Genera of non-algebraic leaves of polynomial foliations of $\mathbb{C}^2$

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In this article, we prove two results. First, we construct a dense subset in the space of polynomial foliations of degree n such that each foliation from this subset has a leaf with at least $\frac{(n+1)(n+2)}{2} - 4$ handles. Next, we prove that for a generic foliation invariant under the map $(x, y) \mapsto (x, -y)$ all leaves (except for a finite set of algebraic leaves) have infinitely many handles.

Contents

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1 Introduction

Consider a polynomial differential equation in $\mathbb{C}^2$ (with complex time),

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1}$$

where $\max(\deg P, \deg Q) = n$. The splitting of $\mathbb{C}^2$ into trajectories of this vector field defines a singular analytic foliation of $\mathbb{C}^2$. For a typical foliation, each leaf is dense in $\mathbb{C}^2$, see [KhV62, M75, Il78, Shch84].

Denote by $\mathcal{A}_n$ the space of foliations of $\mathbb{C}^2$ defined by vector fields (1) of degree at most n with coprime P and Q. Two vector fields define the same foliation if they are proportional, hence $\mathcal{A}_n$ is a Zariski open subset of the projective space of dimension[4] $(n+1)(n+2)-1$. $\mathcal{A}_n$ is equipped with a natural topology induced from this projective space.

Denote by $\mathcal{B}_n$ the space of foliations of $\mathbb{C}P^2$ defined by a polynomial vector field (1) of degree at most n in each affine chart. It is easy to show that $\mathcal{A}_n \subset \mathcal{B}_{n+1} \subset \mathcal{A}_{n+1}$.

Numerous studies in this field are devoted to the properties of generic foliations from $\mathcal{A}_n$ and $\mathcal{B}_n$, see [Shch06] for a survey. Another classical question concerns degree and genus of an algebraic leaf of a polynomial foliation, see [LN02]. We study genera of non-algebraic leaves.

For a generic *analytic* foliation, the question about the topology of a leaf was studied by T. Firsova and T. Golenishcheva–Kutuzova.

Theorem ([F06, K06])**.** *Among leaves of a generic analytic foliation, countably many are topological cylinders, and the rest are topological discs.*

For a generic polynomial foliation, the analogous result is not known. The fact that almost all leaves are topological discs would follow from Anosov conjecture on identical cycles.

Definition. An *identical cycle* on a leaf L is a non-trivial element $[\gamma]$ of the free homotopy group of L such that the holonomy along one (and hence any) its representative γ is identical.

Conjecture (D. Anosov)**.** *A generic polynomial foliation has no identical cycles.*

In Section "A leaf with many handles", we give a partial answer to the question: "What topological structures of the leaves can arise in a *dense* subset of $\mathcal{A}_n$?". Namely, we prove the following theorem.

Theorem 1. *For each $n \geqslant 2$, the set of polynomial foliations having a leaf with at least $\frac{(n+1)(n+2)}{2} - 4$ handles is dense in $\mathcal{A}_n$.*

This theorem is inspired by the following theorem due to D. Volk [V06].

Theorem (Density of foliations with separatrix connection)**.** *For each $n \geqslant 2$, the set of polynomial foliations having a separatrix connection is dense in $\mathcal{A}_n$.*

We shall discuss the latter theorem in more details in Section "Volk's Theorem" below.

In Section "Leaves of infinite genus", we get the following result:

Theorem 2. *Let $\mathcal{A}_n^{sym}$ (resp., $\mathcal{B}_n^{sym}$) be the subspace of $\mathcal{A}_n$ (resp., $\mathcal{B}_n$) given by*

$$P(x,-y) = -P(x,y), \quad Q(x,-y) = Q(x,y). \tag{2}$$

Take $n \geqslant 2$. For any foliation $\mathcal{F}$ from some open dense subset of $\mathcal{A}_n^{sym}$ (resp., $\mathcal{B}_{n+1}^{sym}$), all leaves of $\mathcal{F}$ (except for a finite set of algebraic leaves) have infinite genus.

[4] From now on, "dimension" means "complex dimension".

There are some unpublished earlier results in this direction. For generic homogeneous vector fields, almost all leaves have infinite genus; the proof is due to Yu.Ilyashenko, but it was never written down. We write it in Section "Proof of Ilyashenko's Theorem".

In the unpublished version of his thesis, V. Moldavskis [MolTh] proves that for a generic vector field of degree $n \geq 5$ with *real coefficients* and the symmetry (2) each leaf has infinitely-generated first homology group. However this is only a draft text, so the proof lacks some details and has some gaps.

2 Preliminaries

In this section we shall recall some results and introduce required notions and notation. In some cases we formulate refined versions of earlier results or provide explicit constructions.

2.1 Genus of a non-compact leaf

A leaf of a foliation is a (usually non-compact) Riemann surface. Since it is not necessarily homeomorphic to the sphere with some handles and holes, we shall provide two equivalent definitions of its genus we shall use in this paper.

Definition. A Riemann surface is said to have *at least g handles*, if it has a subset homeomorphic to a sphere with g handles and one hole. A Riemann surface has *infinite genus*, if it has at least g handles for any integer g.

Definition. A Riemann surface is said to have *at least g handles*, if there exist g pairs of closed loops (c_1, c_2), (c_3, c_4), $\ldots$, (c_{2g-1}, c_{2g}) on this surface, such that c_{2j-1} and c_{2j} intersect transversally at exactly one point, and the loops from different pairs do not intersect. Each pair (c_{2j-1}, c_{2j}) is called *a pair of generating cycles* of a handle.

It is easy to show that these definitions are equivalent.

2.2 Extension to infinity

Let us extend a polynomial foliation $\mathcal{F} \in \mathcal{A}_n$ given by (1) to $\mathbb{C}P^2$. For this end, make the coordinate change $u = \frac{1}{x}, v = \frac{y}{x}$, and the time change $d\tau = -u^{n-1}dt$. The vector field takes the form

$$\begin{aligned} \dot{u} &= u\widetilde{P}(u,v) \\ \dot{v} &= v\widetilde{P}(u,v) - \widetilde{Q}(u,v) \end{aligned} \tag{3}$$

where $\widetilde{P}(u,v) = P\left(\frac{1}{u}, \frac{v}{u}\right) u^n$ and $\widetilde{Q}(u,v) = Q\left(\frac{1}{u}, \frac{v}{u}\right) u^n$ are two polynomials of degree at most n.

Since $\dot{u}(0,v) \equiv 0$, the infinite line $\{\, u = 0 \,\}$ is invariant under this vector field. Denote by $h(v)$ the polynomial $\dot{v}(0,v) = v\widetilde{P}(0,v) - \widetilde{Q}(0,v)$. There are two cases.

Dicritical case, $h(v) \equiv 0$ In this case (3) vanishes identically on $\{\, u = 0 \,\}$. Thus it is natural to consider the time change $d\tau = -u^n dt$ instead of $d\tau = -u^{n-1}dt$, and study the vector field

$$\begin{aligned} \dot{u} &= \widetilde{P}(u,v) \\ \dot{v} &= \frac{v\widetilde{P}(u,v) - \widetilde{Q}(u,v)}{u} \end{aligned}$$

whose trajectories are almost everywhere transverse to the infinite line. This case corresponds to $\mathcal{B}_n \subset \mathcal{A}_n$.

