



Quantization of Drinfeld zastava in type C

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To Vladimir Drinfeld on his 60th birthday

ABSTRACT

A Drinfeld zastava is a certain closure of the moduli space of maps from the projective line to the Kashiwara flag scheme of an affine Lie algebra $\hat{\mathfrak{g}}$. In case \mathfrak{g} is the symplectic Lie algebra \mathfrak{sp}_N , we introduce an affine, reduced, irreducible, normal quiver variety Z which maps to the zastava space isomorphically in characteristic 0. The natural Poisson structure on the zastava space Z can be described in terms of the Hamiltonian reduction of a certain Poisson subvariety of the dual space of a (nonsemisimple) Lie algebra. The quantum Hamiltonian reduction of the corresponding quotient of its universal enveloping algebra produces a quantization Y of the coordinate ring of Z . The same quantization was obtained in the finite (as opposed to the affine) case generically in the work of Gerasimov-Kharchev-Lebedev-Oblezin (2005). We prove that Y is a quotient of the affine Borel Yangian. The analogous results for $\mathfrak{g} = \mathfrak{sl}_N$ were obtained in our previous work.

1. Introduction

1.1. This note is a continuation of [FR14], where we have studied the Drinfeld zastava spaces $Z^{\underline{d}}(\widehat{\mathfrak{sl}}_N)$ from the viewpoint of invariant theory. Given a collection of complex vector spaces $(V_l)_{l \in \mathbb{Z}/N\mathbb{Z}}$ with dimension vector $\underline{\dim}(V_l)_{l \in \mathbb{Z}/N\mathbb{Z}} = (d_l)_{l \in \mathbb{Z}/N\mathbb{Z}} = \underline{d}$ along with a collection of vector spaces $(W_l)_{l \in \mathbb{Z}/N\mathbb{Z}}$ with dimension vector $\underline{\dim}(W_l)_{l \in \mathbb{Z}/N\mathbb{Z}} = (1, \dots, 1)$, we consider the space $M_{\underline{d}} = \{(A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/N\mathbb{Z}}\} =$

$$\bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} \text{End}(V_l) \oplus \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} \text{Hom}(V_l, V_{l+1}) \oplus \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} \text{Hom}(W_{l-1}, V_l) \oplus \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} \text{Hom}(V_l, W_l)$$

of representations of the following chainsaw quiver:

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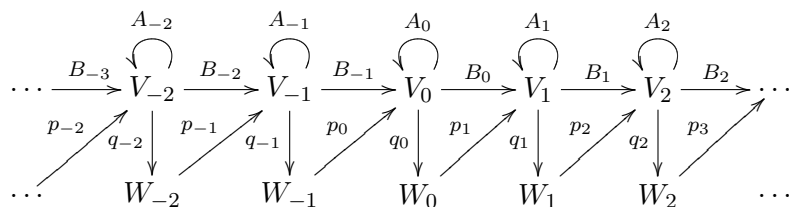
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Furthermore, we consider the closed subscheme $M_{\underline{d}} \subset M_{\underline{d}}$ cut out by the equations $A_{l+1}B_l - B_lA_l + p_{l+1}q_l = 0$ for all l , and two open subschemes $M_{\underline{d}}^s \subset M_{\underline{d}}$ (resp. $M_{\underline{d}}^c \subset M_{\underline{d}}$) formed by all $\{(A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/N\mathbb{Z}}\}$ such that for any $\mathbb{Z}/N\mathbb{Z}$ -graded subspace $V'_\bullet \subset V_\bullet$ with $A_lV'_l \subset V'_l$, and $B_lV'_l \subset V'_{l+1}$ for all l , if $p_l(W_{l-1}) \subset V'_l$ for all l , then $V'_\bullet = V_\bullet$ (resp. if $V'_l \subset \text{Ker } q_l$ for all l , then $V'_\bullet = 0$). The group $G(V_\bullet) = \prod_{l \in \mathbb{Z}/N\mathbb{Z}} GL(V_l)$ acts naturally on $M_{\underline{d}}$ preserving the subschemes $M_{\underline{d}}, M_{\underline{d}}^s, M_{\underline{d}}^c$.

According to [FR14], [BF11], the action of $G(V_\bullet)$ on $M_{\underline{d}} \cap M_{\underline{d}}^s \cap M_{\underline{d}}^c$ is free, and the quotient $(M_{\underline{d}} \cap M_{\underline{d}}^s \cap M_{\underline{d}}^c)/G(V_\bullet)$ is naturally isomorphic to the moduli space $\overset{\circ}{Z}^{\underline{d}}$ of based maps of degree \underline{d} from the projective line to the Kashiwara flag scheme of the affine Lie algebra $\widehat{\mathfrak{sl}}_N$. Moreover, the categorical quotient $M_{\underline{d}}//G(V_\bullet)$ is naturally isomorphic to the Drinfeld zastava closure $Z^{\underline{d}}(\widehat{\mathfrak{sl}}_N) \supset \overset{\circ}{Z}^{\underline{d}}$. Furthermore, the scheme $M_{\underline{d}}//G(V_\bullet) \simeq Z^{\underline{d}}(\widehat{\mathfrak{sl}}_N)$ is reduced, irreducible and normal. One of the crucial points in proving this consists in checking that $(M_{\underline{d}} \cap M_{\underline{d}}^s \cap M_{\underline{d}}^c) \subset M_{\underline{d}}$ is dense, while $M_{\underline{d}} \subset M_{\underline{d}}$ is a complete intersection.

1.2. One of the goals of this note is to extend the above results to the case of zastava spaces for the affine *symplectic* Lie algebra $\widehat{\mathfrak{sp}}_N$ (in case N is even). We prove that any zastava scheme for $\widehat{\mathfrak{sp}}_N$ is reduced, irreducible and normal. Note that these properties of zastava schemes were established in [BF11] for all *finite dimensional* simple Lie algebras.

To this end we again invoke invariant theory. Following [BS00], we equip $W := \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} W_l$ with a symplectic form such that W_l and W_k are orthogonal unless $l + k = N - 1$. We equip $V := \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} V_l$ with a nondegenerate *symmetric* bilinear form such that V_l and V_k are orthogonal unless $l + k = 0$. In particular, we must have $d_{-l} = d_l$ for all l , so that the collection \underline{d} is encoded by $\bar{d} := (d_0, d_1, \dots, d_{N/2})$. We denote by $O(V_\bullet)$ the Levi subgroup of the orthogonal group $O(V)$ preserving the decomposition $V := \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} V_l$. We consider the space $M_{\underline{d}}^{-1} \subset M_{\underline{d}}$ of representations of the *quadratic chainsaw quiver* formed by all the selfadjoint collections $A_l^* = A_l$, $B_l^* = B_{-l-1}$, $p_l^* = q_{-l}$. We denote by $M_{\underline{d}}^{-1} \subset M_{\underline{d}}^{-1}$ the scheme-theoretic intersection $M_{\underline{d}} \cap M_{\underline{d}}^{-1}$.

