AROUND THE ABHYANKAR–SATHAYE CONJECTURE

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ABSTRACT. A "rational" version of the strengthened form of the Commuting Derivation Conjecture, in which the assumption of commutativity is dropped, is proved. A systematic method of constructing in any dimension greater than 3 the examples answering in the negative a question by M. El Kahoui is developed.

1. INTRODUCTION

Throughout this paper k stands for an algebraically closed field of characteristic zero which serves as domain of definition for each of the algebraic varieties considered below.

Recall that an element c of the polynomial ring $k[x_1, \ldots, x_n]$ in variables $x_1, \ldots, x_n$ with coefficients in k is called a *coordinate* if there are the elements $t_1, \ldots, t_{n-1} \in k[x_1, \ldots, x_n]$ such that

$$k[c, t_1, \ldots, t_{n-1}] = k[x_1, \ldots, x_n]$$

(see, e.g., [vdE 00]). Every coordinate is irreducible and, if $x_1, \ldots, x_n$ are the standard coordinate functions on the affine space $\mathbf{A}^n$, then the zero locus $\{c = 0\}$ of c in $\mathbf{A}^n$ is isomorphic to $\mathbf{A}^{n-1}$. The converse is claimed by the classical

Abhyankar–Sathaye Conjecture. *If $f \in k[x_1, \ldots, x_n]$ is an irreducible element whose zero locus in $\mathbf{A}^n$ is isomorphic to $\mathbf{A}^{n-1}$, then f is a coordinate.*

This conjecture is equivalent to the claim that every closed embedding $\iota\colon \mathbf{A}^{n-1} \hookrightarrow \mathbf{A}^n$ is *rectifiable*, i.e., there is an automorphism $\sigma \in \operatorname{Aut} \mathbf{A}^n$ such that $\sigma \circ \iota\colon \mathbf{A}^{n-1} \hookrightarrow \mathbf{A}^n$ is the standard embedding $(a_1, \ldots, a_{n-1}) \mapsto (a_1, \ldots, a_{n-1}, 0)$ (see [vdE 00, Lemma 5.3.13]).

For $n = 2$ the Abhyankar–Sathaye conjecture is true (the Abhyankar–Moh–Suzuki theorem). For $n \geqslant 3$ it is still open, though there is a belief that in general it is false [vdE 00, p. 103].

Exploration of this conjecture leads to the problem of constructing closed hypersurfaces in $\mathbf{A}^n$ isomorphic to $\mathbf{A}^{n-1}$, and irreducible polynomials in $k[x_1, \ldots, x_n]$ whose zero loci in $\mathbf{A}^n$ are such hypersurfaces. The following

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two facts lead, in turn, to the idea of linking this problem with unipotent group actions:

(i) Every homogeneous space U/H, where U is a unipotent algebraic group and H its closed subgroup, is isomorphic to $\mathbf{A}^{\dim U/H}$ (see, e.g., [Gr 58, Prop. 2(ii)]).

(ii) All orbits of every morphic unipotent algebraic group action on a quasi-affine variety X are closed in X (see [Ro 61_2, Thm. 2]).

In view of (i) and (ii), every orbit of a morphic unipotent algebraic group action on $\mathbf{A}^n$ is the image of a closed embedding of some $\mathbf{A}^d$ in $\mathbf{A}^n$. In particular, orbits of dimension $n-1$ are the hypersurfaces of the sought-for type. Such actions, with a view of getting an approach to the Abhyankar–Sathaye conjecture, have been the object of study during the last decade, see [Ma 03], [EK 05], [DEM 08], [DEFM 11]. In particular, for commutative unipotent algebraic group actions, the following conjecture (whose formulation uses the equivalent language of locally nilpotent derivations, see [Fr 06]) has been put forward:

Commuting Derivations Conjecture ([Ma 03]). *Let D be a set of $n-1$ commuting locally nilpotent k-derivations of $k[x_1,\ldots,x_n]$ linearly independent over $k[x_1,\ldots,x_n]$. Then*

$$\{f\in k[x_1,\ldots,x_n]\mid \partial(f)=0 \textit{ for every derivation } \partial\in D\}=k[c], \tag{1}$$

where c is a coordinate in $k[x_1,\ldots,x_n]$.

This conjecture is open for $n>3$, proved in [Ma 03] for $n=3$, and follows from Rentschler's theorem [Re 68] for $n=2$. In [EK 05, Cor. 4.1] is shown that it is equivalent to a weak version of the Abhyankar–Sathaye conjecture.

On the other hand, in [EK 05] is raised the question as to which extent $k[x_1,\ldots,x_n]$ is characterized by property (1). Namely, let $\mathcal{A}$ be a commutative associative unital k-algebra of transcendence degree $n>0$ over k such that

(a) $\mathcal{A}$ is a unique factorization domain;

(b) there is a set D of $n-1$ commuting linearly independent over $\mathcal{A}$ locally nilpotent k-derivations of $\mathcal{A}$.

Consider the invariant algebra of D, i.e., the k-algebra

$$\mathcal{A}^D:=\{a\in\mathcal{A}\mid \partial(a)=0 \text{ for every } \partial\in D\}.$$

Question 1 ([EK 05, p. 449]). *Does the equality*

$$\mathcal{A}^D=k[c] \textit{ for some element } c\in\mathcal{A} \tag{2}$$

imply the existence of elements $s_1,\ldots,s_{n-1}\in\mathcal{A}$ and $c_1,\ldots,c_{n-1}\in k[c]$ such that $\mathcal{A}$ is the polynomial k-algebra $k[c,s_1,\ldots,s_{n-1}]$ and $D=\{c_i\partial_{s_i}\}_{i=1}^{n-1}$?

