ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Boltzmanngasse 9
A-1090 Wien, Austria

Problem of Erdös – Vershik for Golden Ratio

Z.I. Bezhaeva
V.I. Oseledets

Vienna, Preprint ESI 2087 (2008)

December 10, 2008

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

Problem of Erdös - Vershik for golden ratio

Z.I. Bezhaeva * V.I. Oseledets †

December 15, 2008

Abstract

We study the properties of the Erdös measure and the invariant Erdös measure in the case of the golden ratio and for all values of the Bernoulli parameter. We prove that the two - sided shift on the Fibonacci compact set with the invariant Erdös measure is isomorphic to an integral automorphism over the Bernoulli shift with a countable alphabet. We provide an effective algorithm for the calculations of the entropy of the invariant Erdös measure. We show that for certain values of the Bernoulli parameter that algorithm gives the Hausdorff dimension of the Erdös measure to the fifteen decimal places.

1 Introduction

Almost seventy years ago Erdös posed the following problem:
What one can say about the distribution function of the random variable :

$\zeta = \zeta_1\rho + \zeta_2\rho^2 + \dots$, *where* $\zeta_1, \zeta_2, \dots$ *are independent, identically distributed random variables taking values* 0, 1 *and* $0 < P(\zeta_i = 0) = 1/2$, $(0 < \rho < 1)$.

We will call this distribution of the random variable ζ the *Erdös measure* on the real line .

The problem of Erdös has been the subject of the large number of papers. In [2] the authors gave the definitions of the *Erdös measure* on the unit interval $[0, 1]$, on the Fibonacci compactum and the invariant *Erdös measure* on the Fibonacci compactum for the case $\rho = (\sqrt{5} - 1)/2$ (the inverse β), where $\beta = (\sqrt{5} + 1)/2$ is the golden ratio. In [2] the authors proved that the *Erdös measure* is equivalent the invariant *Erdös measure* on the Fibonacci compactum.

Vershik posed the problem about the ergodic properties of the invariant *Erdös measure* on the Fibonacci compactum. This problem was solved in [2] .

*MIEM

†Moscow State University.

The work of the authors was partially supported by RFBR grant 07-01-00203, the work of second author was partially supported by RFBR-CNRS grant 07-01-92215.

2000 Mathematics Subject Classification. Primary: 28D05. Secondary: 69J10.

Key words and phrases: hodden Markov chain, Erdös measure, invariant Erdös measure, goldenshift, integral automorphism, entropy, Hausdorff dimension of a measure.

In the previous paper [1] we discovered the connection of the Erdös - Vershik problem with the hidden Markov chains for more general case $0 < P(\zeta_i = 0) = q < 1,\ P(\zeta_i = 1) = p$, $\rho = (\sqrt{5}-1)/2$. With the help of this connection we shall prove the result, analogous to one of the main results of the paper [2]. Namely, we consider the shift on the two-sided Fibonacci compactum with the invariant *Erdös measure.* It is isomorphic to theintegral automorphism over the Bernoulli automorphism with the countable alphabet. Also we obtain the formula for the entropy of the invariant *Erdös measure.*

The ratio of the entropy of the invariant *Erdös measure* and $\ln(\beta)$ is the Hausdorff dimension of the invariant *Erdös measure* on the Fibonacci compactum with the metric $d(x, y) = \rho^{n(x,y)}$, where $n(x, y)$ is the length of the longest common prefix of the words x and y. This dimension is equal to the Hausdorff dimension of the *Erdös measure* on the real line.

The formula for the Hausdorff dimension of the *Erdös measure* on the real line was obtained by Feng in [5](theorem 4.29). Our formula coincides with the Feng's formula. Therefore we gave a new derivation of the Feng's formula. The direct calculation of the Hausdorff dimension with the help of Feng's formula is impossible because the series for the Hausdorff dimension converges too slowly for the effective computation.

In our case Lalley [4] obtained another formula for the Hausdorff dimension of the *Erdös measure* on the real line. Using this formula and Monte Carlo method he was able to obtain the estimates of the Hausdorff dimension of the *Erdös measure* with the accuracy (to confidence level .99) to within $\pm .002$ for the various values of p. For those values of p that are in [4] we calculate the Hausdorff dimension of the *Erdös measure* with a larger number of decimal digits. This allows to estimate the accuracy of Lalley's estimates. In our calculations we use the acceleration of the convergence of the series in the formula for the Hausdorff dimension. This acceleration is analogues to the acceleration from the paper of Alexander - Zagier [3]. Remark that the Lalley's calculations are the calculations of the Lyapunov exponent for some sequence of random matrices(see [4]). One can say the same about our calculations but our sequence of random matrices is another sequence.

