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MAPPINGS OF BOUNDED Φ-VARIATION WITH ARBITRARY FUNCTION Φ

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ABSTRACT. We develop the general theory of mappings of bounded Φ-*variation in the sense* of L. C. Young that are defined on a subset of the real line and take values in metric or normed spaces. We single out the characterizing properties for these mappings, prove the structural theorem for them, and study their continuity properties. We obtain the existence of a geodesic path of bounded Φ-variation between two points of a compact set with certain regularity of its modulus of continuity. The classical Helly selection principle from the theory of functions of bounded variation is generalized for mappings of bounded Φ-variation. Under natural restrictions on the function Φ, we show that the space of all normed space-valued mappings under consideration can be endowed with a metric. Finally, we consider the problem of existence of selections of a continuous set-valued mapping F of bounded Φ-variation with respect to the Hausdorff distance. We show that if $\Phi'(0)$ is finite > 0, then F has a continuous selection of bounded Φ-variation; if $\Phi'(0) = \infty$, then F is a constant mapping; and if $\Phi'(0) = 0$, then, under additional assumptions on Φ, we give examples of mappings F with no continuous selection and with no selection of bounded Φ-variation.

1. INTRODUCTION

In the theory of functions of bounded variation [17], Ch. 8 the following criterion is well known (Jordan's decomposition): a real-valued function on an interval of the real line is of bounded variation if and only if it is the difference of two bounded nondecreasing functions. Most of the classical facts of the theory follow from this criterion, among them Helly's selection principle, etc. Considering problems of existence of selections of set-valued

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mappings in the optimal control theory [14], [15], [16], we encounter mappings of bounded variation with values in a metric space with respect to the Hausdorff distance. Hence, it is important to know if these mappings can be characterized in a similar manner. It was shown in [4], Theorem 3.19 and [5], Theorem 3.1 that, if X is a metric space, then $f : E \subset \mathbb{R} \to X$ is of bounded variation if and only if it is the composition $f = g \circ \varphi$ of a bounded nondecreasing function $\varphi : E \to \mathbb{R}$ and an X-valued mapping g defined on the image of φ and satisfying the Lipschitz condition with the Lipschitz constant ≤ 1. From this, the main results of the theory follow (among others, the existence of Lipschitz continuous geodesic paths and Helly's selection principle) in a parallel fashion to the classical theory; and, moreover, it becomes clear that the Jordan decomposition theorem is a very specific feature of real-valued functions. In addition, under certain natural assumptions (continuous) set-valued mappings of bounded variation (also, Lipschitz or absolutely continuous mappings) with respect to the Hausdorff distance admit (continuous) selections of bounded variation (respectively, Lipschitz or absolutely continuous selections) [5], Theorem 9.1, [6], Theorem 6.1.

In [5] a number of properties was singled out which lead to almost complete characterization of mappings of bounded variation. In particular, in [7] by checking these properties for mappings of bounded p-variation, $p > 1$ (see Sec. 2 below), it was shown that any mapping $f : E \subset \mathbb{R} \to X$ of bounded p-variation is decomposable as $f = g \circ \varphi$ with $\varphi : E \to \mathbb{R}$ a bounded nondecreasing function and $g : \varphi(E) \to X$ a Hölder continuous mapping of exponent $1/p$ and the Hölder constant ≤ 1. Up to now, this is the only criterion known for mappings of bounded p-variation with $p > 1$; and it gives the complete picture of the theory of these mappings. However, in this case the situation is different for selections of set-valued mappings, since, contrary to Lipschitz continuous set-valued mappings, Hölder continuous set-valued mappings of (any) exponent $0 < \gamma < 1$ do not, in general, have continuous selections, [7], Proposition 8.2. It is not known whether set-valued mappings of bounded p-variation with $p > 1$ admit selections of bounded p-variation.

The purpose of the present paper is to develop the general theory of mappings of bounded Φ-variation valued in metric or normed spaces with "arbitrary" function Φ (see Sec. 2 for definitions). The plan of presentation is as follows. In Sec. 2 we establish the main properties of mappings of bounded Φ-variation and obtain relations between various spaces of mappings of bounded variation with different functions Φ. In Sec. 3 we prove the decomposition theorem: a mapping $f : E \subset \mathbb{R} \to X$ is of bounded Φ-variation if and only if it can be written in the form $f = g \circ \varphi$, where $\varphi : E \to \mathbb{R}$ is a bounded nondecreasing function and $g : \varphi(E) \to X$ is a mapping having the modulus of continuity bounded from above by the inverse function Φ^{-1}. If, in addition, X is a Banach space, then we show that g can be extended from E onto $\mathbb{R}$ with the preservation of certain properties of its modulus of continuity sufficient for Helly's selection principle to be held without the continuity assumption on the family of mappings. Also, we show that if the derivative $\Phi'(0) = \infty$, then any continuous mapping $f : [a, b] \to X$ of bounded Φ-variation is a constant mapping. In Sec. 4 we prove that any mapping of bounded Φ-variation is continuous almost everywhere (more precisely, outside of a subset which is at most countable) and provide estimates for the shocks in terms of the moduli of continuity of Φ and Φ^{-1}. In Sec. 5 we obtain the existence of a geodesic path of bounded Φ-variation between two given points of a compact subset of the metric space X having certain regularity properties with respect to its modulus of continuity. In Sec. 6 we prove the following generalization of the Helly selection principle: any infinite family of mappings of uniformly bounded Φ-variation defined on the compact interval in $\mathbb{R}$ with values in the compact subset of a Banach space contains a sequence which converges pointwise to a mapping of bounded Φ-variation. Under more restrictive assumption (7.1), in Sec. 7 we show that the space of all mappings of bounded Φ-variation with values in a normed vector space X can be endowed with a metric in such a way that it becomes a complete metric vector space if X is complete. Finally, in Sec. 8 we treat the problem of existence of selections of a continuous set-valued mapping F of bounded Φ-variation with respect to the Hausdorff metric. Roughly speaking, we show that if $\Phi'(0)$ is finite > 0, then F has a continuous selection of bounded Φ-variation (which is, in fact, of bounded variation); if $\Phi'(0)$ is infinite, then F is a constant set-valued mapping, and if $\Phi'(0) = 0$, then, under additional assumptions on Φ, we give examples of continuous mappings F with no continuous selection and with no selection of bounded Φ-variation.

2. Properties of the Φ-variation

The following notation will be used throughout this paper:

$\varnothing \neq E \subset \mathbb{R}$,

$E_t^- = \{ s \in E \mid s \leq t \}$ and $E_t^+ = \{ s \in E \mid t \leq s \}$ if $t \in E$,

$E_a^b = E_a^+ \cap E_b^- = (E_b^-)_a^+$ if a, $b \in E$, $a \leq b$ (in particular, $[a, b] = \mathbb{R}_a^b$ and $\mathbb{R}_0^+ = \{ t \in \mathbb{R} \mid 0 \leq t \} = [0, \infty[$),

X is a metric space with a fixed metric (or distance function) $d = d(\cdot\,, \cdot)$,

X^E is the set of all mappings $f : E \to X$ from E into X.

If $f \in X^E$, we denote by

$f(E) = \{ f(t) \mid t \in E \}$ the image of f in X,

$D(f, E) = \operatorname{diam} f(E) = \sup\{ d(f(t), f(s)) \mid t, s \in E \}$ the diameter of the image $f(E)$ (in other words, $D(f, E)$ is the oscillation of f on E).

Given two mappings $f : E \to X$ and $\varphi : E_1 \to E$, the composition $f \circ \varphi : E_1 \to X$ is defined (as usual) by $(f \circ \varphi)(\tau) = f(\varphi(\tau))$ for all $\tau \in E_1$.

We write $A := B$ or $B =: A$ to indicate that A is defined by means of B.

We denote by $\mathcal{F}$ the set of all functions $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that Φ is continuous, strictly increasing, $\Phi(0) = 0$, and $\Phi(\infty) = \infty$. The set $\mathcal{F}$ is closed under sums, products, multiplication by positive constants, compositions, and taking inverse mappings, i. e., if Φ, $\Psi \in \mathcal{F}$ and $c > 0$, then $\Phi + \Psi$, $\Phi \cdot \Psi$, $c\Phi$, $\Phi \circ \Psi$ and Φ^{-1} belong to $\mathcal{F}$; also, we have $\Phi \wedge \Psi = \min\{\Phi, \Psi\}$ and $\Phi \vee \Psi = \max\{\Phi, \Psi\}$ belong to $\mathcal{F}$. A generic function from $\mathcal{F}$ will usually be denoted by Φ.

Definition. Let

$$\mathcal{T}(E) = \big\{ T = \{t_i\}_{i=0}^m \subset E \mid m \in \mathbb{N} \cup \{0\},\ t_{i-1} \le t_i,\ i = 1, \ldots, m \big\} \quad (2.1)$$

be the set of all partitions of E by finite ordered collections of points from E. For given $\Phi \in \mathcal{F}$, $f : E \to X$, and $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$ we set

$$V_\Phi[f, T] \equiv V_{\Phi,d}[f, T] = \sum_{i=1}^m \Phi \circ d\big(f(t_i), f(t_{i-1})\big), \quad (2.2)$$

where

$$\Phi \circ d\big(f(t_i), f(t_{i-1})\big) = \Phi\Big(d\big(f(t_i), f(t_{i-1})\big)\Big), \quad (2.3)$$

and define $V_\Phi(f, E) \in [0, \infty]$ by

$$V_\Phi(f, E) = \sup\big\{ V_\Phi[f, T] \mid T \in \mathcal{T}(E) \big\}. \quad (2.4)$$

The value $V_\Phi(f, E)$ is called the *total Φ-variation of f on E.* If $V_\Phi(f, E) < \infty$, the mapping f is said to be *of bounded Φ-variation.* The set of all mappings of bounded Φ-variation from E into X is denoted by $\mathcal{V}_\Phi(E; X)$. If $\varnothing \ne A \subset E$, we set $V_\Phi(f, A) = V_\Phi(f|_A, A)$, where $f|_A$ is the restriction of f to A, and we set $\mathcal{T}(\varnothing) = \varnothing$ and $V_\Phi(f, \varnothing) = 0$. The functional $V_\Phi : X^E \times 2^E \to \mathbb{R}_0^+ \cup \{\infty\}$ is called the *Φ-variation.*

For real-valued functions the definition of $V_\Phi(f, E)$ was introduced by Young in [20] and [21]. If $\Phi(t) = t^p$, $t \ge 0$, $p > 1$, the p-variation $V_p(f, E) = V_\Phi(f, E)$ was originally considered by Wiener [19], and if $p = 1$, the total variation $V(f, E) = V_1(f, E)$ was classically defined by Jordan [12] (see also Schwartz [18], Ch. 4, Sec. 9. Note that this definition is suitable for mappings defined on any linearly ordered set E. In this respect, some results of this paper are valid also in the case where $\le$ is a linear ordering on E.

Recently the notion of $V(f, E)$ of Jordan was revisited by the first author, [4], [5], and [6], so as to obtain the main properties of the variation in the general situation and to get a structural theorem for (various classes of) mappings of bounded variation. Then the general properties were reformulated (almost without change) for mappings of bounded p-variation in the sense of Wiener with $p > 1$ in [7]. The aim of this section is to reformulate the general properties for the Φ-variation in the sense of Young (for the motivation of these properties see [5], the remarks after (P7), p. 264, or [7], Remarks 2.2–2.4).