We prove that $M_{\underline{d}}^{-1} \cap M_{\underline{d}}^s = M_{\underline{d}}^{-1} \cap M_{\underline{d}}^c \subset M_{\underline{d}}^{-1}$ is dense, while $M_{\underline{d}}^{-1} \subset M_{\underline{d}}^{-1}$ is a complete intersection. We deduce that the categorical quotient $M_{\underline{d}}^{-1}//O(V_\bullet)$ is reduced, irreducible and normal. Furthermore, we prove that the action of $O(V_\bullet)$ on $M_{\underline{d}}^{-1} \cap M_{\underline{d}}^s = M_{\underline{d}}^{-1} \cap M_{\underline{d}}^c$ is free and that the quotient $(M_{\underline{d}}^{-1} \cap M_{\underline{d}}^s)/O(V_\bullet)$ is naturally isomorphic to the moduli space $\overset{\circ}{Z}^{\bar{d}}(\widehat{\mathfrak{sp}}_N)$ of based maps of degree \bar{d} from the projective line to the Kashiwara flag scheme of the affine Lie algebra $\widehat{\mathfrak{sp}}_N$. Moreover, the categorical quotient $M_{\underline{d}}^{-1}//G(V_\bullet)$ is naturally isomorphic to the Drinfeld zastava closure $Z^{\bar{d}}(\widehat{\mathfrak{sp}}_N) \supset \overset{\circ}{Z}^{\bar{d}}(\widehat{\mathfrak{sp}}_N)$.

1.3. Quite naturally, we would like to extend the above results to the case of the affine orthogonal Lie algebra $\widehat{\mathfrak{so}}_N$. To this end we change the parities of the bilinear forms in 1.2. That is, we equip W with a nondegenerate *symmetric* bilinear form, and we equip V with a *symplectic* form. The corresponding space of representations of the quadratic chainsaw quiver is denoted by $M_{\underline{d}}^1$, and the corresponding Levi subgroup of $Sp(V)$ is denoted by $Sp(V_{\bullet})$. It is still true that the action of $Sp(V_{\bullet})$ on $M_{\underline{d}}^1 \cap M_{\underline{d}}^s = M_{\underline{d}}^1 \cap M_{\underline{d}}^c$ is free, and the quotient $(M_{\underline{d}}^1 \cap M_{\underline{d}}^s)/Sp(V_{\bullet})$ is naturally isomorphic to $\overset{\circ}{Z}^{\bar{d}}(\widehat{\mathfrak{so}}_N)$.

However, we encounter the following mysterious obstacle: $M_{\underline{d}}^1$ is *not* irreducible in general, and $M_{\underline{d}}^1 \cap M_{\underline{d}}^s = M_{\underline{d}}^1 \cap M_{\underline{d}}^c$ is only dense in one of its irreducible components. For this reason the categorical quotient $M_{\underline{d}}^1//Sp(V_{\bullet})$ is *not* isomorphic to the zastava space $Z^{\bar{d}}(\widehat{\mathfrak{so}}_N)$ in general. The simplest example occurs when \bar{d} is the affine simple coroot of $\widehat{\mathfrak{so}}_N$, that is, $\underline{d} = (\dots, 0, 1, 2, 1, 0, \dots)$.

1.4. Following [FR14], we describe the natural Poisson structure on $Z^{\bar{d}}(\widehat{\mathfrak{sp}}_N)$ in quiver terms. It is obtained by the Hamiltonian reduction of a Poisson subvariety of the dual vector space of a (non-semisimple) Lie algebra $\mathfrak{a}_{\underline{d}}^{-1}$ with its Lie-Kirillov-Kostant bracket. Now the ring of functions $\mathbb{C}[Z^{\bar{d}}(\widehat{\mathfrak{sp}}_N)]$ admits a natural quantization $\mathcal{Y}_{\underline{d}}^{-1}$ as the quantum Hamiltonian reduction of a quotient algebra of the universal enveloping algebra $U(\mathfrak{a}_{\underline{d}}^{-1})$. The algebra $\mathcal{Y}_{\underline{d}}^{-1}$ admits a homomorphism from the Borel subalgebra \mathcal{Y}^{-1} of the Yangian of type C in the case of a finite zastava space. We prove that this homomorphism is surjective. In the affine situation, there is an affine analog $\widehat{\mathcal{Y}}^{-1}$ of \mathcal{Y}^{-1} (it is no longer a subalgebra in the Yangian of $\widehat{\mathfrak{sp}}_N$), and we define it explicitly by generators and relations. We prove that there is a surjective homomorphism $\widehat{\mathcal{Y}}^{-1} \rightarrow \mathcal{Y}_{\underline{d}}^{-1}$. Moreover, we write down certain elements in the kernel of this homomorphism and conjecture that they generate the kernel (as a two-sided ideal). These elements are similar to the generators of the kernel of the Kamnitzer-Webster-Weekes-Yacobi homomorphism from the *shifted Yangian* to the quantization of the transversal slices in the affine Grassmannian. In fact, as explained in [KWWY12], $\mathcal{Y}_{\underline{d}}^{-1}$ as a filtered algebra is the limit of a sequence of quantum coordinate rings of transversal slices.

2. A quiver approach to Drinfeld zastava for symplectic groups

2.1 Quadratic spaces

We will recall the convenient terminology introduced in [KP82]. Let U be an N -dimensional complex vector space equipped with a nondegenerate bilinear form $(,)$ such that $(u, v) = \varepsilon(v, u)$. It will be called a *quadratic space of type ε* (shortened to *orthogonal space* if $\varepsilon = 1$ and *symplectic space* if $\varepsilon = -1$). We denote by $G_{\varepsilon}(U)$ the subgroup of $GL(U)$ leaving the form invariant. So we have $G_{\varepsilon}(U) = O(N)$ or $Sp(N)$ according to whether $\varepsilon = 1$ or $\varepsilon = -1$.

Let $A \mapsto A^*$, $\text{End}(U) \rightarrow \text{End}(U)$ be the canonical involution associated with the form, that is, $(Au, v) = (u, A^*v)$ for any $u, v \in U$. More generally, for a linear operator $B \in \text{Hom}(U, 'U)$ we denote by B^* the adjoint (or transposed) operator $B^* \in \text{Hom}('U^*, U^*)$.

We choose a basis w_0, \dots, w_{N-1} in a quadratic space W of type $\varepsilon = -1$ such that for $0 \leq l < N/2$ we have $(w_l, w_m) = \delta_{m, N-1-l}$ (note that N is necessarily even). The linear span of w_l will be denoted by $W_l \cong \mathbb{C}$. We will often parametrize the base vectors by the elements of $\mathbb{Z}/N\mathbb{Z}$. We define $I := \{0, 1, \dots, N/2\} \subset \{0, \dots, N-1\} = \mathbb{Z}/N\mathbb{Z}$. We set $I = I_0 \sqcup I_1$ where $I_0 = \{1, \dots, \frac{1}{2}N - 1\}$, $I_1 = \{0, \frac{1}{2}N\}$.

We choose another quadratic space V of type $-\varepsilon = 1$ decomposed into a direct sum $V = \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} V_l$ such that V_l is orthogonal to V_m unless $l + m = 0 \in \mathbb{Z}/N\mathbb{Z}$. We denote the dimension of V_l by d_l . We set $\underline{d} := (d_l)_{l \in I}$.

