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A Banach algebra of functions of several variables of finite total variation and Lipschitzian superposition operators. I☆

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Abstract

It is shown that the space of functions of n real variables with finite total variation in the sense of Vitali, Hardy and Krause, defined on a rectangle $I_a^b \subset \mathbb{R}^n$, is a Banach algebra under the pointwise operations and Hildebrandt–Leonov's norm. This result generalizes the classical case of functions of bounded Jordan variation on an interval $I_a^b = [a, b]$ for $n = 1$ and a previous result of the author in [Monatsh. Math. 137(2) (2002) 99–114] for $n = 2$.

Keywords: Functions of several variables; Finite total variation; Banach algebra property; Superposition operator; Lipschitz condition

1. Introduction

The purpose of this paper is to characterize Lipschitzian superposition (Nemytskii) operators in the space of functions of n real variables (with arbitrary $n \in \mathbb{N}$) having finite total variation in the sense of Vitali, Hardy and Krause as a continuation of the studies in [10] for $n = 2$. All main results of this paper have been announced in [11,15].

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Let I be a nonempty set (a rectangle in $\mathbb{R}^n$ below), $\mathbb{R}^I$ be the algebra of all functions $f : I \to \mathbb{R}$ mapping I into the reals $\mathbb{R}$ equipped with the usual pointwise operations and $h : I \times \mathbb{R} \to \mathbb{R}$ be a given function. The *superposition* (*Nemytskii*) *operator* $H = H_h : \mathbb{R}^I \to \mathbb{R}^I$ generated by h is defined by

$$Hf(x) \equiv H(f)(x) = h(x, f(x)), \quad x \in I, \ f \in \mathbb{R}^I. \tag{1.1}$$

The function h is called the *generator* of H.

If $I = I_a^b = [a, b]$ is a closed interval on $\mathbb{R}$, let $\mathrm{BV}(I; \mathbb{R})$ be the subset of $\mathbb{R}^I$ of all functions f of bounded (i.e., finite) Jordan total variation

$$V_a^b(f) = \sup_{\mathscr{P}} \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken over all $m \in \mathbb{N}$ and all partitions $\mathscr{P} = \{x_i\}_{i=0}^m$ of the interval I (i.e., $a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$). Classically, it is known that $\mathrm{BV}(I; \mathbb{R})$ is a Banach algebra with respect to the norm $\|f\| = |f(a)| + V_a^b(f)$, $f \in \mathrm{BV}(I; \mathbb{R})$, and $\|f \cdot g\| \leqslant 2\,\|f\| \cdot \|g\|$ for all functions f and g of bounded variation on I. This inequality is a straightforward consequence of the following two inequalities (e.g., [30, VIII.3.3]):

$$|f|_{\mathrm{s}} \equiv \sup_{x \in [a,b]} |f(x)| \leqslant \|f\| \quad \text{and} \quad V_a^b(f \cdot g) \leqslant V_a^b(f)|g|_{\mathrm{s}} + |f|_{\mathrm{s}} V_a^b(g). \tag{1.2}$$

The Banach algebra property of $\mathrm{BV}(I; \mathbb{R})$ implies immediately that if the generator $h : I \times \mathbb{R} \to \mathbb{R}$ is given by $h(x, u) = h_1(x)u + h_0(x)$, $x \in I$, $u \in \mathbb{R}$, for some functions $h_0, h_1 \in \mathrm{BV}(I; \mathbb{R})$, then the corresponding superposition operator H maps $\mathrm{BV}(I; \mathbb{R})$ into itself and is Lipschitzian in the sense that there exists a nonnegative constant L (actually, $L = 2\,\|h_1\|$) such that

$$\|Hf_1 - Hf_2\| \leqslant L\|f_1 - f_2\| \tag{1.3}$$

for all $f_1, f_2 \in \mathrm{BV}(I; \mathbb{R})$. Of course, property (1.3) with $L < 1$ is closely connected with the solution of the functional equation $f = Hf$ via the classical Banach contraction theorem.

Conversely, Matkowski and Miś [29] proved that if the superposition operator H, generated by a function $h : I \times \mathbb{R} \to \mathbb{R}$, maps $\mathrm{BV}(I; \mathbb{R})$ into itself and is Lipschitzian in the sense of (1.3), then its generator h satisfies the condition (*Matkowski's representation*):

$$h^*(x, u) = h_1(x)u + h_0(x) \quad \text{for all } x \in (a, b] \text{ and } u \in \mathbb{R}, \tag{1.4}$$

where $h^*(x, u) = \lim_{y \to x-0} h(y, u)$ is the left regularization of h in the first variable and functions $h_0, h_1 \in \mathrm{BV}(I; \mathbb{R})$ are continuous from the left on $(a, b]$.

Note that the representation (1.4) is not the case for Lipschitzian superposition operators, e.g., in the space of continuous functions $C(I; \mathbb{R})$ with the supremum norm $|\cdot|_{\mathrm{s}}$ or in the space $L^p(I; \mathbb{R})$ of Lebesgue p-summable ($p \geqslant 1$) functions on I with the standard norm: as an example, consider $h(x, u) = \sin u$, $x \in I$, $u \in \mathbb{R}$. On the other hand, in the case of functions of one variable (1.4) holds in a large number of functional spaces involving certain types of bounded variation property as is shown in [6,7,9,12,16,26] (representation (1.4) was found in [26] in the space of Lipschitz functions on I), [27,28] and others; see the references in these papers.

In [8,10], the author showed that the Banach algebra property and representation (1.4) are valid for a certain space of functions of two variables with finite total variation (cf. also [13] for the case of functions with more general values than the real numbers). It is the aim of this paper to obtain the Banach algebra property and representation (1.4) for a space of functions of n real variables with finite variation when $n \in \mathbb{N}$ is arbitrary (see Section 2). For this, we establish and make use of exact equalities for mixed differences and variations in the multiindex notation and exact estimates for them when appropriate (not at all employing arguments involving induction). We adopt and restrict ourselves to the definition of bounded variation in the multidimensional case originally due to Vitali, Hardy and Krause ([1,18,20], [21, Section III.4], [31,32]), which was redefined by Leonov [25] to get the notions of total variation and norm (for functions of two variables these two notions were employed in [21, III.6.3] and [22]). Other definitions of bounded variation for functions of several variables may be found, e.g., in [2,23,33,34].

Although the superposition operator H is well studied in many classical functional spaces (ideal, Lebesgue, Orlicz, Hölder, Sobolev, etc., cf. [3]), little is known about its properties in spaces of functions of bounded variation even of one variable (except for certain partial results [3, Section 6.5], [5,17,19,24]). So, properties of boundedness, continuity, compactness, local Lipschitz continuity, differentiability, ..., of H are yet to be studied, and our results are the first step into the general theory of superposition operators in a bounded variation context for functions of several real variables.

The paper is divided into two Parts, I and II. Part I is organized as follows. In Section 2, we introduce definitions and notations, present our main results (Theorems 1 and 2) and briefly comment on the main results of Part II (Theorem 4). Section 3 and the first part of Section 4 (on mixed differences) are preparatory for the proof of the main results. In Section 5, we give some generalizations of our results when functions under consideration have their values in normed linear spaces.

2. Definitions and main results

Let $\mathbb{N}$ be the set of positive integers, $n \in \mathbb{N}$ be fixed and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. The coordinate representation of a point $x \in \mathbb{R}^n$ will be written as $x = (x_i \mid i \in \{1, \dots, n\}) = (x_1, \dots, x_n)$. If $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we say that $x = y$, $x \leqslant y$, $y \geqslant x$ or $x < y$ (in $\mathbb{R}^n$) provided $x_i = y_i$, $x_i \leqslant y_i$, $y_i \geqslant x_i$ or $x_i < y_i$ for all $i \in \{1, \dots, n\}$, respectively, and we set $x + y = (x_1 + y_1, \dots, x_n + y_n)$ and $x - y = (x_1 - y_1, \dots, x_n - y_n)$. If $x \leqslant y$ in $\mathbb{R}^n$, we define the n-dimensional rectangle I_x^y (possibly degenerated) with the endpoints x and y as the Cartesian product of n closed intervals:

$$I_x^y = \prod_{i=1}^{n} [x_i, y_i] = [x_1, y_1] \times \cdots \times [x_n, y_n] = \{z \in \mathbb{R}^n \mid x \leqslant z \leqslant y\}.$$

In what follows $a, b \in \mathbb{R}^n$ with $a < b$ are fixed and the rectangle I_a^b is the domain of most functions under consideration, called the *basic rectangle*.

Elements of $\mathbb{N}_0^n = (\mathbb{N}_0)^n$ are said to be *multiindices* and will be denoted, as a rule, by Greek letters. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$ we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ (the order of α) and $\alpha x = (\alpha_1 x_1, \dots, \alpha_n x_n) \in \mathbb{R}^n$. The zero multiindex $0_n = (0, \dots, 0) \in \mathbb{N}_0^n$ and the unit multiindex $1_n = (1, \dots, 1) \in \mathbb{N}_0^n$ will be denoted simply by 0 and 1, respectively; each time the dimension of the zero/unit multiindex will be clear from the context. Setting $\mathscr{A}_0(n)=\{\alpha \in \mathbb{N}_0^n \mid \alpha \leqslant 1\}$ and $\mathscr{A}(n)=\mathscr{A}_0(n)\setminus\{0\}$, we have $\#\mathscr{A}_0(n)=2^n$ and $\#\mathscr{A}(n)=2^n-1$ where $\#\mathscr{A}$ designates the number of elements in the set $\mathscr{A}$.

For the sake of brevity we adopt the following *convention*: unless otherwise stated, a summation over multiindices will be understood over *n-dimensional* multiindices, the range of the summation being usually specified under the summation sign. For instance, the sum $\sum_{\alpha\in\mathscr{A}(n)}$ will be written as $\sum_{0\neq\alpha\leqslant 1}$.