Non-dicritical case, $h(v) \not\equiv 0$ In this case (3) has isolated singular points $a_j \in \{\, u = 0 \,\}$ at the roots of h, and $L_\infty = \{\, u = 0 \,\} \setminus \{a_1, a_2, \ldots\}$ is a leaf of the extension of $\mathcal{F}$ to $\mathbb{C}P^2$.

Denote by $\mathcal{A}'_n$ the set of foliations $\mathcal{F} \in \mathcal{A}_n$ such that h has $n+1$ distinct roots a_j, $j = 1, \ldots, n+1$. In particular, all these foliations are non-dicritical.

For each j, let λ_j be the ratio of the eigenvalues of the linearization of (3) at a_j (the eigenvalue corresponding to L_∞ is in the denominator). One can show that $\sum \lambda_j = 1$, and this is the only relation on λ_j.

2.3 Monodromy group and rigidity

For $\mathcal{F} \in \mathcal{A}'_n$, fix a non-singular point $O \in L_\infty$ and a cross-section S at O given by $v = \text{const}$. Let ΩL_∞ be the loop space of (L_∞, O), i. e., the space of all continuous maps $(S^1, pt) \to (L_\infty, O)$. For a loop $\gamma \in \Omega L_\infty$, denote by $\mathbf{M}_\gamma : (S, O) \to (S, O)$ (a germ of) the monodromy map along γ. It is easy to see that $\mathbf{M}_\gamma$ depends only on the class $[\gamma] \in \pi_1(L_\infty, O)$, and the map $\gamma \mapsto \mathbf{M}_\gamma$ reverses the order of multiplication,

$$\mathbf{M}_{\gamma\gamma'} = \mathbf{M}_{\gamma'} \circ \mathbf{M}_\gamma.$$

The set of all possible monodromy maps $\mathbf{M}_\gamma$, $\gamma \in \Omega L_\infty$, is called the *monodromy pseudogroup* $G = G(\mathcal{F})$. The word "pseudogroup" means that there is no common domain where all elements of G are defined. However we will follow the tradition and write "monodromy group" instead of "monodromy pseudogroup".

Remark. This construction heavily relies on the fact that the infinite line is an algebraic leaf of $\mathcal{F}$. Since a generic foliation from $\mathcal{B}_n$ has no algebraic leaves, this construction does not work for foliations from $\mathcal{B}_n$. This is why the analogues of many results on $\mathcal{A}_n$ are not proved for $\mathcal{B}_n$.

Choose $n+1$ loops $\gamma_j \in \Omega L_\infty$, $j = 1, 2, \ldots, n+1$, passing around the points a_j, respectively. Then the pseudogroup $G(\mathcal{F})$ is generated by the monodromy maps $\mathbf{M}_j = \mathbf{M}_{\gamma_j}$. It is easy to see that the multipliers $\mu_j = \mathbf{M}'_j(0)$ are equal to $\exp 2\pi i\lambda_j$. Recall that $\sum \lambda_j = 1$, hence, $\prod \mu_j = 1$.

A generic foliation of $\mathbb{R}^2$ is structurally stable, i.e. any its small perturbation is topologically conjugate to the initial foliation. For a generic polynomial foliation of $\mathbb{C}^2$, we have the opposite property, called *rigidity*. Informally, a foliation sufficiently close to $\mathcal{F}$ is topologically conjugate to it only if it is affine conjugate to $\mathcal{F}$.

There are few different theorems of the form "topological conjugacy of polynomial foliations plus some assumptions imply affine conjugacy of these foliations", see [Il78, Shch84, N94] . These theorems are called Rigidity Theorems with various adjectives that depend on the extra assumptions on the foliations and conjugating homeomorphism. We shall need the following theorem.

Theorem 3. *There exists an open dense subset $\mathcal{A}^R_n \subset \mathcal{A}'_n$ such that for each $\mathcal{F}_0 \in \mathcal{A}^R_n$ the following holds. There exists a neighborhood $U \ni \mathcal{F}_0$ such that for $\mathcal{F} \in U$ the analytic conjugacy of the monodromy groups $G(\mathcal{F}_0)$, $G(\mathcal{F})$ at infinity (as groups with marked generators) implies the affine conjugacy of foliations.*

This theorem easily follows from the proof of Theorem 28.32 in [IY07]. Theorem 28.32 states that the analogue of Theorem 3 holds even if we require only a topological conjugacy of monodromy groups (as groups with marked generators), but for a full-measure subset of $\mathcal{A}_n$ instead of an open dense subset. The authors split the proof into two steps:

Step 1. Under their assumptions on $\mathcal{F}_0$, the topological conjugacy of monodromy groups of $\mathcal{F}_0$ and $\mathcal{F}$ implies the analytic conjugacy.

Step 2. Under their assumptions on $\mathcal{F}_0$, the analytic conjugacy of monodromy groups implies the affine conjugacy of foliations.

However Step 1 is the only place in the proof where they use the assumption that the multipliers μ_i of $\mathcal{F}_0$ generate a dense multiplicative subgroup in $\mathbb{C}^*$. Other assumptions on $\mathcal{F}_0$ define an open dense set in $\mathcal{A}_n$, thus Step 2 is actually proved for an open dense subset $\mathcal{A}_n^R \subset \mathcal{A}'_n$. This coincides with the statement of Theorem 3.

We shall also need the following theorem, see [Shch84].

Theorem. *For $n \geqslant 2$, there exists an open dense subset $\mathcal{A}_n^{NC} \subset \mathcal{A}'_n$ such that for each $\mathcal{F} \in \mathcal{A}_n^{NC}$ the monodromy group at infinity is not commutative.*

Note that $\mathbf{M}_i \circ \mathbf{M}_j = \mathbf{M}_j \circ \mathbf{M}_i$ defines an analytic subset of $\mathcal{A}'_n$, and $\mathcal{A}_n^{NC}$ is the complement to the intersection of these analytic subsets for $1 \leqslant i < j \leqslant n$. Due to the previous theorem, at least one of the sets $\mathbf{M}_i \circ \mathbf{M}_j = \mathbf{M}_j \circ \mathbf{M}_i$ has positive codimension. On the other hand, it is easy to find a loop in $\mathcal{A}'_n$ that swaps this set with any other set of the same form. Therefore, *all* sets $\mathbf{M}_i \circ \mathbf{M}_j = \mathbf{M}_j \circ \mathbf{M}_i$ have positive codimension.

Corollary 4. *There exists an open dense subset of $\mathcal{A}'_n$ such that for each $\mathcal{F}$ from this subset none of $\mathbf{M}_i$, $\mathbf{M}_j$ commute.*

2.4 Infinite number of limit cycles

The following definition generalizes the notion of a limit cycle of a foliation of $\mathbb{R}^2$.

Definition. *Limit cycle* on a leaf L is an element $[\gamma]$ of the free homotopy group of L such that the holonomy along (any) its representative γ is non-identical (cf. with the definition of identical cycle).

Note that each isolated fixed point z_0 of some monodromy map $\mathbf{M}_\gamma \in G$ gives us a limit cycle, namely we can take the lifting of the loop γ that starts at z_0.

Definition. A set of limit cycles of a foliation is called *homologically independent*, if for any leaf L all the cycles located in this leaf are linearly independent in $H_1(L)$.

The following result was obtained in [SRO98].