This question is inspired by one of the main results of [EK 05], Theorem 3.1, claiming that for $n=2$ the answer is affirmative. By [Mi 95, Thm. 2.6], given properties (a) and (b), equality (2) holds and the answer to Question 1 is affirmative if $\mathcal{A}$ is finitely generated over k, the multiplicative group $\mathcal{A}^\star$ of invertible elements of $\mathcal{A}$ coincides with $k^\star$, and $n=2$.

The present paper contributes to the Commuting Derivation Conjecture and Question 1. In Section 2 is proved a "rational" version of the strengthened form of the Commuting Derivation Conjecture, in which the assumption of commutativity is dropped (see Theorem 2). Here "rational" means that the notion of "coordinate" is replaced by that of "rational coordinate" (see Definition 1 below). Geometrically, the latter means the existence of a birational (rather than biregular) automorphism of the ambient affine space that rectifies the corresponding hypersurface into the standard coordinate hyperplane. In Section 3, for every $n \geqslant 4$, is given a systematic method of constructing the pairs $(\mathcal{A}, D)$, for which the answer to Question 1 is negative. Section 4 contains some remarks.

Notation, conventiones, and some generalities

Below, as in [Bor 91], [Sp 98], "variety" means "algebraic variety" in the sense of Serre. The standard notation and conventions of [Bor 91], [Sp 98], and [PV 94] are used freely. In particular, the algebra of functions regular on a variety X is denoted by $k[X]$ (not by $\mathcal{O}(X)$ as in [DEFM 11], [DEM 08]).

Given an algebraic variety Z, below we denote the Zariski tangent space of Z at a point $z \in Z$ by $\mathrm{T}_{Z,z}$.

Let G be an algebraic group and let X be a variety. Given an action

$$\alpha\colon G \times X \to X \tag{3}$$

of G on X and the elements $g \in G$, $x \in X$, we denote $\alpha(g,x) \in X$ by $g \cdot x$. The G-orbit and the G-stabilizer of x are denoted resp. by $G \cdot x$ and G_x. If (3) is a morphism, then α is called a *regular* (or *morphic*) *action*. A regular action α is called *locally free* if there is a dense open subset U of X such that the G-stabilizer of every point of U is trivial.

Assume that X is irreducible. The map

$$\mathrm{Bir}\, X \to \mathrm{Aut}_k\, k(X), \qquad \varphi \mapsto (\varphi^*)^{-1}, \tag{4}$$

is a group isomorphism. We identify $\mathrm{Bir}\, X$ and $\mathrm{Aut}_k\, k(X)$ by means of (4) when we consider action of a subgroup of $\mathrm{Bir}\, X$ by k-automorphisms of $k(X)$ and, conversely, action of a subgroup of $\mathrm{Aut}_k k(X)$ by birational automorphisms of X.

Let $\theta\colon G \to \mathrm{Bir}\, X$ be an abstract group homomorphism. It determines an action of G on X by birational isomorphisms. If the domain of definition of the partially defined map $G \times X \to X$, $(g,x) \mapsto \theta(g)(x)$ contains a dense open subset of $G \times X$ and coincides on it with a rational map $\varrho\colon G \times X \dashrightarrow X$, then ϱ is called a *rational action* of G on X.

By [Ro 56, Thm. 1], for every rational action ϱ there is a regular action of G on an irreducible variety Y, the open subsets X_0 and Y_0 of resp. X and Y, and an isomorphism $Y_0 \to X_0$ such that the induced field isomorphism $k(X) = k(X_0) \to k(Y_0) = k(Y)$ is G-equivariant.

If ϱ is a rational action of G on X, then by

$$\pi_{G,X}\colon X \dashrightarrow X/\!\!/G$$

is denoted a rational quotient of ϱ, i.e., $X/\!\!/G$ and $\pi_{G,X}$ are resp. a variety and a dominant rational map such that $\pi_{G,X}^*(k(X/\!\!/G)) = k(X)^G$ (see [PV 94, Sect. 2.4]). Depending on the situation we choose $X/\!\!/G$ as a suitable variety within the class of birationally isomorphic ones. A *rational section* for ϱ is a rational map $\sigma\colon X/\!\!/G \dashrightarrow X$ such that $\pi_{G,X} \circ \sigma = \mathrm{id}$.

2. Rational coordinate

Definition 1. An irreducible element c of the polynomial ring $k[x_1, \ldots, x_n]$ in variables $x_1, \ldots, x_n$ with coefficients in k is called a *rational coordinate* if there are the elements $f_1, \ldots, f_{n-1} \in k(x_1, \ldots, x_n)$ such that

$$k(c, f_1, \ldots, f_{n-1}) = k(x_1, \ldots, x_n). \tag{5}$$

If c is a rational coordinate and (5) holds, then the hypersurface $\{c = 0\}$ is birationally isomorphic to $\mathbf{A}^{n-1}$ and the rational map

$$\tau\colon \mathbf{A}^n \dashrightarrow \mathbf{A}^n, \quad a \mapsto (c(a), f_1(a), \ldots, f_{n-1}(a))$$

is an element of Bir $\mathbf{A}^n$. Since $k(c, f_1, \ldots, f_{n-1}) = k(c, f_1c^{d_1}, \ldots, f_{n-1}c^{d_{n-1}})$ for any $d_1, \ldots, d_{n-1} \in \mathbf{Z}$, we may replace in (5) every f_i by an appropriate $f_ic^{d_i}$ and assume that the intersection of $\{c = 0\}$ with the domain of definition of τ is nonempty. Then the image of $\{c = 0\}$ under τ is defined and its closure is the standard coordinate hyperplane $\{x_1 = 0\}$. In other words, the hypersurface $\{c = 0\}$ is rectified by the birational automorphism $\tau \in \mathrm{Bir}\,\mathbf{A}^n$.