General properties of the Φ-variation V_Φ. Let $f : E \to X$ be an arbitrary mapping, and let $\Phi \in \mathcal{F}$. Then we have:

(P1) if $t, s \in E$, then $\Phi\big(d(f(t), f(s))\big) \le \Phi\big(D(f, E)\big) \le V_\Phi(f, E)$ (*minimality*);

(P2) if $a, t, s, b \in E$ and $a \le t \le s \le b$, then $V_\Phi(f, E_t^-) \le V_\Phi(f, E_s^-)$, $V_\Phi(f, E_s^+) \le V_\Phi(f, E_t^+)$ and $V_\Phi(f, E_t^s) \le V_\Phi(f, E_a^b)$ (*monotonicity*);

(P3) if $t \in E$, then $V_\Phi(f, E_t^-) + V_\Phi(f, E_t^+) \le V_\Phi(f, E)$ (*semi-additivity*);

(P4) if $E_1 \subset \mathbb{R}$ and $\varphi : E_1 \to E$ is a (not necessarily strictly) monotone function, then $V_\Phi(f, \varphi(E_1)) = V_\Phi(f \circ \varphi, E_1)$ (*change of a variable*);

(P5) $V_\Phi(f, E) = \sup\{ V_\Phi(f, E_a^b) \mid a, b \in E, a \le b \}$ (*regularity*);

(P6) if $s = \sup E \in \mathbb{R} \cup \{\infty\}$ and $i = \inf E \in \mathbb{R} \cup \{-\infty\}$, we have (*limit properties*):

($P6_1$) if $s \notin E$, then $V_\Phi(f, E) = \lim_{E \ni t \to s} V_\Phi(f, E_t^-)$,

($P6_2$) if $i \notin E$, then $V_\Phi(f, E) = \lim_{E \ni t \to i} V_\Phi(f, E_t^+)$,

($P6_3$) if $s \notin E$ and $i \notin E$, then, in addition to ($P6_1$) and ($P6_2$), we have

$$V_\Phi(f, E) = \lim_{\substack{E \ni a \to i \\ E \ni b \to s}} V_\Phi(f, E_a^b) = \lim_{E \ni b \to s} \lim_{E \ni a \to i} V_\Phi(f, E_a^b) =$$
$$= \lim_{E \ni a \to i} \lim_{E \ni b \to s} V_\Phi(f, E_a^b);$$

(P7) if, as $n \to \infty$, the sequence of functions $\{\Phi_n\}_{n=1}^\infty \subset \mathcal{F}$ converges pointwise to $\Phi \in \mathcal{F}$ and the sequence of mappings $\{f_n\}_{n=1}^\infty \subset X^E$ converges pointwise on E (in the metric d) to $f \in X^E$, then

$$V_\Phi(f, E) \le \liminf_{n \to \infty} V_{\Phi_n}(f_n, E)$$

(*sequential lower semi-continuity*).

Proof. Properties (P1) and (P2) are obvious. The proof of (P4)–(P6) is implied by the same lines of reasoning as in [6], Sec. 2 or [7], Sec. 2.

(P3) For any two partitions $T_1 \in \mathcal{T}(E_t^-)$ and $T_2 \in \mathcal{T}(E_t^+)$ we have

$$V_\Phi[f, T_1] + V_\Phi[f, T_2] \le V_\Phi[f, T_1 \cup \{t\}] + V_\Phi[f, T_2 \cup \{t\}] =$$
$$= V_\Phi[f, T_1 \cup \{t\} \cup T_2] \le V_\Phi(f, E).$$

It suffices to take the supremum over all T_1 and T_2 as above.

(P7) For any $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$ and $n \in \mathbb{N}$ we have $V_{\Phi_n}[f_n, T] \le V_{\Phi_n}(f_n, E)$. If $\rho_{i,n} = d\big(f_n(t_i), f_n(t_{i-1})\big)$ and $\rho_i = d\big(f(t_i), f(t_{i-1})\big)$, then

$$V_{\Phi_n}[f_n, T] - V_\Phi[f, T] = \sum_{i=1}^{m} \big(\Phi_n(\rho_{i,n}) - \Phi(\rho_i)\big).$$

Since $d(\,\cdot\,,\,\cdot\,)$ is continuous and $f_n \to f$ pointwise on the finite set T, we have $\rho_{i,n} \to \rho_i$ as $n \to \infty$ for all $i = 1, \ldots, m$, and since $\Phi_n \to \Phi$ pointwise on $\mathbb{R}_0^+$, (as is shown below) $\Phi_n(\rho_{i,n}) \to \Phi(\rho_i)$ as $n \to \infty$. It follows that $V_{\Phi_n}[f_n, T] \to V_\Phi[f, T]$ as $n \to \infty$; and, hence,

$$V_\Phi[f, T] \le \liminf_{n\to\infty} V_{\Phi_n}(f_n, E).$$

As T is arbitrary, we get (P7).

Now we show that $\Phi_n(\rho_n) \to \Phi(\rho)$ if $\rho_n \to \rho > 0$ as $n \to \infty$ (deliberately omitting the subscript i in $\rho_{i,n}$ and ρ_i). Given $\varepsilon > 0$, by continuity of Φ choose $\delta = \delta(\varepsilon) > 0$ such that $\delta < \rho$ and

$$|\Phi(t) - \Phi(\rho)| \le \varepsilon/2 \quad \text{for all } t \ge 0 \text{ with } |t - \rho| \le \delta.$$

Let α, $\beta \in \mathbb{R}_0^+$ be such that $\rho - \delta < \alpha < \rho < \beta < \rho + \delta$. Since $\rho_n \to \rho$ and $\Phi_n \to \Phi$ pointwise as $n \to \infty$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for all $n \ge N$ we have

$$\alpha < \rho_n < \beta, \quad |\Phi_n(\alpha) - \Phi(\alpha)| \le \varepsilon/2 \quad \text{and} \quad |\Phi_n(\beta) - \Phi(\beta)| \le \varepsilon/2.$$

As Φ_n is nondecreasing, for $n \ge N$ it follows that

$$\Phi_n(\rho_n) \le \Phi_n(\beta) \le \Phi(\beta) + \varepsilon/2 \le \Phi(\rho) + \varepsilon/2 + \varepsilon/2,$$
$$\Phi_n(\rho_n) \ge \Phi_n(\alpha) \ge \Phi(\alpha) - \varepsilon/2 \ge \Phi(\rho) - \varepsilon/2 - \varepsilon/2,$$

and hence, $|\Phi_n(\rho_n) - \Phi(\rho)| \le \varepsilon$ for all $n \ge N$.

The case $\rho_n \to \rho = 0$ as $n \to \infty$ is now obvious. □

Proposition 2.1 (*minimality of* V_Φ). *Let $\Phi \in \mathcal{F}$ be fixed, and assume that the mapping $W : X^E \times 2^E \to [0, \infty]$ satisfies, for all $f \in X^E$ and $\emptyset \ne A \subset E$, the following conditions ($W(f, \varnothing) = 0$):*

(a) $\Phi\big(d(f(t), f(s))\big) \le W(f, A)$ for all t, $s \in A$;
(b) $W(f, A_t^s) \le W(f, A)$ for all t, $s \in A$, $t \le s$;
(c) $W(f, A_t^-) + W(f, A_t^+) \le W(f, A)$ for all $t \in A$.

Then $V_\Phi(f, A) \le W(f, A)$ for all $f : E \to X$ and $A \subset E$.

Proof. The proof is analogous to the one given for the p-variation V_p ($p \ge 1$) in [7], Proposition 2.1. □

Property (P1) implies that if $f \in \mathcal{V}_\Phi(E; X)$, then f is bounded in the sense that $D(f, E) < \infty$. A refinement of this property is the following.

Proposition 2.2. *If $f \in \mathcal{V}_\Phi(E;X)$, then the image $f(E) \subset X$ is totally bounded and separable. If, in addition, X is a complete metric space, then $f(E)$ is precompact (i.e., the closure of $f(E)$ in X is compact).*

Proof. The proof proceeds as in [5], Proposition 2.1 or [7], Proposition 2.2. □

Proposition 2.3. *Let Φ, $\Psi \in \mathcal{F}$ be two functions such that there exist two positive constants $C > 0$ and $\delta > 0$ for which $\Psi(t) \le C\Phi(t)$ for all $t \in [0,\delta]$. Then $\mathcal{V}_\Phi(E;X) \subset \mathcal{V}_\Psi(E;X)$.*

Proof. Let $f \in \mathcal{V}_\Phi(E;X)$. If $T = \{t_i\}_{i=0}^m$ is a partition of E, then

$$V_\Phi[f,T] = \sum_{i=1}^m \Phi(\rho_i) \le V_\Phi(f,E) < \infty, \quad \text{where} \quad \rho_i = d\big(f(t_i), f(t_{i-1})\big).$$

There are less than $V_\Phi(f,E)/\Phi(\delta)$ terms in the above sum which are greater than $\Phi(\delta)$; and hence, the number of ρ_i's which are greater than δ is less than $V_\Phi(f,E)/\Phi(\delta)$. If $\rho_i \le \delta$, then $\Psi(\rho_i) \le C\Phi(\rho_i)$ according to the assumption; and if $\rho_i > \delta$, we have $\Psi(\rho_i) \le \Psi(D(f,E))$. It follows that

$$\begin{aligned} V_\Psi[f,T] = \sum_{i=1}^m \Psi(\rho_i) &\le C\sum_{i=1}^m \Phi(\rho_i) + \Psi(D(f,E))V_\Phi(f,E)/\Phi(\delta) \le \\ &\le CV_\Phi(f,E) + \Psi(D(f,E))V_\Phi(f,E)/\Phi(\delta) < \infty. \end{aligned}$$

Since the last inequality holds for all $T \in \mathcal{T}(E)$, we have $f \in \mathcal{V}_\Psi(E;X)$. □

Remark. In the case of real- (or complex-) valued functions Proposition 2.3 is due to Golubov [9], the remark after Definition 1, where he presented it without proof.

A function $\Phi \in \mathcal{F}$ is said to be *superadditive* (resp., *subadditive*) if for all t, $s \ge 0$ we have

$$\Phi(t) + \Phi(s) \le \Phi(t+s) \quad (\text{resp., } \Phi(t+s) \le \Phi(t) + \Phi(s)). \tag{2.5}$$

Note that $\Phi \in \mathcal{F}$ is superadditive if and only if $\Phi^{-1} (\in \mathcal{F})$ is subadditive. Also recall that the function $\Phi(t) = t^p$, $t \ge 0$, satisfies the following inequalities: if t, $s \ge 0$, then

$$\begin{aligned} &t^p + s^p \le (t+s)^p \le 2^{p-1}(t^p + s^p), \quad p \ge 1, \\ &(t+s)^p \le t^p + s^p \le 2^{1-p}(t+s)^p, \quad 0 < p < 1. \end{aligned} \tag{2.6}$$

Proposition 2.4. *If $f : E \to \mathbb{R}$ is a bounded monotone function and $\Phi \in \mathcal{F}$ is superadditive, then*

$$V_\Phi(f,E) = \Phi(D(f,E)) = \Phi\Big(\sup_{t\in E} f(t) - \inf_{t\in E} f(t)\Big).$$

Proof. By virtue of (P1), we have $\Phi(D(f,E)) \le V_\Phi(f,E)$. On the other hand, if $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$ and $t \in E$ is such that $t_{k-1} \le t \le t_k$ for some $1 \le k \le m$, then the monotonicity of f implies that

$$|f(t_k) - f(t_{k-1})| = |f(t_k) - f(t)| + |f(t) - f(t_{k-1})|,$$

so that applying the first inequality in (2.5), we obtain

$$V_\Phi[f, T \cup \{t\}] = V_\Phi[f,T] + \Phi\big(|f(t) - f(t_{k-1})|\big) + \\ + \Phi\big(|f(t_k) - f(t)|\big) - \Phi\big(|f(t_k) - f(t_{k-1})|\big) \le V_\Phi[f,T].$$

It follows that

$$V_\Phi[f,T] \le V_\Phi[f, \{t_0, t_m\}] = \Phi\big(|f(t_m) - f(t_0)|\big) \le \Phi\big(D(f,E)\big)$$

for all $T \in \mathcal{T}(E)$; and hence, $V_\Phi(f,E) \le \Phi\big(D(f,E)\big)$. □

Proposition 2.5. *Let Φ, $\Psi \in \mathcal{F}$, $c > 0$, and $f : E \to X$. Then we have:*

(a) $V_{\Phi+\Psi}(f,E) = V_\Phi(f,E) + V_\Psi(f,E)$;
(b) $V_{\Phi\cdot\Psi}(f,E) \le V_\Phi(f,E) \cdot V_\Psi(f,E)$;
(c) $V_{c\Phi}(f,E) = cV_\Phi(f,E)$;
(d) *if $\Phi \in \mathcal{F}$ is superadditive (subadditive), then*

$$V_{\Phi\circ\Psi}(f,E) \le \Phi\big(V_\Psi(f,E)\big)$$

(resp., $V_{\Phi\circ\Psi}(f,E) \ge \Phi\big(V_\Psi(f,E)\big)$);
(e) *if $\Phi \in \mathcal{F}$ is superadditive (subadditive), then $V_\Phi(f,E) \le \Phi\big(V_1(f,E)\big)$ (resp., $\Phi\big(V_1(f,E)\big) \le V_\Phi(f,E)$); and in particular, $\mathcal{V}_1(E;X) \subset \mathcal{V}_\Phi(E;X)$ (resp., $\mathcal{V}_\Phi(E;X) \subset \mathcal{V}_1(E;X)$).*

Proof. Properties (a), (b), (c), and (d) follow by a direct verification. To prove (e), let Φ be superadditive; since $(\Phi^{-1} \circ \Phi)(t) = t$, $t \ge 0$, and $\Phi^{-1} \in \mathcal{F}$ is subadditive, by virtue of (d), we have

$$V_1(f,E) = V_{\Phi^{-1}\circ\Phi}(f,E) \ge \Phi^{-1}\big(V_\Phi(f,E)\big). \quad \square$$

3. A decomposition theorem

First of all we recall that the *modulus of continuity* of a bounded mapping $f : E \to X$ is the function $\omega_{f,E} :]0,\infty[\to [0,\infty[$ defined by

$$\omega_{f,E}(\rho) = \sup\{\, d\big(f(t), f(s)\big) \mid t,\ s \in E \text{ and } |t-s| \le \rho \,\} = \\ = \sup_{s\in E} \sup\{\, d\big(f(t), f(s)\big) \mid t \in E \text{ and } |t-s| \le \rho \,\}, \quad \rho > 0.$$

Clearly, $\omega_{f,E}(\rho) \le D(f,E)$ for all $\rho > 0$. It is a classical fact that the modulus of continuity $\omega_{f,E}$ has the following properties.