2 Invariant Erdös measure on Fibonacci compactum.

We shall give the definition of invariant *Erdös measure* on Fibonacci compactum (see [1]). In [1] the problem about the ergodic properties of the invariant *Erdös measure* was reduced to the study of the hidden Markov chain $\{\eta_i = f(\xi_i)\}$, with generating Markov chain $\{\xi_i\}$ with 5 states $1, 2, 3, 4, 5$ and the transition matrix P of form

$$P = \begin{pmatrix} q & 0 & 0 & pq & p^2 \\ q & 0 & qp & 0 & p^2 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The initial distribution l is the stationary distribution, the “gluing” function f equals 0 for the states $1, 2, 3$ and equals 1 for the states $4, 5$. The hidden Markov chain generates the distribution of probabilities μ on the space of its realizations.

A useful fact is that the distribution μ is the distribution of the infinite random $0-1$ word $\eta_1\eta_2....\eta_n... = f(\xi_1)f(\xi_2)....f(\xi_n)....$ Its support is the Fibonacci compactum of the infinite $0-1$ Fibonacci words of without the subwords 11. This set is the compact set with the respect to the metric $d(x, y) = \rho^{n(x,y)}$, where $n(x, y)$ is the length of the longest common prefix of words x and y. The measure μ is the invariant *Erdös measure* on the Fibonacci compactum [1].

In the matrix P we take blocks $P(00), P(01), P(10), P(11)$, corresponding to the partition of the set $\{1, 2, 3, 4, 5\}$ into two subsets, $\{1, 2, 3\} and \, \{4, 5\}$:

$$P(00) = \begin{pmatrix} q & 0 & 0 \\ q & 0 & qp \\ 0 & 1 & 0 \end{pmatrix},$$

$$P(01) = \begin{pmatrix} pq & p^2 \\ 0 & p^2 \\ 0 & 0 \end{pmatrix},$$

$$P(10) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Denote by $l(0)$ the row whose elements are first three elements of the row l, where the row l is the stationary distribution of the Markov chain with the transition matrix P. Also denote by $r(0)$ the column $(1, 1, 1)^\top$ and denote by $r(1)$ the column $(1, 1)^\top$.

Let $n \geq 2$ and $a = a_1...a_n$ be a finite Fibonacci word. We set

$$P(a) = P(a_1a_2)...P(a_{n-1}a_n).$$

Then

$$\mu(\{x : x_1x_2...x_n = a\}) = \mu(a) = l(a_1)P(a)r(a_n), \mu(a_1) = l(a_1)r(a_1).$$

Let $\widetilde{X}$ be the two-sided Fibonacci compactum of the infinite two-sided $0-1$ Fibonacci words without subwords 11 and with the fixed first position. Let T be a two-sided shift on the space $\widetilde{X}$. Define the measure $\widetilde{\mu}$ on the two-sided Fibonacci compactum :

$$\widetilde{\mu}(\{x : x_{1+j}x_{2+j}...x_{n+j} = a_1...a_n = a\}) = \mu(a), \ \forall j \in \mathbb{Z}, n \geq 1$$

The measure $\widetilde{\mu}$ is the invariant *Erdös measure* on the space $\widetilde{X}$.

3 Golden shift.

A finite Fibonacci word is called elementary word if it has form $10^{k+1}, k = 0, 1, 2, ...$ An elementary word 10^{k+1} is called even word if k is even and it is called odd word if k is odd.

In [2] the authors introduce the subset of regular words. Recall the definition of the regular word which was given in [2]. Let $\widetilde{X}_1$ be the subset of Fibonacci words from the space $\widetilde{X}$ with $x_1 = 1$, containing 1-s infinitely many times both to the left and to the right with respect to the first position. For any x from the space $\widetilde{X}_1$ we introduce the numbers $y_i(x), i \in \mathbb{Z}$, where $y_i(x) + 1$ is the number of zeros between $i - th$ 1 and $(i + 10 - th$ 1 in the word x (the 1 with number 1 stands on the first position).