Let I_a^b be the basic rectangle, $f : I_a^b \to \mathbb{R}$ and $x, y \in \mathbb{R}^n$, $x < y$. The *Vitali nth mixed difference of f* on the subrectangle $I_x^y \subset I_a^b$ is the quantity [32]:

$$\mathrm{md}_n(f, I_x^y) = \sum_{0\leqslant\theta\leqslant 1} (-1)^{|\theta|} f(x + \theta(y - x)). \tag{2.1}$$

For example, if $n = 1$, we have $\mathrm{md}_1(f, I_x^y) = f(x) - f(y)$; for $n = 2$ we find $\mathscr{A}_0(2) = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$, and so,

$$\mathrm{md}_2(f, I_{x_1,x_2}^{y_1,y_2}) = f(x_1, x_2) - f(y_1, x_2) - f(x_1, y_2) + f(y_1, y_2);$$

and if $n = 3$, then $\mathscr{A}_0(3) = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$, which implies

$$\begin{aligned}\mathrm{md}_3(f, I_{x_1,x_2,x_3}^{y_1,y_2,y_3}) = {} & f(x_1, x_2, x_3) - f(y_1, x_2, x_3) - f(x_1, y_2, x_3) - f(x_1, x_2, y_3)\\ & + f(y_1, y_2, x_3) + f(y_1, x_2, y_3) + f(x_1, y_2, y_3) - f(y_1, y_2, y_3).\end{aligned}$$

We say that $\mathscr{P}$ is a (*net*) *partition* of I_a^b if there exist a multiindex $\kappa=(\kappa_1, \dots, \kappa_n) \in \mathbb{N}^n$ and a collection of points $x[\sigma] \equiv (x_1(\sigma_1), \dots, x_n(\sigma_n))$ from I_a^b indexed by $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{N}_0^n$ with $\sigma \leqslant \kappa$ and satisfying $x[0] = a$, $x[\kappa] = b$ and $x[\sigma - 1] < x[\sigma]$ in $\mathbb{R}^n$ for all $\sigma \in \mathbb{N}^n$, $1 \leqslant \sigma \leqslant \kappa$, such that $\mathscr{P} = \{x[\sigma] \mid \sigma \in \mathbb{N}_0^n,\ \sigma \leqslant \kappa\}$ (in other words, $\mathscr{P}$ is the Cartesian product of ordinary partitions of intervals $[a_i, b_i]$, $i = 1, \dots, n$). We will denote such a partition by $\mathscr{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$. Note that I_a^b is the union of nonoverlapping subrectangles $I_{x[\sigma-1]}^{x[\sigma]}$ over all $1 \leqslant \sigma \leqslant \kappa$.

The *Vitali nth variation* [25,32] of $f : I_a^b \to \mathbb{R}$ is defined by

$$V_n(f, I_a^b) = \sup_{\mathscr{P}} \sum_{1\leqslant\sigma\leqslant\kappa} |\mathrm{md}_n(f, I_{x[\sigma-1]}^{x[\sigma]})|, \tag{2.2}$$

where the supremum is taken over all multiindices $\kappa \in \mathbb{N}^n$ and all partitions $\mathscr{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ of I_a^b. If $n = 1$, $V_1(f, I_a^b)$ is the usual Jordan variation $V_a^b(f)$.

In order to define the notion of the total variation of *f* we need the notion of lower order (less than *n*) variation of *f*, which is initially due to Hardy and Krause [18,20,25]. Given $\alpha \in \mathscr{A}(n)$ and $x \in \mathbb{R}^n$, we define the *truncation of x by* α by

$$x\lfloor\alpha = (x_i \mid i \in \{1, \dots, n\}, \alpha_i = 1) \in \mathbb{R}^{|\alpha|}$$

and set

$$I_a^b\lfloor\alpha = I_{a\lfloor\alpha}^{b\lfloor\alpha} = \prod_{i\in\{1,\dots,n\},\alpha_i=1}[a_i, b_i].$$

Clearly, $x\lfloor 1 = x$, $I_a^b\lfloor 1 = I_a^b$, and if $x \in I_a^b$, then $x\lfloor\alpha \in I_a^b\lfloor\alpha$ for all $\alpha \in \mathscr{A}(n)$. For example, if $x = (x_1, x_2, x_3, x_4)$ and $\alpha = (0, 1, 0, 1)$, we have $x\lfloor\alpha = (x_2, x_4)$ and $I_a^b\lfloor\alpha = [a_2, b_2] \times [a_4, b_4]$.

If $f : I_a^b \to \mathbb{R}$, $z \in I_a^b$ and $\alpha \in \mathscr{A}(n)$, we define the *truncated function* $f_\alpha^z : I_a^b\lfloor\alpha \to \mathbb{R}$ *with the base at z* as follows:

$$f_\alpha^z(x\lfloor\alpha) = f(z + \alpha(x - z)), \quad x \in I_a^b. \tag{2.3}$$

The idea behind this is that f_α^z depends only on $|\alpha|$ variables $x_i \in [a_i, b_i]$ (i.e., for which $\alpha_i = 1$), and the other variables are fixed and equal to z_i (if $\alpha_i = 0$). For example, if $\alpha = (0, 1, 0, 1)$ and $x = (x_1, x_2, x_3, x_4)$, then $f_\alpha^z(x_2, x_4) = f_\alpha^z(x\lfloor\alpha) = f(z_1, x_2, z_3, x_4)$. It is clear that $f_1^z = f$ on I_a^b for any $z \in I_a^b$.

Now, given $f : I_a^b \to \mathbb{R}$ and $\alpha \in \mathscr{A}(n)$, the function $f_\alpha^a : I_a^b\lfloor\alpha \to \mathbb{R}$ with the base at $z = a$ depends only on $|\alpha|$ variables, and so, making use of the above definition (2.2) (and (2.1)) with n replaced by $|\alpha|$, f replaced by f_α^a and I_a^b replaced by $I_a^b\lfloor\alpha$, we get the notion of the (Hardy–Krause) $|\alpha|$-*variation of f*, which we denote by $V_{|\alpha|}(f_\alpha^a, I_a^b\lfloor\alpha)$.

The *total variation* of $f : I_a^b \to \mathbb{R}$ in the sense of Hildebrandt [21, III.6.3] and Leonov [25] (see also [10,11]) is defined by

$$\mathrm{TV}(f, I_a^b) = \sum_{0\neq\alpha\leqslant 1} V_{|\alpha|}(f_\alpha^a, I_a^b\lfloor\alpha). \tag{2.4}$$

For the first three dimensions $n = 1, 2, 3$ we have, respectively,

$$\mathrm{TV}(f, I_a^b) = V_a^b(f), \quad \text{the usual Jordan total variation,}$$

$$\mathrm{TV}(f, I_a^b) = V_{a_1}^{b_1}(f(\cdot, a_2)) + V_{a_2}^{b_2}(f(a_1, \cdot)) + V_2(f, I_a^b),$$

$$\begin{aligned}\mathrm{TV}(f, I_a^b) = {} & V_{a_1}^{b_1}(f(\cdot, a_2, a_3)) + V_{a_2}^{b_2}(f(a_1, \cdot, a_3)) + V_{a_3}^{b_3}(f(a_1, a_2, \cdot)) \\ & + V_2(f(\cdot, \cdot, a_3), I_{a_1,a_2}^{b_1,b_2}) + V_2(f(\cdot, a_2, \cdot), I_{a_1,a_3}^{b_1,b_3}) \\ & + V_2(f(a_1, \cdot, \cdot), I_{a_2,a_3}^{b_2,b_3}) + V_3(f, I_a^b).\end{aligned}$$

We define the *space of functions of finite total variation* (in the sense of Vitali, Hardy and Krause) as $\mathrm{BV}(I_a^b; \mathbb{R}) = \{f : I_a^b \to \mathbb{R} \mid \mathrm{TV}(f, I_a^b) < \infty\}$ and equip it with the norm ([21, III.6.3] if $n = 2$, [25] for all $n \in \mathbb{N}$):

$$\|f\| = |f(a)| + \mathrm{TV}(f, I_a^b), \quad f \in \mathrm{BV}(I_a^b; \mathbb{R}). \tag{2.5}$$

It was shown by Hildebrandt [21, III.4.2] (for $n = 2$) and Leonov [25, Corollary 4] (for $n \in \mathbb{N}$) that the space $\mathrm{BV}(I_a^b; \mathbb{R})$ above coincides with the set of all functions of finite variation in the modification of Hardy and Krause (cf. [1,18,20]). Also, in [21, III.6.3] (for $n = 2$) and [25, Corollary 2] (for arbitrary n) it was proved that $\mathrm{BV}(I_a^b; \mathbb{R})$ is a Banach space with the norm (2.5). The two notions (2.4) and (2.5) were effectively used in [4, Theorem 2], [21, III.6.5], [22, Theorem 3.2] (for $n=2$), [25, Theorem 4] and [14] (if $n \in \mathbb{N}$) in order to obtain a pointwise Helly's selection principle for functions from $\mathrm{BV}(I_a^b;\mathbb{R})$ (even for more general values of functions than the real numbers).

The main result of Part I of this paper, to be proved in Section 4, is a theorem, generalizing Theorem 1 from [10] for $n=2$:

Theorem 1. *The space* $\mathrm{BV}(I_a^b;\mathbb{R})$ *is a Banach algebra with respect to the usual pointwise operations and norm* (2.5), *and the following inequality holds*:

$$\|f\cdot g\| \leqslant 2^n \|f\|\cdot\|g\|, \qquad f,g\in \mathrm{BV}(I_a^b;\mathbb{R}).$$

This theorem is based on the following fundamental estimate for the variation of any order of the product fg in terms of appropriate variations of f and g separately (for the sake of convenience we set $V_0(f_0^a, I_a^b\lfloor 0)\equiv |f(a)|$):

Theorem 2. *If* $f,g\in \mathrm{BV}(I_a^b;\mathbb{R})$ *and* $\gamma\in\mathscr{A}(n)$, *then*

$$V_{|\gamma|}((f\cdot g)_\gamma^a, I_a^b\lfloor\gamma) \leqslant \sum_{\substack{0\leqslant\alpha,\beta\leqslant\gamma,\\ \alpha+\beta\geqslant\gamma}} 2^{|\alpha|+|\beta|-|\gamma|}\, V_{|\alpha|}(f_\alpha^a, I_a^b\lfloor\alpha) V_{|\beta|}(g_\beta^a, I_a^b\lfloor\beta). \tag{2.6}$$

As a corollary of Theorem 1 for superposition operators we have:

Corollary 3. *If* $h_0,h_1\in \mathrm{BV}(I_a^b;\mathbb{R})$ *and* $h(x,u)=h_1(x)u+h_0(x)$, $x\in I_a^b$, $u\in\mathbb{R}$, *then the superposition operator H generated by h maps* $\mathrm{BV}(I_a^b;\mathbb{R})$ *into itself and is Lipschitzian*: $\|Hf-Hg\|\leqslant 2^n\|h_1\|\cdot\|f-g\|$ *for all* $f,\ g\in \mathrm{BV}(I_a^b;\mathbb{R})$.