Proposition 3.1. *For any bounded mapping $f : E \to X$ we have:*

(a) $\omega_{f,E}$ *is nondecreasing on* $]0,\infty[$;
(b) $\omega_{f,E}(0) := \lim_{\rho\to+0}\omega_{f,E}(\rho) = \inf_{\rho>0}\omega_{f,E}(\rho) \geq 0$ *is finite*;
(c) f *is uniformly continuous on* E *if and only if* $\omega_{f,E}(0) = 0$;
(d) *if* E *is an interval (open or closed), then* $\omega_{f,E}$ *is subadditive:*

$$\omega_{f,E}(\rho_1+\rho_2) \leq \omega_{f,E}(\rho_1) + \omega_{f,E}(\rho_2) \quad \forall\, \rho_1,\, \rho_2 \geq 0\,;$$

(e) *if* E *is an interval (open or closed) and* $f : E \to X$ *is uniformly continuous on* E, *then* $\omega_{f,E}$ *is continuous on* $[0,\infty[$;
(f) *for a nondecreasing function* $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *the following two conditions are equivalent:*
 (i) $d\big(f(t), f(s)\big) \leq \omega(|t-s|)$ *for all* $t,\, s \in E$;
 (ii) $\omega_{f,E} \leq \omega$ *on* $\mathbb{R}_0^+$ *(i.e.,* $\omega_{f,E}(\rho) \leq \omega(\rho)$ *for all* $\rho \geq 0$*).*

From now on in this section we assume that $\Phi \in \mathcal{F}$ is a fixed function. The main result of this section is the following decomposition theorem.

Theorem 3.2. *The mapping* $f : E \to X$ *is of bounded* Φ*-variation on* E *if and only if there exist a bounded nondecreasing function* $\varphi : E \to \mathbb{R}$ *and a mapping* $g : E_1 := \varphi(E) \to X$ *such that* $\omega_{g,E_1} \leq \Phi^{-1}$ *on* $\mathbb{R}_0^+$ *and* $f = g\circ\varphi$ *on* E.

Moreover, if X *is a Banach space, the mapping* $g : E_1 \to X$ *can be extended to a mapping* $g^* : \mathbb{R} \to X$ *such that* $\omega_{g^*,\mathbb{R}} \leq \Omega_{\Phi,f}$ *on* $\mathbb{R}_0^+$, *where* $\Omega_{\Phi,f} \in \mathcal{F}$ *is defined by* $\Omega_{\Phi,f} := 3\Phi^{-1} + \omega_{\Phi^{-1},[0,L]}$ *with* $L := V_\Phi(f,E)$.

We divide the proof of this theorem into three steps which constitute the following three lemmas. The first lemma (sufficiency) gives typical examples of mappings of bounded Φ-variation.

Lemma 3.3. *Let* $\varphi : E \to \mathbb{R}$ *be bounded monotone,* $g : \varphi(E) \to X$ *be a mapping such that* $\omega_{g,\varphi(E)} \leq \Phi^{-1}$ *on* $\mathbb{R}_0^+$, *and let* $f := g\circ\varphi$ *on* E. *Then* $f \in \mathcal{V}_\Phi(E;X)$.

Proof. For any partition $T = \{t_i\}_{i=0}^m$ of E we have

$$V_\Phi[f,T] = \sum_{i=1}^m \Phi\circ d\Big(g\big(\varphi(t_i)\big), g\big(\varphi(t_{i-1})\big)\Big).$$

Now the estimate $\omega_{g,\varphi(E)} \leq \Phi^{-1}$ and monotonicity and boundedness of φ imply that

$$V_\Phi[f,T] \leq \sum_{i=1}^m |\varphi(t_i)-\varphi(t_{i-1})| = |\varphi(t_m)-\varphi(t_0)| \leq$$
$$\leq \sup_{t\in E}\varphi(t) - \inf_{t\in E}\varphi(t) = D(\varphi,E) < \infty.$$

The conclusion of Lemma 3.3 follows by taking the supremum over all partitions T of E. $\square$

Remark. In particular, if $f : E \to X$ is such that $\omega_{f,E} \le \Phi^{-1}$ on $\mathbb{R}_0^+$ and $E \subset \mathbb{R}$ is a *bounded* set, then (choosing $\varphi(t) = t$ in Lemma 3.3) we have: f is of bounded Φ-variation on E and $V_\Phi(f, E) \le \sup E - \inf E$.

Next, in the second lemma (necessity) we obtain the canonical decomposition of a mapping of bounded Φ-variation.

Lemma 3.4. *If $f : E \to X$ is a mapping of bounded Φ-variation, then there exist a bounded nondecreasing nonnegative function $\varphi : E \to \mathbb{R}$ and a mapping $g : E_1 = \varphi(E) \to X$ satisfying $\omega_{g,E_1} \le \Phi^{-1}$ on $\mathbb{R}_0^+$ such that*
(a) $f = g \circ \varphi$ *on* E; (b) $g(E_1) = f(E)$ *in* X, *and* (c) $V_\Phi(g, E_1) = V_\Phi(f, E)$.

Proof. The function $\varphi : E \to \mathbb{R}$ defined by $\varphi(t) = V_\Phi(f, E_t^-)$ for $t \in E$ is nonnegative, bounded (since $\varphi(t) \le V_\Phi(f, E)$), and nondecreasing due to (P2). For t, $s \in E$, $t \le s$, by virtue of (P1) and (P3), we have

$$\Phi\big(d(f(s), f(t))\big) \le V_\Phi(f, E_t^s) \le \varphi(s) - \varphi(t). \tag{3.1}$$

If $\varphi(t) = \varphi(s)$, then (3.1) implies that $f(t) = f(s)$, so that the mapping $g : E_1 \to X$ given by

$$g(\tau) := f(t) \text{ for any } t \in E \text{ such that } \varphi(t) = \tau \tag{3.2}$$

for $\tau \in E_1 = \varphi(E)$ is well defined on E_1. Now (a) follows from (3.2), and the assertions (b) and (c) follow from (a) and (P4).

It remains to prove that $\omega_{g,E_1} \le \Phi^{-1}$. Indeed, for α, $\beta \in E_1$, we have $\alpha = \varphi(t)$ and $\beta = \varphi(s)$ for some t, $s \in E$; and hence, by virtue of (3.1) and (3.2) it follows that

$$\Phi\big(d(g(\alpha), g(\beta))\big) = \Phi\big(d(f(t), f(s))\big) \le |\varphi(t) - \varphi(s)| = |\alpha - \beta|. \quad \square$$

Remark. If $\varphi : E \to E_1$ in the proof of Lemma 3.4 is strictly increasing, it is a bijection, so that the equality $f = g \circ \varphi$ on E is equivalent to the equality $g = f \circ \varphi^{-1}$ on E_1, where $\varphi^{-1} : E_1 \to E$ is the inverse function of φ. An algebraic aspect in the construction of the mapping such as g in Lemma 3.4 was considered in [5], at the end of Sec. 3.

The second part of Theorem 3.2 follows from Lemma 3.5 below by setting $L = V_\Phi(f, E)$ and $\Psi = \Phi^{-1}$.

Lemma 3.5. *Suppose that $\Psi \in \mathcal{F}$, $E_1 \subset [0, L]$ with $0 < L < \infty$, X is a Banach space* (*over the field $\mathbb{R}$ or $\mathbb{C}$*) *with the norm $\|\cdot\|$, and $g : E_1 \to X$ is a mapping satisfying*

$$\| g(t) - g(s) \| \le \Psi(|t - s|) \qquad \forall t,\ s \in E_1. \tag{3.3}$$

Then there exists a mapping $g^* : \mathbb{R} \to X$ *extending* g *such that*

$$\| g^*(t) - g^*(s) \| \le \Psi^*(|t-s|) \qquad \forall t,\, s \in \mathbb{R}, \tag{3.4}$$

where the function $\Psi^* \in \mathcal{F}$ *is defined by* $\Psi^* = 3\Psi + \omega_{\Psi,[0,L]}$.

Proof. Since g is uniformly continuous on E_1, it admits an extension $\overline{g}$ to the closure $\overline{E_1}$ of E_1 such that $\overline{g} : \overline{E_1} \to X$ satisfies (3.3) on $\overline{E_1}$. We define g^* to be equal to $\overline{g}$ on $\overline{E_1}$. The complement $\mathbb{R} \setminus \overline{E_1}$ of $\overline{E_1}$ in $\mathbb{R}$ is open; and hence, it is at most a countable union of nonempty disjoint open intervals $]a_k, b_k[$ for $k \in J$ with at most countable $J \subset \mathbb{N}$. On intervals $]a_k, b_k[$ with $b_k - a_k < \infty$ we define g^* as follows:

$$g^*(t) = \overline{g}(a_k) + c_k \Psi(t - a_k), \quad c_k := \frac{\overline{g}(b_k) - \overline{g}(a_k)}{\Psi(b_k - a_k)}, \quad t \in]a_k, b_k[, \tag{3.5}$$

where $\| c_k \| \le 1$. If $a_k = -\infty$, we set $g^*(t) = \overline{g}(b_k)$ for all $t \in]-\infty, b_k[$, and if $b_k = \infty$, we set $g^*(t) = \overline{g}(a_k)$ for all $t \in]a_k, \infty[$.

It is clear that g^* extends g to $\mathbb{R}$, so that we have to show that g^* satisfies (3.4). For $t,\, s \in \mathbb{R}$, $s \le t$, we have the following four possibilities: (1) $t \in \overline{E_1}$, $s \in \overline{E_1}$; (2) $t \notin \overline{E_1}$, $s \in \overline{E_1}$; (3) $t \in \overline{E_1}$, $s \notin \overline{E_1}$; (4) $t \notin \overline{E_1}$, $s \notin \overline{E_1}$.

Case (1) is clear since $g^* = \overline{g}$ on $\overline{E_1}$ and $\overline{g}$ satisfies (3.3) on $\overline{E_1}$.

