ISSN 1061-9208, Russian Journal of Mathematical Physics, 2015, Vol. 22, No. 4, pp. 463–468.

Planar Penning Trap with Combined Resonance and Top Dynamics on Quadratic Algebra

M. V. Karasev and E. M. Novikova

Laboratory for Mathematical Methods in Natural Science, National Research University HSE, Russia

E-mail: mkarasev@hse.ru, e.m.novikova@gmail.com

Received October 12, 2015

Abstract. We study the dynamics of a charge in the planar Penning trap in which the direction of the magnetic field does not coincide with the trap axis. Under a certain combined resonance condition on the deviation angle and magnitudes of magnetic and electric fields, the trajectories of a charge are near-periodic. We describe the reduction to a top-like system with one degree of freedom on the space with quadratic Poisson brackets and study the stability of the equilibrium points of this system.

DOI 10.1134/S1061920815040056

1. INTRODUCTION

Penning traps of different kinds [1–6] are the simplest and easily realizable devices which represent hyperbolic dynamical systems with noncompact energy levels and with bounded trajectories.

The "ideal" Penning trap in $\mathbb{R}^3$ is designed by using a quadratic electric potential of saddle type and a homogeneous magnetic field parallel to the saddle axis.[1] The trajectories of a charge in such a trap, in general, ergodically cover 3D tori; however, under some resonance conditions on the electric voltage and the magnetic field strength, the trajectories become periodic. We shall refer to this situation as *EM-resonance*.

In more general traps, the magnetic field direction can (slightly) deviate from the saddle potential axis, creating a new option for control. After a deviation, the effect of EM-resonance is destroyed, and the trajectories are not periodic or even near-periodic. However, if the deviation angle and the magnitude of the magnetic field are subjected to a certain resonance condition, then the "periodic picture" comes back; one can see the fibration of the phase space by 2D tori with near-periodic trajectories.

This phenomenon happens due to the additional resonance occurring over the symmetry algebra of the main EM-resonance [7, 8]. We refer to this situation as the combined EM+*angular resonance* (EMA-resonance).

Note that, up to now, we were refereing to harmonic systems described by quadratic Hamiltonians. In real physical Penning traps, along with the harmonic part, there can exist a small anharmonicity. It can appear because of the magnetic field inhomogeneity (e.g., the Ioffe field [9, 10]) or of the spatial asymmetry of the electrodes creating the potential (e.g., cylindrical or cubic boxes instead of hyperbolic caps [11, 12]).

The newest devices try to be of smaller size, aiming to reach nanoscales in order to become quantum. An interesting design of micro and nano Penning traps uses ring electrodes placed on a plane [13–18]. The electric potential in this case has equilibrium point(s) outside the plane, and its saddle axis is directed perpendicularly to the plane. For such a planar design, there always exist cubic, quartic, etc., corrections to the main quadratic part of the potential near the center of the trap. Thus, the problem is to take into account anharmonic perturbations of the Hamiltonian and to keep the property of near-periodicity for the trajectories (this is the key property in order to get a quantum trap with visible spectral gaps [19]).

In the given paper, we consider the simplest planar Penning trap model in the presence of a deviation of the magnetic field from the perpendicular direction. Under the condition of combined

[1]The potential is a nondegenerate quadratic form in Cartesian coordinates with signature $+,-,-$; and the saddle axis is directed along its eigenvector corresponding to the positive eigenvalue.

EMA-resonance in the harmonic part of the Hamiltonian, the influence of the anharmonic part of the electric potential is reduced to top-like dynamics on the secondary symmetry algebra.

The symmetry algebra turns out to be of nonlinear type and is described by quadratic relations [7, 8] (i.e., the Poisson tensor is quadratic in some convenient coordinates and cannot be made linear). We study the equilibrium points of the reduced top Hamiltonian on 2D-symplectic leaves of the symmetry algebra. We describe an algorithm checking the stability of these points. The corresponding energy levels and Hess matrices are computed.