Definition. A word $x \in \widetilde{X}_1$ called a regular word if in this word the odd numbers $y_i(x)$ occur infinitely many times both to the left and to the right with respect to the first position and the number $y_0(x)$ is odd. (A word $x \in \widetilde{X}_1$ called a regular word if in this word the odd elementary Fibonacci words occur infinitely many times both to the left and to the right with respect to the first position and the elementary Fibonacci word to the left of the first position is odd, $y_0(x)$ is odd.)

Following [2] denote the set of the regular words as $\widetilde{X}_0$. Let $\widetilde{\mu}_0$ be the conditional *Erdös measure* on the space $\widetilde{X}_0$. This measure is proportional to the measure $\widetilde{\mu}$ and $\widetilde{\mu}_0(\widetilde{X}_0) = 1$.

According to [2] the finite Fibonacci word b is called a block if it is an odd elementary word or the concatenation $c_1c_2...c_{s-1}c_s, s \geq 2$, where $c_i, i \leq s-1$ are even elementary words and the elementary word c_s is odd.

Let B be the set of all blocks. We can identify this set with the set B' of the finite words $b' = k_1k_2...k_s$, such that if $s = 1$ then k_1 is an odd number, if $s > 1$ then $k_1, ..., k_{s-1}$ are even numbers and k_s is an odd number. The length of the block b with $b' = k_1...k_s$ is equal to $\phi(b) = k_1 + ... + k_s + 2s$. The correspondence $b \to b'$ gives for us an important parameterization of the blocks. In [2] authors used the another parameterization of blocks.

Any regular word $x \in \widetilde{X}_0$ has a unique block expansion with blocks $b_i(x), i \in \mathbb{Z}$. This expansion starts from the first position. It is just the beginning of the first block $b_1(x)$. The block $b_2(x)$ starts after the first block, etc. The block $b_0(x)$ ends on the place with item 0. The block $b_{-1}(x)$ ends before the block $b_0(x)$, etc.

Let $x \in \widetilde{X}_0$ and j is the least positive integer,such that$T^j x \in \widetilde{X}_0$.Then j equals to the length of the block $b_1(x)$. Denote the length of the block $b_1(x)$ by $F(x) = \phi(b_1(x))$, then the derivative automorphism $S = T'^{F(x)}x$. This map of the set of regular words is the two - sided goldenshift from [2]. It is clear that $b_i(x) = b_1(S^{i-1}(x)), i \in \mathbb{Z}$. Remark that the measure $\widetilde{\mu}_0$ on the space $\widetilde{X}_0$ is the invariant measure with respect to the two-sided golden shift.

From previous construction we see that the goldenshift S can be identified with the shift $\widetilde{S}$ on the space $\widetilde{Z}$ of two-sided words with distinguished first position and with the alphabet B. The isomorphism is given by rule :

$$x \mapsto ...b_{-1}(x)b_0(x)b_1(x)....$$

Now let $\widetilde{Z} = \{\widetilde{z} = ...z_{-1}z_0z_1...\}$ be the space two-sided infinite words with distinguished first position and the alphabet B.

Recall that the function $\phi(b)$ on the space B is defined by the formula : $\phi(b) = k_1 + ... + k_s + 2s,\ b' = k_1k_2..k_s$. Define the function $\widetilde{F}(\widetilde{z}) = \phi(z_1(\widetilde{z}))$.

We call that $10^{k+1}1$ is an elementary cycle. An elementary cycle is odd if k is odd number.

In [2] the subset $\widetilde{X}^{reg} \subset \widetilde{X}$ was introduced.

Definition. The subset $\widetilde{X}^{reg} \subset \widetilde{X}$ is the subset of Fibonacci words $x = ..., x_{-1}x_0x_1...$, in which odd cycles occur infinitely many times both to the left and to the right with respect to the first position.

The subset $\widetilde{X}^{reg}$ is the invariant set with respect to the two-sided shift T and $\widetilde{\mu}(\widetilde{X}^{reg}) = 1$.

The integral automorphism $\widehat{T}$, defined by the shift $\widetilde{S}$, and the N - valued functions $\widetilde{F}(\widetilde{z})$ is a transformation of the space $\widehat{Z}$ of the pairs $(\widetilde{z}, j),\ j = 0, ..., F(\widetilde{z}) - 1, \widetilde{z} \in \widetilde{Z}$ defined by

$$(\widetilde{z}, j) \longrightarrow (\widetilde{z}, j+1),$$

if $j < \widetilde{F}(\widetilde{z}) - 1$ 438 $(\widetilde{z}, \widetilde{F}(\widetilde{z}) - 1) \longrightarrow (\widetilde{S}\widetilde{z}, 0)$.