Theorem 5. *For $n \geqslant 3$, for an open dense set $\mathcal{A}_n^{LC} \subset \mathcal{A}_n$, each $\mathcal{F} \in \mathcal{A}_n^{LC}$ possesses an infinite number of homologically independent limit cycles.*

The next two lemmas were proved in [Il78], and were heavily used in the same paper to establish new properties of generic polynomial foliations. In particular, they were used in the proof of a weaker version of Theorem 5.

Lemma 6. *Let $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be an expanding analytic germ, $|g'(0)| > 1$. Let μ, $|\mu| < 1$, be a number such that the multiplicative semigroup generated by μ and $g'(0)$ is dense in $\mathbb{C}^*$. Then for each $\nu \in \mathbb{C}^*$ and any neighborhood of the origin, the linear map $z \mapsto \nu z$ can be approximated by a map of the form $z \mapsto \mu^{-s} g^t(\mu^{r+s} z)$ uniformly in this neighborhood. Moreover, if g depends analytically on some parameter $\varepsilon \in (\mathbb{C}^n, 0)$, then this approximation is uniform in ε.*

Idea of the proof. Due to the condition on μ and $g'(0)$, one can approximate ν by a number of the form $(g'(0))^t \mu^r$. Thus the multiplier of the map $z \mapsto g^t(\mu^r z)$ is close to ν. Then we conjugate this map by a strongly contracting linear map $z \mapsto \mu^s z$. The obtained map has the form $z \mapsto \mu^{-s} g^t(\mu^{r+s} z)$ and is close to its linear part $z \mapsto (g'(0))^t \mu^r z$, hence to the map $z \mapsto \nu z$. For details including dependence on ε see, e.g., [V06]. □

Lemma 7. *Suppose that two monodromy maps* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *do not commute, and their multipliers satisfy*

- $|\mu_1| < 1, |\mu_2| > 1$*;*
- *the multiplicative semigroup generated by* μ_1 *and* μ_2 *is dense in* $\mathbb{C}^*$*.*

Then the set of hyperbolic fixed points of compositions of the form $\mathbf{M}_1^{-s}\mathbf{M}_2^t\mathbf{M}_1^{r+s}\mathbf{M}_2$ *is dense in a small neighborhood of the origin.*

Proof. We will work in a linearizing chart for $\mathbf{M}_1$. Let z_0 be close enough to zero; since $\mathbf{M}_1$ and $\mathbf{M}_2$ do not commute, we can assume, after a small perturbation of z_0, that $|\mathbf{M}_2'(z_0)| \neq \left|\frac{\mathbf{M}_2(z_0)}{z_0}\right|$.

Note that the map $z \mapsto \frac{z_0}{\mathbf{M}_2(z_0)}\mathbf{M}_2(z)$ has an isolated hyperbolic fixed point at z_0. Due to Lemma 6, we can approximate the linear map $z \mapsto \frac{z_0}{\mathbf{M}_2(z_0)}z$ by a map of the form $\mathbf{M}_1^{-s}\mathbf{M}_2^t\mathbf{M}_1^{r+s}$. If this map is close enough to $z \mapsto \frac{z_0}{\mathbf{M}_2(z_0)}z$, then the map $\mathbf{M}_1^{-s}\mathbf{M}_2^t\mathbf{M}_1^{r+s}\mathbf{M}_2$ also has a hyperbolic fixed point close to z_0. □

2.5 Volk's Theorem

In [V06] D. Volk proves that foliations with separatrix connections are dense in $\mathcal{A}_n$. Actually, his arguments work in a more general settings.

Theorem. *Let* $\tilde{\mathcal{F}}$ *be a polynomial foliation of degree* $n \geqslant 2$*. Let* A, B *be holomorphic maps of a neighborhood of* $\tilde{\mathcal{F}}$ *in* $\mathcal{A}_n$ *to* $\mathbb{C}^2$*. Then there exists* $\mathcal{F}$ *arbitrarily close to* $\tilde{\mathcal{F}}$ *such that the points* $A(\mathcal{F})$ *and* $B(\mathcal{F})$ *belong to the same leaf of* $\mathcal{F}$*.*

The original Volk's Theorem follows from this theorem if $A(\mathcal{F})$ and $B(\mathcal{F})$ belong to separatrices of two different singular points of $\mathcal{F}$. However, we shall need a more precise statement.

Take an analytic submanifold $\mathcal{M} \subset \mathcal{A}_n^R$ such that

- $\dim \mathcal{M} > \dim \mathrm{Aff}(\mathbb{C}^2) = 6$;
- $\mu_1 = \mathrm{const}$ and $\mu_2 = \mathrm{const}$ on $\mathcal{M}$;
- $|\mu_1| < 1$ and $|\mu_2| < 1$;
- the multiplicative semigroup generated by μ_1 and μ_2^{-1} is dense in $\mathbb{C}^*$.

Let S be a cross-section at infinity that is included by the Schröder chart of $\mathbf{M}_1$ for all foliations from $\mathcal{M}$; here we diminish $\mathcal{M}$ if necessary.

Theorem 8. *Let* $\mathcal{M}$ *and* S *be as above. Let* $A, B : \mathcal{M} \to S$ *be two non-vanishing holomorphic functions. Then there exist two loops* $\gamma, \gamma' \in \Omega L_\infty$ *not depending on the foliation such that the condition* $\mathbf{M}_{\gamma\gamma'}(A(\mathcal{F})) = B(\mathcal{F})$ *defines a non-empty submanifold* $\mathcal{M}' \subset \mathcal{M}$ *of codimension one.*

Moreover, the loops can be constructed in the following way. There exists an index i *such that for each sufficiently large* p *we can choose either* $\gamma = \gamma_1^p\gamma_i$ *or* $\gamma = \gamma_1^p$*. After* i *and* p *are fixed, there exists a triple of arbitrarily large numbers* (r, s, t) *such that we can take* $\gamma' = \gamma_1^{r+s}\gamma_2^{-t}\gamma_1^{-s}$*.*

Let $z = z_{\mathcal{F}} : S \to (\mathbb{C}, 0)$ be the Schröder chart for $\mathbf{M}_1$ such that the change of coordinates with respect to some fixed chart is parabolic. Then $z_{\mathcal{F}}$ depends analytically on $\mathcal{F}$. In the rest of this section, $\mathbf{M}_i$, $A(\mathcal{F})$, $B(\mathcal{F})$ etc. are written in the corresponding chart $z = z_{\mathcal{F}}$. In particular, $\mathbf{M}_1(z) = \mu_1 z$.

Theorem 8 is an immediate consequence of the following two lemmas.

Lemma 9. *In the assumptions of Theorem 8, suppose that the ratio* $\frac{A}{B} = \frac{z_{\mathcal{F}}(A(\mathcal{F}))}{z_{\mathcal{F}}(B(\mathcal{F}))}$ *is a non-constant function of* $\mathcal{F}$. *Then we can choose a triple of arbitrarily large numbers* (r, s, t) *such that for* $\gamma' = \gamma_1^{r+s}\gamma_2^{-t}\gamma_1^{-s}$ *the condition* $\mathbf{M}_{\gamma'}(A(\mathcal{F})) = B(\mathcal{F})$ *defines a non-empty submanifold* $\mathcal{M}' \subset \mathcal{M}$ *of codimension one.*

Fix some foliation $\tilde{\mathcal{F}} \in \mathcal{M}$. The objects corresponding to $\tilde{\mathcal{F}}$ will be denoted by the tilde above, e.g., $\tilde{A}$, $\tilde{B}$, $\tilde{\mathbf{M}}_i$.