In neighborhoods of these energy levels, the trajectories of the whole trap dynamical system are bounded (near-periodic) and stable. The knowledge of the Hessians allows one to compute the semiclassical asymptotics of quantum eigenvalues near the boundaries of spectral clusters generated by the anharmonicity of the trap, following, for instance, the methods of [20].

2. RESONANCE HAMILTONIANS

In the planar Penning traps, the electric potential (a solution of the Laplace equation $\Delta V = 0$) is created by concentric ring electrodes placed on a plane. Under certain conditions on the electric voltages applied to the electrodes, the potential acquires a saddle type with two mutually symmetric equilibrium (saddle) points [14, 16, 18].

Denote by x, y the Cartesian coordinates[2] in the plane and put the origin $x = y = 0$ at the center of the ring. Denote by z the coordinate in the perpendicular direction, and put the origin $z = 0$ at one of the equilibrium points of the electric potential (not at the center of rings). Then the potential is given by the formula

$$V = \frac{1}{2}\omega_0^2 V^{(2)} + \varepsilon\beta V^{(3)} + \varepsilon^2\gamma V^{(4)} + \cdots \tag{2.1}$$

Here $V^{(j)}$ represent harmonic functions of different orders:

$$V^{(2)} = z^2 - \frac{1}{2}(x^2+y^2), \qquad V^{(3)} = z\Big(z^2 - \frac{3}{2}(x^2+y^2)\Big), \qquad V^{(4)} = z^4 - 3z^2(x^2+y^2) + \frac{3}{8}(x^2+y^2)^2,$$

the parameters ω_0, β, γ are determined by the radii of the rings and the electric voltages, and the small parameter ε characterizes the scale of anharmonicity (in reality, it can be not necessarily very small, e.g., $\varepsilon \sim 1/3$). Explicit formulas for the parameters in (2.1) can be found, for instance, in [14, 16, 18].

Along with the electric potential, there is a homogeneous magnetic field $\mathcal{B}$ in the trap. Assume that it consists of two components $\mathcal{B} = \mathcal{B}_0 + \varepsilon\mathcal{B}'$. Here the vector

$$\mathcal{B}_0 = \frac{3}{2}\omega_0(0, 0, 1) \tag{2.2}$$

is directed along the axis z and has magnitude $\frac{3}{2}\omega_0$. The second component $\varepsilon\mathcal{B}'$ is small, i.e., of the first order in the parameter ε, and the direction of the vector $\mathcal{B}'$ is chosen as follows:

$$\mathcal{B}' = \frac{1}{3}(4, 0, 1). \tag{2.3}$$

The special choice of constants in (2.2) and (2.3) generates a *combined resonance situation* in the leading Hamiltonian H_0 of the trap, as well as in the Hamiltonian H_1, which controls the corrections of order ε (for details, see below).

The total Hamiltonian of the trap is

$$H = \frac{1}{2}(p - \mathcal{A})^2 + V, \qquad \operatorname{curl}\mathcal{A} = \mathcal{B},$$

where $p = (p_x, p_y, p_z)$ is the momentum vector corresponding to Cartesian coordinates. The Hamiltonian can be represented in the form of leading and perturbing terms with respect to the small parameter,

$$H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + O(\varepsilon^3). \tag{2.4}$$

The leading part is given by

$$H_0 = \frac{1}{2}\Big[p_x^2 + p_y^2 + p_z^2 - \frac{3}{2}\omega_0(xp_y - yp_x) + \omega_0^2 z^2 + \frac{1}{16}\omega_0^2(x^2+y^2)\Big]. \tag{2.5}$$

[2]Everywhere below, we assume that all physical quantities and coordinates are made dimensionless and are rescaled to be of order 1, i.e., not small or large in the domains on which we study the system.