We denote by $G_{-\varepsilon}(V_\bullet)$ the Levi subgroup of $G_{-\varepsilon}(V)$ preserving the decomposition $V = \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} V_l$. It is isomorphic to $O(V_0) \times O(V_{N/2}) \times \prod_{0 < l < N/2} GL(V_l)$.

2.2 Quadratic chainsaw quivers

Following [FR14, 2.3], we consider the affine space $M_{\underline{d}}^\varepsilon$ of collections $(A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/N\mathbb{Z}}$ where $A_l \in \text{End}(V_l)$, $B_l \in \text{Hom}(V_l, V_{l+1})$, $p_l \in \text{Hom}(W_{l-1}, \tilde{V}_l)$, $q_l \in \text{Hom}(V_l, W_l)$ satisfy the following selfadjointness conditions: $A_l^* = A_{-l}$, $B_l^* = B_{-l-1}$, $p_l^* = q_{-l}$. Here we view p_l (resp. q_l) as a vector (resp. covector) of V_l using the identification of all W_m with \mathbb{C} .

Following *loc. cit.*, we consider the subscheme $M_{\underline{d}}^\varepsilon \subset M_{\underline{d}}^\varepsilon$ parametrizing the \underline{d} -dimensional representations of the chainsaw quiver with bilinear form (or the *quadratic chainsaw quiver* for short), cut out by the equations $A_{l+1}B_l - B_lA_l + p_{l+1}q_l = 0$ for all l .

Clearly, $M_{\underline{d}}^\varepsilon$ is acted upon by the Levi subgroup $G_{-\varepsilon}(V_\bullet)$. We denote by $\mathfrak{Z}_{\underline{d}}^\varepsilon$ the categorical quotient $M_{\underline{d}}^\varepsilon // G_{-\varepsilon}(V_\bullet)$.

Assumption. From now on we set $\varepsilon = -1$.

2.3 Examples

We consider three basic examples in types C_1, C_2, \tilde{C}_1 .

2.3.1 Type C_1 . We take $N \geq 2$, $d_0 = \dots = d_{N/2-1} = 0$, $d_{N/2} = d$. We have $V_{N/2} = V = \mathbb{C}^d$, $A_{N/2} = A = A^* \in \text{End}(V)$, $B_1 = 0$, $p_1 = p \in V$, $q_1 = q \in V^*$, $q(v) = (p, v)$. Thus $M_{\underline{d}}^\varepsilon = \text{End}^+(V) \oplus V$ and $\mathfrak{Z}_{\underline{d}}^\varepsilon = (\text{End}^+(V) \oplus V) // O(V)$, where $\text{End}^+(V) \subset \text{End}(V)$ stands for the linear subspace of selfadjoint operators (symmetric matrices). By classical invariant theory, the ring of $O(V)$ -invariant functions on $\text{End}^+(V) \oplus V$ is freely generated by the functions $a_1, \dots, a_d, b_0, \dots, b_{d-1}$, where $a_m := \text{Tr}(A^m)$, and $b_m = (p, A^m p)$. Hence $\mathfrak{Z}_{\underline{d}}^\varepsilon \simeq \mathbb{A}^{2d}$.

2.3.2 Type C_2 . We take $N = 4$, $d_0 = 0$, $d_1 = d_2 = d_3 = 1$. We have $V_1 = V_2 = V_3 = \mathbb{C}$, and hence all our linear operators act between 1-dimensional vector spaces and can be written just as numbers. Hence $M_{\underline{d}}^\varepsilon$ has coordinates $A_1 = A_3, A_2, B_1 = B_2, q_2 = -p_2, q_1 = p_3, q_3 = p_1$, and $M_{\underline{d}}^\varepsilon$ is cut out by a single equation $B_1(A_1 - A_2) = p_2 p_3$. The group $G_{-\varepsilon}(V_\bullet)$ is the product $GL(V_1) \otimes O(V_2) \simeq \mathbb{C}^* \times \{\pm 1\}$ with coordinates $c \in \mathbb{C}^*$, $s = \pm 1$. It acts on $M_{\underline{d}}^\varepsilon$ as follows:

$$(c, s) \cdot (A_1, A_2, B_1, p_1, p_2, p_3) = (A_1, A_2, csB_1, c^{-1}p_1, sp_2, cp_3).$$

The ring of $\mathbb{C}^* \times \{\pm 1\}$ -invariant functions on $M_{\underline{d}}^\varepsilon$ is generated by the functions $A_1, A_2, b_{12} := p_2^2$, $b_{01} := p_1 p_3$, $b_{02} := p_2 B_1 p_1$, $b_{03} := B_1^2 p_1^2$ with three quadratic relations: $b_{02}(A_1 - A_2) = b_{01} b_{12}$, $b_{03}(A_1 - A_2) = b_{01} b_{02}$, $b_{02}^2 = b_{12} b_{03}$. Thus $\mathfrak{Z}_{\underline{d}}^\varepsilon$ is a 4-dimensional (noncomplete) intersection of three quadrics in \mathbb{A}^6 . According to S. Kovács (private communication) and computations by G. Leuschke and A. Verbitsky with the package Macaulay2, $\mathfrak{Z}_{\underline{d}}^\varepsilon$ is reduced, not \mathbb{Q} -Gorenstein, but is Cohen-Macaulay and normal, and has rational singularities.

2.3.3 Type \tilde{C}_1 . We take $N = 2$, $d_0 = d_1 = 1$. We have $V_1 = \mathbb{C} = V_2$, and hence all our linear operators act between 1-dimensional vector spaces and can be written just as numbers. Hence $M_{\underline{d}}^\varepsilon$ has coordinates $A_1, A_2, B_0 = B_1, q_0 = p_0, q_1 = -p_1$ and $M_{\underline{d}}^\varepsilon$ is cut out by a single equation

$B_0(A_1 - A_0) + p_1q_0 = 0$. The group $G_{-\varepsilon}(V_\bullet)$ is the product $O(V_0) \times O(V_1) \simeq \{\pm 1\} \times \{\pm 1\}$ with coordinates (s_1, s_2) . The ring of $\{\pm 1\} \times \{\pm 1\}$ -invariant functions on M_d^ε is generated by the functions $A_0, A_1, b_1 := p_1^2, b_0 := p_0^2, s := B_0^2$ with a single relation $b_1b_0 - s(A_0 - A_1)^2 = 0$. Note the coincidence with the output of [FR14, Example 2.8.3].

2.4 The dimension of M_d^ε

We define the *factorization morphism* $\Upsilon : M_d^\varepsilon \rightarrow \mathbb{A}^d = \prod_{l \in I} (\mathbb{A}^{(1)})^{(d_l)}$ so that the component Υ_l is just $\text{Spec } A_l$.

