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Analysis of the Impact of the Number of Edges in Connected Graphs on the Time Complexity of an Independent Set Problem

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Abstract—Under study is the complexity status of the independent set problem in a class of connected graphs that are defined by functional constraints on the number of edges depending on the number of vertices. For every natural number C, this problem is shown to be polynomially solvable in the class of graphs

$$\bigcup_{n=1}^{\infty}\{G\,:\,|V(G)|=n,\ |E(G)|\leq n+C[\log_2(n)]\}.$$

On the other hand, this problem is proved not polynomially solvable in the class of graphs

$$\bigcup_{n=1}^{\infty}\{G\,:\,|V(G)|=n,\ |E(G)|\leq n+f^2(n)\}$$

for every unbounded nondecreasing function $f(n):\mathbb{N}\to\mathbb{N}$, such that exponent of this function grows faster than polynomial function of n. The latter result is true if there exists non subexponential algorithm for solving the independent set problem.

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INTRODUCTION

This paper continues [1], where the relation was studied between the number of edges in the graphs and the complexity of the independent set problem. Recall that a subset of vertices of a graph is called *independent* if these vertices are pairwise nonadjacent. An independent set with the maximal cardinality in a graph G is called the *largest independent set in graph*, and its cardinality is called the *independence number of graph* and is denoted by $\alpha(G)$. The independent set problem (called *Problem* IS in the following) is to find the largest independent set in a given graph. It is known [2] that Problem IS is polynomially equivalent to the problem of determining the value of the independence number. We will deal with the latter problem and call it Problem IS.

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In [1], we considered some special subsets of the set $\mathcal{G}$ of all graphs. Each of these subsets (denoted by $\mathcal{G}_{f(n)}$) was defined by the function $f(n) : \mathbb{N} \to \mathbb{N}$ as the set of all graphs from

$$\bigcup_{n=1}^{\infty} \{G \, : \, |V(G)| = n, \, |E(G)| \leq f(n)\}.$$

Thus, the function $f(n)$ serves an upper bound for the growth of the number of edges in graphs. In [1], the problem was stated of finding the function $f'(n)$ such that, for every $\epsilon > 0$, Problem IS is polynomially solvable in the class $\mathcal{G}_{[(1-\epsilon)f'(n)]}$ and NP-hard in the class $\mathcal{G}_{[(1+\epsilon)f'(n)]}$. The corresponding partitioning function was found: It was shown that we can put $f'(n) = n$.

In this paper, we consider the effect of choosing $f(n)$ on the complexity of the independence number in connected graphs from $\mathcal{G}_{f(n)}$ (denoted by $\mathcal{G}^*_{f(n)}$). The subject of study is a "no man's land" of previous results; i.e., functions $f(n)$ that are equal to n at $n \to \infty$. The purpose of the study is to obtain the information about the growth of the second member of the partitioning function of the polynomial and non-polynomial cases. The paper has the two main results: First result states that, for every natural number C, Problem IS is polynomially solvable for the graphs from $\mathcal{G}^*_{n+C[\log_2(n)]}$ (Theorem 1). The interpretation of the second result assumes that there does not exist a subexponential algorithms for solving Problem IS (recall that the algorithm for solving this problem is called *subexponential*, if its time complexity is bounded with $2^{o(n)}$, where n is the number of vertices in graph). The belief in the absence of such an algorithm for IS problem is wide-spread [3] and is shared by the author of this paper. The statement of Theorem 2 uses the notion of *over-logarithmic function* $g(n) : \mathbb{N} \to \mathbb{N}$; i.e., unbounded nondecreasing function such that the exponent of this function grows faster than a polynomial function of n. Theorem 2 states that the absence of subexponential algorithm for Problem IS implies the absence of polynomial algorithm for solving this problem is the class $\mathcal{G}^*_{n+g^2(n)}$ for any over-logarithmic function $g(n)$.

1. THE RESULTS

Theorem 1. *For every natural number C, Problem* IS *is polynomially solvable in the class*

$$\mathcal{G}^*_{n+C[\log_2(n)]}.$$

Proof. Consider an arbitrary graph $G \in \mathcal{G}^*_{n+C[\log_2(n)]}$ and choose its arbitrary spanning tree (it exists since G is a connected graph). It is clear that the construction of spanning tree can be done in polynomial time. Consider the edges of G that does not belong to its spanning tree. Denote the set of vertices of G that are incident to these edges by V'. It is clear that

$$|V'| \leq 2C[\log_2(|V(G)|)] + 2.$$

For each independent set $S \subseteq V'$ of G, consider the graph G_S that is generated by the set of vertices $V(G) \setminus \{V' \cup N(S)\}$, where $N(S) = \{y \mid \exists x \in S \, : \, (x, y) \in E(G)\}$. It is clear that the independent sets of this form can be constructed in polynomial of $|V(G)|$ time. It is also clear that each component of G_S is a tree. Hence, the calculation of $\alpha(G_S)$ can be done in polynomial of $|V(G_S)|$ time. The independence number of G can be calculated by

$$\max_{S \subseteq V'} (|S| + \alpha(G_S)),$$

and this procedure is easily done in polynomial time. The proof of Theorem 1 is complete. □