The first perturbing term is

$$H_1 = \frac{1}{6}\Big[yp_x + (4z - x)p_y - 4yp_z - 3\omega_0 zx + \frac{3}{4}\omega_0(x^2 + y^2)\Big] + \beta z\Big[z^2 - \frac{3}{2}(x^2 + y^2)\Big], \tag{2.6}$$

and the second term is determined by the formula

$$H_2 = \frac{1}{72}[17y^2 + (x - 4z)^2] + \gamma\Big[z^4 - 3z^2(x^2 + y^2) + \frac{3}{8}(x^2 + y^2)^2\Big]. \tag{2.7}$$

Now, after the canonical transformation

$x=\sqrt{\frac{2}{\omega_0}}(q_+ + q_-)$, $p_x=\frac{1}{2}\sqrt{\frac{\omega_0}{2}}(p_+ + p_-)$, $y=\sqrt{\frac{2}{\omega_0}}(p_+ - p_-)$, $p_y=-\frac{1}{2}\sqrt{\frac{\omega_0}{2}}(q_+ - q_-)$, $z=\frac{1}{\sqrt{\omega_0}}q_3$, $p_z=\sqrt{\omega_0}p_3$,

the leading Hamiltonian becomes

$$H_0 = \frac{1}{2}\omega_0(2S_+ - S_- + 2S_3), \tag{2.8}$$

where

$$S_\pm = \frac{1}{2}(q_\pm^2 + p_\pm^2), \qquad S_3 = \frac{1}{2}(q_3^2 + p_3^2). \tag{2.9}$$

The actions $S_\pm$ and S_3 have 2π-periodic trajectories and are in involution with each other. Thus, in the Hamiltonian, we see the resonance 2:(-1):2 between three normal modes; we called it above the EM-resonance. For this reason, the symmetry algebra of H_0 is noncommutative and is generated by actions and by additional symmetries

$$A_\rho = \frac{1}{2}(q_+ - ip_+)(q_3 + ip_3), \qquad A_\sigma = \frac{1}{2\sqrt{2}}(q_+ - ip_+)(q_- - ip_-)^2, \tag{2.10}$$

which obey the relations $|A_\rho|^2 = S_+S_3$ and $|A_\sigma|^2 = S_+S_-^2$.

Since the trajectories of H_0 are periodic, one can apply the averaging procedure [7, 8] to (2.4), i.e., one can make a canonical transformation which reduces (2.4) to a Hamiltonian

$$H_0 + \varepsilon H_{10} + \varepsilon^2 H_{20} + O(\varepsilon^3), \tag{2.11}$$

in which the perturbing terms are in involution with the leading part, i.e., the following Poisson brackets vanish: $\{H_0, H_{10}\} = \{H_0, H_{20}\} = 0$.

The explicit formula for H_{10} is obtained just by averaging H_1 along the periodic trajectories of H_0, namely,

$$H_{10} = \frac{1}{3}(2S_+ + S_-) - \frac{2\sqrt{2}}{3}(A_\rho + \overline{A_\rho}). \tag{2.12}$$

The formula for H_{20} is obtained by averaging in two steps (see, in [7, 8]):

$$\begin{aligned} H_{20} = {}& f_{++}S_+^2 + f_{--}S_-^2 + f_{33}S_3^2 + f_{+-}S_+S_- + f_{3+}S_3S_+ + f_{3-}S_3S_- \\ &+ f_-S_- + f_3S_3 + g_\rho(A_\rho + \overline{A_\rho}) + g_\sigma(A_\sigma + \overline{A_\sigma}), \end{aligned} \tag{2.13}$$

where

$$\begin{gathered} f_{++} = f_{--} = \frac{6}{\omega_0^4}(\omega_0^2\gamma - 3\beta^2),\ f_{33} = \frac{3}{4\omega_0^4}(2\omega_0^2\gamma - 5\beta^2),\ f_{+-} = -2f_{3+} = -2f_{3-} = \frac{12}{\omega_0^4}(2\omega_0^2\gamma - 7\beta^2), \\ f_- = -\frac{28}{27\omega_0}, \qquad f_3 = -\frac{20}{27\omega_0}, \qquad g_\rho = \frac{4\sqrt{2}}{9\omega_0}, \qquad g_\sigma = -\frac{16\beta}{3\omega_0^{5/2}}. \end{gathered} \tag{2.14}$$