PROPOSITION 2.1. *Every fiber of Υ has dimension $\dim G_{-\varepsilon}(V_\bullet) + \sum_{l \in I} d_l$.*

Proof. We use the same argument as in the proof of [FR14, Prop. 2.11]. Let us list the minor changes necessary in the quadratic case. The dimension estimate for a general fiber is reduced to the dimension estimate for the zero fiber where all A_l are nilpotent. By the adjointness condition, $(A_\bullet, B_\bullet, p_\bullet, q_\bullet)$ is determined by its components A_l for $0 \leq l \leq N/2$, B_l for $0 \leq l < N/2$, p_l for $0 < l \leq N/2$, q_l for $0 \leq l < N/2$, and $A_0 \in \text{End}^+(V_0)$, $A_{N/2} \in \text{End}^+(V_{N/2})$. Note that $\dim \text{End}^+(V_l) = d_l(d_l + 1)/2$ for $l = 0, N/2$, while $\dim O(V_l) = d_l(d_l - 1)/2$ for $l = 0, N/2$. The dimension of the space of nilpotent selfadjoint operators in $\text{End}^+(V_l)$ equals $d_l(d_l - 1)/2$ for $l = 0, N/2$. More generally, the space $\mathbb{O}_\lambda^+ \subset \text{End}^+(V_l)$ of nilpotent selfadjoint operators of Jordan type λ (a partition of d_l) is a finite union of $O(V_l)$ -orbits, all of the same dimension $\dim \mathbb{O}_\lambda^+ = \dim \mathbb{O}_\lambda/2$ (where \mathbb{O}_λ is the nilpotent $GL(V_l)$ -orbit consisting of the nilpotent matrices of Jordan type λ), see [KR71, Prop. 5]. The argument of the proof of [FR14, Prop. 2.11] implies that, for example, the dimension of the space of $(A_0, A_1, B_0, p_1, q_0)$ subject to $A_1B_0 - B_0A_0 + p_1q_0 = 0$ is at most $d_0(d_0 - 1)/2 + d_1^2 - d_1 + \min(d_0, d_1) + \max(d_0, d_1)$. Summing up the similar estimates over $0 \leq l < N/2$, we obtain the desired inequality $\dim \Upsilon^{-1}(0, \dots, 0) \leq \dim G_{-\varepsilon}(V_\bullet) + \sum_{l \in I} d_l$. The opposite inequality follows from the computation of the generic fiber of Υ , and the proposition follows. \square

The following corollary is proved the same way as [FR14, Cor. 2.12].

COROLLARY 2.2. *The subscheme M_d^ε is an irreducible reduced complete intersection in M_d^ε .*

THEOREM 2.3. *The categorical quotient $\mathfrak{Z}_d^\varepsilon$ is a reduced irreducible normal scheme.*

Proof. We use the same argument as in the proof of [FR14, Thm. 2.7.a)]. We just list the minor changes necessary in the quadratic case. As in *loc. cit.*, we have to check the normality of the open subscheme $U \subset \mathfrak{Z}_d^\varepsilon$ defined as the preimage under the factorization morphism $\Phi : \mathfrak{Z}_d^\varepsilon \rightarrow \mathbb{A}^d$ of an open subset $\hat{U} \subset \mathbb{A}^d$ formed by all the colored configurations where at most two points collide. As in *loc. cit.*, this reduces to a few basic checks we already performed in the examples: **2.3.1** when two points of the same color 0 or $N/2$ (*outmost color* for short) collide; [FR14, 2.8.1] when two points of the same color l , $0 < l < N/2$ (*innermost color* for short) collide; **2.3.2** when a point of an outmost color collides with a point of an innermost color; **2.3.3** when two points of different outmost colors collide; [FR14, 2.8.2] when two points of different innermost colors collide. This completes the proof of the theorem. \square

2.5 Symplectic zastava

For $\vec{d} = (d_0, \dots, d_{N/2}) \in \mathbb{N}^I$ we consider the affine zastava space for the symplectic Lie group $G = Sp_N$ introduced in [BFG06], where it is denoted by $\mathfrak{U}_{G;B}^{\vec{d}}$. In the present paper we will

denote it by $Z_\varepsilon^{\bar{d}}$. It is a reduced irreducible affine scheme containing as an open subscheme the (smooth) moduli space $\overset{\circ}{Z}_\varepsilon^{\bar{d}}$ of degree \bar{d} based maps from \mathbb{P}^1 to the affine flag scheme of $G = Sp_N$.

Recall that Sp_N is the fixed point subgroup of the involutive pinning-preserving outer automorphism $\sigma : SL_N \rightarrow SL_N$. This automorphism acts on the affine flag scheme of SL_N , and on the zastava spaces $Z^{\underline{d}}$ for SL_N . More precisely, we define $\underline{d} = (\tilde{d}_0, \dots, \tilde{d}_{N-1}) \in \mathbb{N}^N$ as follows: for $0 \leq l \leq N/2$ we have $\tilde{d}_l = d_l$, and for $0 < l < N/2$ we set $\tilde{d}_{N-l} = d_l$. Then σ acts on $Z^{\underline{d}}$, and the fixed point subscheme (with the reduced closed subscheme structure) is isomorphic to $Z_\varepsilon^{\bar{d}}$. In other words, $Z_\varepsilon^{\bar{d}}$ is the closure in $Z^{\underline{d}}$ of $\overset{\circ}{Z}_\varepsilon^{\bar{d}} \cong (\overset{\circ}{Z}^{\underline{d}})^\sigma$.

Recall the chainsaw quiver variety $\mathfrak{Z}_{\underline{d}}$ introduced in [FR14]. Thm. [FR14, 2.7.b)] constructs a morphism $\eta : \mathfrak{Z}_{\underline{d}} \rightarrow Z^{\underline{d}}$, and [BF11, Thm. 3.5] proves that η is an isomorphism. We will identify $\mathfrak{Z}_{\underline{d}}$ and $Z^{\underline{d}}$ via η . Under this identification the open subscheme $\overset{\circ}{Z}^{\underline{d}}$ corresponds to the open subscheme $\overset{\circ}{\mathfrak{Z}}_{\underline{d}}$ formed by the closed orbits of stable and costable quadruples $(A_\bullet, B_\bullet, p_\bullet, q_\bullet)$, that is, such that V_\bullet is generated by the action of A_\bullet, B_\bullet from the image of p_\bullet , and V_\bullet contains no nonzero subspaces in $\text{Ker } q_\bullet$ closed with respect to A_\bullet, B_\bullet . The fixed point subscheme $(\overset{\circ}{\mathfrak{Z}}_{\underline{d}})^\sigma$ coincides with $\overset{\circ}{\mathfrak{Z}}_{\underline{d}}^\varepsilon$, the open subscheme of $\mathfrak{Z}_{\underline{d}}^\varepsilon$ formed by the closed orbits of stable (equivalently, costable) quadruples $(A_\bullet, B_\bullet, p_\bullet, q_\bullet) \in M_{\underline{d}}^\varepsilon$, cf. [BS00, Table 1 and Prop. 3.3].