Lemma 10. *In the assumptions of Theorem 8, we can find an index* i *such that for each sufficiently large* p *either for* $\gamma = \gamma_1^p\gamma_i$ *or for* $\gamma = \gamma_1^p$, *the equality* $\frac{\mathbf{M}_\gamma(A)}{B} \equiv \frac{\tilde{\mathbf{M}}_\gamma(\tilde{A})}{\tilde{B}}$ *does not hold.*

Indeed, it is sufficient to take γ from Lemma 10, and substitute $\mathbf{M}_\gamma(A)$ for A in Lemma 9.

Remark. Lemma 10 is a refined version of the union of Lemmas 6 and 7 in [V06]. The proof of Lemma 6 in [V06] deals separately with $n \geqslant 3$ and $n = 2$; unfortunately, the proof for the case $n = 2$ has a gap. We give another proof which works for all $n \geqslant 2$.

Now let us prove the lemmas.

Proof of Lemma 9. Since the equality $\frac{A}{B} = \frac{\tilde{A}}{\tilde{B}}$ is not trivial, it defines a codimension-one submanifold in $(\mathcal{M}, \tilde{\mathcal{F}})$. This submanifold is non-empty, because it contains $\tilde{\mathcal{F}}$.

Let us approximate the linear map $z \mapsto \frac{\tilde{B}}{\tilde{A}}z$ in the chart z by a map of the form $\mathbf{M}_{\gamma'} = \mathbf{M}_1^{-s}\mathbf{M}_2^{-t}\mathbf{M}_1^{r+s}$. This approximation is uniform with respect to $\mathcal{F}$ (see Lemma 6). If $\mathbf{M}_{\gamma'}$ is sufficiently close to this linear map, then the holomorphic function $\mathbf{M}_{\gamma'}(A) - B$ on $(\mathcal{M}, \tilde{\mathcal{F}})$ is close to the function $\frac{\tilde{B}}{\tilde{A}}A - B$, thus the condition $\mathbf{M}_{\gamma'}(A) - B = 0$ also defines a codimension-one nonempty submanifold $\mathcal{M}' \subset \mathcal{M}$. □

Remark. In the chart z, $\mathbf{M}_{\gamma'}$ approximates the linear map $z \mapsto \frac{\tilde{B}}{\mathbf{M}_\gamma(\tilde{A})}z$ in the C^0 topology in some fixed neighborhood of $\mathbf{M}_\gamma(A)$. In a slightly smaller domain, we can use Cauchy estimates and prove that $\mathbf{M}_{\gamma'}$ approximates this linear map in the C^r topology. In particular, the derivative $\frac{d}{dz}\mathbf{M}_{\gamma'}|_{\mathbf{M}_\gamma(A)}$ can be made arbitrarily close to $\frac{\tilde{B}}{\mathbf{M}_\gamma(\tilde{A})}$ uniformly in $\mathcal{F} \in \mathcal{M}'$. Further, $\mathbf{M}_i'(\mu_1^p A) \approx \mu_i$ and $\frac{\mathbf{M}_i(\mu_1^p A)}{\mu_1^p A} \approx \mu_i$ for large p, hence $\mathbf{M}'_{\gamma\gamma'}(A)$ can be made arbitrarily close to $\frac{\tilde{B}}{\tilde{A}}$.

Proof of Lemma 10. Since $\dim \mathcal{M} > \dim \mathrm{Aff}(\mathbb{C}^2)$, there exists $\mathcal{F} \in \mathcal{M}$ close to $\tilde{\mathcal{F}}$ which is not affine conjugated to $\tilde{\mathcal{F}}$. Since $\tilde{\mathcal{F}} \in \mathcal{A}_n^R$, Theorem 3 implies that the monodromy groups at infinity of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are not analytically conjugated as groups with marked generators. Hence there exists i such that $\mathbf{M}_i$ is not conjugate to $\tilde{\mathbf{M}}_i$ by the map $z \mapsto \frac{\tilde{A}}{A}z$. Fix a punctured neighborhood U of the origin such that

$$\frac{\tilde{A}}{A}\mathbf{M}_i(z) \neq \tilde{\mathbf{M}}_i\left(\frac{\tilde{A}}{A}z\right) \quad \forall z \in U. \tag{4}$$

Let p be a large integer number such that $\mu_1^p A \in U$. Note that $\mathbf{M}_1^p(A) = \mu_1^p A$, $\tilde{\mathbf{M}}_1^p(\tilde{A}) = \mu_1^p \tilde{A}$. If the assertion of the lemma fails for such p, then we have both $\frac{\mu_1^p A}{B} = \frac{\mu_1^p \tilde{A}}{\tilde{B}}$ and $\frac{\mathbf{M}_i(\mu_1^p A)}{B} = \frac{\tilde{\mathbf{M}}_i(\mu_1^p \tilde{A})}{\tilde{B}}$, thus $\frac{\tilde{A}}{A}\mathbf{M}_i\left(\mu_1^p A\right) = \tilde{\mathbf{M}}_i\left(\frac{\tilde{A}}{A} \cdot \mu_1^p A\right)$, and this contradicts the inequality above. □

2.6 Intersections with lines

We shall need to prove that a generic leaf of a generic foliation from $\mathcal{A}_n^{sym}$ or $\mathcal{B}_n^{sym}$ intersects the line $x = 0$ in infinitely many points. The proof will be based on the following two statements. First, we use theorem due to Jouanolou to estimate the number of algebraic leaves.

Theorem 11. [J79]

If a polynomial foliation $\mathcal{F} \in \mathcal{A}_n$ has at least $\frac{1}{2}n(n+1)+2$ algebraic irreducible invariant curves, then it has a rational first integral.

Then we prove that a non-algebraic leaf intersects a generic line in infinitely many points.

Lemma 12. *Consider a polynomial foliation $\mathcal{F}$ of $\mathbb{C}P^2$, its non-algebraic leaf L and a line $T \subset \mathbb{C}P^2$ such that there are no singular points of $\mathcal{F}$ in T. Then $\#(L \cap T) = \infty$.*

The proof is completely analogous to the proof of Lemma 28.10 in [IY07]. This lemma states that a non-algebraic leaf of a foliation $\mathcal{F} \in \mathcal{A}'_n$ cannot approach the infinite line only along the separatrices of singular points. However, we repeat the proof here for completeness.

Proof. Suppose the contrary, i.e. L is not algebraic and $\#(L \cap T) < \infty$. Make a projective coordinate change such that T is mapped to the infinite line $\{u = 0\}$, and the point $v = \infty$ of the infinite line does not belong to L. It is easy to show that L cannot be bounded, hence it must intersect $\{u = 0\}$ in at least one point.

Suppose that the leaf L is given by $y = \varphi_j(x), j = 1, 2, \dots, k$, in neighborhoods of k points of $L \cap T$. Note that each φ_j has a linear growth at infinity. Consider the product $\prod_{j=1}^{k}(y - \varphi_j(x))$; this is a polynomial in y, with symmetric functions $\sigma_1 = \sum_{j=1}^{k} \varphi_j$, $\sigma_2 = \sum_{1 \leqslant j < l \leqslant k} \varphi_j \varphi_l$, $\dots$, $\sigma_k = \prod_{j=1}^{k} \varphi_j$ as coefficients.