In case (2) we have $s \le a_k < t < b_k$ for some $k \in J$. If $b_k = \infty$, then $g^*(t) = \overline{g}(a_k)$, so that

$$\| g^*(t) - g^*(s) \| = \| \overline{g}(a_k) - \overline{g}(s) \| \le \Psi(a_k - s) \le \Psi(t - s).$$

If $b_k < \infty$, then (3.5) and properties of $\overline{g}$ yield

$$\begin{aligned} \| g^*(t) - g^*(s) \| &\le \| g^*(t) - g^*(a_k) \| + \| g^*(a_k) - g^*(s) \| = \\ &= \| g^*(t) - \overline{g}(a_k) \| + \| \overline{g}(a_k) - \overline{g}(s) \| \le \\ &\le \| c_k \| \Psi(t - a_k) + \Psi(a_k - s) \le 2\Psi(t - s). \end{aligned}$$

In case (3) we have $a_k < s < b_k \le t$ for some $k \in J$. If $a_k = -\infty$, then $g^*(s) = \overline{g}(b_k)$, so that

$$\| g^*(t) - g^*(s) \| = \| \overline{g}(t) - \overline{g}(b_k) \| \le \Psi(t - b_k) \le \Psi(t - s).$$

If $a_k > -\infty$, then $[a_k, b_k] \subset [0, L]$; and hence, (3.5) and properties of $\overline{g}$ imply that

$$\begin{aligned} \| g^*(t) - g^*(s) \| &\le \| g^*(t) - g^*(b_k) \| + \| g^*(b_k) - g^*(s) \| = \\ &= \| \overline{g}(t) - \overline{g}(b_k) \| + \| c_k \big(\Psi(b_k - a_k) - \Psi(s - a_k) \big) \| \le \\ &\le \Psi(t - b_k) + \omega_{\Psi,[0,L]}(b_k - s) \le \\ &\le \Psi(t - s) + \omega_{\Psi,[0,L]}(t - s). \end{aligned}$$

In case (4) we have $s \in]a_k, b_k[$ and $t \in]a_m, b_m[$ for some k, $m \in J$. Suppose that $m = k$, so that $a_k < s \le t < b_k$. If $a_k = -\infty$ or $b_k = \infty$, then $\| g^*(t) - g^*(s) \| = 0$; and if $a_k > -\infty$ and $b_k < \infty$, then $[a_k, b_k] \subset [0, L]$, and by virtue of (3.5) we have

$$\| g^*(t) - g^*(s) \| \le \| c_k \| \cdot |\Psi(t - a_k) - \Psi(s - a_k)| \le \omega_{\Psi,[0,L]}(t - s).$$

Now suppose that $m \ne k$, so that $a_k < s < b_k < t$. If $a_k = -\infty$, then $g^*(s) = g^*(b_k)$; and as in case (2), we have

$$\| g^*(t) - g^*(s) \| = \| g^*(t) - g^*(b_k) \| \le 2\Psi(t - b_k) \le 2\Psi(t - s).$$

If $a_k > -\infty$, then $[a_k, b_k] \subset [0, L]$; and as in cases (2) and (3), we have

$$\begin{aligned} \| g^*(t) - g^*(s) \| &\le \| g^*(t) - g^*(b_k) \| + \| g^*(b_k) - g^*(s) \| \le \\ &\le 2\Psi(t - b_k) + \big(\Psi(b_k - s) + \omega_{\Psi,[0,L]}(b_k - s)\big) \le \\ &\le 3\Psi(t - s) + \omega_{\Psi,[0,L]}(t - s) = \Psi^*(t - s). \end{aligned}$$

Thus, in all the cases we have obtained the desired estimate (3.4). It remains to note that $\Psi^* \in \mathcal{F}$ which is a consequence of the fact that $\Psi \in \mathcal{F}$ and Proposition 3.1(a), (b), (c), and (e). $\square$

Corollary 3.6. *Suppose that $\Phi \in \mathcal{F}$ is such that $\lim_{t\to+0} t/\Phi(t) = 0$ (or, equivalently, that $\Phi'(+0) = \infty$, where $\Phi'(+0)$ denotes the right derivative of Φ at $t = 0$). Then any continuous mapping from $[a, b]$ into X of bounded Φ-variation is a constant mapping on $[a, b]$.*

In order to prove the corollary we need a lemma. The following lemma (which is interesting in its own right) seems to be known; however, we could not find an appropriate reference, and hence, we present it with the proof.

Lemma 3.7. *Let $\Psi \in \mathcal{F}$ be such that $\Psi'(+0) = \lim_{t\to+0} \Psi(t)/t = 0$. If for $g : [a, b] \to X$ we have $d(g(t), g(s)) \le \Psi(|t - s|)$ for all t, $s \in [a, b]$, then g is a constant mapping on $[a, b]$.*

Proof. Let $\mathcal{B}(X; \mathbb{R})$ be the Banach space of all bounded mappings from X into $\mathbb{R}$ endowed with the sup-norm $\| R \|_u = \sup_{x\in X} |R(x)|$, $R \in \mathcal{B}(X; \mathbb{R})$. The mapping

$$X \ni x \longmapsto R_x - R_{x_0} \in \mathcal{B}(X; \mathbb{R}),$$

where $R_x(y) := d(x, y)$, $y \in X$, and $x_0 \in X$ is a fixed element, defines an imbedding of X into $\mathcal{B}(X; \mathbb{R})$ such that

$$\| R_x - R_y \|_u = d(x, y) \qquad \forall\, x,\, y \in X.$$

The composed mapping $[a,b] \ni t \mapsto G(t) := R_{g(t)} - R_{x_0} \in \mathcal{B}(X;\mathbb{R})$ (see the diagram)

$$\begin{array}{ccccc} [a,b] & \longrightarrow & X & \longrightarrow & \mathcal{B}(X;\mathbb{R}), \\ t & \longmapsto & g(t) & \longmapsto & R_{g(t)} - R_{x_0} \end{array}$$

satisfies the inequality

$$\| G(t) - G(s) \|_u = d\big(g(t), g(s)\big) \le \Psi(|t-s|) \qquad \forall t,\, s \in [a,b].$$

Hence, for any fixed $t_0 \in [a,b]$ we have

$$\left\| \frac{G(t) - G(t_0)}{t - t_0} \right\|_u \le \frac{\Psi(|t-t_0|)}{|t-t_0|} \to 0 \quad \text{as} \quad t \to t_0.$$

Since the left-hand side tends to the sup-norm of the derivative $G'(t_0)$, it follows that $G'(t) = 0$ for all $t \in [a,b]$, so that, by the mean value theorem, G is a constant mapping from $[a,b]$ into $\mathcal{B}(X;\mathbb{R})$: $\exists\, G_0 \in \mathcal{B}(X;\mathbb{R})$ such that $G(t) = R_{g(t)} - R_{x_0} = G_0$ for all $t \in [a,b]$. Thus, for any $t,\, s \in [a,b]$ we have

$$d\big(g(t), g(s)\big) = \| (R_{g(t)} - R_{x_0}) - (R_{g(s)} - R_{x_0}) \|_u = \| G_0 - G_0 \|_u = 0,$$

so that g is a constant mapping from $[a,b]$ into X. □

Proof of Corollary 3.6. Let $f : [a,b] \to X$ be a continuous mapping of bounded Φ-variation. According to Theorem 4.2(a) below (see also (4.7)), the function $\varphi(t) = V_\Phi(f,[a,t])$, $t \in [a,b]$, is continuous on $[a,b]$, so that $E_1 = \varphi([a,b]) = [0,\ell]$, where $\ell = V_\Phi(f,[a,b])$. Thanks to Theorem 3.2, we have $f = g \circ \varphi$ on $[a,b]$, where $g : [0,\ell] \to X$ is such that $\omega_{g,[0,\ell]} \le \Phi^{-1}$ on $\mathbb{R}_0^+$. We are going to show that $\ell = 0$, so that the assertion of Corollary 3.6 is obvious from (P1). On the contrary, suppose that $\ell > 0$. Since

$$\left(\frac{1}{\Phi'(+0)} = (\Phi^{-1})'(+0) = \right) \lim_{t \to +0} \frac{\Phi^{-1}(t)}{t} = \lim_{s \to +0} \frac{s}{\Phi(s)} = 0,$$

we can apply Lemma 3.7 with $\Psi = \Phi^{-1}$ and conclude that g is constant on $[0,\ell]$, which implies that $f = g \circ \varphi$ is a constant mapping on $[a,b]$. Thus, $\ell = 0$, a contradiction. □

4. Continuity properties of mappings of bounded Φ-variation

In this section we address continuity properties of mappings of bounded Φ-variation. We show that these mappings have the same continuity properties as mappings of bounded variation in the sense of Jordan (cf. [5], Sec. 4), but the method of the proof is closer to the one given for mappings of bounded p-variation ($p > 1$) in the sense of Wiener, [7], Sec. 4.

Throughout this section we assume that (X,d) is a metric space, $\Phi \in \mathcal{F}$ is a given function, $f : E \subset \mathbb{R} \to X$ is a fixed mapping of bounded Φ-variation, and the function $\varphi : E \to \mathbb{R}$ is defined by $\varphi(t) = V_\Phi(f, E_t^-)$ for $t \in E$.

We begin with a preliminary lemma.

Lemma 4.1. *If $t \in E$ is a limit point of the set E_t^-, then $d(f(t), f(s))$ has a limit in $[0,\infty[$ as $E \ni s \to t-0$. If, in addition, X is complete, then, as $E \ni s \to t-0$, $f(s)$ has a one-sided limit in X denoted by $f(t-)$, and $d(f(s), f(t))$ tends to $d(f(t-), f(t))$.*

Analogous assertion holds if $t \in E$ is a limit point of E_t^+, with the one-sided limit $f(t+) := \lim_{E \ni s \to t+0} f(s) \in X$ if X is complete.

Proof. If s_1, $s_2 \in E$, $s_1 \le s_2 < t$, then, using continuity of the metric d and applying (P1) and (P3), we have

$$\begin{aligned} &\Phi\big(|d(f(t), f(s_1)) - d(f(t), f(s_2))|\big) \le \Phi\big(d(f(s_1), f(s_2))\big) \le \\ &\le V_\Phi(f, E_{s_1}^{s_2}) \le V_\Phi(f, E_{s_2}^-) - V_\Phi(f, E_{s_1}^-) = \varphi(s_2) - \varphi(s_1). \end{aligned} \tag{4.1}$$

Since φ is nondecreasing and bounded, the limit

$$\varphi(t-) := \lim_{E \ni s \to t-0} \varphi(s) = \sup\{\, \varphi(s) \mid s \in E_t^-,\ s \ne t \,\}$$

is finite; and hence, applying the Cauchy criterion in $\mathbb{R}$ and using (4.1), we obtain the existence of the limit of $d(f(t), f(s))$ as $E \ni s \to t-0$.

If X is complete and s_1, $s_2 \in E$, $s_1 \le s_2 < t$, then, as in (4.1) above, we have $d(f(s_1), f(s_2)) \le \Phi^{-1}\big(\varphi(s_2) - \varphi(s_1)\big)$; and hence, the existence of the limit $f(t-)$ follows from the Cauchy criterion in the complete metric space X. From the continuity of d we have, as $E \ni s \to t-0$,

$$|d(f(s), f(t)) - d(f(t-), f(t))| \le d(f(s), f(t-)) \to 0.$$

The case where $t \in E$ is a limit point of the set E_t^+ is completely analogous. □

Theorem 4.2. (a) *f is continuous at the point $t \in E$ if and only if the function φ is continuous at t.* (b) *f is continuous on E outside, possibly, of a subset of E which is at most countable.* (c) *If X is complete and $\{t_i\}_{i=1}^N \subset E$, $N \in \mathbb{N} \cup \{\infty\}$, is the set of points of discontinuity of f such that every t_i is the limit point of each of the sets $E_{t_i}^-$ and $E_{t_i}^+$, then*

$$\sum_{i=1}^N \Phi\big(d(f(t_i-), f(t_i+))\big) \le V_\Phi(f, E). \tag{4.2}$$

Proof. (a) The only interesting case is where $t \in E$ is a limit point of E. In addition, we suppose that t is a limit point of both E_t^- and E_t^+.

Sufficiency follows from the inequality (cf., (4.1)):

$$d(f(s), f(t)) \le \Phi^{-1}\big(|\varphi(s) - \varphi(t)|\big), \quad s \in E. \tag{4.3}$$

Necessity (which is the most difficult hard part of this theorem) follows from Lemma 4.3 below and Proposition 3.1.

(b) The function φ is nondecreasing on E; and hence, it has at most countably many points of discontinuity. It remains to note that, by (a), the sets of discontinuity points of f and φ are the same.

(c) If $\{s_i, \tau_i\}_{i=1}^n \subset E$ and $s_1 < t_1 < \tau_1 < s_2 < t_2 < \tau_2 < \cdots < s_n < t_n < \tau_n$, then

$$\sum_{i=1}^n \Phi\big(d(f(s_i), f(\tau_i))\big) \le V_\Phi(f, E),$$

so that passing to the limits as $s_i \to t_i - 0$, $\tau_i \to t_i + 0$ and $n \to N$ (if $N = \infty$) and applying Lemma 4.1, we arrive at (4.2). □

Remark. In the case where $\Phi(t) = t^p$, $t \ge 0$, $p > 1$, N. Wiener [19] was the first who proved the inequality (4.2) for real-valued functions f under the additional condition that f has at most countably many points of discontinuity.