The Hamiltonian (2.12) can be represented as the sum of two summands,

$$H_{10} = \frac{1}{3}\Big(6S - \frac{2}{\omega_0}H_0\Big), \tag{2.15}$$

where we introduce the new quadratic action function

$$S = \frac{1}{3}[2S_+ + S_3 - \sqrt{2}(A_\rho + \overline{A_\rho})]. \tag{2.16}$$

The summands S and $(2/\omega_0)H_0$ in (2.15) are in involution with each other and have 2π-periodic trajectories. Therefore, in (2.15), we obtain the resonance 6:(-1), which was called above the "angular resonance." Because of the resonance, the symmetry algebra of H_{10} is noncommutative. It is generated by the additional symmetries $A = S_-$ and $B = \sqrt{\frac{2}{3}}(A_\sigma - \sqrt{2}i\{\overline{A_\rho}, A_\sigma\})$ which obey the relation

$$|B|^2 = A^2(A+d), \tag{2.17}$$

where

$$d \stackrel{\text{def}}{=} \frac{4}{3\omega_0} H_0 - H_{10} = \frac{2}{\omega_0} H_0 - 2S.$$

The commutation relations in the symmetry algebra are as follows:

$$\{A, B\} = 2iB, \qquad \{\overline{B}, B\} = i(6A^2 + 4dA). \tag{2.18}$$

Since the trajectories of H_{10} are periodic, one can apply the averaging procedure to (2.11), i.e., make a canonical transformation reducing (2.11) to a Hamiltonian

$$H_0 + \varepsilon H_{10} + \varepsilon^2 H_{200} + O(\varepsilon^3), \tag{2.19}$$

where the second perturbing term is in involution with both H_0 and H_{10},

$$\{H_0, H_{200}\} = \{H_{10}, H_{200}\} = 0. \tag{2.20}$$

An explicit formula for H_{200} is obtained just by averaging H_{20} along the periodic trajectories of H_{10}, namely,

$$H_{200} = \lambda\Big(aA^2 + bA + c - \frac{1}{2}(B + \overline{B})\Big), \tag{2.21}$$

where

$$\lambda = 16\beta\sqrt{\frac{2}{27\omega_0^5}}, \qquad a = \frac{298\omega_0^2\gamma - 881\beta^2}{48\omega_0^4\,\lambda} \neq 0,$$

and b, c are functions in H_0 and S,

$$b = \frac{1}{216\omega_0^5\lambda}\Big[1260\omega_0(2\omega_0^2\gamma - 7\beta^2)S + 18(10\omega_0^2\gamma - 17\beta^2)H_0 - 640\omega_0^4 - 234\omega_0^3\gamma + 765\omega_0\beta^2\Big],$$

$$c = \frac{1}{108\omega_0^6\lambda}\Big[9\omega_0^2(26\omega_0^2\gamma - 85\beta^2)S^2 - 36\omega_0(2\omega_0^2\gamma - 7\beta^2)SH_0$$
$$+ 9(10\omega_0^2\gamma - 17\beta^2)H_0^2 - 2\omega_0^2\big(152\omega_0^3 - 234\omega_0^2\gamma + 765\beta^2\big)S + \omega_0\big(32\omega_0^3 - 234\omega_0^2\gamma + 765\beta^2\big)H_0\Big].$$

Thus, by a canonical transformation, it is possible to reduce the original trap Hamiltonian (2.4) to the form (2.19) up to $O(\varepsilon^3)$. The accuracy can be made $O(\varepsilon^N)$ with any $N \geqslant 3$, but this refinement will not introduce new critical (topological) feature into the dynamics. The character of the trajectories is completely determined by the averaged cubic+quartic anharmonic part H_{200} of the Hamiltonian.