LEMMA 2.4. *The closed embedding $\overset{\circ}{\mathfrak{Z}}_{\underline{d}}^\varepsilon \cong (\overset{\circ}{\mathfrak{Z}}_{\underline{d}})^\sigma \hookrightarrow \overset{\circ}{\mathfrak{Z}}_{\underline{d}}$ extends to the closed embedding $\mathfrak{Z}_{\underline{d}}^\varepsilon \hookrightarrow \mathfrak{Z}_{\underline{d}}$.*

Proof. It suffices to check that any $G_{-\varepsilon}(V_\bullet)$ -invariant function on $M_{\underline{d}}^\varepsilon$ extends to a $\prod_{l=0}^{N-1} GL(V_l)$ -invariant function on $M_{\underline{d}}$. This is immediately seen on the generators provided by classical invariant theory. \square

Now, since $\overset{\circ}{\mathfrak{Z}}_{\underline{d}}^\varepsilon$ is dense in $\mathfrak{Z}_{\underline{d}}^\varepsilon = (\mathfrak{Z}_{\underline{d}})^\sigma$, we conclude that $\mathfrak{Z}_{\underline{d}}^\varepsilon \subset \mathfrak{Z}_{\underline{d}} = Z^{\underline{d}}$ coincides with the closure of $(\overset{\circ}{\mathfrak{Z}}_{\underline{d}})^\sigma$ in $\mathfrak{Z}_{\underline{d}} = Z^{\underline{d}}$. Since the symplectic zastava scheme $Z_\varepsilon^{\bar{d}}$ also coincides with this closure, we arrive at the following result.

THEOREM 2.5. *There is a canonical isomorphism $\eta : \mathfrak{Z}_{\underline{d}}^\varepsilon \xrightarrow{\sim} Z_\varepsilon^{\bar{d}}$ making the following diagram commutative:*

$$\begin{array}{ccc} \mathfrak{Z}_{\underline{d}}^\varepsilon & \longrightarrow & \mathfrak{Z}_{\underline{d}} \\ \eta \downarrow & & \downarrow \eta, \\ Z_\varepsilon^{\bar{d}} & \longrightarrow & Z_{\underline{d}} \end{array}$$

where the horizontal morphisms are the closed embeddings of the σ -fixed points subschemes.

COROLLARY 2.6. *The symplectic zastava scheme $Z_\varepsilon^{\bar{d}}$ is normal.*

3. Hamiltonian reduction

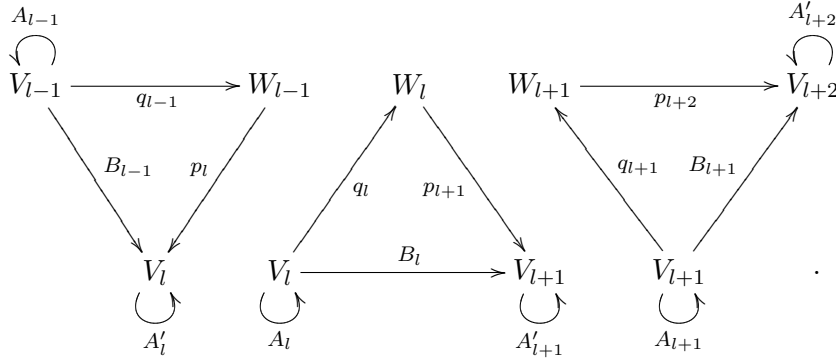
3.1 Poisson structures

According to [FKMM99] (cf. [FR14, 3.1,3.3]), the smooth scheme $\overset{\circ}{Z}^{\underline{d}}$ carries a canonical symplectic structure which extends as a Poisson structure to $\mathfrak{Z}_{\underline{d}} = Z^{\underline{d}}$. This Poisson structure was constructed in [FR14] via Hamiltonian reduction. The restriction of the symplectic form on $\overset{\circ}{Z}^{\underline{d}}$ to $\overset{\circ}{Z}_\varepsilon^{\bar{d}} \cong (\overset{\circ}{Z}^{\underline{d}})^\sigma$ coincides with the canonical symplectic form [FKMM99] on $\overset{\circ}{Z}_\varepsilon^{\bar{d}}$. We conclude that

the canonical symplectic structure on $\overset{\circ}{Z}_\varepsilon^{\bar{d}}$ extends as a Poisson structure to $Z_\varepsilon^{\bar{d}} \cong \mathfrak{Z}_d^\varepsilon$ and that the σ -fixed point embedding $\mathfrak{Z}_d^\varepsilon \hookrightarrow \mathfrak{Z}_d$ is Poisson. In the next subsection we will construct this Poisson structure on $\mathfrak{Z}_d^\varepsilon$ via Hamiltonian reduction.

3.2 Classical reduction

Recall the Hamiltonian reduction definition of the Poisson bracket on zastava spaces in type A (see [FR14]). We “triangulate” the chainsaw quiver in the following way:



For a pair of vector spaces U, V define the following 2-step nilpotent Lie algebra:

$$\mathfrak{n}(U, V) := U \oplus V^* \oplus (U \otimes V^*),$$

where the space $U \otimes V^*$ is central, $[U, U] = [V^*, V^*] = 0$, and for $u \in U$, $v^\vee \in V^*$ one has $[u, v^\vee] = u \otimes v^\vee$.

To define the Poisson structure, we attach to each triangle of our graph the following Lie algebra:

$$\mathfrak{a}_l := (\mathfrak{gl}(V_l) \oplus \mathfrak{gl}(V_{l+1})) \rtimes \mathfrak{n}(V_l, V_{l+1})$$

(the semidirect sum is with respect to the tautological action of $\mathfrak{gl}(V_l)$ on V_l and $\mathfrak{gl}(V_{l+1})$ on V_{l+1}^*).

Consider the Lie algebra

$$\mathfrak{a}_d := \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} \mathfrak{a}_l = \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} (\mathfrak{gl}(V_l) \oplus \mathfrak{gl}(V_{l+1})) \rtimes \mathfrak{n}(V_l, V_{l+1}).$$

The coadjoint representation of \mathfrak{a}_d is the space $\mathfrak{a}_d^* = \{(A_l, A'_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/N\mathbb{Z}}\}$, where

$$A_l \in \text{End}(V_l), \quad A'_l \in \text{End}(V_l), \quad B_l \in \text{Hom}(V_l, V_{l+1}), \quad p_l \in V_l, \quad q_l \in V_l^*.$$

Consider the subvariety $S_d \subset \mathfrak{a}_d^*$ defined by the following equations:

$$B_l A_l + A'_{l+1} B_l + p_{l+1} q_l = 0, \quad l \in \mathbb{Z}/N\mathbb{Z}. \quad (1)$$

Let $\mathfrak{gl}(V_l)_{\text{diag}}$ be the diagonal $\mathfrak{gl}(V_l)$ inside $\mathfrak{gl}(V_l) \oplus \mathfrak{gl}(V_l) \subset \mathfrak{a}_d$ and let $\pi : \mathfrak{a}_d^* \rightarrow \mathfrak{gl}(V_l)_{\text{diag}}^*$ be the projection. Then the Drinfeld zastava space \mathfrak{Z}_d is identified with the Hamiltonian reduction

$$S_d // \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} \mathfrak{gl}(V_l)_{\text{diag}} = \pi^{-1}(0) \cap S_d // \prod_{l \in \mathbb{Z}/N\mathbb{Z}} GL(V_l)_{\text{diag}}.$$

This provides a natural Poisson bracket on \mathfrak{Z}_d .

The involution σ acts on the space \mathfrak{a}_d^* as follows:

$$A_l \mapsto -A'_{N-l}, \quad A'_l \mapsto -A_{N-l}, \quad B_l \mapsto B_{N-l}^*, \quad p_l \mapsto q_{N-l}^*, \quad q_l \mapsto p_{N-l}^*.$$

Remark 3.1. Strictly speaking, σ is not an involution on \mathfrak{a}_d^* since $p^{**} = -p$ and $q^{**} = -q$, but becomes an involution after the Hamiltonian reduction.