Let P_i be projections of finite singularities of $\mathcal{F}$ to x-plane. It is possible to extend σ_j holomorphically to $\mathbb{C} \setminus \{ P_1, \dots \}$ by the symmetric combinations of intersections $L \cap \{ x = c \}$, with multiplicities. Indeed, the number of these intersections stays locally the same, thus equals k for any c. The intersections depend holomorphically on c and stay bounded, otherwise the leaf L would approach the infinite line along $x = \text{const}$, hence the point $v = \infty, u = 0$ would belong to L.

Since σ_j are bounded in any compact, P_i are removable singularities of σ_j.

So, the symmetric combinations of φ_j extend holomorphically to $\mathbb{C}$ and have a polynomial growth at infinity. Thus they are polynomials in x, and the function $F = \prod_{j=1}^{k}(y - \varphi_j(x))$ is a polynomial in x, y. Hence $F = 0$ is a polynomial equation defining the leaf L, and L is algebraic. Contradiction shows that $\#(L \cap T) = \infty$ for a non-algebraic L. □

3 A leaf with many handles

Consider an open subset $\mathcal{U} \subset \mathcal{A}_n$. Shrinking $\mathcal{U}$ if necessary, we may and will assume that

- $\mathcal{U} \subset \mathcal{A}_n^R$ (due to Theorem 3, $\mathcal{U} \subset \mathcal{A}_n^R$ is open and dense);
- one can enumerate the singular points $a_1, \dots, a_{n+1}$ at the infinite line so that a_i depend analytically on $\mathcal{F} \in \mathcal{U}$;
- ranges of $a_i(\mathcal{U})$ are small enough so that we can and shall fix a point O and paths γ_i as in Section "Monodromy group and rigidity" independently on $\mathcal{F} \in \mathcal{U}$;

- none of μ_i belongs to the unit circle.

Due to Corollary 4, we can and shall assume that

- $\mathbf{M}_i \circ \mathbf{M}_j \neq \mathbf{M}_j \circ \mathbf{M}_i$ for $\mathcal{F} \in \mathcal{U}$ and $i \neq j$.

Note that passing to the conjugated coordinates $(\bar{x}, \bar{y})$ in $\mathbb{C}^2$ replaces all $|\mu_j| = |\exp(2\pi i \lambda_j)| = \exp(-2\pi \operatorname{Im} \lambda_j)$ by $\exp(-2\pi \operatorname{Im}(\bar{\lambda}_j)) = |\mu_j|^{-1}$. Therefore, we can assume that at least two of $|\mu_j|$ are less that one (otherwise we just pass to the conjugated coordinates). Recall that $\prod \mu_j = 1$, hence at least one of $|\mu_j|$ is greater than one. Let us reenumerate the singularities at the infinite line so that the multipliers satisfy

- $|\mu_1| < 1$, $|\mu_2| > 1$ and $|\mu_3| < 1$.

Let $\mathcal{M}_0 \subset \mathcal{U}$ be a non-empty submanifold given by $\mu_1 = \text{const}$, $\mu_2 = \text{const}$ and $\mu_3 = \text{const}$. Slightly perturbing constants in these equations, assume that for $\mathcal{F} \in \mathcal{M}_0$,

- the multiplicative semigroup generated by μ_1 and μ_2 is dense in $\mathbb{C}^*$;
- the multiplicative semigroup generated by μ_1^{-1} and μ_3 is dense in $\mathbb{C}^*$.

For $n = 2$, $\operatorname{codim} \mathcal{M}_0 = 2$: the equations $\mu_1 = \text{const}$, $\mu_2 = \text{const}$ imply that $\mu_3 = \text{const}$ since $\mu_1 \mu_2 \mu_3 = 1$. For $n \geqslant 3$, $\operatorname{codim} \mathcal{M}_0 = 3$. The following lemma is a key step in the proof of Theorem 1.

Lemma 13. *Let $\mathcal{M}_0$ and μ_i, $i = 1, 2, 3$ be as above. Let $\mathcal{M} \subset \mathcal{M}_0$ be an analytic submanifold of dimension at least 7. Then for any $\varepsilon > 0$ there exists a submanifold $\mathcal{M}' \subset \mathcal{M}$ of codimension one such that each $\mathcal{F} \in \mathcal{M}'$ has a leaf with a handle ε-close to L_∞.*

More precisely, there exist two curves $\gamma^{(1)}, \gamma^{(2)} \subset L_\infty$ such that $\mathbf{M}_{\gamma^{(1)}}$ and $\mathbf{M}_{\gamma^{(2)}}$ have a common hyperbolic fixed point $B = B(\mathcal{F}) \in S$, the lifts c_1, c_2 of the curves $\gamma^{(1)}, \gamma^{(2)}$ starting from B intersect transversely at exactly one point, and c_1 and c_2 are included by ε-neighborhood of the infinite line.

We shall postpone the proof of this lemma till the end of this section. Now let us deduce Theorem 1 from this lemma. First, we obtain many handles on different leaves.

Corollary 14. *For each $0 \leqslant g \leqslant \dim \mathcal{M}_0 - 6$, there exists an analytic submanifold $\mathcal{M}_g \subset \mathcal{M}_0$ of codimension at most g such that the leaves of each $\mathcal{F} \in \mathcal{M}_g$ possess g handles (possibly on different leaves of $\mathcal{F}$) with hyperbolic generating cycles $(c_1, c_2), (c_3, c_4), \ldots, (c_{2g-1}, c_{2g})$. The generators of different handles do not intersect (even if they are located in the same leaf), and c_{2j-1} intersects c_{2j} at a point $B_j \in S$.*

Proof. Let us prove the assertion by induction. For $g = 0$, we just take $\mathcal{M}_0$. Suppose that we already have $\mathcal{M}_g$, $g \leqslant \dim \mathcal{M}_0 - 7$. Then $\dim \mathcal{M}_g \geqslant 7$. Using Lemma 13, we get a submanifold $\mathcal{M}_{g+1} \subset \mathcal{M}_g$ of codimension 1 such that each $\mathcal{F} \in \mathcal{M}_{g+1}$ possesses a handle generated by (c_{2g+1}, c_{2g+2}) which is closer to L_∞ than all the loops guaranteed by $\mathcal{M}_g$. Hence, $\mathcal{M}_{g+1}$ satisfies the assertion of this corollary. This completes the proof. □

Proof of Theorem 1. Let us apply the previous corollary to $g = \frac{(n+1)(n+2)}{2} - 4$. This is possible since for $n \geqslant 2$ we have

$$\frac{(n+1)(n+2)}{2} - 4 \leqslant (n+1)(n+2) - 10 \leqslant \dim \mathcal{M}_0 - 6.$$

Note that the cycles c_{2j-1} and c_{2j} correspond to hyperbolic fixed points of the germs of monodromy maps $\mathbf{M}_{c_{2j-1}}, \mathbf{M}_{c_{2j}} : (S, B_j) \to (S, B_j)$. Hence they survive under a small perturbation. The manifold $\mathcal{M}_g$ is defined by g equations of the form "Hyperbolic fixed points of $\mathbf{M}_{c_{2i-1}}$ and $\mathbf{M}_{c_{2i}}$ coincide".