Note that H_{200} is a function of fourth, third, and second orders with respect to the coordinates x, y, z, p_x, p_y, p_z on the phase space. This Hamiltonian has two quadratic integrals of motion: H_0 and H_{10}. Thus, the dynamics of the trap is reduced up to $O(\varepsilon^3)$ to the common algebra of symmetries of H_0 and H_{10}. Therefore, the number of degrees of freedom is reduced from 3 to 1. Now one must study the system with quadratic top-like Hamiltonian (2.21) in the 3D space with coordinates A, B and quadratic Poisson brackets (2.18) possessing the 2D symplectic leaves (2.17).

3. EQUILIBRIUM POINTS OF TOP DYNAMICS

Let us introduce the real and imaginary parts of the complex coordinate B, namely, $\mathcal{Y}_1 = \frac{1}{2}(B + \overline{B})$ and $\mathcal{Y}_2 = \frac{1}{2i}(B - \overline{B})$. Then the Poisson brackets (2.18) read

$$\{A, \mathcal{Y}_1\} = -2\mathcal{Y}_2, \qquad \{A, \mathcal{Y}_2\} = 2\mathcal{Y}_1, \qquad \{\mathcal{Y}_1, \mathcal{Y}_2\} = 3A^2 + 2dA. \tag{3.1}$$

The symplectic leaves (2.17) are given by

$$\mathcal{Y}_1^2 + \mathcal{Y}_2^2 = A^2(A+d). \tag{3.2}$$

Note that, in our realization of the Poisson algebra (3.1), there are restrictions

$$A > 0, \qquad A \geqslant -d. \tag{3.3}$$

The Hamiltonian (2.21) has the form $H_{200} = \lambda\mathcal{H}$, where

$$\mathcal{H} = aA^2 + bA + c - \mathcal{Y}_1. \tag{3.4}$$

The Hamiltonian system corresponding to (3.4), (3.1) reads

$$\begin{cases} \dot{A} = -2\mathcal{Y}_2, \\ \dot{\mathcal{Y}}_1 = -2\mathcal{Y}_2(2aA + b), \\ \dot{\mathcal{Y}}_2 = 2(2aA + b)\mathcal{Y}_1 - A(3A + 2d). \end{cases} \tag{3.5}$$

Here by the upper dot we denote the "time" derivative and include the normalization constant λ into the time variable.

Now our problem is to study the equilibrium points of (3.5) on the surface (3.2).

The coordinates $A = A_0$, $\mathcal{Y}_1 = \mathcal{Y}_{10}$, $\mathcal{Y}_2 = \mathcal{Y}_{20}$ of equilibrium points are given by the equations

$$\begin{cases} \mathcal{Y}_{20} = 0, \\ 2(2aA_0 + b)\mathcal{Y}_{10} = A_0(3A_0 + 2d). \end{cases}$$

By extracting $\mathcal{Y}_{10}$ from (3.2) and from (3.3), we obtain

$$\begin{cases} 2\varkappa(2aA_0 + b)\sqrt{A_0 + d} = 3A_0 + 2d, \\ A_0 > 0, \qquad A_0 \geqslant -d, \end{cases} \tag{3.6}$$

where $\varkappa = \operatorname{sgn} \mathcal{Y}_{10} \in \{\pm 1\}$.