Remark. The inequality (4.3) and Lemma 4.1 imply that

$$\lim_{E \ni s \to t-0} d(f(t), f(s)) \le \Phi^{-1}\big(\varphi(t) - \varphi(t-)\big),$$
$$\lim_{E \ni s \to t+0} d(f(s), f(t)) \le \Phi^{-1}\big(\varphi(t+) - \varphi(t)\big).$$

Lemma 4.3. *If $t \in E$ is a limit point of each of the sets E_t^- and E_t^+, then*

$$\varphi(t+) - \varphi(t-) \le \omega(A) + \omega(B), \tag{4.4}$$

where $\omega(\rho) = \omega_{\Phi,[0,L]}(\rho)$ for $\rho > 0$, $\omega(+0) = 0$, $L = D(f, E)$ and

$$A = A(f, t) = \lim_{E \ni s \to t-0} d(f(t), f(s)), \tag{4.5}$$

$$B = B(f, t) = \lim_{E \ni s \to t+0} d(f(s), f(t)). \tag{4.6}$$

Before the proof of this lemma two remarks are in order.

Remark. If X is complete, then (4.4) assumes the form

$$\varphi(t+) - \varphi(t-) \le \omega\big(d(f(t), f(t-))\big) + \omega\big(d(f(t+), f(t))\big).$$

If, in addition, $\Phi(t) = t$, then $\varphi(t+) - \varphi(t-) \le d(f(t), f(t-)) + d(f(t+), f(t))$. The last inequality turns out to be the equality (see [5], Lemma 5.2(a,b)).

Remark. If $t = \inf E \in E$ is a limit point of E, then (4.4) holds if we replace A by zero and $\varphi(t-)$ by $\varphi(t)$. If $t \in E$ is a limit point of E_t^-, then (4.4) holds as well if we replace B by zero and $\varphi(t+)$ by $\varphi(t)$. In particular, this and Lemma 4.3 imply that if $f : [a, b] \to X$ is a continuous mapping of bounded Φ-variation, then

$$[a,b] \ni t \mapsto \varphi(t) = V_\Phi(f,[a,t]) \text{ is also continuous.} \tag{4.7}$$

In order to prove Lemma 4.3, we need one more lemma.

Lemma 4.4. *If a, s, $b \in E$, then*

$$\Phi \circ d(f(b), f(a)) \leq \Phi \circ d(f(s), f(a)) + \omega\big(d(f(b), f(s))\big),$$

where the function ω is the same as in Lemma 4.3.

Proof. Clearly,

$$\begin{aligned} \Phi \circ d(f(b), f(a)) \leq \Phi \circ d(f(s), f(a)) + {}& \\ + |\Phi \circ d(f(b), f(a)) - \Phi \circ d(f(s), f(a))|&, \end{aligned} \tag{4.8}$$

and since $d(f(t), f(\tau)) \leq D(f, E)$ for all t, $\tau \in E$, by definition of ω and continuity of d, we have

$$\begin{aligned} |\Phi \circ d(f(b), f(a)) - \Phi \circ d(f(s), f(a))| &\leq \\ &\leq \omega\big(|d(f(b), f(a)) - d(f(s), f(a))|\big) \leq \\ &\leq \omega\big(d(f(b), f(s))\big). \end{aligned}$$

The last inequality and (4.8) prove the lemma. □

Proof of Lemma 4.3. Let $\varepsilon > 0$ be fixed. By virtue of (4.5) and (4.6), choose a_0, $b_0 \in E$, $a_0 < t < b_0$, such that

$$|d(f(t), f(s)) - A| \leq \varepsilon \qquad \forall\, s \in E,\ a_0 \leq s < t, \tag{4.9}$$

$$|d(f(s), f(t)) - B| \leq \varepsilon \qquad \forall\, s \in E,\ \ t < s \leq b_0. \tag{4.10}$$

Let $T = \{\, t_0 < t_1 < \ldots < t_{m-1} < t_m \,\}$ be a partition of the set $E_{b_0}^-$ with the property (following from the definition of $\varphi(b_0) = V_\Phi(f, E_{b_0}^-)$)

$$\varphi(b_0) - \varepsilon \leq V_\Phi[f, T]. \tag{4.11}$$

We will consider only the case where $t_0 < t < t_m$ since it is clear how to (similarly) handle the cases where $t \leq t_0$ and $t_m \leq t$.

We have two possibilities: (I) $t \notin T$ and (II) $t \in T$.

(I) Let $t \notin T$. Then there exists $k \in \{1, \dots, m\}$ such that $t_{k-1} < t < t_k$, so that from the properties of the Φ-variation we have

$$\begin{aligned} V_\Phi[f,T] &= \sum_{i=1}^{k-1} \Phi \circ d(f(t_i), f(t_{i-1})) + \Phi \circ d(f(t_k), f(t_{k-1})) + \\ &+ \sum_{i=k+1}^{m} \Phi \circ d(f(t_i), f(t_{i-1})) \le \\ &\le \varphi(t_{k-1}) + \Phi \circ d(f(t_k), f(t_{k-1})) + V_\Phi(f, E_{t_k}^{b_0}) \le \\ &\le \varphi(t_{k-1}) + \Phi \circ d(f(t_k), f(t_{k-1})) + \varphi(b_0) - \varphi(t+); \end{aligned} \tag{4.12}$$

in the last inequality we have used the fact that, by (P3),

$$V_\Phi(f, E_{t_k}^{b_0}) \le \varphi(b_0) - \varphi(t_k) \le \varphi(b_0) - \varphi(t+).$$

There are two cases: (a) $a_0 \le t_{k-1}$ and (b) $t_{k-1} < a_0$.

(a) If $a_0 \le t_{k-1}$, then taking into account the definition of ω, (4.9), (4.10), and the triangle inequality for d, we have

$$\begin{aligned} \Phi \circ d(f(t_k), f(t_{k-1})) &\le \omega\big(d(f(t_k), f(t_{k-1}))\big) \le \\ &\le \omega\big(d(f(t), f(t_{k-1})) + d(f(t_k), f(t)) \big) \le \\ &\le \omega(A + \varepsilon + B + \varepsilon). \end{aligned}$$

By virtue of (4.12), it follows that

$$V_\Phi[f,T] \le \varphi(t-) + \omega(A + B + 2\varepsilon) + \varphi(b_0) - \varphi(t+). \tag{4.13}$$

(b) If $t_{k-1} < a_0$, then applying Lemma 4.4 with $a = t_{k-1}$, $s = a_0$, and $b = t_k$, using the triangle inequality for d, (4.9) and (4.10), we have

$$\begin{aligned} \Phi \circ d(f(t_k), f(t_{k-1})) &\le \Phi \circ d(f(a_0), f(t_{k-1})) + \omega\big(d(f(t_k), f(a_0))\big) \le \\ &\le \Phi \circ d(f(a_0), f(t_{k-1})) + \omega\big(d(f(t), f(a_0)) + \\ &+ d(f(t_k), f(t)) \big) \le \\ &\le \Phi \circ d(f(a_0), f(t_{k-1})) + \omega(A + B + 2\varepsilon). \end{aligned}$$

By virtue of (4.12) and (P3), it follows that

$$\begin{aligned} V_\Phi[f,T] &\le \varphi(t_{k-1}) + \Phi \circ d(f(a_0), f(t_{k-1})) + \omega(A + B + 2\varepsilon) + \\ &+ \varphi(b_0) - \varphi(t+) \le \\ &\le \varphi(a_0) + \omega(A + B + 2\varepsilon) + \varphi(b_0) - \varphi(t+) \le \\ &\le \varphi(t-) + \omega(A + B + 2\varepsilon) + \varphi(b_0) - \varphi(t+). \end{aligned} \tag{4.14}$$

(II) Consider the second possibility $t \in T$. There exists $1 \le k \le m-1$ such that $t = t_k$; and we have

$$V_\Phi[f,T] = \sum_{i=1}^{k-1} \Phi \circ d(f(t_i), f(t_{i-1})) + \Phi \circ d(f(t), f(t_{k-1})) + \\ + \Phi \circ d(f(t_{k+1}), f(t)) + \sum_{i=k+2}^{m} \Phi \circ d(f(t_i), f(t_{i-1})) \le \\ \le \varphi(t_{k-1}) + \Phi \circ d(f(t), f(t_{k-1})) + \omega(B+\varepsilon) + \\ + \varphi(b_0) - \varphi(t+), \tag{4.15}$$

where we have used the definition of ω and (4.10) in the last inequality. As above, the following two cases are possible: (a) $a_0 \le t_{k-1}$ and (b) $t_{k-1} < a_0$.

(a) If $a_0 \le t_{k-1}$, then taking into account the definition of ω and (4.9), we have

$$\Phi \circ d(f(t), f(t_{k-1})) \le \omega\big(d(f(t), f(t_{k-1}))\big) \le \omega(A+\varepsilon),$$

so that by virtue of (4.15), it follows that

$$V_\Phi[f,T] \le \varphi(t-) + \omega(A+\varepsilon) + \omega(B+\varepsilon) + \varphi(b_0) - \varphi(t+). \tag{4.16}$$

(b) If $t_{k-1} < a_0$, then applying Lemma 4.4 with $a = t_{k-1}$, $s = a_0$, and $b = t_k = t$, and using (4.9), we have

$$\Phi \circ d(f(t), f(t_{k-1})) \le \Phi \circ d(f(a_0), f(t_{k-1})) + \omega\big(d(f(t), f(a_0))\big) \le \\ \le \Phi \circ d(f(a_0), f(t_{k-1})) + \omega(A+\varepsilon).$$

By virtue of (4.15), it follows that

$$V_\Phi[f,T] \le \varphi(t_{k-1}) + \Phi \circ d(f(a_0), f(t_{k-1})) + \omega(A+\varepsilon) + \\ + \omega(B+\varepsilon) + \varphi(b_0) - \varphi(t+) \le \\ \le \varphi(t-) + \omega(A+\varepsilon) + \omega(B+\varepsilon) + \varphi(b_0) - \varphi(t+). \tag{4.17}$$

Summing up, from (4.13), (4.14), (4.16), (4.17), and Proposition 3.1(d) we conclude that in both cases (I) and (II) we have the inequality

$$V_\Phi[f,T] \le \varphi(b_0) + \varphi(t-) - \varphi(t+) + \omega(A+\varepsilon) + \omega(B+\varepsilon). \tag{4.18}$$

(Now it is clear from the above that (4.18) can be similarly proved if $t \le t_0$ or $t_m \le t$.) From (4.18) and (4.11) we find that

$$\varphi(t+) - \varphi(t-) \le \omega(A+\varepsilon) + \omega(B+\varepsilon) + \varepsilon \quad \forall \varepsilon > 0.$$

It remains to let ε tend to $+0$ and take into account Proposition 3.1(e). □

5. Paths of minimal Φ-variation

The following theorem asserts the existence of a geodesic path between two points in a compact set with respect to the Φ-variation; it extends the results which were previously given in [2], Ch. 1, (5.18), [5], Theorem 6.1, and [7], Theorem 5.1.