More corollaries of the main result are presented in Sections 4 and 5.

Now we briefly comment on the main results of Part II. It turns out that Corollary 3 almost characterizes Lipschitzian superposition operators H on the space $\mathrm{BV}(I_a^b;\mathbb{R})$. Essentially, this may be stated as follows (see the complete formulation in Part II, which is to appear in this journal):

Theorem 4. *Let* $H:\mathbb{R}^I\to\mathbb{R}^I$ *be the superposition operator generated by a function* $h: I\times\mathbb{R}\to\mathbb{R}$ *according to* (1.1) *with* $I=I_a^b$. *If H maps* $\mathrm{BV}(I_a^b;\mathbb{R})$ *into itself and is Lipschitzian* (*in the sense of* (1.3)), *then there exist two functions* $h_0,h_1\in \mathrm{BV}(I_a^b;\mathbb{R})$ *such that the Matkowski representation holds*: $\lim_{y\to x-0} h(y,u)=h_1(x)u+h_0(x)$ *for all* $x\in I_a^b$, $x>a$, *and* $u\in\mathbb{R}$.

Taking this and Corollary 3 into account we obtain a complete characterization of Lipschitzian superposition operators generated by functions h which are left-continuous as functions $x\mapsto h(x,u)$.

The above results are extensions of the results in [29] (for $n=1$) and [10] (for $n=2$) to the case of arbitrary $n\in\mathbb{N}$.

3. Properties of mixed differences

Two fundamental properties of md_n and V_n are known, namely, *additivity*, i.e., if $\mathscr{P}=\{x[\sigma]\}_{\sigma=0}^{\kappa}$ is a partition of I_a^b and $f: I_a^b\to\mathbb{R}$, then

$$\mathrm{md}_n(f,I_a^b)=\sum_{1\leqslant\sigma\leqslant\kappa}\mathrm{md}_n(f,I_{x[\sigma-1]}^{x[\sigma]})\quad\text{and}\quad V_n(f,I_a^b)=\sum_{1\leqslant\sigma\leqslant\kappa}V_n(f,I_{x[\sigma-1]}^{x[\sigma]});$$

and (*sequential*) *lower semicontinuity*, i.e., if a sequence of functions $f_j: I_a^b\to\mathbb{R}$, $j\in\mathbb{N}$, converges pointwise on I_a^b to a function $f: I_a^b\to\mathbb{R}$ as $j\to\infty$, then

$$V_n(f,I_a^b)\leqslant\liminf_{j\to\infty}V_n(f_j,I_a^b).$$

Consequently, these two properties are also valid for the $|\alpha|$-variation $V_{|\alpha|}$ for each $\alpha\in\mathscr{A}(n)$ (in their formulation one should take into account obvious modifications), and the second one is valid for the total variation TV.

In order to get more properties of the total variation, we need a number of relations between mixed differences of all orders. First, let us explicitly calculate the lower order mixed difference of a function $f: I_a^b\to\mathbb{R}$ at any base $z\in I_a^b$.

Lemma 5. *If $f: I_a^b\to\mathbb{R}$, $x,y\in I_a^b$, $x\leqslant y$, $z\in I_a^b$ and $\alpha\in\mathscr{A}(n)$, then*

$$\mathrm{md}_{|\alpha|}(f_\alpha^z,I_x^y\lfloor\alpha)=\sum_{0\leqslant\theta\leqslant\alpha}(-1)^{|\theta|}f(z+\alpha(x-z)+\theta(y-x)).\tag{3.1}$$

In particular, if $z=a$ or $z=x$, we have, respectively,

$$\mathrm{md}_{|\alpha|}(f_\alpha^a,I_x^y\lfloor\alpha)=\mathrm{md}_{|\alpha|}(f_\alpha^{a+\alpha(x-a)},I_{a+\alpha(x-a)}^y\lfloor\alpha),\tag{3.2}$$

$$\mathrm{md}_{|\alpha|}(f_\alpha^x,I_x^y\lfloor\alpha)=\sum_{0\leqslant\theta\leqslant\alpha}(-1)^{|\theta|}f(x+\theta(y-x)).\tag{3.3}$$

Proof. To prove (3.1), we observe that an $\eta\in\mathscr{A}_0(|\alpha|)$ if and only if there exists a unique $\theta\in\mathscr{A}_0(n)$ such that $\theta\leqslant\alpha$ and $\eta=\theta\lfloor\alpha$, in which case $|\eta|=|\theta|\leqslant|\alpha|$; in fact, if $\eta=(\eta_1,\dots,\eta_{|\alpha|})$ and $i\in\{1,\dots,n\}$, we set $\theta_i=0$ if $\alpha_i=0$ and $\theta_i=\eta_j$ if α_i is the jth unit in α, $j=1,\dots,|\alpha|$. Taking into account that

$$x\lfloor\alpha+\eta(y\lfloor\alpha-x\lfloor\alpha)=x\lfloor\alpha+(\theta\lfloor\alpha)(y\lfloor\alpha-x\lfloor\alpha)=(x+\theta(y-x))\lfloor\alpha$$

and that, by (2.3),

$$\begin{aligned}f_\alpha^z((x+\theta(y-x))\lfloor\alpha)&=f(z+\alpha(x+\theta(y-x)-z))\\&=f(z+\alpha(x-z)+(\alpha\theta)(y-x))\end{aligned}$$

and since $\mathrm{md}_{|\alpha|}(f_\alpha^z, I_x^y\lfloor\alpha) = \mathrm{md}_{|\alpha|}(f_\alpha^z, I_{x\lfloor\alpha}^{y\lfloor\alpha})$, we have according to (2.1):

$$\mathrm{md}_{|\alpha|}(f_\alpha^z, I_x^y\lfloor\alpha) = \sum_{\eta\in\mathscr{A}_0(|\alpha|)} (-1)^{|\eta|} f_\alpha^z(x\lfloor\alpha + \eta(y\lfloor\alpha - x\lfloor\alpha))$$
$$= \sum_{0\leqslant\theta\leqslant\alpha} (-1)^{|\theta|} f(z + \alpha(x-z) + \theta(y-x)),$$

where we have used the fact that $\alpha\theta = \theta$ if and only if $\theta\leqslant\alpha$.

Equality (3.3) is a consequence of (3.1) with $z = x$.

Now, (3.2) follows from (3.1) and (3.3): in fact, setting $x' = a + \alpha(x-a)$ we find for $\theta\leqslant\alpha$ (i.e., $\theta\alpha = \theta$) that $\theta(y-x') = \theta(y-x)$ and

$$\mathrm{md}_{|\alpha|}(f_\alpha^a, I_x^y\lfloor\alpha) = \sum_{0\leqslant\theta\leqslant\alpha} (-1)^{|\theta|} f(x' + \theta(y-x')) = \mathrm{md}_{|\alpha|}(f_\alpha^{x'}, I_{x'}^y\lfloor\alpha). \qquad \square$$

The next lemma is a variant of Theorem 2 (and remark following it) from [25] written in different (and easier) notation and provided with an easier proof.

Lemma 6. *Given $f : I_a^b \to \mathbb{R}$, $x, y \in I_a^b$, $x < y$, and $\gamma \in \mathscr{A}(n)$, we have*

$$f(x + \gamma(y-x)) - f(x) = \sum_{0\neq\alpha\leqslant\gamma} (-1)^{|\alpha|} \mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y\lfloor\alpha).$$

Proof. As it is seen from (3.3), the right-hand side of the equality can be written as the sum of terms of the form $c_\beta f(x + \beta(y-x))$ over all multiindices $\beta \in \mathbb{N}_0^n$ with $\beta\leqslant\gamma$. Let us calculate the factors c_β for all $0\leqslant\beta\leqslant\gamma$. By (3.3), the value $(-1)^{|\beta|} f(x+\beta(y-x))$ is contained in the mixed difference $\mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y\lfloor\alpha)$ for each $\alpha \in \mathscr{A}(n)$ with $\beta\leqslant\alpha\leqslant\gamma$. So, if $\alpha \in \mathscr{A}(n)$, $\beta\leqslant\alpha\leqslant\gamma$ and $i = |\alpha|$, then $\max\{1, |\beta|\}\leqslant i\leqslant|\gamma|$, and since (as usual we set $0! = 1$)

$$\#\{\alpha \in \mathscr{A}(n) \mid \beta\leqslant\alpha\leqslant\gamma, |\alpha| = i\} = C_{|\gamma|-|\beta|}^{i-|\beta|}, \quad \text{where } C_m^i = \frac{m!}{i!(m-i)!}, \tag{3.4}$$

then taking into account the right-hand side of the equality in lemma we get

$$c_\beta = \sum_{i=\max\{1,|\beta|\}}^{|\gamma|} (-1)^{i+|\beta|} C_{|\gamma|-|\beta|}^{i-|\beta|}.$$

Applying the binomial formula we find $c_0 = -1$, $c_\beta = 0$ if $0 < |\beta| < |\gamma|$ and $c_\gamma = 1$. $\square$