Theorem 2. *If Problem* IS *is not solvable in subexponential time then, for any over-logarithmic function $g(n)$, this problem is not solvable in polynomial time for the graphs in the class $\mathcal{G}^*_{n+g^2(n)}$.*

Proof. Consider the set of graphs

$$\mathcal{G}^{(k)} = \{G\,:\, g(k) \le |V(G)| \le g(k+1)\}, \qquad k \in \{0, 1, 2, \dots\},$$

where we suppose that $g(0) = 0$. It is obvious that

$$\bigcup_{k=1}^{\infty} \mathcal{G}^{(k)} = \mathcal{G}.$$

There are the two possible variants: either there exist constants $C > 0$ and $\alpha > 0$ independent of k such that, for any k, Problem IS for the graphs from $\mathcal{G}^{(k)}$ can be solved in time at most Ck^α or there are no such constants. Let us show that the first variant is impossible.

Consider the function

$$k(n) = \max_k \{k\,:\, g(k) \le n\}.$$

This function is defined for every n due to unboundedness and monotonicity of the function $g(k)$. Interest to $k(n)$ is aroused due to the fact that all graphs from $\mathcal{G}$ with n vertices are contained in $\mathcal{G}^{(k(n))}$ (this is immediate from the maximality of $k(n)$; i.e., from the inequality $n < g(k(n)+1)$). It is clear that $k(n) \to \infty$ as $n \to \infty$. Since $g(k)$ is over-logarithmic function, $g(k) = \log_2(k)\beta(k)$, where $\beta(k) \to \infty$ as $k \to \infty$. Exponentiating the inequality $g(k(n)) \le n$, we obtain $(k(n))^{\beta(k(n))} \le 2^n$. Hence,

$$k(n) \le 2^{\frac{n}{\beta(k(n))}}.$$

Since $\beta(k(n)) \to \infty$ as $n \to \infty$, we have $k(n) \in 2^{o(n)}$. Then, for the graphs from $\mathcal{G}^{(k(n))}$, Problem IS can be solved in $C(k(n))^\alpha \in 2^{o(n)}$ time. Hence,

$$\bigcup_{k=1}^{\infty} \mathcal{G}^{(k)} = \mathcal{G}$$

implies that Problem IS for the graphs from $\mathcal{G}$ is solvable in subexponential time of the number of vertices; this gives a contradiction. Thus, there are no constants C and α with the above-mentioned properties.

For each graph from $\mathcal{G}$ with n vertices, we add $k(n)$ isolated vertices and obtain the family of graphs $\mathcal{G}'$. Every isolated vertex of the graph belongs to each of its largest independent sets. This and the fact obtained in previous paragraph imply that, for the independent set problem for the graphs from $\mathcal{G}'$, there does not exist polynomial of the number of vertices algorithm for solving this problem. It is easy to check that the number of edges in each graph with n' vertices from this set does not exceed $g^2(n')$. Let us add one vertex to each $G \in \mathcal{G}'$ and connect it with edges to all its vertices. The so-constructed graph G' has exactly $|V(G)| + 1$ vertices and $|E(G)| + |V(G)|$ edges. Since

$$|E(G)| \le g^2(|V(G)|), \qquad g(|V(G)|) \le g(|V(G)| + 1),$$

we have $|E(G')| \le |V(G')| + g^2(|V(G')|)$ and, therefore, $G' \in \mathcal{G}^*_{n+g^2(n)}$. Graph G' contains an unique vertex of $|V(G)| + 1$ degree which, as it is easy to show, does not belong to any of its largest independent sets (except the case when G is complete). Hence, either $\alpha(G') = \alpha(G)$ (if G is not complete) or $\alpha(G') = \alpha(G) + 1$ (if G is complete). Thus, Problem IS for the graphs from $\mathcal{G}'$ is polynomially reduced to the same problem for the graphs from $\mathcal{G}^*_{n+g^2(n)}$. This implies the statement of Theorem 2. □

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