Theorem 1. *Let $\varrho\colon S \times X \dashrightarrow X$ be a rational action of a connected solvable affine algebraic group S on an irreducible algebraic variety X. Let*

$$\pi_{S,X}\colon X \dashrightarrow X/\!\!/S \tag{6}$$

be a rational quotient of this action. Then there are an integer $m \geqslant 0$ and a birational isomorphism $\varphi\colon X/\!\!/S \times \mathbf{A}^m \dashrightarrow X$ such that the following diagram is commutative

$$\begin{tikzcd} X/\!\!/S \times \mathbf{A}^m \arrow[rr, dashed, "\varphi"] \arrow[dr, "\mathrm{pr}_1"'] & & X \arrow[dl, dashed, "\pi_{S,X}"] \\ & X/\!\!/S & \end{tikzcd}.$$

Proof. Replacing X by a birationally isomorphic variety, we may (and shall) assume that the action ϱ is regular. Put

$$m_{S,X} := \max_{x \in X} \dim S \cdot x. \tag{7}$$

First, consider the case

$$\dim S = 1. \tag{8}$$

In this case $m_{S,X} \leqslant 1$. If $m_{S,X} = 0$, the action ϱ is trivial, hence $X/\!\!/S = X$, $\pi_{S,X} = \mathrm{id}$, and the claim is clear. Now let $m_{S,X} = 1$. This means that S-stabilizers of points of a dense open subset are finite. In this case, we may assume that

$$\text{the action } \varrho \text{ is locally free.} \tag{9}$$

To prove this claim, recall (see, e.g., [Sp 98, Thm. 3.4.9]) that, given (8), we have $S = \mathbf{G}_a$ or $\mathbf{G}_m$. If $S = \mathbf{G}_a$, then the claim follows from the fact that, due to the assumption $\operatorname{char} k = 0$, there are no nontrivial finite subgroups in S. If $S = \mathbf{G}_m$, then S/F is isomorphic to S for any finite subgroup F, see, e.g., [Sp 98, 2.4.8(ii) and 6.3.6]. Therefore, taking as F the kernel of ϱ, we may assume that ϱ is faithful. Since S is a torus, this, in turn, implies that ϱ is locally free, see [Po 13, Lemma 2.4]. Thus (9) holds.

Given (9), by [CTKPR 11, Thm. 2.13] we may replace X by an appropriate S-invariant open subset and assume that (6) is a torsor. Since S is a connected solvable affine algebraic group, by [Ro 56, Thm. 10] this torsor admits a rational section and therefore is trivial over an open subset of X/S. As the group variety of S is birationally isomorphic to $\mathbf{A}^1$, this completes the proof of theorem in the case when (8) holds.

In the general case we argue by induction on $\dim S$. If $\dim S > 0$, then solvability of S yields the existence of a closed connected normal subgroup N in S such that the (connected solvable affine) algebraic group $G := S/N$ is one-dimensional. Put $Y := X/N$. By the inductive assumption, there are an integer $r \geqslant 0$ and a birational isomorphism $\lambda\colon Y \times \mathbf{A}^r \dashrightarrow X$ such that the following diagram is commutative

$$\begin{tikzcd} Y \times \mathbf{A}^r \arrow[rr, dashed, "\lambda"] \arrow[dr, "\mathrm{pr}_1"'] & & X \arrow[dl, dashed, "\pi_{N,X}"] \\ & Y & \end{tikzcd} \quad . \tag{10}$$

Since $N \lhd S$ and $\pi^*_{N,X}(k(Y)) = k(X)^N$, the action ϱ induces a rational action of G on Y such that

$$Y/G = X/S, \tag{11}$$

$$\pi_{S,X} = \pi_{G,Y} \circ \pi_{N,X}. \tag{12}$$

Given (11) and using the proved validity of theorem for one-dimensional groups, we obtain that there are an integer $t \geqslant 0$ and a birational isomorphism $\gamma\colon X/S \times \mathbf{A}^t \dashrightarrow Y$ such that the following diagram is commutative

$$\begin{tikzcd} X/S \times \mathbf{A}^t \arrow[rr, dashed, "\gamma"] \arrow[dr, "\mathrm{pr}_1"'] & & Y \arrow[dl, dashed, "\pi_{G,Y}"] \\ & X/S & \end{tikzcd} \quad . \tag{13}$$

From (12) and diagrams (10), (13) we see that one can take $m = r + t$ and $\varphi = \lambda \circ (\gamma \times \mathrm{id}_{\mathbf{A}^r})$. This completes the proof. $\square$

Remark 1. The number m in the formulation of Theorem 1 is equal to the number $m_{S,X}$ given by (7).