As a consequence of Lemma 6 and definition (2.4), we get the following Leonov's inequality [25, Corollary 5]:

$$|f(y) - f(x)| \leqslant \mathrm{TV}(f, I_x^y), \quad f \in \mathrm{BV}(I_a^b; \mathbb{R}), \; x, y \in I_a^b, \; x\leqslant y. \tag{3.5}$$

Since the total variation (2.4) is defined via truncated functions with the base at a, we need a formula (if it exists!) expressing $\mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y\lfloor\alpha)$ by means of mixed differences of functions f^a_β for some $\beta \in \mathscr{A}(n)$. Fortunately, such a formula, of fundamental importance for the whole subsequent material, can be given as follows:

Lemma 7. *If $f : I^b_a \to \mathbb{R}$, $x,\ y \in I^b_a$, $x < y$, and $\alpha \in \mathscr{A}(n)\backslash\{1\}$, then*

$$\mathrm{md}_{|\alpha|}(f^x_\alpha, I^y_x \lfloor \alpha) = (-1)^{|\alpha|} \sum_{\alpha \leqslant \beta \leqslant 1} (-1)^{|\beta|}\, \mathrm{md}_{|\beta|}(f^a_\beta, I^{x+\alpha(y-x)}_{a+\alpha(x-a)} \lfloor \beta). \tag{3.6}$$

Proof. We will compare the right-hand sides of equalities (3.6) and (3.3). Applying (3.1) to $\mathrm{md}_{|\beta|}(f^a_\beta, I^{x+\alpha(y-x)}_{a+\alpha(x-a)} \lfloor \beta)$ and noting that $\alpha\beta = \alpha$ for $\beta \geqslant \alpha$ we get the following expression for the right-hand side of (3.6):

$$(-1)^{|\alpha|} \sum_{\alpha \leqslant \beta \leqslant 1} (-1)^{|\beta|} \sum_{0 \leqslant \eta \leqslant \beta} (-1)^{|\eta|} f(a+(\alpha+\eta-\alpha\eta)(x-a)+(\alpha\eta)(y-x)). \tag{3.7}$$

Since $1 \in \{\beta \in \mathscr{A}(n) \mid \beta \geqslant \alpha\}$, (3.7) can be written as the sum over all $\overline{\eta} \in \mathscr{A}_0(n)$ of terms of the form $c_{\overline{\eta}} f(a + (\alpha + \overline{\eta} - \alpha\overline{\eta})(x - a) + (\alpha\overline{\eta})(y - x))$. In order to calculate constants $c_{\overline{\eta}}$, we note that if the above term is contained in (3.7), then there exist β and η satisfying $\alpha \leqslant \beta \leqslant 1$ and $0 \leqslant \eta \leqslant \beta$ such that $\alpha + \eta - \alpha\eta = \alpha + \overline{\eta} - \alpha\overline{\eta}$ and $\alpha\eta = \alpha\overline{\eta}$, whence $\eta = \overline{\eta}$. So, in the rest of the proof we write η in place of $\overline{\eta}$ and calculate c_η. We set $\mu = \max\{\alpha, \eta\} \equiv (\max\{\alpha_i, \eta_i\} \mid i \in \{1, \ldots, n\})$, so that $1 \leqslant |\mu| \leqslant n$. Since for $|\mu| \leqslant i \leqslant n$ we have (cf. also (3.4))

$$\#\{\beta \in \mathscr{A}(n) \mid \beta \geqslant \alpha, \beta \geqslant \eta, |\beta| = i\} = \#\{\beta \in \mathscr{A}(n) \mid \beta \geqslant \mu, |\beta| = i\} = C^{i-|\mu|}_{n-|\mu|},$$

it follows from (3.7) that

$$\begin{aligned} c_\eta &= (-1)^{|\alpha|+|\eta|} \sum_{i=|\mu|}^{n} (-1)^i C^{i-|\mu|}_{n-|\mu|} = (-1)^{|\alpha|+|\eta|+|\mu|} \sum_{j=0}^{n-|\mu|} (-1)^j C^j_{n-|\mu|} \\ &= \begin{cases} (-1)^{|\alpha|+|\eta|+|\mu|} & \text{if } |\mu| = n, \\ (-1)^{|\alpha|+|\eta|+|\mu|}(1-1)^{n-|\mu|} = 0 & \text{if } |\mu| < n. \end{cases} \end{aligned} \tag{3.8}$$

Let us show that $|\mu| = n$ if and only if $\mu = \max\{\alpha, \eta\} = 1$ if and only if there exists a unique

$$\theta \in \mathscr{A}_0(n) \text{ such that } \theta \leqslant \alpha \text{ and } \eta = 1 - \alpha + \theta. \tag{3.9}$$

In fact, given $\theta \leqslant \alpha$, $\eta = 1 - \alpha + \theta$ and $i \in \{1, \ldots, n\}$, we have: if $\alpha_i = 1$, then $\mu_i = 1$, and if $\alpha_i = 0$, then $\theta_i = 0$, and so, $\mu_i = \eta_i = 1$. Conversely, condition $\max\{\alpha, \eta\} = 1$ implies $\alpha_i = 1$ or $\eta_i = 1$, so that $\alpha_i + \eta_i \geqslant 1$ for all $i \in \{1, \ldots, n\}$, and it suffices to set $\theta = \alpha + \eta - 1$.

Now, let $\theta \in \mathscr{A}_0(n)$, $\theta \leqslant \alpha$ (cf. (3.3)). Setting $\eta = 1 - \alpha + \theta$, by (3.9) and (3.8), we have: $|\eta| = n - |\alpha| + |\theta|$, $|\mu| = n$, $\alpha\eta = \theta$, $\alpha + \eta - \alpha\eta = 1$ and $c_\eta = (-1)^{|\theta|}$, and so,

$$c_\eta f(a + (\alpha + \eta - \alpha\eta)(x - a) + (\alpha\eta)(y - x)) = (-1)^{|\theta|} f(x + \theta(y - x)).$$

As it is seen now from (3.9) and (3.8), $c_\eta = 0$ in the other cases. $\square$

4. Banach algebra $\mathrm{BV}(I_a^b; \mathbb{R})$

As the first step in the proof that $\mathrm{BV}(I_a^b; \mathbb{R})$ is a Banach algebra, in the next lemma we show that the nth mixed difference of the product fg of two functions f and g can be split into the sum of certain products with factors depending on f and g separately.

Lemma 8. *If $f, g : I_a^b \to \mathbb{R}$ and $x,\ y \in I_a^b$, $x < y$, then*

$$\begin{aligned} \mathrm{md}_n(fg, I_x^y) = \mathrm{md}_n(f, I_x^y)g(x) &+ f(y)\mathrm{md}_n(g, I_x^y) \\ &+ \sum_{\theta \in \mathscr{A}(n)\setminus\{1\}} (-1)^{|\theta|}[f(x+\theta(y-x)) - f(y)] \\ &\times [g(x+\theta(y-x)) - g(x)]. \end{aligned}$$

Proof. Taking into account definition (2.1), we have:

$$\begin{aligned} \mathrm{md}_n(fg, I_x^y) &= \sum_{0 \leqslant \theta \leqslant 1} (-1)^{|\theta|} f(x+\theta(y-x))g(x+\theta(y-x)) \\ &= \mathrm{md}_n(f, I_x^y)g(x) + (-1)^n f(y)[g(y) - g(x)] \\ &\quad + \sum_{\theta \in \mathscr{A}(n)\setminus\{1\}} (-1)^{|\theta|} f(x+\theta(y-x))[g(x+\theta(y-x)) - g(x)]. \end{aligned}$$

Noting that, according to the binomial formula,

$$\begin{aligned} \sum_{\theta \in \mathscr{A}(n)\setminus\{1\}} (-1)^{|\theta|} &= \sum_{i=1}^{n-1} (-1)^i C_n^i = -1 + \left(\sum_{i=0}^{n} (-1)^i C_n^i \right) - (-1)^n \\ &= -1 - (-1)^n, \end{aligned} \tag{4.1}$$

we get

$$\begin{aligned} \sum_{\theta \in \mathscr{A}(n)\setminus\{1\}} (-1)^{|\theta|}[g(x+\theta(y-x)) - g(x)] &= \sum_{\theta \in \mathscr{A}(n)\setminus\{1\}} (-1)^{|\theta|} g(x+\theta(y-x)) \\ &\quad + g(x) + (-1)^n g(x) \\ &= \mathrm{md}_n(g, I_x^y) - (-1)^n[g(y) - g(x)]. \end{aligned}$$

Writing out the expression for $(-1)^n[g(y)-g(x)]$ and substituting it into the above equality we obtain the desired equality. □

Actually, the mixed difference $\mathrm{md}_n(fg, I_x^y)$ can be developed as the sum of products of certain mixed differences of f and g separately.