Theorem1([2].) *The shift T on the space $\widetilde{X}^{reg}$ is isomorphic to the integral automorphism $\hat{T}$ on the space $\widehat{Z}$. The isomorphism is given by formula $T^j x \mapsto (...b_{-1}(x)b_0(x)b_1(x)...., j), 0 \le j \le F(x) - 1, x \in \widetilde{X}_0$.*

4 Golden shift and the invariant Erdös measure

Define the matrix $M(k)$ by

$$M(k) = P(10)P^k(00)P(01), k = 0, 1, 2, ...$$

The following lemma is valid.

Lemma 1. *The following is true:*

$$M(k) = \begin{cases} M_o(n), k = 2n+1 \\ M_e(n), k = 2n \end{cases},$$

$$M_o(n) = \begin{pmatrix} pq^{2n+2} & p^2q^{2n+1} \\ pq^{n+2}\frac{p^{n+1}-q^{n+1}}{p-q} & p^2q^{n+1}\frac{p^{n+1}-q^{n+1}}{p-q} \end{pmatrix},$$

$$M_e(n) = \begin{pmatrix} pq^{2n+1} & p^2q^{2n} \\ pq^{n+2}\frac{p^n-q^n}{p-q} & p^2q^n\frac{p^{n+1}-q^{n+1}}{p-q} \end{pmatrix}.$$

Proof. The characteristic polynomial of the matrix $P(00)$ equals $x^3 - qx^2 - pqx + pq^2$. (By the Hamilton - Cayley theorem). Dyley gives the recurrent relation for the sequence of matrices $M(k) = P(10)(P(00))^kP(01),\ k = 0, 1, 2, ..$:

$$M(k+3) = qM(k+2) + pqM(k+1) - pq^2M(k),\ k = 0, 1, ..$$

Direct check shows that the sequence of matrices

$$\widehat{M}(k) = \begin{cases} M_o(n), k = 2n+1 \\ M_e(n), k = 2n \end{cases}$$

satisfies the same recurrent relation. Moreover, direct check gives that $M(0) = \widehat{M}(0),\ M(1) = \widehat{M}(1),\ M(2) = \widehat{M}(2)$. Hence $M(k) = \widehat{M}(k),\ k \geq 0$. End of proof.

Matrices $M_e(n)$ are nonsingular and matrices $M_o(n)$ are singular. We can write matrices $M_o(n)$ in the following form: $M_o(n) = u(n)v$, where the column

$$u(n) = \begin{pmatrix} q^{2n+1} \\ q^{n+1}\frac{p^{n+1}-q^{n+1}}{p-q} \end{pmatrix},$$

and the row $v = (pq, p^2)$.

Consider the set B' of finite words $b' = k_1k_2...k_s$ such that if $s = 1$ then $k_1 = 2n_1 + 1$, if $s > 1$ then $k_j = 2n_j,\ j \leq s - 1$, 4̈30 $k_s = 2n_s + 1$.

For any block $b \in B$ with $b' = k_1k_2...k_s$ define the matrix

$$M(b) = M(k_1k_2...k_s) = M(k_1)...M(k_s) =$$

$$= M_e(n_1)...M_e(n_{s-1})M_o(n_s).$$

Introduce the notations : $u(b) = u(n_s)$, $M_e(b) = M_e(n_1)...M_e(n_{s-1})$ if $s > 2$, and if $s = 1$, $M_e(b)$ is the identity matrix. Hence

$$M(b) = M_e(b)u(b)v$$

Also define $p(b)$ as

$$p(b) = vM_e(b)u(b).$$

Now calculate the distribution of the random variable $b_1(x)$ on the set $\widetilde{X}_0$ with the measure $\widetilde{\mu}_0$:

$$\widetilde{\mu}_0(\{x \in \widetilde{X}_0 : b_1(x) = b\}).$$

From the definition of the m easure $\widetilde{\mu}_0$ follow, that

$$\widetilde{\mu}_0(\{x \in \widetilde{X}_0 : b_1(x) = b\}) = \sum_{n=0}^{\infty} \widetilde{\mu}(\{x \in \widetilde{X}_0 : y_0(x) = 2n+1, b_1(x) = b\})/\widetilde{\mu}(\widetilde{X}_0) =$$