Corollary. *In the notation of Theorem* 1*, there are the elements $f_1, \ldots, f_m$ of $k(X)$ such that*

(i) *$f_1, \ldots, f_m$ are algebraically independent over $k(X)^S$;*

(ii) $k(X) = k(X)^S(f_1, \ldots, f_m)$.

Theorem 2. *Let a unipotent affine algebraic group U regularly act on $\mathbf{A}^n$. If*

$$\max_{a \in \mathbf{A}^n} \dim U \cdot a = n - 1, \tag{14}$$

then $k[\mathbf{A}^n]^U = k[c]$, where c is a rational coordinate in $k[\mathbf{A}^n]$.

Proof. By Rosenlicht's theorem [Ro 56, Thm. 2] and the fiber dimension theorem, (14) implies that the transcendence degree of $k(\mathbf{A}^n)^U$ over k is 1 (cf. [PV 94, Sect. 2.3, Cor.]). Since U is unipotent, $k(\mathbf{A}^n)^U$ is the field of fractions of $k[\mathbf{A}^n]^U$, see [Ro 61_2, p. 220, Lemma]. By [Za 54] these properties imply that $k[\mathbf{A}^n]^U$ is a finitely generated k-algebra. Integral closedness of $k[\mathbf{A}^n]$ yields integral closedness of $k[\mathbf{A}^n]^U$, see [PV 94, Thm. 3.16]. Thus $\mathbf{A}^n /\!\!/ U := \operatorname{Spec} k[\mathbf{A}^n]^U$ is an irreducible smooth affine algebraic curve. This curve is rational by Lüroth's theorem because $k(\mathbf{A}^n)^U$ is a subfield of $k(\mathbf{A}^n)$. We then conclude that $\mathbf{A}^n /\!\!/ U$ is obtained from $\mathbf{P}^1$ by removing $s \geqslant 1$ points. Since $k[\mathbf{A}^n /\!\!/ U]^\star = k^\star$, we have $s = 1$, i.e., $\mathbf{A}^n /\!\!/ U = \mathbf{A}^1$, or, equivalently, $k[\mathbf{A}^n]^U = k[c]$ for an element $c \in k[\mathbf{A}^n]$. Since the group U is unipotent, it is connected (in view of $\operatorname{char} k = 0$) and admits no nontrivial algebraic homomorphisms $U \to \mathbf{G}_m$. By [PV 94, Thm. 3.1], this implies that every nonconstant irreducible element of $k[\mathbf{A}^n]$ dividing c lies in $k[\mathbf{A}^n]^U$, which, in turn, easily implies irreducibility of c.

We now claim that c is a rational coordinate in $k[\mathbf{A}^n]$. Indeed, since $k(\mathbf{A}^n)^U$ is the field of fractions of $k[\mathbf{A}^n]^U$, we have $k(\mathbf{A}^n)^U = k(c)$. Hence by (14), Remark 1, and Corollary of Theorem 1, there are the elements $f_1, \ldots, f_{n-1} \in k(\mathbf{A}^n)$ such that $k(\mathbf{A}^n) = k(c, f_1, \ldots, f_{n-1})$. Whence the claim by Definition 1. $\square$

3. Commuting derivations of unique factorization domains

First, we shall introduce the notation. Let G be a connected simply connected semisimple algebraic group. Fix a maximal torus T of G. Let X and $\mathrm{X}^\vee$ be respectively the character lattice and the cocharacter lattice of T in additive notation, and let $\langle\ ,\ \rangle\colon \mathrm{X} \times \mathrm{X}^\vee \to \mathbf{Z}$ be the natural pairing. The value of an element $\varphi \in \mathrm{X}$ at a point $t \in T$ denote by t^φ. Let Φ and $\Phi^+ \subset \mathrm{X}$ respectively be the root system of G with respect to T and the system of positive roots of Φ determined by a fixed Borel subgroup B of G containing T. Given a root $\alpha \in \Phi$, denote by $\alpha^\vee\colon \mathbf{G}_m \to T$ and U_α respectively the coroot and the one-dimensional unipotent root subgroup of G corresponding to α.

Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the system of simple roots of Φ_+ indexed as in [Bou 68]. If I is a subset of Δ, let Φ_I be the set of elements of Φ that are linear combinations of the roots in I. Denote by L_I be the subgroup of G generated by T and all the U_α's with $\alpha \in \Phi_I$. Let U_I (respectively, U_I^-) be the subgroup of G generated by all the U_α's with $\alpha \in \Phi^+ \setminus \Phi_I$ (respectively, $-\alpha \in \Phi^+ \setminus \Phi_I$). Then $P_I := L_I U_I$ and $P_I^- := L_I U_I^-$ are parabolic subgroups of G opposite to one another, U_I and U_I^- are the unipotent radicals of P_I

and P_I^- respectively, L_I is a Levi subgroup of P_I and P_I^-, and

$$\dim U_I = \dim U_I^- = |\Phi^+ \setminus \Phi_I|, \tag{15}$$

$$\dim G = \dim L_I + 2\dim U_I^-. \tag{16}$$

Every closed subgroup of G containing B is of the form P_I for some I. Every parabolic subgroup of G is conjugate to a unique P_I, called *standard* (with respect to T and B); see, e.g., [Sp 98, 8.4.3].

Let $\mathcal{D} \subset \mathrm{X}$ be the monoid of highest weights (with respect to T and B) of simple G-modules. Given a weight $\varpi \in \mathcal{D}$, let $E(\varpi)$ be a simple G-module with ϖ as the highest weight.