Lemma 9. *If $f, g : I_a^b \to \mathbb{R}$ and $x, y \in I_a^b$, $x < y$, then*

$$\begin{aligned}\mathrm{md}_n(fg, I_x^y) &= \mathrm{md}_n(f, I_x^y)g(x) + f(y)\mathrm{md}_n(g, I_x^y)\\&\quad + (-1)^n \sum_{\theta\in\mathscr{A}(n)\setminus\{1\}} \left[\sum_{1-\theta\leqslant\alpha\leqslant 1} (-1)^{|\alpha|}\,\mathrm{md}_{|\alpha|}(f_\alpha^a, I_{a+(1-\theta)(x-a)}^y \lfloor\alpha)\right]\\&\quad \times\left[\sum_{\theta\leqslant\beta\leqslant 1} (-1)^{|\beta|}\,\mathrm{md}_{|\beta|}(g_\beta^a, I_{a+\theta(x-a)}^{x+\theta(y-x)}\lfloor\beta)\right].\end{aligned}$$

Proof. We first show that the following equality holds:

$$\begin{aligned}&\sum_{\theta\in\mathscr{A}(n)\setminus\{1\}} (-1)^{|\theta|}[f(x+\theta(y-x)) - f(y)][g(x+\theta(y-x)) - g(x)]\\&= \sum_{\theta\in\mathscr{A}(n)\setminus\{1\}} \mathrm{md}_{n-|\theta|}(f_{1-\theta}^{x+\theta(y-x)}, I_{x+\theta(y-x)}^y\lfloor(1-\theta))\mathrm{md}_{|\theta|}(g_\theta^x, I_x^y\lfloor\theta).\end{aligned} \tag{4.2}$$

It will be proved in two steps, and for the sake of clarity the right-hand side of this equality will be written with θ replaced by η:

$$\sum_{\eta\in\mathscr{A}(n)\setminus\{1\}} \mathrm{md}_{n-|\eta|}(f_{1-\eta}^{x+\eta(y-x)}, I_{x+\eta(y-x)}^y\lfloor(1-\eta))\mathrm{md}_{|\eta|}(g_\eta^x, I_x^y\lfloor\eta). \tag{4.3}$$

1. We transform the sum (4.3) by applying equality (3.3) to the mixed differences, noting that $\theta(1-\eta) = \theta$ for $\theta\leqslant 1-\eta$ and changing the summation index in the third sum below:

$$\begin{aligned}&\mathrm{md}_{n-|\eta|}(f_{1-\eta}^{x+\eta(y-x)}, I_{x+\eta(y-x)}^y\lfloor(1-\eta))\\&\quad= \sum_{0\leqslant\theta\leqslant 1-\eta} (-1)^{|\theta|} f(x+\eta(y-x)+\theta[y-x-\eta(y-x)])\\&\quad= \sum_{0\leqslant\theta\leqslant 1-\eta} (-1)^{|\theta|} f(x+(\eta+\theta)(y-x))\\&\quad= (-1)^{|\eta|}\left[(-1)^n f(y) + \sum_{\eta\leqslant\lambda\leqslant 1,\lambda\neq 1} (-1)^{|\lambda|} f(x+\lambda(y-x))\right]\end{aligned}$$

and also,

$$\mathrm{md}_{|\eta|}(g_\eta^x, I_x^y\lfloor\eta) = g(x) + \sum_{0\neq\mu\leqslant\eta} (-1)^{|\mu|} g(x+\mu(y-x))$$

and so, (4.3) is equal to

$$\left[\sum_{\eta\in\mathscr{A}(n)\setminus\{1\}}(-1)^{|\eta|+n}\right]f(y)g(x)$$
$$+f(y)\left[\sum_{\eta\in\mathscr{A}(n)\setminus\{1\}}(-1)^{|\eta|+n}\sum_{0\neq\mu\leqslant\eta}(-1)^{|\mu|}g(x+\mu(y-x))\right]$$
$$+\left[\sum_{\eta\in\mathscr{A}(n)\setminus\{1\}}(-1)^{|\eta|}\sum_{\eta\leqslant\lambda\leqslant 1,\lambda\neq 1}(-1)^{|\lambda|}f(x+\lambda(y-x))\right]g(x)$$
$$+\sum_{\eta\in\mathscr{A}(n)\setminus\{1\}}(-1)^{|\eta|}\left[\sum_{\eta\leqslant\lambda\leqslant 1,\lambda\neq 1}(-1)^{|\lambda|}f(x+\lambda(y-x))\right]$$
$$\times\left[\sum_{0\neq\mu\leqslant\eta}(-1)^{|\mu|}g(x+\mu(y-x))\right]. \tag{4.4}$$

2. In steps 2a–2d we compare the respective coefficients (in square brackets) in (4.4) and the expression on the left in (4.2) and show that they are equal.

2a. The factor by $f(y)g(x)$ on the left in (4.2) is, by virtue of (4.1), equal to $\sum_{\theta\in\mathscr{A}(n)\setminus\{1\}}(-1)^{|\theta|}=-1-(-1)^n$. The respective coefficient in (4.4) is

$$\sum_{\eta\in\mathscr{A}(n)\setminus\{1\}}(-1)^{|\eta|+n}=(-1)^n\sum_{\eta\in\mathscr{A}(n)\setminus\{1\}}(-1)^{|\eta|}=(-1)^n[-1-(-1)^n]$$
$$=-(-1)^n-1.$$

2b. The value $-\sum_{\theta\in\mathscr{A}(n)\setminus\{1\}}(-1)^{|\theta|}g(x+\theta(y-x))$ is the factor by $f(y)$ on the left in (4.2). In the respective coefficient in (4.4) $\mu\neq 0$ can assume any value $\mu\leqslant\eta$ for $\eta\in\mathscr{A}(n)\setminus\{1\}$, i.e., $\mu\in\mathscr{A}(n)\setminus\{1\}$, and so, this coefficient is of the form $\sum_{\gamma\in\mathscr{A}(n)\setminus\{1\}}c_\gamma g(x+\gamma(y-x))$. In order to calculate c_γ, we fix $\gamma\in\mathscr{A}(n)\setminus\{1\}$ and note that γ corresponds to those $\eta\in\mathscr{A}(n)\setminus\{1\}$, for which $\eta\geqslant\gamma$. So, if $\gamma\leqslant\eta\leqslant 1$, $\eta\neq 1$ and $i=|\eta|$, then $|\gamma|\leqslant i\leqslant n-1$. Since (cf. (3.4)) $\#\{\eta\in\mathscr{A}(n)\mid\gamma\leqslant\eta,\ |\eta|=i\}=C_{n-|\gamma|}^{i-|\gamma|}$, from the second term in (4.4) and the binomial formula we find

$$c_\gamma=\sum_{i=|\gamma|}^{n-1}(-1)^{i+n+|\gamma|}C_{n-|\gamma|}^{i-|\gamma|}=(-1)^n\sum_{j=0}^{n-1-|\gamma|}(-1)^jC_{n-|\gamma|}^{j}$$
$$=(-1)^n\left[\sum_{j=0}^{n-|\gamma|}(-1)^jC_{n-|\gamma|}^{j}-(-1)^{n-|\gamma|}\right]=-(-1)^n(-1)^{n-|\gamma|}=-(-1)^{|\gamma|},$$

and so, the respective coefficients are equal.

2c. We have $-\sum_{\theta\in\mathscr{A}(n)\setminus\{1\}}(-1)^{|\theta|}f(x+\theta(y-x))$ times $g(x)$ on the left in (4.2). In the respective coefficient in (4.4) $\lambda\neq 1$ can assume any value $\lambda\geqslant\eta$ for $\eta\in\mathscr{A}(n)\setminus\{1\}$, i.e.,

$\lambda \in \mathscr{A}(n)\setminus\{1\}$, and so, this coefficient is of the form $\sum_{\gamma\in\mathscr{A}(n)\setminus\{1\}} c_\gamma f(x+\gamma(y-x))$. The multiindex $\gamma \in \mathscr{A}(n)\setminus\{1\}$ corresponds to those $\eta \in \mathscr{A}(n)\setminus\{1\}$, for which $\eta \leqslant \gamma$. So, if $0 \leqslant \eta \leqslant \gamma$, $\eta \neq 0$ and $i=|\eta|$, then $1 \leqslant i \leqslant |\gamma|$. Since (again cf. (3.4)) $\#\{\eta \in \mathscr{A}(n) \mid \eta \leqslant \gamma, |\eta| = i\} = C^i_{|\gamma|}$, from the third term in (4.4) and the binomial formula we get

$$c_\gamma = \sum_{i=1}^{|\gamma|} (-1)^{i+|\gamma|} C^i_{|\gamma|} = (-1)^{|\gamma|}\left[\sum_{i=0}^{|\gamma|}(-1)^i C^i_{|\gamma|} - 1\right] = -(-1)^{|\gamma|}.$$

2d. The remaining term on the left in (4.2) is the sum over all $\theta \in \mathscr{A}(n)\setminus\{1\}$ of terms of the form $(-1)^{|\theta|} f(x+\theta(y-x))g(x+\theta(y-x))$. We will show that it is equal to the fourth term in (4.4). It is easily seen that the fourth term in (4.4) can be rewritten in the form

$$\sum_{\gamma,\delta\in\mathscr{A}(n)\setminus\{1\},\delta\leqslant\gamma} c_{\gamma\delta} f(x+\gamma(y-x))g(x+\delta(y-x)).$$

Let us fix γ, $\delta \in \mathscr{A}(n)\setminus\{1\}$, $\delta \leqslant \gamma$, and evaluate $c_{\gamma\delta}$. For this, we note that the term $f(x+\gamma(y-x))g(x+\delta(y-x))$ is contained in the fourth term of (4.4) for those $\eta \in \mathscr{A}(n)\setminus\{1\}$, for which $\delta \leqslant \eta \leqslant \gamma$. Since for $i=|\eta|$ we have $|\delta| \leqslant i \leqslant |\gamma|$ and $\#\{\eta \in \mathscr{A}(n) \mid \delta \leqslant \eta \leqslant \gamma, |\eta| = i\} = C^{i-|\delta|}_{|\gamma|-|\delta|}$, from the fourth term in (4.4) we find

$$\begin{aligned} c_{\gamma\delta} &= \sum_{i=|\delta|}^{|\gamma|} (-1)^{i+|\gamma|+|\delta|} C^{i-|\delta|}_{|\gamma|-|\delta|} = (-1)^{|\gamma|} \sum_{j=0}^{|\gamma|-|\delta|} (-1)^j C^j_{|\gamma|-|\delta|} \\ &= \begin{cases} (-1)^{|\gamma|} & \text{if } |\delta| = |\gamma| \text{ or, equivalently, } \delta=\gamma, \\ (-1)^{|\gamma|}(1-1)^{|\gamma|-|\delta|} = 0 & \text{if } |\delta| < |\gamma| \text{ or, equivalently, } \delta\neq\gamma. \end{cases} \end{aligned}$$

Thus, the proof of (4.2) is complete.