Theorem 5.1. *Let $\Phi \in \mathcal{F}$, K be a compact subset of X, and let x, $y \in K$. Suppose that there is a continuous mapping $f_0 : [0,1] \to K$ of finite Φ-variation such that $f_0(0) = x$ and $f_0(1) = y$. Then there exists a continuous mapping $h : [0,1] \to K$ of minimal Φ-variation such that $h(0) = x$, $h(1) = y$, and for some $0 \le L < \infty$,*

$$\Phi\big(d(h(t), h(s))\big) \le L|t-s| \quad \forall\, t,\ s \in [0,1].$$

Proof. In the proof below we borrow some ideas from [5], Theorem 6.1. Let $x \ne y$, $W(x,y) = \{\, f : [0,1] \to K \mid f \text{ continuous},\ f(0) = x,\ f(1) = y \,\}$, and let

$$\ell = \inf\{\, V_\Phi(f, [0,1]) \mid f \in W(x,y) \,\}. \tag{5.1}$$

By (P1), $V_\Phi(f,[0,1]) \ge \Phi\big(d(f(0), f(1))\big) = \Phi\big(d(x,y)\big) > 0$ for $f \in W(x,y)$, so that $\ell \ge \Phi\big(d(x,y)\big)$. By assumption, $\ell \le V_\Phi(f_0,[0,1]) < \infty$; and hence, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of mappings in $W(x,y)$ such that $\ell_n := V_\Phi(f_n,[0,1]) \to \ell$ as $n \to \infty$, where $\ell_n > 0$, so that $L := \sup_{n\in\mathbb{N}} \ell_n$ is finite > 0. By Lemma 3.4, for any $n \in \mathbb{N}$ there exists a continuous mapping $g_n : [\,0,\ell_n\,] \to K$ such that $f_n = g_n \circ \varphi_n$ on $[0,1]$ and

$$d\big(g_n(\alpha), g_n(\beta)\big) \le \Phi^{-1}(|\alpha-\beta|), \quad \alpha,\ \beta \in [\,0,\ell_n],$$

where $\varphi_n(t) = V_\Phi(f_n,[0,t])$ for $0 \le t \le 1$ is continuous by Theorem 4.2(a) (or (4.7)), $g_n(0) = x$, $g_n(\ell_n) = y$ and $V_\Phi(g_n,[0,\ell_n]) = V_\Phi(f_n,[0,1]) = \ell_n$, by (P4). Setting $h_n(\tau) = g_n(\tau\ell_n)$ for $\tau \in [0,1]$, we have $h_n \in W(x,y)$, $V_\Phi(h_n,[0,1]) = \ell_n \to \ell$ as $n \to \infty$ (by (P4)) and

$$d\big(h_n(\alpha), h_n(\beta)\big) \le \Phi^{-1}(\ell_n|\alpha-\beta|) \le \Phi^{-1}(L|\alpha-\beta|), \quad \alpha, \beta \in [0,1]. \tag{5.2}$$

The last estimate shows that the sequence $\{h_n\}$ of mappings $h_n : [0,1] \to K$ is equicontinuous, so that by Ascoli–Arzelà's theorem (see [8], Theorem (4.44)) it has a uniformly convergent subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ with the uniform limit $h : [0,1] \to K$. Clearly, $h \in W(x,y)$ and h satisfies (5.2). Now (P7) implies that

$$V_\Phi(h,[0,1]) \le \liminf_{k\to\infty} V_\Phi(h_{n_k},[0,1]) = \lim_{k\to\infty} \ell_{n_k} = \ell.$$

From (5.1) we know that $\ell \le V_\Phi(h,[0,1])$, so that $\ell = V_\Phi(h,[0,1])$; this is what is required. □

6. Helly selection principle

The following result is a compactness theorem relative to the Φ-variation, which, in the theory of mappings of bounded p-variation with $p \geq 1$, is known as the *E. Helly selection principle* (cf., [17], Ch. 8, Sec. 4, Helly theorem, [5], Sec. 7, [6], Sec. 5, and [7], Sec. 6).

Theorem 6.1. *Let $\Phi \in \mathcal{F}$, K be a compact subset of the metric space X, and let $\mathfrak{F}$ be an infinite family of continuous mappings from the interval $[a, b]$ into K of uniformly bounded Φ-variation, i.e., $\sup_{f\in\mathfrak{F}} V_\Phi(f, [a, b]) < \infty$. Then there exists a sequence of mappings from $\mathfrak{F}$ that converges pointwise on $[a, b]$ to a mapping from $[a, b]$ into K of bounded Φ-variation.*

Moreover, if X is a Banach space, then the theorem holds without the continuity assumption on the family $\mathfrak{F}$.

Proof. We adapt the proof from [5], Theorem 7.1 for the case considered, and divide it into three steps.

Step 1 (common auxiliary part). According to Theorem 3.2 we can write any mapping $f \in \mathfrak{F}$ in the form $f = g_f \circ \varphi_f$ on $[a, b]$, where $\varphi_f(t) = V_\Phi(f, [a, t])$ for $t \in [a, b]$ and $g_f : E_{1f} = \varphi_f([a, b]) \to K$ is a mapping such that $\omega_{g_f,E_{1,f}} \leq \Phi^{-1}$ on $\mathbb{R}_0^+$. The family of nondecreasing nonnegative functions $\{\varphi_f \mid f \in \mathfrak{F}\}$ (with $\varphi_f(a) = 0$) is infinite and uniformly bounded on $[a, b]$ since

$$D(\varphi_f, [a, b]) = \varphi_f(b) = V_\Phi(f, [a, b]);$$

and hence, it contains a sequence of functions $\{\varphi_n\}_{n=1}^\infty$ corresponding to the decompositions $f_n = g_n \circ \varphi_n$ (i.e., $\varphi_n = \varphi_{f_n}$ and $g_n = g_{f_n}$) for all $n \in \mathbb{N}$, which converges pointwise on $[a, b]$ to a nondecreasing and bounded function $\varphi : [a, b] \to \mathbb{R}$ (see [17], Ch. 8, Sec. 4, Lemma 2). If $\ell = \varphi(b)$ and $\ell_n = \varphi_n(b)$, then $0 \leq \ell < \infty$ and $\ell_n \to \ell$ as $n \to \infty$.

Step 2. Suppose that the family $\mathfrak{F}$ consists of continuous mappings. Since $f_n \in \mathfrak{F}$ is continuous, then φ_n is continuous as well by Theorem 4.2(a) (or (4.7)), so that the mapping g_n is defined on the interval $E_{1n} = \varphi_n([a, b]) = [0, \ell_n]$. If $\ell_n \geq \ell$, then we consider g_n only on the interval $[0, \ell]$, and if $\ell_n < \ell$, then we extend g_n onto $]\ell_n, \ell]$ by setting $g_n(\tau) = g_n(\ell_n)$ for all $\tau \in]\ell_n, \ell]$. It follows that $\omega_{g_n,[0,\ell]} \leq \Phi^{-1}$ on $\mathbb{R}_0^+$, so that the sequence $\{g_n\}_{n=1}^\infty \subset K^{[0,\ell]}$ is equicontinuous, and hence, by the Ascoli–Arzelà theorem, it is precompact in the space $\mathcal{C}([0, \ell]; K)$ of all continuous mappings from $[0, \ell]$ into K, and hence, it has a uniformly convergent subsequence $\{g_{n_k}\}_{k=1}^\infty$ with the uniform limit $g : [0, \ell] \to K$ such that $\omega_{g,[0,\ell]} \leq \Phi^{-1}$ on $\mathbb{R}_0^+$. By virtue of Lemma 3.3, the composed mapping $f := g \circ \varphi : [a, b] \to K$ is of bounded Φ-variation. Now, if $t \in [a, b]$, we have

$$d(f_{n_k}(t), f(t)) = d((g_{n_k} \circ \varphi_{n_k})(t), (g \circ \varphi)(t)) \leq$$

$$\leq d(g_{n_k}(\varphi_{n_k}(t)), g_{n_k}(\varphi(t))) + d(g_{n_k}(\varphi(t)), g(\varphi(t))) \leq$$
$$\leq \Phi^{-1}(|\varphi_{n_k}(t) - \varphi(t)|) + d(g_{n_k}(\varphi(t)), g(\varphi(t))).$$

The terms in the last sum tend to zero as $k \to \infty$, so that the sequence $\{f_{n_k}\}_{k=1}^{\infty} \subset \mathfrak{F}$ converges pointwise on $[a,b]$ to $f \in \mathcal{V}_\Phi([a,b];K)$.

Step 3. Let X be a Banach space and $\mathfrak{F}$ be an infinite family of mappings from $[a,b]$ into K of uniformly bounded Φ-variation. Using the argument of step 1, in this case we have $E_{1n} = \varphi_n([a,b]) \subset [0,\ell_n]$. If $L = \sup_{n\in\mathbb{N}} \ell_n$, then $0 \leq L < \infty$ and $\ell = \lim_{n\to\infty} \ell_n \leq L$. Denote by $\overline{g_n}$ the restriction to $[0,L]$ of the mapping g_n^* given by Lemma 3.5 with $\Psi = \Phi^{-1}$. For all $n \in \mathbb{N}$ we have $\omega_{\overline{g_n},[0,L]} \leq \Psi^*$ on $[0,L]$, and hence, by the Ascoli–Arzelà theorem, the equicontinuous sequence $\{\overline{g_n}\}_{n=1}^{\infty} \subset K^{[0,L]}$ has a uniformly convergent subsequence $\{\overline{g_{n_k}}\}_{k=1}^{\infty}$, whose uniform limit we denote by $\overline{g}$. Setting $f = \overline{g} \circ \varphi : [a,b] \to K$, for all $t \in [a,b]$ we have (as at the end of step 2)

$$d(f_{n_k}(t), f(t)) = d(\overline{g_{n_k}}(\varphi_{n_k}(t)), \overline{g}(\varphi(t)) \leq$$
$$\leq \Psi^*(|\varphi_{n_k}(t) - \varphi(t)|) + d(\overline{g_{n_k}}(\varphi(t)), \overline{g}(\varphi(t))).$$

Thus, f_{n_k} converges pointwise on $[a,b]$ to f as $k \to \infty$. Applying property (P7), we obtain $f \in \mathcal{V}_\Phi([a,b];K)$. □

Open question. Is it possible to replace the condition "$\mathfrak{F} \subset K^{[a,b]}$ with a compact subset $K \subset X$" in Theorem 6.1 by the following weaker condition: "for every $t \in [a,b]$ the set $\mathfrak{F}(t) := \{ f(t) \mid f \in \mathfrak{F} \}$ is precompact in X"?

7. Mappings with values in a normed vector space

In this section we assume that X is a normed vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the norm $\|\cdot\|$, and as usual, $\varnothing \neq E \subset \mathbb{R}$. We are going to prove that if the function $\Phi \in \mathcal{F}$ satisfies the condition (see (2.6)):

$$\exists C \in \mathbb{R} \text{ such that } \Phi(t+s) \leq C(\Phi(t) + \Phi(s)) \ \forall t,\, s \geq 0, \qquad (7.1)$$

then the convergence of sequences of mappings from $\mathcal{V}_\Phi(E;X)$ in (the sense of) variation is given by a metric $\Delta = \Delta_\Phi$ on $\mathcal{V}_\Phi(E;X)$; moreover, the metric space $(\mathcal{V}_\Phi(E;X), \Delta)$ is *complete* if X is a Banach space. (Note that condition (7.1) can be replaced by the following: $\exists C \in \mathbb{R}$ such that $\Phi(2t) \leq C\Phi(t)$ $\forall t \geq 0$; in fact, the latter condition implies that for all $t,\, s \geq 0$

$$\Phi(t+s) \leq \Phi(2\max\{t,s\}) \leq C\Phi(\max\{t,s\}) \leq C\big(\Phi(t) + \Phi(s)\big).\ )$$

Choosing $s = 0$ and $t > 0$ in (7.1), we have $1 \leq C < \infty$. Since E is fixed throughout this section, for brevity, we will write $V_\Phi(f) := V_\Phi(f,E)$.

From (7.1) we have that if $f,\, g \in \mathcal{V}_\Phi(E;X)$ and $\alpha \in \mathbb{K}$, then

$$V_\Phi(f+g) \le C\bigl(V_\Phi(f)+V_\Phi(g)\bigr), \qquad (7.2)$$
$$V_\Phi(\alpha f) \le C_\alpha V_\Phi(f),$$

where

$$C_\alpha = \begin{cases} |\alpha| C^{|\alpha|-1}, & \text{if} \quad \alpha = 0 \text{ or } |\alpha| \ge 1, \\ 1, & \text{if} \quad 0 < |\alpha| < 1, \end{cases}$$

and hence, $\mathcal{V}_\Phi(E;X)$ is a *vector* space over $\mathbb{K}$.

In what follows, $t_0 \in E$ is a fixed point.

Definition. A sequence of mappings $\{f_n\}_{n=1}^\infty \subset \mathcal{V}_\Phi(E;X)$ is said to be

(a) *convergent* to a mapping $f \in X^E$ *in* (the sense of Φ-) *variation* if

$$\| f_n(t_0) - f(t_0) \| + V_\Phi(f_n - f) \to 0 \quad \text{as} \quad n \to \infty;$$

(b) *Cauchy in* (the sense of Φ-) *variation* if

$$\| f_n(t_0) - f_m(t_0) \| + V_\Phi(f_n - f_m) \to 0 \quad \text{as} \quad n,\, m \to \infty.$$

If $C = 1$ in (7.1), then from the properties of the Φ-variation and (7.2) it follows that the mapping

$$\Delta(f,g) = \| f(t_0) - g(t_0) \| + V_\Phi(f-g), \qquad f,\, g \in \mathcal{V}_\Phi(E;X),$$

is a metric on $\mathcal{V}_\Phi(E;X)$ which induces the convergence in variation.