To describe the fixed point set, we consider the half of the chainsaw quiver formed by the vertices $l \in I$. Define the Lie algebra

$$\mathfrak{a}_d^\varepsilon := \bigoplus_{l=0}^{\frac{N}{2}-1} \mathfrak{a}_l = \bigoplus_{l=1}^{\frac{N}{2}-1} (\mathfrak{gl}(V_l) \oplus \mathfrak{gl}(V_{l+1})) \rtimes \mathfrak{n}(V_l, V_{l+1}).$$

The coadjoint representation of $\mathfrak{a}_d^\varepsilon$ is the space

$$\mathfrak{a}_d^{\varepsilon*} = \{(A_l, A'_l, B_l, p_l, q_l)_{l \in I_0}, A_0, q_0, B_0, A'_{\frac{N}{2}}, p_{\frac{N}{2}}\},$$

where

$$A_l \in \text{End}(V_l), \quad A'_l \in \text{End}(V_l), \quad B_l \in \text{Hom}(V_l, V_{l+1}), \quad p_l \in V_l, \quad q_l \in V_l^*.$$

The invariant subvariety $S_d^\varepsilon \subset \mathfrak{a}_d^{\varepsilon*}$ is again defined by the equations (1):

$$B_l A_l + A'_{l+1} B_l + p_{l+1} q_l = 0, \quad l = 0, \dots, \frac{N}{2} - 1.$$

Let $\mathfrak{o}(V_l) \subset \mathfrak{gl}(V_l) \subset \mathfrak{a}_d^\varepsilon$ for $l \in I_1$ be the orthogonal Lie subalgebra and let

$$\pi : \mathfrak{a}_d^{\varepsilon*} \rightarrow \bigoplus_{l \in I_0} \mathfrak{gl}(V_l)_{\text{diag}}^* \oplus \bigoplus_{l \in I_1} \mathfrak{o}(V_l)^*$$

be the projection. Then the symplectic Drinfeld zastava space $\mathfrak{Z}_d^\varepsilon$ is identified (as a Poisson variety) with the Hamiltonian reduction

$$S_d^\varepsilon // \bigoplus_{l \in I_0} \mathfrak{gl}(V_l)_{\text{diag}} \oplus \bigoplus_{l \in I_1} \mathfrak{o}(V_l) = (\pi^{-1}(0) \cap S_d^\varepsilon) // \prod_{l \in I_0} GL(V_l)_{\text{diag}} \times \prod_{l \in I_1} O(V_l).$$

We denote the group $\prod_{l \in I_0} GL(V_l)_{\text{diag}} \times \prod_{l \in I_1} O(V_l)$ simply by G_d , and the corresponding Lie algebra $\bigoplus_{l \in I_0} \mathfrak{gl}(V_l)_{\text{diag}} \oplus \bigoplus_{l \in I_1} \mathfrak{o}(V_l)$ by \mathfrak{g}_d .

3.3 Quantum reduction

The natural quantization of the coordinate ring $\mathbb{C}[\mathfrak{a}_d^{\varepsilon*}]$ is the enveloping algebra $U(\mathfrak{a}_d^\varepsilon)$. It will be convenient to gather the generators of $U(\mathfrak{a}_d^\varepsilon)$ (that is, the basis elements of the Lie algebra $\mathfrak{a}_d^\varepsilon$) into the following $U(\mathfrak{a}_d^\varepsilon)$ -valued matrices:

$$A_k, B_k, q_k, A'_l, p_l, \quad 0 \leq k < \frac{N}{2}, \quad 0 < l \leq \frac{N}{2}.$$

According to [FR14], the coefficients of the following matrices form a subspace $R \subset U(\mathfrak{a}_d^\varepsilon)$ invariant with respect to the adjoint action:

$$B_l A_l + A'_{l+1} B_l + p_{l+1} q_l, \quad l = 0, \dots, \frac{N}{2} - 1, \quad i = 1, \dots, d_{l+1}, \quad j = 1, \dots, d_l. \quad (2)$$

Equivalently, $U(\mathfrak{a}_d^\varepsilon)R$ is a two-sided ideal in $U(\mathfrak{a}_d^\varepsilon)$.

The natural quantization of the coordinate ring of the space $\mathfrak{Z}_d^\varepsilon$ is the *quantum Hamiltonian reduction* $\mathcal{Y}_d^\varepsilon := \left(U(\mathfrak{a}_d^\varepsilon) / U(\mathfrak{a}_d^\varepsilon)(R + \mathfrak{g}_d) \right)^{G_d}$. The ring $\mathcal{Y}_d^\varepsilon$ has a natural filtration coming from the PBW filtration on $U(\mathfrak{a}_d^\varepsilon)$.

PROPOSITION 3.2 (PBW property). *We have $\text{gr } \mathcal{Y}_d^\varepsilon = \mathbb{C}[\mathfrak{Z}_d^\varepsilon]$.*

Proof. The proof is a word for word repetition of that of Proposition 3.28 from [FR14]. \square

We consider the following elements of $\mathcal{Y}_d^\varepsilon$:

$$a_{l,r} := \text{Tr } A_l^r, \quad r = 1, 2, \dots, \quad l \in I;$$

$$b_{l,s} := q_l A_l^s p_l, \quad s = 0, 1, \dots, \quad l \in I.$$

We also introduce the following elements:

$$b_{k,l;s_k,\dots,s_l} := q_l A_l^{s_l} B_{l-1} A_{l-1}^{s_{l-1}} B_{l-2} \dots B_k A_k^{s_k} p_k, \quad k \leq l \in \mathbb{Z}, \quad s_i \in \mathbb{Z}_{\geq 0}. \quad (3)$$

$$c_{k,l;s_k,\dots,s_l} := B_l A_l^{s_l} B_{l-1} A_{l-1}^{s_{l-1}} B_{l-2} \dots B_k A_k^{s_k}, \quad l = k + mN, \quad s_i \in \mathbb{Z}_{\geq 0}. \quad (4)$$

From the definitions we get the following relations.