3. In order to establish the equality in Lemma, let us rewrite the mixed differences $\mathrm{md}_{n-|\theta|}(\cdots)$ and $\mathrm{md}_{|\theta|}(\cdots)$ on the right from (4.2) in accordance with Lemma 7. Setting $\alpha = 1-\theta$ and replacing x by $x+\theta(y-x)$ in Lemma 7 and noting that $|\alpha| = n-|\theta|$ and

$$I^{x+\theta(y-x)+(1-\theta)(y-x-\theta(y-x))}_{a+(1-\theta)(x+\theta(y-x)-a)} = I^y_{a+(1-\theta)(x-a)},$$

we have

$$\begin{aligned} &\mathrm{md}_{n-|\theta|}(f^{x+\theta(y-x)}_{1-\theta}, I^y_{x+\theta(y-x)} \lfloor (1-\theta)) \\ &\quad = (-1)^{n-|\theta|} \sum_{1-\theta\leqslant\alpha\leqslant 1} (-1)^{|\alpha|}\, \mathrm{md}_{|\alpha|}(f^a_\alpha, I^y_{a+(1-\theta)(x-a)} \lfloor \alpha), \end{aligned}$$

and similarly,

$$\mathrm{md}_{|\theta|}(g^x_\theta, I^y_x \lfloor \theta) = (-1)^{|\theta|} \sum_{\theta\leqslant\beta\leqslant 1} (-1)^{|\beta|}\, \mathrm{md}_{|\beta|}(g^a_\beta, I^{x+\theta(y-x)}_{a+\theta(x-a)} \lfloor \beta).$$

It remains to substitute these expressions into (4.2) and apply Lemma 8. □

Remark 10. Let N_n be the number of terms on the right in the equality in Lemma 9. Since $\#(\mathscr{A}(n)\backslash\{1\}) = 2^n - 2$, $\#\{\alpha \in \mathscr{A}(n) \mid \alpha \geqslant 1-\theta, |\alpha| = i\} = C_{n-(n-|\theta|)}^{i-(n-|\theta|)}$ for $n-|\theta| \leqslant i \leqslant n$ and $\#\{\beta \in \mathscr{A}(n) \mid \beta \geqslant \theta, |\beta| = i\} = C_{n-|\theta|}^{i-|\theta|}$ for $|\theta| \leqslant i \leqslant n$, we have:

$$N_n = 2 + (2^n - 2)\left(\sum_{i=n-|\theta|}^{n} C_{n-(n-|\theta|)}^{i-(n-|\theta|)}\right)\left(\sum_{i=|\theta|}^{n} C_{n-|\theta|}^{i-|\theta|}\right) = 2 + (2^n-2)2^{|\theta|}2^{n-|\theta|},$$

or $N_n = (2^n - 1)^2 + 1$. In particular, $N_1 = 2$, $N_2 = 10$, $N_3 = 50$, $N_4 = 226$, $N_5 = 962$, $N_6 = 3970$. If $n = 1$, the equality in Lemma 9 is well known:

$$\mathrm{md}_1(fg, I_x^y) = (fg)(x) - (fg)(y) = (f(x) - f(y))g(x) + f(y)(g(x) - g(y))$$

and it implies the second inequality in (1.2); for $n = 2$ it is explicitly written in [10] (proof of Theorem 1). We have found the equality in Lemma 9 by studying the cases $n = 2$ and 3 (obviously, the case $n = 1$ gives nothing).

Now we are in a position to prove the fundamental estimate in Theorem 2.

Proof of Theorem 2. 1. In the first step we will prove estimate (2.6) for $\gamma = 1$:

$$V_n(fg, I_a^b) \leqslant \sum_{\substack{0\leqslant\alpha,\beta\leqslant 1,\\ \alpha+\beta\geqslant 1}} 2^{|\alpha|+|\beta|-n} V_{|\alpha|}(f_\alpha^a, I_a^b\lfloor\alpha) V_{|\beta|}(g_\beta^a, I_a^b\lfloor\beta). \tag{4.5}$$

Let $\mathscr{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ be an arbitrary partition of I_a^b, so that $\kappa \in \mathbb{N}^n$, $x[0] = a$, $x[\kappa] = b$ and $x[\sigma - 1] < x[\sigma]$ for all $\sigma \in \mathbb{N}^n$, $\sigma \leqslant \kappa$. Setting $x = x[\sigma-1]$ and $y = x[\sigma]$ in Lemma 9, taking the absolute values, summing over all $1 \leqslant \sigma \leqslant \kappa$ and applying (2.5) and (3.5), we find

$$\begin{aligned}
&\sum_{1\leqslant\sigma\leqslant\kappa} |\mathrm{md}_n(fg, I_{x[\sigma-1]}^{x[\sigma]})| \\
&\quad \leqslant \sum_{1\leqslant\sigma\leqslant\kappa} |\mathrm{md}_n(f, I_{x[\sigma-1]}^{x[\sigma]})| \cdot |g(x[\sigma-1])| \\
&\qquad + \sum_{1\leqslant\sigma\leqslant\kappa} |f(x[\sigma])| \cdot |\mathrm{md}_n(g, I_{x[\sigma-1]}^{x[\sigma]})| \\
&\qquad + \sum_{1\leqslant\sigma\leqslant\kappa} \sum_{\theta\in\mathscr{A}(n)\backslash\{1\}} \left[\sum_{1-\theta\leqslant\alpha\leqslant 1} |\mathrm{md}_{|\alpha|}(f_\alpha^a, I_{a+(1-\theta)(x[\sigma-1]-a)}^{x[\sigma]}\lfloor\alpha)|\right]
\end{aligned}$$

$$
\begin{aligned}
&\quad\times\left[\sum_{\theta\leqslant\beta\leqslant 1}|\mathrm{md}_{|\beta|}(g^a_\beta, I^{x[\sigma-1]+\theta(x[\sigma]-x[\sigma-1])}_{a+\theta(x[\sigma-1]-a)}\lfloor\beta)|\right]\\
&\leqslant V_n(f,I^b_a)\|g\|+\|f\|V_n(g,I^b_a)\\
&\quad+\sum_{\theta\in\mathscr{A}(n)\setminus\{1\}}\;\sum_{1-\theta\leqslant\alpha\leqslant 1}\;\sum_{\theta\leqslant\beta\leqslant 1}\;\sum_{1\leqslant\sigma\leqslant\kappa}|\mathrm{md}_{|\alpha|}(f^a_\alpha, I^{x[\sigma]}_{a+(1-\theta)(x[\sigma-1]-a)}\lfloor\alpha)|\\
&\quad\times|\mathrm{md}_{|\beta|}(g^a_\beta, I^{x[\sigma-1]+\theta(x[\sigma]-x[\sigma-1])}_{a+\theta(x[\sigma-1]-a)}\lfloor\beta)|.
\end{aligned}\tag{4.6}
$$

Let us estimate the term $|\mathrm{md}_{|\beta|}(\cdots)|$ in (4.6). Since $I^{x+\theta(y-x)}_{a+\theta(x-a)}\subset I^{b+\theta(y-b)}_{a+\theta(x-a)}$ for $a\leqslant x<y\leqslant b$, we have

$$
I^{x[\sigma-1]+\theta(x[\sigma]-x[\sigma-1])}_{a+\theta(x[\sigma-1]-a)}\subset I^{b+\theta(x[\sigma]-b)}_{a+\theta(x[\sigma-1]-a)}
$$

and so, the definition of $V_{|\beta|}$ and the monotonicity of $V_{|\beta|}$ imply

$$
\begin{aligned}
|\mathrm{md}_{|\beta|}(g^a_\beta, I^{x[\sigma-1]+\theta(x[\sigma]-x[\sigma-1])}_{a+\theta(x[\sigma-1]-a)}\lfloor\beta)|&\leqslant V_{|\beta|}(g^a_\beta, I^{x[\sigma-1]+\theta(x[\sigma]-x[\sigma-1])}_{a+\theta(x[\sigma-1]-a)}\lfloor\beta)\\
&\leqslant V_{|\beta|}(g^a_\beta, I^{b+\theta(x[\sigma]-b)}_{a+\theta(x[\sigma-1]-a)}\lfloor\beta).
\end{aligned}
$$

As $\bigcup_{\sigma\lfloor\theta} I^{b+\theta(x[\sigma]-b)}_{a+\theta(x[\sigma-1]-a)}=I^b_a$ is the union of nonoverlapping rectangles (here the union over $\sigma\lfloor\theta$ being understood in the sense that it is taken only over those σ_i, $1\leqslant\sigma_i\leqslant\kappa_i$, for which $\theta_i=1$), the additivity property of $V_{|\beta|}$ yields:

$$
\begin{aligned}
\sum_{\sigma\lfloor\theta}|\mathrm{md}_{|\beta|}(g^a_\beta, I^{x[\sigma-1]+\theta(x[\sigma]-x[\sigma-1])}_{a+\theta(x[\sigma-1]-a)}\lfloor\beta)|&\leqslant\sum_{\sigma\lfloor\theta}V_{|\beta|}(g^a_\beta, I^{b+\theta(x[\sigma]-b)}_{a+\theta(x[\sigma-1]-a)}\lfloor\beta)\\
&=V_{|\beta|}(g^a_\beta, I^b_a\lfloor\beta).
\end{aligned}\tag{4.7}
$$

A similar estimate holds for $|\mathrm{md}_{|\alpha|}(\cdots)|$ from (4.6): we set $\overline{\theta}=1-\theta$, note that $I^y_{a+(1-\theta)(x-a)}\subset I^{b+(1-\theta)(y-b)}_{a+(1-\theta)(x-a)}$ for $a\leqslant x<y\leqslant b$ and, therefore,

$$
I^{x[\sigma]}_{a+(1-\theta)(x[\sigma-1]-a)}\subset I^{b+\overline{\theta}(x[\sigma]-b)}_{a+\overline{\theta}(x[\sigma-1]-a)}\quad\text{and}\quad\bigcup_{\sigma\lfloor\overline{\theta}} I^{b+\overline{\theta}(x[\sigma]-b)}_{a+\overline{\theta}(x[\sigma-1]-a)}=I^b_a
$$

and arguing as above we find

$$
|\mathrm{md}_{|\alpha|}(f^a_\alpha, I^{x[\sigma]}_{a+(1-\theta)(x[\sigma-1]-a)}\lfloor\alpha)|\leqslant V_{|\alpha|}(f^a_\alpha, I^{b+\overline{\theta}(x[\sigma]-b)}_{a+\overline{\theta}(x[\sigma-1]-a)}\lfloor\alpha),
$$

which after summing over $1\lfloor\overline{\theta}\leqslant\sigma\lfloor\overline{\theta}\leqslant\kappa\lfloor\overline{\theta}$ gives

$$
\sum_{\sigma\lfloor\overline{\theta}}|\mathrm{md}_{|\alpha|}(f^a_\alpha, I^{x[\sigma]}_{a+(1-\theta)(x[\sigma-1]-a)}\lfloor\alpha)|\leqslant V_{|\alpha|}(f^a_\alpha, I^b_a\lfloor\alpha).\tag{4.8}
$$