$$= \sum_{n=0}^{\infty} l(1)u(n)vM(b)r(1)/\widetilde{\mu}(\widetilde{X}_0) =$$

$$= \sum_{n=0}^{\infty} l(1)u(n)vM_e(b)u(b)vr(1)/\widetilde{\mu}(\widetilde{X}_0) = vM_e(b)u(b)\sum_{n=0}^{\infty} l(1)u(n)vr(1)/\widetilde{\mu}(\widetilde{X}_0) =$$

$$= vM_e(b)u(b)[\sum_{n=0}^{\infty} \widetilde{\mu}(\{x \in \widetilde{X}_0 : y_0(x) = 2n+1\})/\widetilde{\mu}(\widetilde{X}_0)] = p(b).$$

Hence, in particular, we obtain the equality $\sum_{b \in B} p(b) = 1$.

Since golden shift S preserves the measure $\widetilde{\mu}_0$ on the space $\widetilde{X}_0$, then random variables $b_i(x) = b_1(S^{i-1}x),\ i \in \mathbb{Z}$ are identically distributed and

$$\widetilde{\mu}_0(\{x \in \widetilde{X}_0 : b_i(x) = b\}) = p(b).$$

In similar way we can calculate the joint distribution of random variables

$$b_1(x), b_2(x), ..., b_m(x)$$

on the set $\widetilde{X}_0$ with the measure $\widetilde{\mu}_0$.

$$\widetilde{\mu}_0(\{x \in \widetilde{X}_0 : b_1(x) = b^1, ..., b_m(x) = b^m\}) =$$

$$= \sum_{n=0}^{\infty} \widetilde{\mu}_0(\{x \in \widetilde{X}_0 : y_0(x) = 2n+1, b_1(x) = b^1, ..., b_m(x) = b^m\}) =$$

$$= \sum_{n=0}^{\infty} l(1)u(n)vM_e(b^1)u(b^1)vM_e(b^2)u(b^2)...vM_e(b^m)u(b^m)vr(1)/\widetilde{\mu}(\widetilde{X}_0) =$$

$$= p(b^1)...p(b^m).$$

Thus identically distributed random variables $b_j(x)$ (defined on the space $\widetilde{X}_0$ with the measure $\widetilde{\mu}_0$) are independent.

Consider Bernoulli measure $\hat{\nu}$ on the space $\widetilde{Z}$ with one-dimensional distribution

$$p(b) = \hat{\nu}(\{\widetilde{z} : z_j = b\}) = vM_e(b)u(b), b \in B,\ j \in \mathbb{Z}.$$

Recall that $F(x) = \phi(b_1(x)), \widetilde{F}(\widetilde{z}) = \phi(z_1(\widetilde{z}))$.

Define the $\widehat{T}-$ invariant measure $\widehat{\mu}$ on the space of pairs $\{(\widetilde{z}, j), \widetilde{z} \in \widetilde{Z}, 0 \leq j \leq \widetilde{F}(\widetilde{z}) - 1\}$. The set of pairs $(\widetilde{z}, 0), \widetilde{z} \in \widetilde{Z}$ can be identified with the set $\widetilde{Z}$. $\widehat{\nu}$ is the measure on this set. Define the measure $\widehat{\mu}$ as

$$\int f(\widetilde{z}, j)d\widehat{\mu}(\widetilde{z}, j) = \frac{\int \sum_{j=0}^{\widetilde{F}(\widetilde{z})-1} f(\widetilde{z}, j)d\widehat{\nu}(\widetilde{z})}{\int \widetilde{F}(\widetilde{z})d\widehat{\nu}(\widetilde{z})}$$

Theorem 2. *The shift T on the subset $\widetilde{X}^{reg}$ with invariant* Erdös measure *is an isomorphic integral automorphism $\hat{T}$ with the measure $\widehat{\mu}$ The isomorphism* *given by the formula* $T^j x \mapsto (...b_{-1}(x)b_0(x)b_1(x)...., j), 0 \le j \le F(x)-1, x \in \widetilde{X}_0.$

From previous considerations it is easy to prove theorem 2. From this theorem for the case $p = 1/2$ one can obtain a new proof of one of main results of the paper [2].