Denote by $\varpi_1, \ldots, \varpi_r$ the system of all indecomposable elements (i.e., *fundamental weights*) of $\mathcal{D}$ indexed in such a way that

$$\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}. \tag{17}$$

This system freely generates $\mathcal{D}$, i.e., for every weight $\varpi \in \mathcal{D}$ there are uniquely defined nonnegative integers $m_1, \ldots, m_r$ such that $\varpi = m_1\varpi_1 + \cdots + m_r\varpi_r$. By virtue of (17),

$$\langle \varpi, \alpha_i^\vee \rangle = m_i. \tag{18}$$

The integers (18) are called the *numerical labels of* ϖ. The "labeled" Dynkin diagram of $\alpha_1, \ldots, \alpha_r$, in which m_i is the label of the node α_i for every i, is called *the Dynkin diagram of* ϖ.

Given a nonzero $\varpi \in \mathcal{D}$, denote by $\mathbf{P}(E(\varpi))$ the projective space of all one-dimensional linear subspaces of $E(\varpi)$. The natural projection

$$\pi\colon E(\varpi) \setminus \{0\} \to \mathbf{P}(E(\varpi))$$

is G-equivariant with respect to the natural action of G on $\mathbf{P}(E(\varpi))$. The fixed point set of B in $\mathbf{P}(E(\varpi))$ is a single point $p(\varpi)$ and the G-orbit $\mathcal{O}(\varpi)$ of $p(\varpi)$ is the unique closed G-orbit in $\mathbf{P}(E(\varpi))$.

Consider in $E(\varpi)$ the affine cone $X(\varpi)$ over $\mathcal{O}(\varpi)$, i.e.,

$$X(\varpi) = \{0\} \sqcup \pi^{-1}(\mathcal{O}(\varpi)), \tag{19}$$

It is a G-stable irreducible closed subset of $E(\varpi)$. Let $\mathcal{A}(\varpi)$ be the coordinate algebra of $X(\varpi)$:

$$\mathcal{A}(\varpi) = k[X(\varpi)],$$

and let n be the transcendence degree of $\mathcal{A}(\varpi)$ over k, i.e.,

$$n = \dim X(\varpi). \tag{20}$$

Since every U_α is a one-dimensional unipotent group, its natural action on $X(\varpi)$ determines an algebraic vector field $\mathcal{F}_\alpha$ on $X(\varpi)$, which, in turn, determines a locally nilpotent derivation ∂_α of $\mathcal{A}(\varpi)$; see [Fr 06, 1.5]. Actually, ∂_α is induced by a locally nilpotent derivation of $k[E(\varpi)]$. Namely, as above, the natural action of U_α on $E(\varpi)$ determines a locally nilpotent derivation D_α of $k[E(\varpi)]$. Since the ideal $\mathcal{I}(\varpi)$ of $X(\varpi)$ in $k[E(\varpi)]$ is D_α-stable, D_α induces a locally nilpotent derivation of $\mathcal{A}(\varpi) = k[E(\varpi)]/\mathcal{I}(\varpi)$; the latter is ∂_α.

Theorem 3. *For every nonzero weight $\varpi \in \mathcal{D}$, the following hold*:

(i) *The stabilizer $G_{p(\varpi)}$ of $p(\varpi)$ in G is $P_{I(\varpi)}$, where*

$$I(\varpi) = \{\alpha \in \Delta \mid \langle \varpi, \alpha^\vee \rangle = 0\}. \tag{21}$$

(ii) $\dim U^-_{I(\varpi)} = \Phi^+ \setminus \Phi_{I(\varpi)} = n - 1$.

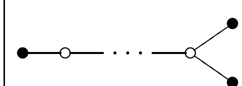
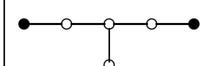
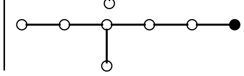
(iii) *The stabilizer of a point in general position for the natural action of $U^-_{I(\varpi)}$ on $X(\varpi)$ is trivial.*

(iv) *The set $\{\partial_{-\alpha} \mid \alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}\}$ of $n-1$ locally nilpotent derivations of the algebra $\mathcal{A}(\varpi)$ is linearly independent over $\mathcal{A}(\varpi)$.*

(v) *The following properties are equivalent*:

(C) $\{\partial_{-\alpha} \mid \alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}\}$ *is the set of commuting derivations. Equivalently, the unipotent group $U^-_{I(\varpi)}$ is commutative.*

(D) *In the Dynkin diagram of ϖ, every connected component S has at most one node with a nonzero label, and if such a node v exists, then S is not of type E_8, F_4, or G_2, and v is a black node of S colored as in the following table*:

type of S	colored S
A_l	
B_l	
C_l	
D_l	
E_6	
E_7	

(vi) $\mathcal{A}(\varpi)^\star = k^\star$.

(vii) *$\mathcal{A}(\varpi)$ is a unique factorization domain if and only if ϖ is a fundamental weight.*

(viii) *The following properties are equivalent*:

(s$_1$) *$X(\varpi)$ is singular*;

(s$_2$) $\dim E(\varpi) > n$;

(s$_3$) $X(\varpi) \neq E(\varpi)$.

The singular locus of every singular $X(\varpi)$ is the vertex 0.