Taking into account that $\sum_{1\leqslant\sigma\leqslant\kappa}=\sum_{\sigma\lfloor\theta}\sum_{\sigma\lfloor\bar{\theta}}$, from (4.6)–(4.8) we have:

$$\begin{aligned}&\sum_{1\leqslant\sigma\leqslant\kappa}|\mathrm{md}_n(fg,I_{x[\sigma-1]}^{x[\sigma]})|\\&\quad\leqslant V_n(f,I_a^b)\|g\|+\|f\|V_n(g,I_a^b)\\&\qquad+\sum_{\theta\in\mathscr{A}(n)\setminus\{1\}}\ \sum_{1-\theta\leqslant\alpha\leqslant 1}\ \sum_{\theta\leqslant\beta\leqslant 1}V_{|\alpha|}(f_\alpha^a,I_a^b\lfloor\alpha)V_{|\beta|}(g_\beta^a,I_a^b\lfloor\beta).\end{aligned}$$

Since we have set $V_0(f_0^a,I_a^b\lfloor 0)=|f(a)|$ (cf. the text preceding Theorem 2), by (2.4) and (2.5) we have

$$\|f\|=|f(a)|+\sum_{\alpha\in\mathscr{A}(n)}V_{|\alpha|}(f_\alpha^a,I_a^b\lfloor\alpha)=\sum_{\alpha\in\mathscr{A}_0(n)}V_{|\alpha|}(f_\alpha^a,I_a^b\lfloor\alpha)\tag{4.9}$$

and so, taking the supremum over all partitions $\mathscr{P}$ of I_a^b in the last estimate, we get:

$$V_n(fg,I_a^b)\leqslant\sum_{\theta\in\mathscr{A}_0(n)}\ \sum_{1-\theta\leqslant\alpha\leqslant 1}\ \sum_{\theta\leqslant\beta\leqslant 1}V_{|\alpha|}(f_\alpha^a,I_a^b\lfloor\alpha)V_{|\beta|}(g_\beta^a,I_a^b\lfloor\beta)\tag{4.10}$$

$$=\sum_{\alpha,\beta\in\mathscr{A}_0(n),\alpha+\beta\geqslant 1}c_{\alpha\beta}V_{|\alpha|}(f_\alpha^a,I_a^b\lfloor\alpha)V_{|\beta|}(g_\beta^a,I_a^b\lfloor\beta).\tag{4.11}$$

Given $\alpha,\ \beta\in\mathscr{A}_0(n)$ with $\alpha+\beta\geqslant 1$, let us evaluate the number $c_{\alpha\beta}$. To such α and β there correspond $\theta\in\mathscr{A}_0(n)$ from (4.10) such that $1-\alpha\leqslant\theta\leqslant\beta$. Supposing $n-|\alpha|=|1-\alpha|\leqslant i\leqslant|\beta|$ and noting that (cf. (3.4))

$$\#\{\theta\in\mathscr{A}_0(n)\mid 1-\alpha\leqslant\theta\leqslant\beta,|\theta|=i\}=C_{|\beta|-(n-|\alpha|)}^{i-(n-|\alpha|)},$$

we find from (4.10) and the binomial formula

$$c_{\alpha\beta}=\sum_{i=n-|\alpha|}^{|\beta|}C_{|\beta|-(n-|\alpha|)}^{i-(n-|\alpha|)}=\sum_{j=0}^{|\beta|-(n-|\alpha|)}C_{|\beta|-(n-|\alpha|)}^{j}=2^{|\alpha|+|\beta|-n}.$$

Now, inequality (4.5) follows immediately from (4.11).

2. In order to prove (2.6), we will apply inequality (4.5). Let us fix $\gamma\in\mathscr{A}(n)$ and note that $(fg)_\gamma^a=f_\gamma^a g_\gamma^a$; in fact, given $y\in I_a^b$, by virtue of (2.3) we have:

$$\begin{aligned}(fg)_\gamma^a(y\lfloor\gamma)&=(fg)(a+\gamma(y-a))=f(a+\gamma(y-a))g(a+\gamma(y-a))\\&=f_\gamma^a(y\lfloor\gamma)g_\gamma^a(y\lfloor\gamma)=(f_\gamma^a g_\gamma^a)(y\lfloor\gamma).\end{aligned}$$

Replacing $1=1_n$ by $1\lfloor\gamma$ in (4.5), so that $|1\lfloor\gamma|=|\gamma|$, we get:

$$\begin{aligned}&V_{|\gamma|}((fg)_\gamma^a,I_a^b\lfloor\gamma)\\&\quad=V_{|1\lfloor\gamma|}(f_\gamma^a g_\gamma^a,I_{a\lfloor\gamma}^{b\lfloor\gamma})\\&\quad\leqslant\sum_{\substack{\alpha',\beta'\in\mathscr{A}_0(|\gamma|),\\ \alpha'+\beta'\geqslant 1\lfloor\gamma}}2^{|\alpha'|+|\beta'|-|1\lfloor\gamma|}V_{|\alpha'|}((f_\gamma^a)_{\alpha'}^{a\lfloor\gamma},(I_{a\lfloor\gamma}^{b\lfloor\gamma})\lfloor\alpha')V_{|\beta'|}((g_\gamma^a)_{\beta'}^{a\lfloor\gamma},(I_{a\lfloor\gamma}^{b\lfloor\gamma})\lfloor\beta').\end{aligned}$$

We employ the observation from the proof of Lemma 5: given $\alpha', \beta' \in \mathscr{A}_0(|\gamma|)$, there exists a unique pair $(\alpha, \beta) \in \mathscr{A}_0(n) \times \mathscr{A}_0(n)$ such that $\alpha \leqslant \gamma$, $\beta \leqslant \gamma$, $\alpha' = \alpha\lfloor\gamma$, $\beta' = \beta\lfloor\gamma$, $|\alpha| = |\alpha'|$ and $|\beta| = |\beta'|$; moreover, condition $\alpha' + \beta' \geqslant 1\lfloor\gamma$ implies $\alpha + \beta \geqslant \gamma$. Let us show that

$$V_{|\alpha'|}((f_\gamma^a)_{\alpha'}^{a\lfloor\gamma}, (I_{a\lfloor\gamma}^{b\lfloor\gamma})\lfloor\alpha') = V_{|\alpha|}(f_\alpha^a, I_a^b\lfloor\alpha). \tag{4.12}$$

Taking into account the equality $(x\lfloor\gamma)\lfloor(\alpha\lfloor\gamma) = x\lfloor\alpha$ for any $x \in \mathbb{R}^n$ (since $\alpha \leqslant \gamma$) and definition (2.3) we find

$$(I_{a\lfloor\gamma}^{b\lfloor\gamma})\lfloor\alpha' = (I_{a\lfloor\gamma}^{b\lfloor\gamma})\lfloor(\alpha\lfloor\gamma) = I_{(a\lfloor\gamma)\lfloor(\alpha\lfloor\gamma)}^{(b\lfloor\gamma)\lfloor(\alpha\lfloor\gamma)} = I_{a\lfloor\alpha}^{b\lfloor\alpha} = I_a^b\lfloor\alpha$$

and if $y \in I_a^b$,

$$\begin{aligned}(f_\gamma^a)_{\alpha'}^{a\lfloor\gamma}(y\lfloor\alpha) &= (f_\gamma^a)_{\alpha\lfloor\gamma}^{a\lfloor\gamma}((y\lfloor\gamma)\lfloor(\alpha\lfloor\gamma)) = f_\gamma^a(a\lfloor\gamma + (\alpha\lfloor\gamma)(y\lfloor\gamma - a\lfloor\gamma)) \\ &= f_\gamma^a([a + \alpha(y-a)]\lfloor\gamma) = f(a + \gamma(a + \alpha(y-a) - a)) \\ &= f(a + \alpha(y-a)) = f_\alpha^a(y\lfloor\alpha)\end{aligned}$$

and so, $(f_\gamma^a)_{\alpha'}^{a\lfloor\gamma} = f_\alpha^a$. In this way, we have proved (4.12) and, along with it, inequality (2.6) as well. $\square$

Proof of Theorem 1. 1. One can easily check that $\mathrm{BV}(I_a^b; \mathbb{R})$ is a linear space and the functional (2.5) is a norm on it (the property '$\|f\|=0$ implies $f=0$' is a consequence of (2.5) and (3.5)). In order to prove the completeness, let $\{f_j\}_{j=1}^\infty$ be a Cauchy sequence in $\mathrm{BV}(I_a^b; \mathbb{R})$, i.e., $\|f_j - f_k\| \to 0$ as $j, k \to \infty$. Then, by (2.5) and (3.5), $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ for all $x \in I_a^b$, and so, there exists a function $f : I_a^b \to \mathbb{R}$ such that $f_j(x) \to f(x)$ as $j \to \infty$ for all $x \in I_a^b$. From inequality $|\mathrm{TV}(f_j, I_a^b) - \mathrm{TV}(f_k, I_a^b)| \leqslant \mathrm{TV}(f_j - f_k, I_a^b)$ and the lower semicontinuity of TV we find $\mathrm{TV}(f, I_a^b) \leqslant \liminf_{j\to\infty} \mathrm{TV}(f_j, I_a^b) < \infty$, i.e., $f \in \mathrm{BV}(I_a^b; \mathbb{R})$. Again, by the lower semicontinuity of TV,

$$\mathrm{TV}(f_j - f, I_a^b) \leqslant \liminf_{k\to\infty} \mathrm{TV}(f_j - f_k, I_a^b) \leqslant \lim_{k\to\infty} \|f_j - f_k\|, \quad k \in \mathbb{N}$$

and the Cauchy property of $\{f_j\}_{j=1}^\infty$ implies

$$\limsup_{j\to\infty} \mathrm{TV}(f_j - f, I_a^b) \leqslant \lim_{j\to\infty} \lim_{k\to\infty} \|f_j - f_k\| = 0$$

so that $\|f_j - f\| \to 0$ as $j \to \infty$.