If $C > 1$ in (7.1), then the definition of a metric on $\mathcal{V}_\Phi(E;X)$ is more complicated. We derive the construction of the metric Δ below from the metrization lemma, [13], Lemma 6.12, which gives a pseudometric corresponding to the uniform structure with a countable base. Clearly, the base of the uniform structure, which gives rise to the convergence in variation, consists of the sets

$$\mathcal{U}_n = \Bigl\{ (f,g) \ \Big|\ \| f(t_0) - g(t_0) \| + V_\Phi(f-g) < \frac{1}{n} \Bigr\}, \quad n \in \mathbb{N},$$

where $f,\, g \in \mathcal{V}_\Phi(E;X)$.

Definition. Let $C > 1$ in (7.1). For $f,\, g \in \mathcal{V}_\Phi(E;X)$ we set

$$\Delta_1(f,g) = \Bigl(\| f(t_0) - g(t_0) \| + V_\Phi(f-g) \Bigr)^{1/\gamma}, \quad \gamma := \log_2(3C^2),$$

and define the mapping $\Delta : \mathcal{V}_\Phi(E;X) \times \mathcal{V}_\Phi(E;X) \to \mathbb{R}_0^+$ by

$$\Delta(f,g) = \inf \sum_{i=1}^{n} \Delta_1(f_i, f_{i-1}),$$

where the infimum is taken over all finite sequences $\{f_i\}_{i=0}^n$ of mappings from $\mathcal{V}_\Phi(E;X)$ such that $f_0 = f$ and $f_n = g$.

The main result of this section is the following theorem.

Theorem 7.1. (a) Δ *is a metric on* $\mathcal{V}_\Phi(E;X)$.

(b) *Let* $\{f_n\}_{n=1}^\infty \subset \mathcal{V}_\Phi(E;X)$. *Then the sequence* f_n *converges to* $f \in \mathcal{V}_\Phi(E;X)$ *in variation (is Cauchy in variation) if and only if* $\lim_{n\to\infty} \Delta(f_n, f) = 0$ *(respectively,* $\lim_{n,m\to\infty} \Delta(f_n, f_m) = 0$*).*

(c) *If* X *is a Banach space, then* $(\mathcal{V}_\Phi(E;X), \Delta)$ *is a* complete *metric vector space.*

In order to prove this theorem, we need two lemmas.

Lemma 7.2. *If* $S \in \mathbb{R}_0^+$ *and* $\Delta_1(f_i, f_{i-1}) \le S$ *for* $i = 1, 2, 3$, *then* $\Delta_1(f_3, f_0) \le 2S$.

Proof. From (7.1), using that $C \ge 1$, we have

$$V_\Phi(f_3 - f_0) \le C^2\big(V_\Phi(f_1 - f_0) + V_\Phi(f_2 - f_1) + V_\Phi(f_3 - f_2)\big)$$

which, by virtue of the triangle inequality for the norm $\|\cdot\|$, implies that

$$\Delta_1(f_3, f_0) \le \Big(\sum_{i=1}^{3}\big(\| f_i(t_0) - f_{i-1}(t_0) \| + C^2 V_\Phi(f_i - f_{i-1})\big)\Big)^{1/\gamma} \le$$

$$\le \Big(C^2 \sum_{i=1}^{3}\big(\Delta_1(f_i, f_{i-1})\big)^\gamma\Big)^{1/\gamma}.$$

It follows that if $\Delta_1(f_i, f_{i-1}) \le S$ for $i = 1, 2, 3$, then the choice of γ yields

$$\Delta_1(f_3, f_0) \le (3C^2 S^\gamma)^{1/\gamma} = 2S.$$

□

Lemma 7.3. $\Delta(f,g) \ge \frac{1}{2}\Delta_1(f,g)$ *for all* f, $g \in \mathcal{V}_\Phi(E;X)$.

Proof. By the definition of $\Delta(f,g)$, it suffices to show that for any finite sequence of mappings $\{f_i\}_{i=0}^n \subset \mathcal{V}_\Phi(E;X)$ we have

$$\sum_{i=1}^{n} \Delta_1(f_i, f_{i-1}) \ge \frac{1}{2}\Delta_1(f_n, f_0). \tag{7.3}$$

This is done by induction in $n \in \mathbb{N}$ as follows. The inequality is obvious if $n = 1$. Let $m \in \mathbb{N}$, and suppose that (7.3) holds for any sequence $\{f_i\}_{i=0}^n$ with $n \le m$. Let us prove that the corresponding inequality also holds for arbitrary sequence $\{f_i\}_{i=0}^{m+1}$. Set $S = \sum_{i=1}^{m+1} \Delta_1(f_i, f_{i-1})$ and choose a number $k \in \{0, 1, \ldots, m\}$ in such a way that

$$\sum_{i=1}^{k} \Delta_1(f_i, f_{i-1}) \le \frac{1}{2}S \le \sum_{i=1}^{k+1} \Delta_1(f_i, f_{i-1})$$

(note that we use the convention that $\sum_{i=1}^{0}\cdots=\sum_{i=m+2}^{m+1}\cdots=0$). Clearly,

$$\sum_{i=k+2}^{m+1}\Delta_1(f_i,f_{i-1})\le\frac{1}{2}S.$$

Using the induction assumption, we have

$$\sum_{i=1}^{k}\Delta_1(f_i,f_{i-1})\ge\frac{1}{2}\Delta_1(f_k,f_0),$$

$$\sum_{i=k+2}^{m+1}\Delta_1(f_i,f_{i-1})\ge\frac{1}{2}\Delta_1(f_{m+1},f_{k+1}),$$

and hence,

$$\Delta_1(f_k,f_0)\le S,\qquad \Delta_1(f_{m+1},f_{k+1})\le S.$$

Since (obviously) $\Delta_1(f_{k+1},f_k)\le S$, by Lemma 7.2 we have

$$\Delta_1(f_{m+1},f_0)\le 2S;$$

this is the desired inequality. □

Now we are in a position to prove Theorem 7.1.

Proof of Theorem 7.1. (a) Clearly, Δ is nonnegative, symmetric, and satisfies the triangle inequality. Let us show that $\Delta(f,g)=0\iff f=g$. Indeed, if $\Delta(f,g)=0$, then $\Delta_1(f,g)=0$ according to Lemma 7.3, or $\|f(t_0)-g(t_0)\|=0$ and $V_\Phi(f-g)=0$. The last equality and property (P1) imply that $f-g$ is a constant mapping, and since $f(t_0)=g(t_0)$, then $f=g$. The other implication is obvious.

(b) The necessity follows from the inequality $\Delta(f_n,f)\le\Delta_1(f_n,f)$, and the sufficiency is a consequence of Lemma 7.3. An analogous argument applies to Cauchy sequences.

(c) Let $\{f_n\}_{n=1}^{\infty}$ be a Δ-Cauchy sequence in $\mathcal{V}_\Phi(E;X)$. By virtue of (b), the sequence $\{f_n\}$ is Cauchy in variation, and hence, as n, $m\to\infty$, $\|f_n(t_0)-f_m(t_0)\|\to 0$ and

$$V_\Phi(f_n-f_m)\to 0. \tag{7.4}$$

For all $t\in E$ and n, $m\in\mathbb{N}$ we have, by (P1),

$$\begin{aligned}\|f_n(t)-f_m(t)\| &\le \|f_n(t_0)-f_m(t_0)\|+\|(f_n-f_m)(t)-(f_n-f_m)(t_0)\|\\ &\le \|f_n(t_0)-f_m(t_0)\|+\Phi^{-1}\big(V_\Phi(f_n-f_m)\big).\end{aligned}$$

Thus, the sequence $\{f_n(t)\}_{n=1}^{\infty}$ is Cauchy in X for all $t \in E$, so that $\{f_n\}_{n=1}^{\infty}$ has a pointwise limit $f \in X^E$, and in particular,

$$\| f_n(t_0) - f(t_0) \| \to 0 \quad \text{as} \quad n \to \infty. \tag{7.5}$$

Let us show that f is of bounded Φ-variation. By (7.4), the sequence $\{V_\Phi(f_n - f_1)\}_{n=1}^{\infty}$ is bounded, so that, $V_\Phi(f_1) < \infty$ and (7.2) imply that $\{V_\Phi(f_n)\}_{n=1}^{\infty}$ is bounded as well. Now property (P7) yields

$$V_\Phi(f) \le \liminf_{n\to\infty} V_\Phi(f_n) < \infty,$$

i.e., $f \in \mathcal{V}_\Phi(E;X)$.

It remains to show that f_n converges to f in variation. In view of (7.5), it suffices to verify that $V_\Phi(f_n - f) \to 0$ as $n \to \infty$. By virtue of (P7) and (7.4), we have

$$\limsup_{n\to\infty} V_\Phi(f_n - f) \le \limsup_{n\to\infty} \liminf_{m\to\infty} V_\Phi(f_n - f_m) = 0;$$

this ensures that $\lim_{n\to\infty} V_\Phi(f_n - f) = 0$. The proof is complete. □

Remark. If $\Phi(t) = t^p$, $t \ge 0$, $p \ge 1$, then $\mathcal{V}_\Phi(E;X)$ is, actually, a normed vector space, which is complete whenever X is complete (see [5], Sec. 8 for $p = 1$ and [7], Sec. 7 for $p > 1$).

8. Set-valued mappings and their selections

We begin with the definitions of the *Hausdorff distance* and *set-valued mappings* (for more detail see [1], Ch. 1, Secs. 1, 5 and [3], Ch. 2, Sec. 1).

Given two nonempty subsets A, $B \subset X$ of a metric space (X, d), the *Hausdorff distance d_H between A and B* is defined by

$$d_H(A,B) = \max\{\mathrm{e}\,(A,B), \mathrm{e}\,(B,A)\}, \quad \text{where} \quad \mathrm{e}\,(A,B) := \sup_{x\in A} \inf_{y\in B} d(x,y).$$

The mapping $d_H(\,\cdot\,,\,\cdot\,)$ is a *metric* on the set of all nonempty closed bounded subsets of X, and hence, on the set of all nonempty compact subsets of X.

Let E and X be two metric spaces, 2^X be the set of all subsets of X and $\dot{2}^X = 2^X \setminus \{\varnothing\}$. A *set-valued mapping* from E into X is a mapping $F : E \to 2^X$, so that $F(t) \subset X$ for every $t \in E$. The set-valued mapping $F : E \to \dot{2}^X$ is said to be

(a) *compact* if its graph $\mathrm{Gr}(F) := \{\, (t,x) \in E \times X \mid x \in F(t) \,\}$ is a compact subset of $E \times X$ (and hence, $F(t)$ is a compact subset of X for every $t \in E$ but not vice versa);

(b) *of bounded Φ-variation on* $E = [a,b] \subset \mathbb{R}$, where $\Phi \in \mathcal{F}$ if (see (2.1)–(2.4))

$$V_{a,\Phi}^b(F) = \sup\{\, V_{\Phi,d_H}[F,T] \mid T \in \mathcal{T}([a,b]) \,\} < \infty;$$

(c) *Hausdorff continuous at* $t_0 \in E$ if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $d_H(F(t), F(t_0)) \leq \varepsilon$ for all $t \in E$ with $d_E(t, t_0) \leq \delta$; *Hausdorff continuous on* E if it is continuous at every point $t_0 \in E$.

The mapping $f : E \to X$ is said to be a (*regular*) *selection* of the set-valued mapping $F : E \to \dot{2}^X$ if $f(t) \in F(t)$ for all $t \in E$.

Continuous selections of convex-valued set-valued mappings under very general conditions are known to exist due to Michael [15]. Here we consider mainly the nonconvex case. The first result of this section addresses the question of existence of continuous regular selections of a set-valued mapping of bounded Φ-variation. It generalizes some results for mappings of bounded variation in the sense of Jordan in [10] and [11] in the finite-dimensional case and [16], Supplement, Theorem 1.8, [5], Theorem 9.1, and [6], Theorem 6.1 in the infinite-dimensional case.

Theorem 8.1. *Suppose that* $\Phi \in \mathcal{F}$ *and there are* $0 < c_1 \leq c_2 < \infty$ *and* $\delta > 0$ *such that* $c_1 t \leq \Phi(t) \leq c_2 t$ *for all* $0 \leq t \leq \delta$ (*in particular, this holds if* $\Phi'(+0) = c$ *for some* $0 < c < \infty$). *Let* X *be a Banach space* (*over the field* $\mathbb{R}$ *or* $\mathbb{C}$), $F : [a, b] \to \dot{2}^X$ *be a compact Hausdorff continuous set-valued mapping of bounded* Φ*-variation on* $[a, b]$, $t_0 \in [a, b]$ *and* $x_0 \in F(t_0)$. *Then there exists a continuous selection* $f : [a, b] \to X$ *of* F *of bounded* Φ*-variation* (*and bounded* 1*-variation*) *such that* $f(t_0) = x_0$.