LEMMA 3.3. *Let $k < l + 1$. Then*

$$[b_{k,l+1;0,\dots,0}, b_{l+1,0}] = \begin{cases} b_{k,l+1;0,\dots,0} & l \pm k \neq 0, 2 \pmod{N}, \quad 2l + 2 \neq 0 \pmod{N}, \\ 2b_{k,l+1;0,\dots,0} & 2l + 2 = 0 \pmod{N}, \\ b_{k,l+1;0,\dots,0} - b_{k-1,l;0,\dots,0} & l + k = 0 \pmod{N}, \\ 2(b_{k,l+1;0,\dots,0} - b_{k-1,l;0,\dots,0}) & l + k = 0 \pmod{N}, \quad 2l + 2 = 0 \pmod{N}, \\ b_{k,l+1;0,\dots,0} - b_{k-1,l;0,\dots,0} & l - k = 2 \pmod{N}. \end{cases}$$

LEMMA 3.4. *For $k \leq m \leq l$, we have $[a_{m,r}, b_{k,l;s_k,\dots,s_l}] = \lambda b_{k,l;s_k,\dots,s_{m+r-1},\dots,s_l} + L$, where $\lambda \in \mathbb{C} \setminus \{0\}$, $L \in \mathcal{Y}_d^\varepsilon$ is expressed in $b_{k',l';s_{k'},\dots,s_{l'}}$ with $l' - k' < l - k$ and $\deg L \leq \deg b_{k,l;s_k,\dots,s_{m+r-1},\dots,s_l}$ with respect to the PBW filtration.*

Proof. Straightforward. \square

LEMMA 3.5. (C_1 case) *Let p_0, p_1, A_0, A_1, B be the (matrices of) generators of the algebra \mathfrak{a}_d for $N = 2$. Then the algebra $\mathcal{Y}_d^\varepsilon$ is generated by $a_{l,r} := \text{Tr } A_l^r$, $b_{l,0} := p_l^* p_l$ with $l = 0, 1$, $r = 1, \dots, d_l$.*

Proof. According to Lemma 3.4, it suffices to check that the invariants of the form $p_0^*(B^*B)^m p_0$, $p_1^*(B^*B)^m p_0$, $p_0^*(B^*B)^m B^* p_1$, $p_1^*(BB^*)^m p_0$ and $\text{Tr}(BB^*)^m$ can be expressed in $a_{l,r}, b_{l,0}$. This is easily checked by induction on m . \square

PROPOSITION 3.6. *The algebra $\mathcal{Y}_d^\varepsilon$ is generated by $a_{l,r}, b_{l,s}$ with $l \in I$, $r = 1, \dots, d_l$ and $s = 0, \dots, d_l - 1$.*

Proof. Arguing in the same way as in Proposition 3.35 of [FR14] we reduce the problem to expressing $b_{k,l;0,0,\dots,0}$ and $c_{0,mN;0,\dots,0}$ in $a_{l,r}, b_{l,s}$.

By Lemma 3.3 for $l - k < N - 1$ we have $b_{k,l;0,0,\dots,0} = \lambda[\dots[b_{k,0} b_{k+1,0}] \dots, b_{l-1,0}], b_{l,0}]$, where λ is a nonzero number. Thus the $b_{k,l;0,0,\dots,0}$ with $l - k < n$ can be expressed in $a_{l,r}, b_{l,s}$. Suppose that $b_{k,k+mN-1;0,0,\dots,0}$ for some $m \in \mathbb{Z}_+$ can be expressed in $a_{l,r}, b_{l,s}$, then for $l - k < N$ we have $b_{k,l+mN;0,0,\dots,0} = \lambda[\dots[b_{k,k+mN-1;0,0,\dots,0} b_{k+N+1,0}] \dots, b_{l+N-1,0}], b_{l+N,0}]$. Thus $b_{k,l+mN;0,0,\dots,0}$ with $l - k < N$ can be expressed in $a_{l,r}, b_{l,s}$ as well. So, the problem reduces to expressing $b_{k,k+mN-1;0,0,\dots,0}$ and $c_{0,mN;0,\dots,0}$ in $a_{l,r}, b_{l,s}$.

Let $\underline{D} = (d_0, d_{\frac{N}{2}})$. Define the homomorphism $\Phi : U(\mathfrak{a}_{\underline{D}}^\varepsilon) \rightarrow U(\mathfrak{a}_d^\varepsilon)$ by

$$\Phi(A_0) = A_0, \quad \Phi(A_1) = A_{\frac{N}{2}}, \quad \Phi(B) = B_{\frac{N}{2}-1} \cdot B_{\frac{N}{2}-2} \cdot \dots \cdot B_0,$$

$$\Phi(p_0) = p_0, \quad \Phi(p_1) = B_{\frac{N}{2}-1} \cdot \dots \cdot B_1 p_1.$$

Note that $\Phi(\mathcal{Y}_D^\varepsilon) \subset \mathcal{Y}_d^\varepsilon$. By Lemma 3.5, $\mathcal{Y}_D^\varepsilon$ is generated by the elements $a_{l,r} := \text{Tr } A_l^r$, $b_{l,0} := p_l^* p_l$ with $l = 0, 1$, $r = 1, \dots, d_l$. We have $\Phi(a_{0,r}) = a_{0,r}$, $\Phi(a_{1,r}) = a_{\frac{N}{2},r}$, $\Phi(b_{0,0}) = b_{0,0}$, $\Phi(b_{1,0}) = b_{1,N-1;0,\dots,0}$. Thus everything from $\Phi(\mathcal{Y}_D^\varepsilon)$ can be expressed in $a_{l,r}, b_{l,s}$. On the other hand, $\Phi(b_{0,2m-1;0,\dots,0}) = b_{0,mN-1;0,\dots,0}$ and $\Phi(c_{0,2m;0,\dots,0}) = c_{0,mN;0,\dots,0}$. \square

4. Yangians

4.1 The Yangian of \mathfrak{sp}_N

Let $(c_{kl})_{k,l=1,2,\dots,N/2}$ stand for the symmetrized Cartan matrix of \mathfrak{sp}_N . That is, $c_{kk} = 4$ for $k = N/2$, $c_{kk} = 2$ for $0 < k < N/2$, $c_{kl} = 0$ for $|k - l| > 1$, $c_{kl} = -1$ for $0 < k, l < N/2$ and $l = k \pm 1$, and $c_{kl} = -2$ otherwise.

The Yangian $Y(\mathfrak{sp}_N)$ is generated by $\mathbf{x}_{k,r}^\pm, \mathbf{h}_{k,r}$, $k = 1, 2, \dots, N/2$, $r \in \mathbb{N}$, with the following relations:

$$[\mathbf{h}_{k,r}, \mathbf{h}_{l,s}] = 0, \quad [\mathbf{h}_{k,0}, \mathbf{x}_{l,s}^\pm] = \pm c_{kl} \mathbf{x}_{l,s}^\pm, \quad (5)$$

$$2[\mathbf{h}_{k,r+1}, \mathbf{x}_{l,s}^\pm] - 2[\mathbf{h}_{k,r}, \mathbf{x}_{l,s+1}^\pm] = \pm c_{kl} (\mathbf{h}_{k,r} \mathbf{x}_{l,s}^\pm + \mathbf{x}_{l,s}^\pm \mathbf{h}_{k,r}), \quad (6)$$

$$[\mathbf{x}_{k,r}^+, \mathbf{x}_{l,s}^-] = \delta_{kl} \mathbf{h}_{k,r+s}, \quad (7)$$

$$2[\mathbf{x}_{k,r+1}^\pm, \mathbf{x}_{l,s}^\pm] - 2[\mathbf{x}_{k,r}^\pm, \mathbf{x}_{l,s+1}^\pm] = \pm c_{kl} (\mathbf{x}_{k,r}^\pm \mathbf{x}_{l,s}^\pm + \mathbf{x}_{l,s}^\pm \mathbf{x}_{k,r}^\pm), \quad (8)$$

$$[\mathbf{x}_{k,r}^\pm, [\mathbf{x}_{k,p}^\pm, \mathbf{x}_{l,s}^\pm]] + [\mathbf{x}_{k,p}^\pm, [\mathbf{x}_{k,r}^\pm, \mathbf{x}_{l,s}^\pm]] = 0, \quad k = l \pm 1, \quad k \in I, l \in I_0, \quad \forall p, r, s \in \mathbb{N}, \quad (9)$$

$$\sum_{\sigma \in S_3} [\mathbf{x}_{k,r_{\sigma(3)}}^\pm, [\mathbf{x}_{k,r_{\sigma(2)}}^\pm, [\mathbf{x}_{k,r_{\sigma(1)}}^\pm, \mathbf{x}_{l,s}^\pm]]] = 0, \quad k = l \pm 1, \quad k \in I, l \in I_1, \quad \forall r_1, r_2, r_3, s \in \mathbb{N}. \quad (10)$$