5 Probabilities of blocks and transition to binary words.

For the probabilities of blocks b and $b' = k_1, ..., k_s, k_j = 2n_j, j < s, k_s = 2n_s + 1$ we have the formula

$$p(b) = vM_e(b)u(n_s).$$

If $s = 1$, then

$$M_e(b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

if $s \ge 2$ then

$$M_e(b) = M_e(n_1)..M_e(n_{s-1}).$$

Introduce the function $[n]_\alpha = 1 + \alpha + \alpha^2 + ... + \alpha^{n-1},\ \alpha = p/q$ and define the matrix

$$A(n) = \begin{pmatrix} \alpha[n+1]_\alpha & [n]_\alpha \\ \alpha & 1 \end{pmatrix}$$

Evidently

$$A(n) = \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} \alpha & 0 \\ \alpha & 1 \end{pmatrix}$$

We shall use this relation.

Rewrite the formula for $p(b)$ in another form by using the following relation :

$$CM_e(n)C = pq^{2n+1}A(n),$$

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $s \ge 2$, then

$$p(b) = vM_e(n_1)...M_e(n_{s-1})u(n_s) = p^{s-1}q^{k_1+...+k_{s-1}+s-1}vCA(n_1)...A(n_{s-1})Cu(n_s).$$

It is clear that $Cu(n_s) = q^{2n_s+1}\frac{1}{\alpha}A(n_s)\{1,0\}^\top$. Hence

$$p(b) = p(n_1...n_s) = \alpha^{s-1}q^{\phi(b)}\{\alpha, 1\}A(n_1)...A(n_s)\{1,0\}^\top,$$

$$\phi(b) = k_1 + ... + k_s + 2s.$$

Now let us obtain a new formula for $p(n_1...n_s)$ with the transition to binary words.

Introduce the matrices $\widetilde{M}(0), \widetilde{M}(1)$:

$$\widetilde{M}(0) = \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix},$$

$$\widetilde{M}(1) = \begin{pmatrix} \alpha^2 & 0 \\ \alpha^2 & \alpha \end{pmatrix}.$$

Then

$$\alpha A(n) = (\widetilde{M}(0))^n \widetilde{M}(1).$$

Set $u = \{\alpha, \alpha\}^\top$. Recall that by definition $q = 1/(1+\alpha)$. Because

$$(1+\alpha)^{\phi(b)} p(b) = \alpha^{s-1}\{\alpha, 1\} A(n_1)...A(n_s)\{1, 0\}^\top =$$

$$= \{\alpha, 1\}\widetilde{M}(0)^{n_1}\widetilde{M}(1)....\widetilde{M}(0)^{n_{s-1}}\widetilde{M}(1)\widetilde{M}(0)^{n_s}\widetilde{M}(1)\{1, 0\}^\top/\alpha =$$

$$= \{\alpha, 1\}\widetilde{M}(0)^{n_1}\widetilde{M}(1)....\widetilde{M}(0)^{n_{s-1}}\widetilde{M}(1)\widetilde{M}(0)^{n_s}u = \{\alpha, 1\}\widetilde{M}(b)u.$$

In this formula

$$\widetilde{M}(b) = \widetilde{M}(0)^{n_1}\widetilde{M}(1)....\widetilde{M}(0)^{n_{s-1}}\widetilde{M}(1)\widetilde{M}(0)^{n_s}.$$

Denote by $D_{n-1}, n = 1, 2, ...$ the set of all binary words of the length $n-1$. D_1 containing the empty word. Let us relate to the block b with $b' = 2n_1.....2n_s+1$ and $n_1 + ... + n_s + s = n,\ n \geq 2$ (the length of the block b equals $2n+1$) the binary word $d \in D_{n-1}$ by rule $d = i_1....i_{n-1} = 0^{n_1}1...0^{n_{s-1}}10^{n_s}$. In this word, if $n_i = 0$, then 0^{n_i} is the empty word.

The product of matrices corresponding to this binary word has the form $\widetilde{M}(d) = \widetilde{M}(i_1)....\widetilde{M}(i_{n-1}) = \widetilde{M}(b)$.

The empty word gives the identity matrix.