Proof. (i): By the definition of $p(\varpi)$, the group B is contained in $G_{p(\varpi)}$. Hence

$$G_{p(\varpi)} = P_I \quad \text{for some } I. \tag{22}$$

In order to prove (i), fix a point $v \in \pi^{-1}(p(\varpi))$ and denote by G_v its stabilizer in G and by ℓ the line $\pi^{-1}(p(\varpi)) \cup \{0\}$ in $E(\varpi)$. We first show that the following properties of a root $\alpha \in \Delta$ are equivalent:

(a) $\alpha \in I$;

(b) $\langle \varpi, \alpha^\vee \rangle = 0$;

(c) the image of $\alpha^\vee$ is contained in G_v.

The definitions of $p(\varpi)$ and v imply that

$$t \cdot v = t^{\varpi} v \quad \text{for every element } t \in T, \tag{23}$$

and the definition of $\langle\ ,\ \rangle$ entails the equality

$$(\alpha^{\vee}(s))^{\varpi} = s^{\langle \varpi, \alpha^{\vee}\rangle} \quad \text{for every element } s \in \mathbf{G}_m. \tag{24}$$

Combining (23) and (24), we obtain the equivalence (b)$\Leftrightarrow$(c).

(a)$\Rightarrow$(c): By (22), the line ℓ is stable with respect to U_α. Being unipotent, the group U_α has no nontrivial characters and, therefore, no nontrivial one-dimensional modules. This proves that U_α is contained in G_v.

If (a) holds, then by (22) the line ℓ is stable with respect to $U_{-\alpha}$ as well. The same argument as for U_α then shows that $U_{-\alpha}$ is contained in G_v. Hence G_v contains the group S_α generated by U_α and $U_{-\alpha}$. But S_α contains the image of $\alpha^{\vee}$. This proves the implication (a)$\Rightarrow$(c).

(c)$\Rightarrow$(a): Assume that (c) holds. Since, as explained above, U_α is contained in G_v, the subgroup of S_α generated by U_α and the image of $\alpha^{\vee}$ is contained in G_v. This subgroup is a Borel subgroup of S_α. Therefore the S_α-orbit of v is a complete subvariety of $E(\varpi)$, i.e., a point. This means that S_α is contained in G_v. Therefore, $U_{-\alpha}$ is contained in G_v; whence (a) holds. This proves the implication (c)$\Rightarrow$(a).

Combining now (22) and (21) with the equivalence (a)$\Leftrightarrow$(c), we obtain the proof of part (i).

(ii): Since $X(\varpi)$ is the affine cone over $\mathcal{O}(\varpi)$, we have

$$\dim X(\varpi) = \dim \mathcal{O}(\varpi) + 1. \tag{25}$$

On the other hand, (15), (16), and (i) entails

$$\dim \mathcal{O}(\varpi) = \dim U^-_{I_\varpi}. \tag{26}$$

Combining (25), (26), and (20), we obtain the proof of part (ii).

(iii): Since $U^-_I \cap P_I = \{e\}$ for every I, the stabilizer of v for the natural action of $U^-_{I_\varpi}$ on $X(\varpi)$ is trivial because of (i). Hence $\dim X(\varpi)$ is the maximum of dimensions of $U^-_{I_\varpi}$-orbits in $X(\varpi)$. Since $U^-_{I_\varpi}$-orbits of points of a dense open subset of $X(\varpi)$ have maximal dimension, this means that the $U^-_{I_\varpi}$-stabilizer of a point in general position in $X(\varpi)$ is finite. But $U^-_{I_\varpi}$ has no nontrivial finite subgroups because it is a connected unipotent group and $\operatorname{char} k = 0$. This proves part (iii).

(iv): Given a point $a \in X(\varpi)$, denote its $U^-_{I(\varpi)}$-orbit by $U^-_{I(\varpi)} \cdot a$. By (iii), taking a suitable a, we may assume that

$$\dim \mathrm{T}_{U^-_{I(\varpi)} \cdot a, a} = \dim U^-_{I(\varpi)}. \tag{27}$$

Since $U^-_{I(\varpi)} = \prod_{\alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}} U_{-\alpha}$ (the product being taken in any order), and $\operatorname{char} k = 0$,

$$\mathrm{T}_{U^-_{I(\varpi)} \cdot a, a} = \text{the linear span of } \{\mathcal{F}_{-\alpha}(a) \mid \alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}\} \text{ over } k. \tag{28}$$

It follows from (27), (28), and (ii) that all the vectors $\mathcal{F}_{-\alpha}(a)$, where $\alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}$ are linearly independent over k. Hence all the vector fields $\mathcal{F}_{-\alpha}$, where $\alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}$, are linearly independent over $A(\varpi)$. This proves part (iv).

(v): Since standard parabolic subgroups of G are products of standard parabolic subgroups of connected simple normal subgroups of G, the proof is reduced to the case, where G is simple. In this case (C)$\Leftrightarrow$(D) follows from (21) and the known classification of parabolic subgroups that have commutative unipotent radical (see, e.g., [RRS 92, Lemma 2.2 and Rem. 2.3]).