Let $\tilde{\mathcal{M}}_0 \subset \mathcal{U}$, $\tilde{\mathcal{M}}_0 \supset \mathcal{M}_0$ be the submanifold defined by $\mu_1 = \text{const}$, $\mu_2 = \text{const}$. It is easy to see that $\dim \tilde{\mathcal{M}}_0 = (n+1)(n+2) - 3$ and $\mathcal{M}_g$ extends to a submanifold $\tilde{\mathcal{M}}_g \subset \tilde{\mathcal{M}}_0$ given by the same g equations. Thus $\tilde{\mathcal{M}}_g$ has codimension g in $\tilde{\mathcal{M}}_0$,

$$\dim \tilde{\mathcal{M}}_g = \dim \tilde{\mathcal{M}}_0 - g = \frac{(n+1)(n+2)}{2} + 1 = (g-1) + 6$$

The leaves of the foliations from $\tilde{\mathcal{M}}_g$ also have at least g handles (possibly these handles are on the different leaves). Indeed, after a small perturbation of a foliation $\mathcal{F} \in \mathcal{M}_g$ inside $\tilde{\mathcal{M}}_g$, all cycles c_j survive and still intersect transversely at the points $B_j = B_j(\mathcal{F})$.

Now, let us apply Theorem 8 $(g-1)$ times to the monodromy maps $\mathbf{M}_1, \mathbf{M}_2$ and the points $B_1, B_2, \ldots, B_g$; we obtain a 6-dimensional submanifold $\hat{\mathcal{M}} \subset \tilde{\mathcal{M}}_0$ such that for each $\mathcal{F} \in \hat{\mathcal{M}}$ all points B_j are located in the same leaf, thus all generating cycles c_j are located in the same leaf. □

Now let us prove Lemma 13.

Proof of Lemma 13. Due to Lemma 7, there exist k, l and m such that $\mathbf{M}_1^{-k}\mathbf{M}_2^m\mathbf{M}_1^{k+l} \circ \mathbf{M}_2$ has a hyperbolic fixed point B near the origin. We require some additional conditions on B. More precisely, we proceed in three steps.

1. First we choose a domain for B sufficiently close to the origin, so that $|\mathbf{M}_2(B)| > |B| > |\mathbf{M}_1(B)|$.

2. Then we shrink this domain so that in the linearizing chart for $\mathbf{M}_3$ we have $|\mathbf{M}_1'(B)| \neq \left|\frac{\mathbf{M}_1(B)}{B}\right|$. This is possible since $\mathbf{M}_1$ does not commute with $\mathbf{M}_3$.

3. Finally, we apply Lemma 7 choosing k, l, m so large that $|\mathbf{M}_1^{k+l}(\mathbf{M}_2(B))| < |B|$, hence $\mathbf{M}_1^{k+l}(\mathbf{M}_2(B)) \neq B$.

Let $\gamma^{(1)}$ be the representative of the class $[\gamma_2\gamma_1^{k+l}\gamma_2^m\gamma_1^{-k}] \in \pi_1(L_\infty)$ shown in Figure "First cycle". Let c_1 be the lifting of $\gamma^{(1)}$ starting at B. Since B is a hyperbolic fixed point of the monodromy along $\gamma^{(1)}$, c_1 is a hyperbolic limit cycle. Clearly, c_1 survives under a small perturbation.

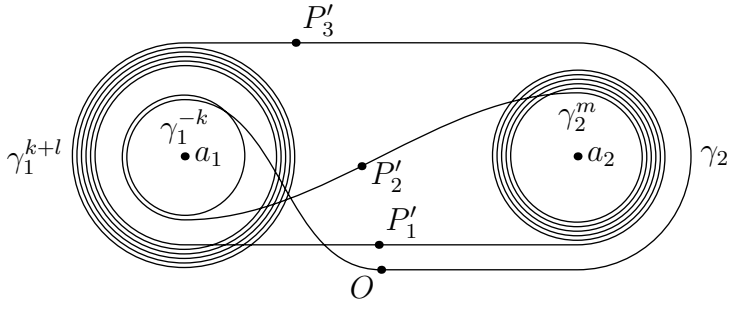

Figure 1: First cycle for $k = 2$, $l = 3$, $m = 6$

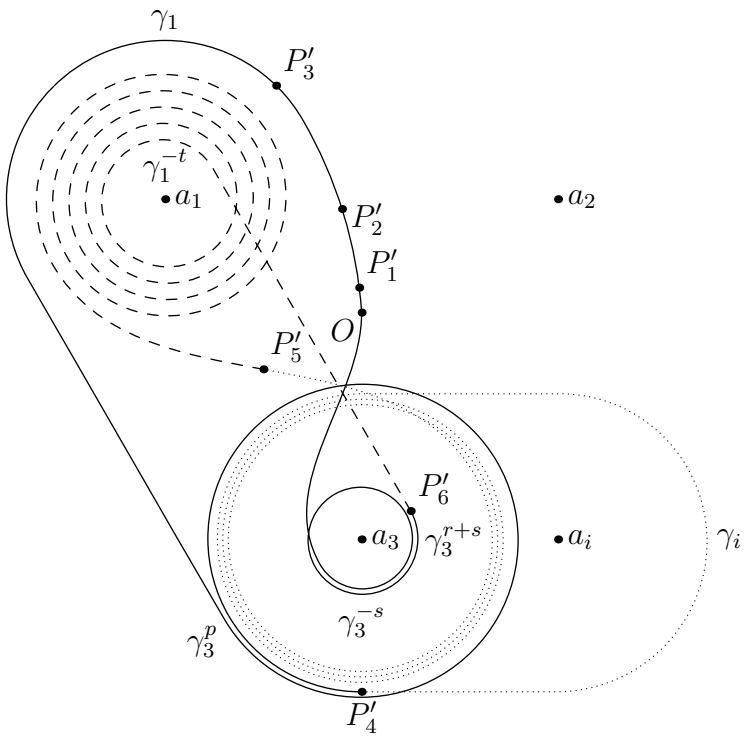

Figure 2: Second cycle for $p = 2$, $q = 1$, $r = 1$, $s = 2$, $t = 5$

Now let us apply Theorem 8 to the points $A(\mathcal{F}) = \mathbf{M}_1(B(\mathcal{F}))$ and $B(\mathcal{F}) = B(\mathcal{F})$ and the maps $\mathbf{M}_3$ and $\mathbf{M}_1$. Due to this theorem, there exist $i \in \{1, \dots, n+1\}$, $p \in \mathbb{N}$, $q \in \{0, 1\}$, $r \in \mathbb{N}$, $s \in \mathbb{N}$, $t \in \mathbb{N}$ such that the equality

$$\mathbf{M}_3^{-s} \circ \mathbf{M}_1^{-t} \circ \mathbf{M}_3^{r+s} \circ \mathbf{M}_i^q \circ \mathbf{M}_3^p \circ \mathbf{M}_1(B(\mathcal{F})) = B(\mathcal{F})$$

defines a codimension one submanifold $\mathcal{M}' \subset \mathcal{M}$. Let $\gamma^{(2)}$ be the representative of $[\gamma_1 \gamma_3^p \gamma_i^q \gamma_3^{r+s} \gamma_1^{-t} \gamma_3^{-s}]$ shown in Figure "Second cycle", let c_2 be the corresponding limit cycle.

Let us prove that one can choose the numbers in Theorem 8 so that c_2 is a hyperbolic cycle and it intersects c_1 transversely at exactly one point $B(\mathcal{F})$.

