} N(v_i) = N(b) \setminus \bigcup_{i=1}^{5} N(v_i)$; otherwise an element of $N(a) \setminus N(b)$ or an element of $N(b) \setminus N(a)$, and a, b, v_{i+2}, v_{i+1} induce a P_5. If $N(a) \setminus \bigcup_{i=1}^{5} N(v_i)$ is empty, then V'''' is empty; otherwise it is a clique separator in G or $V(G) \setminus V''''$ is a non-trivial module, by Lemma 6 (parts 1,2,5). Hence, $|V(G)| \le 5 + |V'''_i| + |V'''_{i-2}| + \sum_{j=1}^{5} |V_j| \le 22$. Suppose that $N(a) \setminus \bigcup_{i=1}^{5} N(v_i)$ is not empty. None of the elements of $N(a) \setminus \bigcup_{i=1}^{5} N(v_i)$ is adjacent to a vertex in $V(G) \setminus (\bigcup_{i=1}^{5} N(v_i) \cup N(a))$; otherwise an element of $V(G) \setminus (\bigcup_{i=1}^{5} N(v_i) \cup N(a))$, an element of $N(a) \setminus \bigcup_{i=1}^{5} N(v_i)$, a, v_{i+2}, v_{i+1} induce a P_5. Every element of V'''' is adjacent to every element of $N(a) \setminus \bigcup_{i=1}^{5} N(v_i)$; otherwise an element of V'''', an element of $N(a) \setminus \bigcup_{i=1}^{5} N(v_i)$, a, b, v_i induce a $\overline{P_3+P_2}$ (see Lemma 6, part 1). Hence, V'''' must be empty; otherwise $V(G) \setminus V''''$ is a non-trivial module or V'''' is a clique separator, by Lemma 6 (parts 1,2). Moreover, $|N(a) \setminus \bigcup_{i=1}^{5} N(v_i)| \le 1$, as $N(a) \setminus \bigcup_{i=1}^{5} N(v_i)$ is a module in G, by Lemma 6 (part 6). Hence, $|V(G)| \le 5 + |V'''_i| + |V'''_{i-2}| + \sum_{j=1}^{5} |V_j| + 1 \le 23$. ∎

3.6. Some complexity results for the weighted coloring problem

Lemma 10. *The weighted coloring problem for an O_3-free graph (G, w) can be solved in $O((\sum_{v\in V(G)} w(v))^3)$ time.*

Proof. First, construct an unweighted graph G' on $(\sum_{v\in V(G)} w(v))^3$ vertices as follows. For each $v \in V(G)$, V'_v is a clique of G' on $w(v)$ vertices. A vertex of V'_v and a vertex of V'_u are adjacent if and only if $(v, u) \in E(G)$. Clearly, $\chi_w(G) = \chi(G')$ and G' is O_3-free. Moreover, $\chi(G') = |V(G')| - \pi(G')$, where $\pi(G')$ is the size of a maximum matching of G'. This size can be computed in $O(|V(G')|^3)$ time [9]. ∎

Lemma 11. *For each fixed C, the weighted coloring problem can be solved in time, bounded by a polynomial on the sum of weights in class of all graphs having at most C vertices.*

Proof. Clearly, the weighted coloring problem for any weighted graph (G, w) on at most C vertices can be solved in $O((\sum_{v\in V(G)} w(v))^{O(1)})$ time, where a hidden exponent constant depends on C. ■

4. Main result

Theorem 5. *For each fixed p, the coloring problem can be solved in polynomial time for Free*$(\{P_5, K_p - e\})$. *The weighted coloring problem can be solved in polynomial time for Free*$(\{P_5, \overline{P_3 + P_2}\})$.

Proof. It is known that the inequality $\chi(G) \leq 4^{w(G)-1}$ holds for any P_5-free graph G [16]. Moreover, for each fixed k, the k-colorability problem can be solved in polynomial time for P_5-free graphs [17]. Hence, by these results, Theorem 2 and Lemma 4, the coloring problem for $\{P_5, K_p - e\}$-free graphs can be polynomially reduced to the same problem for O_3-graphs. The coloring problem for O_3-free graphs is polynomially equivalent to determining the size of maximum matchings in the complement graphs. Hence, for each fixed p, the coloring problem can be solved in polynomial time for *Free*$(\{P_5, K_p - e\})$. The weighted coloring problem can be polynomially solved in the class of perfect graphs [15]. Perfect graphs can be recognized in polynomial time [4]. Any step of the modular decomposition technique keeps the sum of weights. Hence, by these facts, Theorems 2 and 3, and Lemmas 7 and 9–11, the weighted coloring problem can be solved in polynomial time for *Free*$(\{P_5, \overline{P_3 + P_2}\})$. ■

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