(vi): Since $\varpi \neq 0$, the action of T on $\pi^{-1}(p(\varpi))$ is nontrivial and, therefore, transitive. Since the restriction of π to $X(\varpi) \setminus \{0\}$ is a G-equivariant morphism onto the orbit $O(\varpi)$, this entails that

$$G \cdot v = X(\varpi) \setminus \{0\}. \tag{29}$$

By [PV 72, Thm. 2],

$$A(\varpi) \to k[G \cdot v], \quad f \mapsto f|_{G \cdot v}$$

is an isomorphism of k-algebras. On the other hand, the orbit map $G \to G \cdot v$ induces the embedding of $k[G \cdot v]$ into $k[G]$, the coordinate algebra of G. By [Ro 61_1, Thm. 3], every element of $k[G]^\star$ is of the form cf, where $c \in k^\star$ and $f\colon G \to k^\star$ is a character of G. Being connected semisimple, G has no nontrivial characters; whence $k[G]^\star = k^\star$. This proves part (vi).

(vii): This is proved, basing on [Po 72, 74], in [PV 72, Thms. 4 and 5].

(viii): By virtue of (29), the singular locus of $X(\varpi)$ is either $\{0\}$ or empty. In particular, $X(\varpi)$ is singular if and only if 0 is the singular point of $X(\varpi)$, i.e., if and only if $\dim \mathrm{T}_{X(\varpi),0} > n$. The inclusion $X(\varpi) \subseteq E(\varpi)$ yields the inclusion $\mathrm{T}_{X(\varpi),0} \subseteq \mathrm{T}_{E(\varpi),0} = E(\varpi)$, and since 0 is a G-fixed point, $\mathrm{T}_{X(\varpi),0}$ is a submodule of the G-module $E(\varpi)$. As the latter is simple, $\mathrm{T}_{X(\varpi),0} = E(\varpi)$. This proves ($\mathrm{s}_1$)$\Leftrightarrow$($\mathrm{s}_3$). As ($\mathrm{s}_2$)$\Leftrightarrow$($\mathrm{s}_3$) is clear, this completes the proof of (viii). $\square$

Thus, for every fundamental weight ϖ such that

— the property specified in Theorem 3(v)(D) holds;
— the variety $X(\varpi)$ is singular,

the answer to Question 1 for the pair $(\mathcal{A}, D)$, where

$$\begin{aligned} &\mathcal{A} := \mathcal{A}(\varpi),\\ &D := \{\partial_{-\alpha} \mid \alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}\}, \end{aligned}$$

is negative. There are examples of such pairs in any dimension $n \geqslant 4$.

Example 1. Let G be of type D_ℓ, $\ell \geqslant 3$, and $\varpi = \varpi_1$. Denote by V be the underlying vector of $E(\varpi)$ and by $\varphi_\varpi\colon G \to \mathrm{GL}(V)$ the homomorphism determining the G-module structure of $E(\varpi)$. Then $\dim V = 2\ell$ and $\varphi_\varpi(G)$ is the orthogonal group of a nondegenerate quadratic form f on V. There is a basis

$$e_1,\ e_2, \ldots, e_\ell, e_{-\ell}, e_{-\ell+1}, \ldots, e_{-1} \tag{30}$$

of V such that

$$f = x_{-1}x_1 + x_{-2}x_2 + \cdots + x_{-\ell}x_\ell,$$

where x_i is the ith coordinate function on V in basis (30). The variety $X(\varpi)$ coincides with that of all isotropic vectors of f,

$$X(\varpi) = \{v \in V \mid f(v) = 0\},$$

which, in turn, coincides with the closure of the G-orbit of e_1. Hence, if $\mathcal{P}_{2\ell}$ is the polynomial ring in 2ℓ variables $x_1,\ x_2, \ldots, x_\ell, x_{-\ell}, x_{-\ell+1}, \ldots, x_{-1}$ with coefficients in k (i.e., $\mathcal{P}_{2\ell} = k[E(\varpi)]$), then

$$\mathcal{A}(\varpi) = \mathcal{P}_{2\ell}/(f). \tag{31}$$

The k-algebra $\mathcal{A}(\varpi)$ is a unique factorization domain of transcendence degree $n := 2\ell - 1$ over k, and $\mathcal{A}(\varpi)^* = k^*$. The hypersurface of zeros of f in V is not smooth, hence $\mathcal{A}(\varpi)$ is not a polynomial ring over k.

Identifying every element of $\mathrm{GL}(V)$ with its matrix in basis (30), we may assume that $\mathrm{GL}(V) = \mathrm{GL}_{2\ell}$ and that the elements of $\varphi_\varpi(T)$ (resp. $\varphi_\varpi(B)$) are diagonal (resp. upper triangular) matrices (see, e.g., [Bou 75, Chap. VIII, §13, no. 4]). Using the explicit description of Φ, Δ, and U_α's available in this case (see loc.cit.), it is then not difficult to see that all the derivations $D_{-\alpha}$ of $\mathcal{P}_{2\ell}$, where $\alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}$, are precisely the following $n-1$ commuting derivations D_j, $j = 2, 3, \ldots, \ell, -\ell, \ldots, -3, -2$, defined by the formula

$$\begin{aligned} D_j(x_i) &= \begin{cases} 0 & \text{for } i \neq j, \\ x_1 & \text{for } i = j \end{cases} \quad \text{if } i \neq -1, \\ D_j(x_{-1}) &= -x_{-j}. \end{aligned}$$

Let ∂_j be the locally nilpotent derivation of $\mathcal{A}(\varpi)$ induced (in view of $D_j(f) = 0$ and (31)) by D_j. Then $D := \{\partial_j\}$ is the set of $n-1$ commuting derivations that are linearly independent over $\mathcal{A}(\varpi)$; whence (2) holds (see, e.g., [Ma 03, Prop. 3.4], [DEFM 11, Lemma 1]). Thus in this case the answer to Question **1** is negative. □