First, let us prove that for p and s large enough, $c_1 \cap c_2 = \{ B \}$. Since $\gamma^{(1)}$ and $\gamma^{(2)}$ are the projections of c_1 and c_2 to the infinite line, c_1 can intersect c_2 only above the intersection points of $\gamma^{(1)}$ and $\gamma^{(2)}$. Let P_j be the points of c_2 that project to P_j', $j = 1, \dots, 6$ (see figure above). We shall deal separately with arcs P_6BP_4, P_4P_5 and P_5P_6 of c_2.

Arc P_6BP_4 (solid line): The arc $P_6'OP_4'$ of $\gamma^{(2)}$ intersects $\gamma^{(1)}$ in $\{ O, P_1', P_2', P_3' \}$, hence we need to prove that none of P_1, P_2 and P_3 belongs to c_1. If $P_1 \in c_1$, then B is a fixed point of the monodromy map along the union of two arcs, OP_1' on $\gamma^{(1)}$ and $P_1'O$ on $\gamma^{(2)}$. Thus $\mathbf{M}_1^{k+l} \circ \mathbf{M}_2(B) = B$, which contradicts item 3 in the choice of c_1. Analogously, for P_2 we get $\mathbf{M}_1^{-k}(B) = B$, but B belongs to the domain of the linearizing chart of $\mathbf{M}_1$, so this is also impossible. For P_3 we get $\mathbf{M}_2(B) = B$, which contradicts item 1 in the choice of c_1.

Arc P_4P_5 (dotted line): Suppose that the loop c_1 is fixed, and let us prove that for large p the arc P_4P_5 is much closer to L_∞ than c_1. Indeed, the v-coordinate of P_4 is $O(|\mu_3|^p)$ as $p \to \infty$. As we move along γ_i, the v-coordinate is multiplied by a bounded number. Then, as we make $r + s$ turns around a_3, we come even closer to the infinite line. Therefore, P_4P_5 is

$O(|\mu_3|^p)$-close to L_∞. Recall that p can be chosen arbitrarily large *after* the choice of c_1, hence we can choose it so large that all points of P_4P_5 are much closer to L_∞ than all points of c_1, in particular P_4P_5 does not intersect c_1.

Arc P_5P_6 (dashed line): Similarly, going from B along the arc $OP_6'P_5' \subset \gamma^{(2)}$ in the opposite direction, we can see that all points of P_5P_6 are $O(|\mu_3|^s)$-close to L_∞. Since s can be chosen arbitrarily large *after* we fix all other numbers, this part of c_2 can be made much closer to L_∞ than all points of c_1, hence it does not intersect c_1.

Therefore, $B(\mathcal{F})$ is the only intersection point of c_1 and c_2. Note that this intersection is transverse, because the projections of c_1 and c_2 to L_∞ intersect transversely, and (holomorphic) projection to L_∞ preserves angles.

Now let us prove that for sufficiently large numbers in Theorem 8, c_2 is a hyperbolic cycle. Indeed, due to the remark to Volk's Theorem, the derivative $\mathbf{M}'_{\gamma^{(2)}}(B(\mathcal{F}))$ can be made arbitrarily close to $\frac{\mathbf{M}_1'(B(\mathcal{F}))B(\mathcal{F})}{\mathbf{M}_1(B(\mathcal{F}))}$ (the fraction is evaluated in the Schröder chart for $\mathbf{M}_3$). Due to the choice of B, this ratio does not belong to the unit circle, hence one can choose γ and γ' in Theorem 8 so that c_2 is a hyperbolic cycle.

This completes the proof of the lemma, hence the proof of Theorem 1. □

4 Leaves of infinite genus

Consider the map $F_2 : \mathbb{C}^2 \to \mathbb{C}^2$ given by $(x, y) \mapsto (z, w) = (x, y^2)$. Since $\mathcal{F} \in \mathcal{A}_n^{sym}$, the image of $\mathcal{F}$ is a well-defined foliation $(F_2)_*\mathcal{F}$ given by

$$\dot{z} = p(z, w);$$
$$\dot{w} = q(z, w),$$

where $P(x, y) = yp(x, y^2)$ and $Q(x, y) = q(x, y^2)$.

The following lemma explicitly describes the open and dense subset of $\mathcal{A}_n^{sym}$ (or $\mathcal{B}_{n+1}^{sym}$) that satisfies the assertion of Theorem 2.

Lemma 15. *Consider a foliation $\mathcal{F} \in \mathcal{A}_n^{sym}$ such that*

- *$\mathcal{F}$ has no rational first integral;*
- *$(F_2)_*\mathcal{F}$ has no singular points at the projective line $\{w = 0\} \subset \mathbb{C}P^2$.*

Then $\mathcal{F}$ has finitely many (probably, zero) algebraic leaves, and all other leaves have infinite genus.

Remark. We can also take the saturation of the set constructed above by the orbits of affine group. This adds 3 to the dimension, but this saturation will be a more complicated object than an open dense subset of a linear subspace.

Proof. Since $\mathcal{F}$ has no rational first integral, Theorem 11 implies that all but a finite number of leaves are non-algebraic. Let L be a non-algebraic leaf of $\mathcal{F}$. Due to Lemma 12, $F_2(L)$ intersects $\{ w = 0 \}$ in infinitely many points, hence L intersects $\{ y = 0 \}$ in infinitely many points as well.

Note that there is at most finite number of non-transverse intersections, hence there is an infinite number of transverse intersections of L and $\{ y = 0 \}$.

The restriction $F_2|_L : L \to F_2(L)$ is a ramified double covering. It is easy to see that the points of transverse intersection $L \cap \{ y = 0 \}$ are ramification points of $F_2|_L$. Hence the covering $F_2|_L$ has countably many ramification points. Consider a disk $D \subset F_2(L)$ that contains N ramification points. Due to Riemann–Hurwitz Formula,

$$\chi(F_2^{-1}(D)) = 2\chi(D) - N = 2 - N.$$

On the other hand, $F_2^{-1}(\partial D)$ is either a circle (for odd N) or a union of two circles (for even N), hence $F_2^{-1}(D)$ has either one or two holes. Therefore, this preimage has $\lceil \frac{N+1}{2} \rceil$ handles. Finally, L has infinite genus. □

Remark. The last step can be done in a more intuitive manner. Consider infinitely many pairwise disjoint discs D_i, each contains three ramification points. For each disc consider the lifts to the cover of two loops shown in figure below starting at the same lift of O. These lifts intersect transversely at exactly one point (above O), hence they generate a handle.

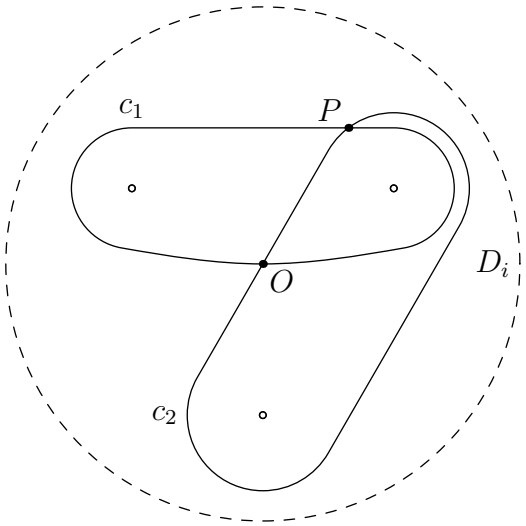

Figure 3: Two cycles whose preimages under F_2 generate a handle

Since $\mathcal{B}_n \subset \mathcal{A}_n$, this lemma is applicable to foliations $\mathcal{F} \in \mathcal{B}_n^{sym}$ as well.