Thus, we have that for any word b with $b' = 2n_1...2n_s + 1$ and with the length of word $\phi(b) = 2n + 1$

$$p(b) = \{\alpha, 1\}\widetilde{M}(b)\{\alpha, \alpha\}^\top q^{2n+1}.$$

6 Generating function of the length of the block.

The length of the block b equals $\phi(b)$. Generating function of length of the block equals

$$\Phi(z) = \sum_b p(b) z^{\phi(b)} =$$

$$= \sum_b \{\alpha, 1\}\widetilde{M}(b)\{\alpha, \alpha\}^\top q^{2n+1} z^{\phi(b)} =$$

$$= \sum_{n=1} \sum_{d \in D_{n-1}} \{\alpha, 1\} \widetilde{M}(d) \{\alpha, \alpha\}^\top q^{2n+1} z^{2n+1} =$$

$$= q^3 z^3 \{\alpha, 1\} (Id - q^2 z^2 \widetilde{M})^{-1} \{\alpha, \alpha\}^\top,$$

where $\widetilde{M} = \widetilde{M}(0) + \widetilde{M}(1), Id$ is the identity matrix.

Hence we obtain:

$$\Phi(z) = \frac{pqz^3}{1 - (1 - pq) z^2}.$$

Now we calculate the mean value of the length of block:

$$E\phi(b) = \Phi'(1) = 1 + \frac{2}{pq}.$$

Expand $\Phi(z)$,

$$\Phi(z) = \sum_{n=0}^{\infty} pq(1 - pq)^{n-1} z^{2n+1},$$

and obtain that the probability that the length equals $2n+1$ is equal to $pq(1-pq)^{n-1}$.

6. Formula for calculation of entropy. In this section for the definition of the entropy we shall use the binary logarithm. It is shown that

$$p(b) = \frac{1}{(1+\alpha)^{\phi(b)}} \{\alpha, 1\} \widetilde{M}(b) \{\alpha, \alpha\}^\top.$$

Hence

$$\log_2 p(b) = -\phi(b) \log_2 (1 + \alpha) + \log_2 \{\alpha, 1\} \widetilde{M}(b) \{\alpha, \alpha\}^\top.$$

Therefore

$$E(-\log_2 p(b)) = E\phi(b) \log_2 (1 + \alpha) - E(\log_2 \{\alpha, 1\} \widetilde{M}(b) \{\alpha, \alpha\}^\top).$$

From the theorem 2 and the Abramov formula [6] for the entropy of the integral automorphism we obtain that the entropy of the invariant Erdös measure equals

$$H = \frac{E(-\log_2 p(b))}{E\phi(b)},$$

$$H = \log_2(1+\alpha) - \frac{1}{E\phi(b)} \sum_{n=1}^{\infty} \left[\sum_{b \in B_n} \log_2 (\{\alpha, 1\} \widetilde{M}(b) \{\alpha, \alpha\}^T) \{\alpha, 1\} \widetilde{M}(b) \{\alpha, \alpha\}^\top \right] \frac{1}{(1+\alpha)^{2n+1}}$$

We know that

$$E\phi(b) = 1 + \frac{2}{pq} = 1 + \frac{2(1+\alpha)^2}{\alpha}.$$

Let B_n be the set of words b with $b' = 2n_1 ... 2n_s + 1, \ n_1 + ... + n_s + s = n$. Introduce the notation

$$k_n = \sum_{b \in B_n} \log_2 (\{\alpha, 1\} \widetilde{M}(b) \{\alpha, \alpha\}^\top)(\{\alpha, 1\} \widetilde{M}(b) \{\alpha, \alpha\}^\top),$$

or

$$k_n = \sum_{d\in D_{n-1}} \log_2(\{\alpha,1\}\widetilde{M}(d)\{\alpha,\alpha\}^{top})(\{\alpha,1\}\widetilde{M}(d)\{\alpha,\alpha\}^{\top}).$$

Then

$$H = \log_2(1+\alpha) - \frac{1}{1+\frac{2(1+\alpha)^2}{\alpha}} \sum_{n=1}^{\infty} k_n \left(\frac{1}{1+\alpha}\right)^{2n+1},$$

If $q = 1/2$, then

$$H = 1 - \frac{1}{9}\sum_{n=1}^{\infty} k_n \frac{1}{2^{2n+1}}.$$

Note that the formula for the Hausdorff dimension of the invariant Erdös measure $H/\log_2(\beta)$ coincides with the Alexander - Zagier formula for the Garsia entropy [3]. The Alexander - Zagier formula was obtained with the help of the combinatorics of the Euclidean tree. It is possible that our formula corresponds to the combinatorics of the $\alpha-$Euclidean tree.