Example 2. Let G be of type B_ℓ, $\ell \geqslant 2$, and $\varpi = \varpi_1$. The argument similar to that in Example 1 shows that if $\mathcal{P}_{2\ell+1}$ is the polynomial ring in $2\ell+1$ variables $x_1,\ x_2, \ldots, x_\ell, x_0, x_{-\ell}, x_{-\ell+1}, \ldots, x_{-1}$ with coefficients in k, then

$$\mathcal{A}(\varpi) = \mathcal{P}_{2\ell+1}/(h), \quad \text{where } h = x_0^2 + x_{-1}x_1 + x_{-2}x_2 + \cdots + x_{-\ell}x_\ell. \tag{32}$$

The k-algebra $\mathcal{A}(\varpi)$ is a unique factorization domain of transcendence degree $n := 2\ell$ over k, which is not a polynomial ring over k, and $\mathcal{A}(\varpi)^* = k^*$. All the derivations $D_{-\alpha}$ of $\mathcal{P}_{2\ell+1}$, where $\alpha \in \Phi^+ \setminus \Phi_{I(\varpi)}$, are precisely the following $n-1$ commuting derivations D_j, $j = 2, 3, \ldots, \ell, 0, -\ell, \ldots, -3, -2$, defined by the formula

$$\begin{aligned} D_j(x_i) &= \begin{cases} 0 & \text{for } i \neq j, \\ x_1 & \text{for } i = j \end{cases} \quad \text{if } i \neq -1, \\ D_j(x_{-1}) &= \begin{cases} -x_{-j} & \text{for } j \neq 0, \\ 2x_0 & \text{for } j = 0. \end{cases} \end{aligned}$$

Let ∂_j be the locally nilpotent derivation of $\mathcal{A}(\varpi)$ induced (in view of $D_j(h)=0$ and (32)) by D_j. Then $D := \{\partial_j\}$ is the set of $n-1$ commuting derivations that are linearly independent over $\mathcal{A}(\varpi)$; whence (2) holds. Therefore, in this case the answer to Question **1** is negative as well. $\square$

In Examples 1 and 2, the algebras $\mathcal{A}(\varpi)$ are hypersurfaces (quadratic cones). In the general case, they are factor algebras of polynomial algebras modulo the ideals generated by finitely many quadratic forms. Namely, the G-module $\mathrm{S}^2(E(\varpi)^*)$ of quadratic forms on $E(\varpi)$ contains a unique submodule (the Cartan component) $C(\varpi)$ isomorphic to $E(2\varpi)^*$; whence there is a unique submodule $M(\varpi)$ such that $\mathrm{S}^2(E(\varpi)^*) = C(\varpi) \oplus M(\varpi)$. It is known that the ideal of $k[E(\varpi)]$ generated by $M(\varpi)$ is then the ideal of elements $k[E(\varpi)]$ vanishing on of $X(\varpi)$. Therefore, $X(\varpi)$ is cut out in $E(\varpi)$ by

$$\frac{\dim E(\varpi)\big(\dim E(\varpi)+1\big)}{2} - \dim E(2\varpi)$$

homogeneous quadrics (cf. [Li 82]).

We note that a pair $(\mathcal{A}, D)$ with $\mathcal{A}$ of transcendence degree 3 over k, for which the answer to Question 1 is negative, exists as well: basing on the famous theorem that the Koras–Russell threefold X is not isomorphic to $\mathbf{A}^3$ (see [M.-L. 96]), in [EK 05] is shown that one may take $\mathcal{A} = k[X]$.

4. Remarks

1. The same arguments as in the proof of Theorem 2 prove the following

Theorem 4. *Let X be an irreducible affine n-dimensional variety endowed with a regular action of a unipotent algebraic group U. Assume that*

(i) *X is unirational;*
(ii) *X is normal;*
(iii) *$k[X]^\star = k^\star$;*
(iv) $\max\limits_{x\in X} \dim U\cdot x = n-1$.

Then there is an irreducible element t of $k[X]$ and the elements $f_1, \dots f_{n-1} \in k(X)$ such that

(a) *$k[X]^U = k[f]$;*
(b) *$k(X) = k(t, f_1, \dots, f_{n-1})$.*

In particular, X is rational.

2. Theorem 1 in [DEFM 11] reads as follows:

Let U be an n-dimensional unipotent group acting faithfully on an affine n-dimensional variety X satisfying $\mathcal{O}(X)^\star = k^\star$. Then $X \cong \mathbf{A}^n$ if one of the following two conditions hold:

(a) *some $x \in X$ has trivial isotropy subgroup, or*
(b) $n=2$, X is factorial, and U acts without fixed points.

The proof shows that, in fact, X is also assumed to be irreducible. We remark that, actually, given (a), the assumption $\mathcal{O}(X)^\star = k^\star$ is superfluous and, changing the proof, one may drop it. Moreover, in this case, more generally, affiness of X may be replaced by quasi-affiness, the assumption $\dim U = n$ may be dropped, and (a) may be replaced by the assumption

$$\dim U_x + \dim X = \dim U. \tag{33}$$

Indeed, (33) implies that $\dim U \cdot x = \dim X$. On the other hand, by [Ro 61_2, Thm. 2], unipotency of U implies that $U \cdot x$ is closed in X. Hence $U \cdot x = X$. Therefore, $X \cong U/U_x$, whence the claim by (i) in Introduction.

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