One must find all solutions $\varkappa$ and A_0 of (3.6). Then the coordinates of equilibrium points read

$$A_0, \qquad \mathcal{Y}_{10} = \varkappa A_0 \sqrt{A_0 + d}, \qquad \mathcal{Y}_{20} = 0 \tag{3.7}$$

and the corresponding energy levels $\mathcal{H} = \mathcal{H}_0$ are

$$\mathcal{H}_0 = aA_0^2 + bA_0 + c - \varkappa A_0 \sqrt{A_0 + d}. \tag{3.8}$$

The Hess matrix of $\mathcal{H}$ at the equilibrium point (e.p.) is given by

$$\frac{\partial^2 \mathcal{H}}{\partial(A, \mathcal{Y}_1)}\bigg|_{\text{e.p.}} = \begin{pmatrix} F'_\varkappa(A_0) & 0 \\ 0 & \frac{\varkappa}{A_0\sqrt{A_0+d}} \end{pmatrix},$$

where we denote by $F_\varkappa(A_0)$ the first derivative of the energy (3.8),

$$F_\varkappa(A_0) \stackrel{\text{def}}{=} \frac{\partial \mathcal{H}_0}{\partial A_0}. \tag{3.9}$$

Since $A_0 > 0$, it follows that the stability of the equilibrium point is equivalent to the positive sign of $\varkappa F'_\varkappa(A_0)$.

The Hessian at the equilibrium point reads

$$\operatorname{Hess}(\mathcal{H})\bigg|_{\text{e.p.}} = \frac{\varkappa F'_\varkappa(A_0)}{A_0\sqrt{A_0 + d}}. \tag{3.10}$$

In order to find A_0 and $\varkappa$, one needs to solve (3.6). Note that (3.6) can be written as

$$F_\varkappa(A_0) = 0. \tag{3.11}$$

Let us make the change of variable from A_0 to y by the formula

$$\sqrt{A_0 + d} = \frac{y + \frac{1}{2}}{2\varkappa a}, \qquad \text{where} \quad \varkappa a\Big(y + \frac{1}{2}\Big) \geqslant 0. \tag{3.12}$$

Then one computes

$$F_\varkappa(A_0) = \frac{G(y)}{a(2y + 1)}, \qquad \varkappa F'_\varkappa(A_0) = \frac{4\varkappa a G'(y)}{(2y + 1)^2}, \tag{3.13}$$

where $G(y) = y^3 + Py + Q$ and

$$P \stackrel{\text{def}}{=} 2ab - 4a^2 d - \frac{3}{4}, \qquad Q \stackrel{\text{def}}{=} ab - \frac{1}{4}. \tag{3.14}$$

Thus equation (3.6) or (3.11) is reduced to the cubic equation

$$y^3 + Py + Q = 0. \tag{3.15a}$$

One needs to find real roots of this equation which ensure the positivity condition $A_0 > 0$, i.e.,

$$\Big(y + \frac{1}{2}\Big)^2 > 2Q - P - \frac{1}{4}. \tag{3.15b}$$

And then, for each of such roots, one can choose the sgn $\varkappa = \pm 1$ from inequality (3.12),

$$\varkappa a\Big(y + \frac{1}{2}\Big) \geqslant 0. \tag{3.15c}$$

Proposition 3.1. *Let* y *and* $\varkappa$ *be found from* (3.15a,b,c). *Then the coordinates of the equilibrium point of system* (3.5) *and the corresponding value of the energy are given by the substitution of* $\varkappa$ *and*

$$A_0 = \frac{1}{4a^2}\Big(y + \frac{1}{2}\Big)^2 - d \tag{3.16}$$

into (3.7), (3.8). *The stability or instability of these equilibrium points is equivalent to the positivity or negativity of the quantity* $\varkappa a(3y^2 + P)$. *The Hessian at the equilibrium point is determined by* (3.10), (3.13).

Now by using this proposition and by studying system (3.15a,b,c), one can describe in detail the equilibrium points of system (3.5) and their stability/instability for all possible values of external parameters P, Q (3.14).

ACKNOWLEDGMENTS

The research of M. V. Karasev was partially supported by the Russian Foundation for Basic Research (grant No. 15-01-08751). The research of E. M. Novikova was supported in 2015 by the Program for Basic Research of HSE.

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