The main difficulty for the calculation of the entropy H is the slow convergence of the corresponding series. The series for H converges too slowly for the effective computation. Following the approach of Alexander - Zagier [3], we use some rearrangement of the series for H.

Introduce

$$\mu_n = k_n - [3]_\alpha k_{n-1}$$

Then

$$(1-[3]_\alpha x)\left(\sum_{n=1}^{\infty} k_n x^n\right) = \sum_{n=1}^{\infty} \mu_n x^n$$

Consider

$$\lambda_n = 2\lambda_{n-1} - \lambda_{n-2} + \mu_n - [3]_\alpha \mu_{n-1}$$

It is clear that

$$\sum_{n=1}^{\infty} k_n x^n = \frac{1}{1-[3]_\alpha x}\sum_{n=1}^{\infty} \mu_n x^n$$

$$(1-x)^2\sum_{n=1}^{\infty} \lambda_n x^n = (1-[3]_\alpha x)\sum_{n=1}^{\infty} \mu_n x^n$$

$$\sum_{n=1}^{\infty} k_n x^n = \frac{(1-x)^2}{(1-[3]_\alpha x)^2}\sum_{n=1}^{\infty} \lambda_n x^n$$

Use this relation and set $x = (\frac{1}{1+\alpha})^2$, obtain

$$H = \log_2(1+\alpha) - \frac{\alpha(2+\alpha)}{(1+2\alpha)}\sum_{n=1}^{\infty} \lambda_n \left(\frac{1}{1+\alpha}\right)^{2n+1}.$$

This series converges more rapidly than the initial series.

The following relation holds: entropy H under the substitution α by $\frac{1}{\alpha}$ is the same. Of course, the series for $\alpha > 1$ converges more rapidly. Below we use this relation for the calculation of the Hausdorff dimension.

7 Results of calculations.

In the following table we give the values of the Hausdorff dimension $H_{dim} = H/\log_2 \beta$ of

the invariant *Erdös measure* on the Fibonacci compactum with the metric $d(x, y) = \rho^{n(x,y)}$, where $n(x, y)$ is the length of the longest common prefix of the words x and y.

In the table the second column gives the values of the Hausdorff dimension of the *Erdös measure* for different probabilities p. In the third column it is shown how many terms of the series are chosen in the formula for the Hausdorff dimension of the *Erdös* measure. In the fourth column shows the results of Lalley calculations.

p	H_{dim}	n	Lalley
0.05	0.392167680782199076	15	0.3877 ± 0.03
0.05	0.392167680782199076	14	
0.1	0.6101383374950678578	20	0.6085 ± 0.008
0.1	0.6101383374950678578	19	
0.2	0.84990339802715197$\underline{6}$	23	0.8499 ± 0.004
0.2	0.84990339802715197$\underline{2}$	22	
0.3	0.95138898022598$\underline{7}$0	24	0.9501 ± 0.002
0.3	0.95138898022598$\underline{6}$9	23	
0.4	0.987545683253293$\underline{8}$	25	0.9868 ± 0.001
0.4	0.987545683253293$\underline{1}$	24	
0.5	0.9957131266855555$\underline{2}$6	24	0.9954 ± 0.0008
0.5	0.99571312668555555$\underline{6}$0	23	

References

[1] Bezhaeva Z.I., Oseledets V.I. Erdös measures, sofic measures, and Markov chains, Zap.Nauch.Sem.POMI, 326, 28-47 (2005).(in Russian)

[2] Sidorov N., Vershik A. Ergodic properties of the Erdos measures, the entropy of the goldenshift, and related problems, Monatsh. Math. 126 No 3, 215-261 (1998).

[3] Alexander I.C., Zagier D. The entropy of certain infinitely convolved Bernoulli measure. J. London Math. Soc. 44, 121-134 (1991).

[4] Lalley Steven P. Random series in powers of algebraic number: Hausdorff dimension of the limit distribution, J. London Math. Soc.(2), 57, n.3, 628-654 (1998).

[5] Feng De-Jun The limited Rademacher function and Bernoulli convolutions associated with Pisot numbers, Advances in Mathematics 195, 24-101,(2005)

[6] Abramov L.M. Entropy of induced automorphism, Dokladi Akad Nauk 4̆214̆214̆21R, 128 , 647 - 650(1959).