Now let us deduce Theorem 2 from the above lemma.

Proof of Theorem 2. It is sufficient to prove that for $n \geqslant 2$ the subset of $\mathcal{A}_n^{sym}$ (resp., $\mathcal{B}_{n+1}^{sym}$) defined by the additional assumptions from Lemma 15 is open and dense in the ambient projective space.

Let us prove that a generic foliation $\mathcal{F} \in \mathcal{A}_n^{sym}$ or $\mathcal{F} \in \mathcal{B}_{n+1}^{sym}$ has no rational first integral. Note that a complex hyperbolic singular point is not *locally* integrable, hence a foliation with a complex hyperbolic singular point cannot have a rational first integral. Since a complex hyperbolic singular point survives under small perturbations, it is sufficient to prove that the set of foliations from $\mathcal{A}_n^{sym}$ (resp., $\mathcal{B}_{n+1}^{sym}$) having a complex hyperbolic singular point is dense in the ambient space.

Consider a foliation $\mathcal{F}_0$ from $\mathcal{A}_n^{sym}$ or $\mathcal{B}_{n+1}^{sym}$, $n \geqslant 2$. Let (x_0, y_0) be one of its singular points with $y_0 \neq 0$. Let $A_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be its linearization matrix at (x_0, y_0). Consider the two-parametric

perturbation $\mathcal{F}_{\varepsilon,\delta}$ of $\mathcal{F}_0$, $\varepsilon, \delta \in (\mathbb{C}, 0)$, given by

$$\dot{x} = P_0(x, y) + y(x - x_0)\varepsilon;$$
$$\dot{y} = Q_0(x, y) + (y^2 - y_0^2)\delta,$$

It is easy to see that the perturbed foliation belongs to the same class ($\mathcal{A}_n^{sym}$ or $\mathcal{B}_{n+1}^{sym}$) and has a singularity at the same point (x_0, y_0). The linearization matrix of $\mathcal{F}_{\varepsilon,\delta}$ at (x_0, y_0) is $A_{\varepsilon,\delta} = \begin{pmatrix} a + y_0\varepsilon & b \\ c & d + 2y_0\delta \end{pmatrix}$. Clearly,

$$\operatorname{tr} A_{\varepsilon,\delta} - \operatorname{tr} A_0 = y_0(\varepsilon + 2\delta);$$
$$\det A_{\varepsilon,\delta} - \det A_0 = y_0(\varepsilon d + 2\delta a + 2y_0\varepsilon\delta).$$

It is easy to see that we can achieve any small perturbation of the trace and determinant of the linearization matrix. Therefore, we can achieve any small perturbation of the eigenvalues. In particular, after some perturbation the singular point at (x_0, y_0) becomes complex hyperbolic.

The line $w = 0$ contains no singular points of $(F_2)_*\mathcal{F}$ for a typical $\mathcal{F}$ since

- $(p|_{\{\, w=0 \,\}}, q|_{\{\, w=0 \,\}})$ may be any pair of polynomials of degrees $\deg \mathcal{F} - 1$ and $\deg \mathcal{F}$, respectively, hence for a generic $\mathcal{F}$ they have no common roots;
- $(F_2)_*\mathcal{F}$ has a singularity at the intersection point of $\{\, w = 0 \,\}$ and the infinite line if and only if $\deg q|_{\{\, w=0 \,\}} < \deg \mathcal{F}$, and this is false for a generic $\mathcal{F}$.

□

Remark. The arguments above do not work for $\mathcal{A}_1^{sym}$ and $\mathcal{B}_2^{sym}$ because generic foliations from these spaces have rational first integrals. Indeed, a generic foliation from the former space is affine equivalent to a foliation of the form

$$\dot{x} = y;$$
$$\dot{y} = ax,$$

which has the first integral $y^2 - ax^2$. A generic foliation from $\mathcal{B}_2^{sym}$ is affine equivalent to a foliation of the form

$$\dot{x} = xy;$$
$$\dot{y} = x + y^2 + a,$$

which has the first integral $\frac{(x+a)^2 - ay^2}{x^2}$.

5 Proof of Ilyashenko's Theorem

In this Section we shall prove the following theorem.

Theorem. *For $n \geq 2$, let $\mathcal{A}_n^h \subset \mathcal{A}_n$ be the space of foliations given by homogeneous polynomials P and Q. For a foliation $\mathcal{F}$ from some open dense subset of $\mathcal{A}_n^h$, all its leaves except for a finite set have infinite genus.*

It seems that this theorem was proved by Ilyashenko many years ago, but he has never written the proof, though he communicated this proof to various people orally.

Proof. Take a homogeneous foliation $\mathcal{F}$. Note that the polynomials $\tilde{P}$ and $\tilde{Q}$ in (3) do not depend on u, hence in the chart $(u, v) = (\frac{1}{x}, \frac{y}{x})$ our foliation $\mathcal{F}$ is given by

$$\dot{u} = u\tilde{P}(v)$$
$$\dot{v} = h(v)$$

Clearly, the monodromy group at infinity is generated by linear maps $\mathbf{M}_j : u \mapsto \mu_j u$.

Fix a cross-section S given by $v = \text{const}$ and a point $p \in S \setminus L_\infty$. Let us find a handle passing through p.

The monodromy maps along loops $[\gamma_2, \gamma_1] = \gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1}$ and $[\gamma_3, \gamma_2^{-1}] = \gamma_3\gamma_2^{-1}\gamma_3^{-1}\gamma_2$ are identity maps, hence the lifts c_1 and c_2 of these loops starting at p are closed loops.

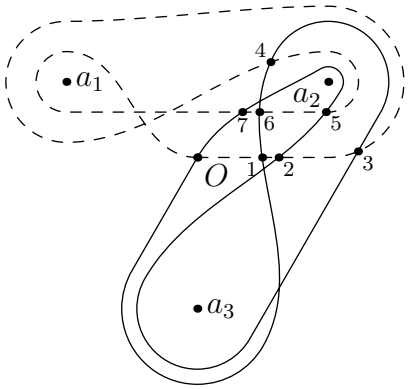

Figure 4: Cycles c_1 (dashed) and c_2

Note that these loops intersect only at p. Indeed, if c_1 and c_2 intersect above one of 7 other intersection points of $[\gamma_2, \gamma_1]$ and $[\gamma_3, \gamma_2^{-1}]$, then p is a fixed point of one of the maps $\mathbf{M}_3$, $\mathbf{M}_2$, $\mathbf{M}_2^{-1} \circ \mathbf{M}_3$, $\mathbf{M}_2^{-1} \circ \mathbf{M}_1^{-1} \circ \mathbf{M}_3$, $\mathbf{M}_1 \circ \mathbf{M}_2$, $\mathbf{M}_1^{-1} \circ \mathbf{M}_3$, $\mathbf{M}_1$, respectively. In a generic case, all these maps are linear non-identical, thus they have no fixed points except zero.

Thus c_1 and c_2 intersect transversely at one point, so we have found a handle passing through p.

Consider a leaf L of $\mathcal{F}$ which is not a separatrix $v = a_j$ of a singular point of $\mathcal{F}$ at the infinite line. In a generic case (say, if $|\mu_j| \neq 1$ for some j), L intersects S arbitrarily close to L_∞. Since each intersection point produces a handle, L has infinite genus. Hence any leaf except separatrices at a_j has infinite genus. □

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