Proof. Proposition 2.3, the inequality $c_1 t \leq \Phi(t)$ on $[0, \delta]$, and Proposition 2.5(c) imply that

$$V^b_{a,\Phi(t)}(F) < \infty \quad \Longrightarrow \quad c_1 V^b_{a,t}(F) = V^b_{a,c_1 t}(F) < \infty.$$

Thus, F is of bounded variation on $[a, b]$ in the sense of Jordan, so that applying Theorem 9.1 from [5], we get a continuous selection $f : [a, b] \to X$ of F of bounded 1-variation such that $f(t_0) = x_0$. Again, thanks to Proposition 2.3 and the inequality $\Phi(t) \leq c_2 t$ on $[0, \delta]$, we have

$$V_{c_2 t}(f, [a, b]) = c_2 V_t(f, [a, b]) < \infty \quad \Longrightarrow \quad V_{\Phi(t)}(f, [a, b]) < \infty,$$

and hence, f is of bounded Φ-variation; this is what is required. □

Remark. If Φ in Theorem 8.1 is such that $\Phi'(+0) = \infty$ (or, equivalently, $\lim_{t \to +0} \Phi(t)/t = \infty$), then F is a *constant* set-valued mapping on $[a, b]$ (see Corollary 3.6), so that F trivially admits continuous selections of bounded (in fact, zero) Φ-variation.

We note that the limiting behavior of the function $\Phi(t)/t$ as $t \to +0$ plays an important rôle. Now we are going to consider the case where $\Phi'(+0) = \lim_{t \to +0} \Phi(t)/t = 0$.

Theorem 8.2. *Let* $\Phi \in \mathcal{F}$, *and set* $\phi(t) := \Phi(t)/t$, $t \in]0, \infty[$.

(a) *If* $\Phi'(+0) = 0$ *and* ϕ *is strictly increasing on* $]0, \infty[$, *then there exists a compact set-valued mapping* $F : [-1, 1] \to \dot{2}^{\mathbb{R}^2}$ *satisfying*

$$d_H(F(t), F(s)) \le \Phi^{-1}(|t - s|) \qquad \forall\, t,\, s \in [-1, 1] \tag{8.1}$$

(*and hence, F is Hausdorff continuous of bounded Φ-variation*), *which admits no continuous selection.*

(b) *If there is a constant* $c_0 > 0$ *such that*

$$\phi(2t) \ge c_0(\phi(t))^{1/2} \quad \textit{for all} \quad 0 < t \le 2, \tag{8.2}$$

then there exists a compact set-valued mapping $F : [-1, 1] \to \dot{2}^{\mathbb{R}^2}$ *which satisfies* (8.1) *and has no selection of bounded Φ-variation.*

Remark. Examples of functions Φ in Theorem 8.2 (a) and (b) are given, respectively, by $\Phi(t) = t^p$ with $p > 1$ and $\Phi(t) = te^{-1/t^q}$ with $q \ge 1$.

Proof of Theorem 8.2. (a) Step 1. The example below is a generalization of Example 1 in [1], Ch. 1, Sec. 6 and Proposition 8.2 in [7]. Let $C = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ be the unit circle in $\mathbb{R}^2$. For $0 < t \le 1$ set

$$\alpha(t) = \frac{1}{t}, \quad \beta(t) = \arcsin \phi^{-1}\Big(\frac{t^2}{a}\Big) \ \text{ with } \ a = 1 + \frac{\pi}{2} + \frac{1}{\phi(1)}.$$

The function $\beta(t)$ is well defined since ϕ is strictly increasing and $t^2/a \le 1/a < \phi(1)$ for $t \in]0, 1]$, so that

$$0 < \beta(t) < \frac{\pi}{2} \quad \forall 0 < t \le 1. \tag{8.3}$$

Since $\phi(t) \to 0$ as $t \to +0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \ge n_0$ the unique positive solution $\tau = \tau_n$ of the equation $\beta(\tau) = \pi/2^{n+2}$, which is of the form $\tau_n = \big(a\phi(\sin(\pi/2^{n+2}))\big)^{1/2}$, lies in $]0, 1]$. Clearly, $\{\tau_n\}_{n=n_0}^{\infty}$ is a decreasing sequence which tends to zero as $n \to \infty$.

Now we define the mapping $F : [-1, 1] \to \dot{2}^{\mathbb{R}^2}$ as follows. Let $t \in]0, 1]$. Then, either $t \in]\tau_{n+1}, \tau_n]$ for some $n = n(t) \ge n_0$, or $t \in [\tau_{n_0}, 1]$. If $t \in]\tau_{n+1}, \tau_n]$, we set $\delta_n(t) = \beta(\tau_n) - \beta(t)$, and for $m = 0, 1, \ldots, 2^n - 1$, we define open arcs $A_m(t)$, $B_m(t) \subset C$ as follows:

$$\begin{aligned} A_m(t) = \{ &(x, y) \in C \mid x = \cos\theta, y = \sin\theta, \\ &\alpha(t) + m\frac{\pi}{2^{n-1}} < \theta < \alpha(t) + m\frac{\pi}{2^{n-1}} + 2\beta(t) \}, \\ B_m(t) = \{ &(x, y) \in C \mid x = \cos\theta, y = \sin\theta, \\ &\alpha(t) + m\frac{\pi}{2^{n-1}} + \frac{\pi}{2^n} + 2\beta(\tau_{n+1}) - 2\delta_n(t) < \theta < \\ &\quad < \alpha(t) + m\frac{\pi}{2^{n-1}} + \frac{\pi}{2^n} + 2\beta(\tau_{n+1}) \}; \end{aligned}$$

and we set

$$A(t) = \bigcup_{m=0}^{2^n-1} A_m(t), \qquad B(t) = \bigcup_{m=0}^{2^n-1} B_m(t). \tag{8.4}$$

If $t \in [\tau_{n_0}, 1]$, we set $A(t) = A(\tau_{n_0})$ and $B(t) = B(\tau_{n_0})$, where $A(\tau_{n_0})$ and $B(\tau_{n_0})$ are defined in (8.4).

We define the set-valued mapping $F : [-1, 1] \to \dot{2}^{\mathbb{R}^2}$ by setting

$$F(t) = \begin{cases} C \setminus \big(A(|t|) \cup B(|t|)\big) & \text{if } t \neq 0, \\ C & \text{if } t = 0. \end{cases}$$

For $t \neq 0$, $F(t)$ is the unit circle in $\mathbb{R}^2$ from which a finite number of sections (depending on t) are removed. As t gets smaller, the arclengths of the holes decrease while the initial angles increase as $1/|t|$, i.e., the holes spin around the origin with increasing angular speed. Any continuous selection $f(t) = (x(t), y(t))$ defined on $[-1, 0[$ or on $]0, 1]$ (for instance, $x(t) = \cos(1/|t|)$, $y(t) = \sin(1/|t|)$) cannot be continuously extended to the whole $[-1, 1]$. In fact, the holes in the circumference should force this selection to rotate around the origin, and hence, the limits $\lim_{t\to\pm 0} f(t)$ cannot exist.

However, F is Hausdorff continuous on $[-1, 1]$ and satisfies (8.1). To see this, let $0 < s < t \le 1$. We have

$$d_H(F(t), F(s)) \le \min\{\, \sin(\alpha(s) - \alpha(t)), \sin\beta(t)\,\}.$$

The inequality $\alpha(s) - \alpha(t) \le \beta(t)$ is equivalent to the inequality $s \ge s_0(t) := t/(1 + t\beta(t))$, so that

$$d_H(F(t), F(s)) \le \begin{cases} \sin\beta(t) & \text{if } 0 < s \le s_0(t), \\ \sin(\alpha(s) - \alpha(t)) & \text{if } s_0(t) \le s \le t. \end{cases}$$

The right-hand side is estimated in steps 2 and 3 below.

Step 2. $\sin\beta(t) \le \Phi^{-1}(t - s_0(t))$ for all $0 < t \le 1$.
In fact, $a > 1$ and $1 + t\beta(t) \le 1 + \pi/2 \le a$ by (8.3), so that

$$t - s_0(t) = \frac{t^2\beta(t)}{1 + t\beta(t)} \ge \frac{t^2\beta(t)}{a} = \phi\big(\sin\beta(t)\big) \cdot \beta(t) \ge \Phi\big(\sin\beta(t)\big),$$

which implies the desired inequality.

Step 3. $\sin(\alpha(s) - \alpha(t)) \le \Phi^{-1}(t - s)$ for all $s_0(t) \le s < t$.
In fact, setting $z = \alpha(s) - \alpha(t) = \frac{1}{s} - \frac{1}{t}$, we have $0 < z \le \beta(t) < \pi/2$, $s = t/(1 + tz)$, and $t - s = t^2z/(1 + tz)$, so that the desired inequality is equivalent to $\sin z \le \Phi^{-1}(t^2z/(1 + tz))$, which follows from

$$\Phi(\sin z) = \phi(\sin z)\sin z \le \phi(\sin\beta(t))z = \frac{t^2}{a}z \le \frac{t^2z}{1 + (\pi/2)} \le \frac{t^2z}{1 + tz}.$$

Step 4. To complete the proof of (a), let $0 < s \le s_0(t)$; then, by step 2, we have

$$d_H(F(t), F(s)) \le \sin\beta(t) \le \Phi^{-1}(t - s_0(t)) \le \Phi^{-1}(t-s).$$

If $s_0(t) \le s \le t$, then, by step 3, we have

$$d_H(F(t), F(s)) \le \sin(\alpha(s) - \alpha(t)) \le \Phi^{-1}(t-s).$$

Thus, $d_H(F(t), F(s)) \le \Phi^{-1}(t-s)$ for all $0 < s < t \le 1$.
If $s = 0 < t \le 1$, then

$$d_H(F(t), F(s)) \le \sin\beta(t) \le \Phi^{-1}(t).$$

The cases $-1 \le t < s \le 0$ and $-1 \le t \le 0 \le s \le 1$ are treated similarly.

(b) Suppose that f is a selection of the mapping F, constructed in the proof of (a). If $t_n := 1/(2\pi n)$, then $\alpha(t_n) = 2\pi n$, $n \in \mathbb{N}$. From the definitions of the sets $A_m(t)$ and $B_m(t)$, it follows that for any $t \in]0, \tau_{n_0}]$ the set $A(t)$ consists of at least $\pi/(8\beta(t))$ arcs $A_m(t)$ of the angle $2\beta(t)$. Hence, for any $n \in \mathbb{N}$ with $n \ge n_1 := 1/\tau_{n_0}$, we have one of the following two possibilities: (1) the interval $[t_{n+1}, t_n]$ contains a pair of points a_n, b_n such that $\| f(b_n) - f(a_n) \|_{\mathbb{R}^2} \ge \sqrt{2}$, or (2) there are $M_n \ge \pi/(16\beta(t_n))$ pairs of points c_{nm}, d_{nm} in the interval $[t_{n+1}, t_n]$, $1 \le m \le M_n$, such that $\| f(c_{nm}) - f(d_{nm}) \|_{\mathbb{R}^2} \ge 2\sin\beta(t_n)$. In this latter case, by virtue of (8.2), we have

$$\begin{aligned}\sum_{m=1}^{M_n} \Phi\big(\| f(c_{nm}) - f(d_{nm}) \|_{\mathbb{R}^2}\big) &\ge \frac{\pi}{16\beta(t_n)}\Phi(2\sin\beta(t_n)) \ge \\ &\ge \frac{\pi}{16\beta(t_n)} \cdot 2\sin\beta(t_n) \cdot \phi(2\sin\beta(t_n)) \ge \\ &\ge \frac{\pi}{8} \cdot \frac{\sin\beta(t_n)}{\beta(t_n)} \cdot c_0 \cdot \big(\phi(\sin\beta(t_n))\big)^{1/2} \ge \\ &\ge \frac{\pi}{8} \cdot \frac{2}{\pi} \cdot c_0 \cdot \left(\frac{t_n^2}{a}\right)^{1/2} = \frac{c_0}{8\pi\sqrt{a}} \cdot \frac{1}{n}.\end{aligned}$$

It follows that

$$V_\Phi(f, [-1,1]) \ge \frac{c_0}{8\pi\sqrt{a}} \sum_{n=n_1}^{\infty} \frac{1}{n} = \infty,$$

and hence, F admits no selection of bounded Φ-variation. □

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