2. In order to prove the inequality in Theorem, given $\gamma \in \mathscr{A}_0(n)$, we set $\mathscr{B}(\gamma) = \{(\alpha, \beta) \in \mathscr{A}_0(n) \times \mathscr{A}_0(n) \mid \alpha \leqslant \gamma, \beta \leqslant \gamma, \alpha + \beta \geqslant \gamma\}$ and note that $\bigcup_{\gamma \in \mathscr{A}_0(n)} \mathscr{B}(\gamma) = \mathscr{A}_0(n) \times \mathscr{A}_0(n) \equiv \mathscr{A}_0(n)^2$ and $\mathscr{B}(\gamma') \cap \mathscr{B}(\gamma'') = \emptyset$ if $\gamma' \neq \gamma''$. In fact, if $(\alpha, \beta) \in \mathscr{A}_0(n)^2$, then $(\alpha, \beta) \in \mathscr{B}(\gamma)$ with $\gamma = \max\{\alpha, \beta\} = \alpha + \beta - \alpha\beta$. Also, if $(\alpha, \beta) \in \mathscr{B}(\gamma') \cap \mathscr{B}(\gamma'')$, then $\max\{\alpha, \beta\} \leqslant \min\{\gamma', \gamma''\}$ and $\alpha + \beta \geqslant \max\{\gamma', \gamma''\}$, whence $\gamma' = \gamma''$.

Setting $v_\alpha(f)=V_{|\alpha|}(f_\alpha^a, I_a^b\lfloor\alpha)$ for $\alpha\in\mathscr{A}_0(n)$ and $f\in\mathrm{BV}(I_a^b;\mathbb{R})$ and applying (4.9) and (2.6) (which trivially holds also for $\gamma=0$) and the remarks above, we get:

$$\begin{aligned}\|fg\| &= \sum_{\gamma\in\mathscr{A}_0(n)} v_\gamma(fg) \leqslant \sum_{\gamma\in\mathscr{A}_0(n)}\ \sum_{(\alpha,\beta)\in\mathscr{B}(\gamma)} 2^{|\alpha|+|\beta|-|\gamma|} v_\alpha(f)\, v_\beta(g)\\ &\leqslant 2^n \sum_{\gamma\in\mathscr{A}_0(n)}\ \sum_{(\alpha,\beta)\in\mathscr{B}(\gamma)} v_\alpha(f) v_\beta(g) = 2^n \sum_{(\alpha,\beta)\in\mathscr{A}_0(n)^2} v_\alpha(f) v_\beta(g)\\ &= 2^n\left[\sum_{\alpha\in\mathscr{A}_0(n)} v_\alpha(f)\right]\left[\sum_{\beta\in\mathscr{A}_0(n)} v_\beta(g)\right] = 2^n\|f\|\cdot\|g\|,\end{aligned}$$

and the proof of Theorem 1 is complete. □

Remark 11. From the theory of Banach algebras one knows that the norm (2.5) can always be replaced by an equivalent norm $|||\cdot|||$ on $\mathrm{BV}(I_a^b;\mathbb{R})$ such that

$$|||f\cdot g|||\leqslant|||f|||\cdot|||g|||,\quad f,g\in\mathrm{BV}(I_a^b;\mathbb{R}). \tag{4.13}$$

In fact, given $f\in\mathrm{BV}(I_a^b;\mathbb{R})$, let $M_f:\mathrm{BV}(I_a^b;\mathbb{R})\to\mathrm{BV}(I_a^b;\mathbb{R})$ be the continuous linear operator defined by $M_f(g)=fg$ and $\|M_f\|=\sup\{\|M_f(g)\|:\|g\|=1\}$ be the operator norm of M_f. Setting $|||f|||=\|M_f\|$ and noting that $M_{fg}=M_f\circ M_g$, we get (4.13), and $\|f\|\leqslant|||f|||\leqslant 2^n\|f\|$ for all $f\in\mathrm{BV}(I_a^b;\mathbb{R})$.

Remark 12. If $h_0,h_1\in\mathrm{BV}(I_a^b;\mathbb{R})$ and $\|h_1\|<1/2^n$, then, by Banach's contraction theorem and Corollary 3, there exists a unique function $f\in\mathrm{BV}(I_a^b;\mathbb{R})$ such that $h_1(x)f(x)+h_0(x)=f(x)$ for all $x\in I_a^b$.

Given $N\in\mathbb{N}$, let $(\mathbb{R}^N)^{I_a^b}=(\mathbb{R}^{I_a^b})^N$ be the algebra of all functions f mapping I_a^b into $\mathbb{R}^N$, $h:I_a^b\times\mathbb{R}^N\to\mathbb{R}$ be a function of $n+N$ variables, $h=h(x_1,\ldots,x_n,u_1,\ldots,u_N)$, and let $H_h:(\mathbb{R}^N)^{I_a^b}\to\mathbb{R}^{I_a^b}$ be the *superposition operator* defined by

$$(H_hf)(x)=h(x,f_1(x),\ldots,f_N(x)),\quad x\in I_a^b,\ \ f=(f_1,\ldots,f_N)\in(\mathbb{R}^N)^{I_a^b}.$$

We endow the Cartesian product

$$\mathrm{BV}(I_a^b;\mathbb{R})^N=\underbrace{\mathrm{BV}(I_a^b;\mathbb{R})\times\cdots\times\mathrm{BV}(I_a^b;\mathbb{R})}_{N\text{ times}}$$

with the product norm $\|f\|_N=\sum_{i=1}^N\|f_i\|$ for $f=(f_1,\ldots,f_N)\in\mathrm{BV}(I_a^b;\mathbb{R})^N$. Clearly, the space $\mathrm{BV}(I_a^b;\mathbb{R})^N$ is a Banach algebra with respect to the componentwise pointwise operations, and for all $f,g\in\mathrm{BV}(I_a^b;\mathbb{R})^N$ the following inequality holds: $\|f\cdot g\|_N\leqslant 2^n\|f\|_N\|g\|_N$.

Corollary 13. *Let $h_0,h_1,\ldots,h_N\in\mathrm{BV}(I_a^b;\mathbb{R})$. Define $h:I_a^b\times\mathbb{R}^N\to\mathbb{R}$ by $h(x,u_1,\ldots,u_N)=h_0(x)+\sum_{i=1}^N h_i(x)u_i$, $x\in I_a^b$, $(u_1,\ldots,u_N)\in\mathbb{R}^N$. Then the superposition operator H_h maps $\mathrm{BV}(I_a^b;\mathbb{R})^N$ into $\mathrm{BV}(I_a^b;\mathbb{R})$ and is Lipschitzian.*

5. Some generalizations

Theorem 1 is valid (with the same proof) if we replace the target space $\mathbb{R}$ in it by any Banach algebra $(\mathbb{U}, |\cdot|)$; the definition of the corresponding space $\mathrm{BV}(I_a^b; \mathbb{U})$ is straightforward.

More generally, let $\mathbb{U}$, $\mathbb{V}$ and $\mathbb{W}$ be normed linear spaces over the same field $\mathbb{R}$ or $\mathbb{C}$ and the norms denoted by the same symbol $|\cdot|$ (which would not lead to ambiguities). Suppose there exists a bilinear mapping $M : \mathbb{U} \times \mathbb{V} \to \mathbb{W}$ such that $|M(u, v)| \leqslant |u| \cdot |v|$ for all $u \in \mathbb{U}$ and $v \in \mathbb{V}$. The following generalization of Theorem 1 holds: *if* $f \in \mathrm{BV}(I_a^b; \mathbb{U})$ *and* $g \in \mathrm{BV}(I_a^b; \mathbb{V})$, *then the product function* $f \cdot g : I_a^b \to \mathbb{W}$ *defined by* $(f \cdot g)(x) = M(f(x), g(x))$, $x \in I_a^b$, *belongs to* $\mathrm{BV}(I_a^b; \mathbb{W})$, *and the inequality in Theorem* 1 *holds*.

If $(\mathbb{U}, |\cdot|)$ and $(\mathbb{V}, |\cdot|)$ are normed linear spaces, we denote by $L(\mathbb{U}; \mathbb{V})$ the normed linear space of all linear continuous operators from $\mathbb{U}$ into $\mathbb{V}$. Denote by $\mathbb{U}^{I_a^b}$ the space of all functions $f : I_a^b \to \mathbb{U}$ mapping I_a^b into $\mathbb{U}$. Given $h : I_a^b \times \mathbb{U} \to \mathbb{V}$, the *superposition operator* $H : \mathbb{U}^{I_a^b} \to \mathbb{V}^{I_a^b}$ is defined as in (1.1) with $x \in I$ and $f \in \mathbb{R}^I$ replaced by $x \in I_a^b$ and $f \in \mathbb{U}^{I_a^b}$, respectively.

Theorem 14. *If* $\mathbb{U}$ *and* $\mathbb{V}$ *are normed linear spaces*, $h_1 \in \mathrm{BV}(I_a^b; L(\mathbb{U}; \mathbb{V}))$, $h_0 \in \mathrm{BV}(I_a^b; \mathbb{V})$ *and* $h(x, u) = h_1(x)u + h_0(x)$, $x \in I_a^b$, $u \in \mathbb{U}$, *then* H *maps* $\mathrm{BV}(I_a^b; \mathbb{U})$ *into* $\mathrm{BV}(I_a^b; \mathbb{V})$ *and is Lipschitzian*.

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