We will consider the ‘‘Borel subalgebra’’ \mathcal{Y}^ε of the Yangian, generated by the $\mathbf{x}_{k,r}^+$ and $\mathbf{h}_{k,r}$. For a formal variable u we introduce the generating series $\mathbf{h}_k(u) := 1 + \sum_{r=0}^\infty \mathbf{h}_{k,r} u^{-r-1}$ and $\mathbf{x}_k^+(u) := \sum_{r=0}^\infty \mathbf{x}_{k,r}^+ u^{-r-1}$.

We also consider a bigger algebra $\mathcal{D}\mathcal{Y}^\varepsilon$, the ‘‘Borel subalgebra of the Yangian double’’, generated by all Fourier components of the series $\mathbf{h}_k(u) := 1 + \sum_{r=0}^\infty \mathbf{h}_{k,r} u^{-r-1}$ and $\mathbf{x}_k^+(u) := \sum_{r=-\infty}^\infty \mathbf{x}_{k,r}^+ u^{-r-1}$ (that is, the generating series $\mathbf{x}_k^+(u)$ are infinite in both positive and negative directions) with the defining relations (5,6,8,9,10). The algebra \mathcal{Y}^ε is then the subalgebra generated by the negative Fourier components of the $\mathbf{x}_k^+(u)$ and $\mathbf{h}_k(u)$ due to the PBW property of the Yangians. We can then rewrite Eqs. (6,8) in the following form:

$$\mathbf{h}_k(u) \mathbf{x}_l^+(v) \frac{2u - 2v - c_{kl}}{2u - 2v + c_{kl}} = \mathbf{x}_l^+(v) \mathbf{h}_k(u). \quad (11)$$

$$\mathbf{x}_k^+(u) \mathbf{x}_l^+(v) (2u - 2v - c_{kl}) = (2u - 2v + c_{kl}) \mathbf{x}_l^+(v) \mathbf{x}_k^+(u). \quad (12)$$

The function $(2u - 2v - c_{kl})/(2u - 2v + c_{kl})$ here is understood as a formal power series in u^{-1} , v^{-1} , $u^{-1}v$, hence Eq. (11) is well defined.

Following [FR14], we will use a little bit different generators of the Cartan subalgebra of the Yangian,

$$\mathbf{A}_k(u) := u^{d_k} + A_{k,0}u^{d_k-1} + \dots + A_{k,r}u^{d_k-r-1} + \dots, \quad (13)$$

obtained as the (unique) solution of the system of functional equations:

$$\mathbf{h}_k(u) = \mathbf{A}_k(u + \frac{1}{2})^{-1} \mathbf{A}_k(u - \frac{1}{2})^{-1} \mathbf{A}_{k-1}(u) \mathbf{A}_{k+1}(u) (u + \frac{1}{2})^{d_k} (u - \frac{1}{2})^{d_k} u^{-d_{k-1}} u^{-d_{k+1}}, \quad (14)$$

for $k = 1, 2, \dots, \frac{N}{2} - 1$, and

$$\begin{aligned} \mathbf{h}_{\frac{N}{2}}(u) &= \mathbf{A}_{\frac{N}{2}}(u+1)^{-1} \mathbf{A}_{\frac{N}{2}}(u-1)^{-1} \mathbf{A}_{\frac{N}{2}-1}(u) \mathbf{A}_{\frac{N}{2}-1}(u + \frac{1}{2}) \\ &\cdot (u+1)^{d_{\frac{N}{2}}} (u-1)^{d_{\frac{N}{2}}} u^{-d_{\frac{N}{2}-1}} (u + \frac{1}{2})^{-d_{\frac{N}{2}-1}}. \end{aligned} \quad (15)$$

Here we take $\mathbf{A}_0(u) = 1$

LEMMA 4.1. *The generators $\mathbf{A}_k(u)$ of \mathcal{DY}^ε satisfy the relations*

$$\mathbf{A}_k(u) \mathbf{x}_l^+(v) \frac{2u - 2v + \frac{1}{2}c_{kk}\delta_{kl}}{2u - 2v - \frac{1}{2}c_{kk}\delta_{kl}} = \mathbf{x}_l^+(v) \mathbf{A}_k(u). \quad (16)$$

LEMMA 4.2. *Let $\mathbf{A}_k(u)$ and $\mathbf{x}_l^+(u)$ be the generating series of \mathcal{DY}^ε . Then the series*

$$\mathbf{a}_k(u) = \frac{\mathbf{A}_k(u - \frac{1}{4}c_{kk})}{\mathbf{A}_k(u + \frac{1}{4}c_{kk})} = 1 - d_k u^{-1} - \sum_{r=1}^{\infty} \mathbf{a}_{k,r} u^{-r-1} \quad \text{and} \quad \mathbf{x}_l^+(u)$$

satisfy the following commutator relations:

$$[\mathbf{a}_k(u), \mathbf{x}_l^+(v)](u-v) = -\frac{\frac{1}{4}c_{kk}^2\delta_{kl}}{u-v} \mathbf{x}_l^+(v) \mathbf{a}_k(u), \quad [\mathbf{a}_k(u), \mathbf{a}_l(v)] = 0. \quad (17)$$

The series $\mathbf{a}_k(u)$, $\mathbf{x}_l^+(u)$ generate \mathcal{DY}^ε with the defining relations (17), (8) and (9), and their negative Fourier components generate \mathcal{Y}^ε .

Proof. For $k \neq l$ the relation is obvious, for $k = l$ we have

$$\mathbf{a}_k(u) \mathbf{x}_k^+(v) \frac{u - \frac{1}{4}c_{kk} - v + \frac{1}{4}c_{kk}}{u - \frac{1}{4}c_{kk} - v - \frac{1}{4}c_{kk}} \cdot \frac{u + \frac{1}{4}c_{kk} - v - \frac{1}{4}c_{kk}}{u + \frac{1}{4}c_{kk} - v + \frac{1}{4}c_{kk}} = \mathbf{x}_k^+(v) \mathbf{a}_k(u).$$

therefore

$$\mathbf{a}_k(u) \mathbf{x}_k^+(v) \frac{(u-v)^2}{(u-v)^2 - \frac{1}{4}c_{kk}^2} = \mathbf{x}_k^+(v) \mathbf{a}_k(u).$$

One can inductively express $\mathbf{A}_{k,r}$ in $\mathbf{a}_{k,s}$ with $s \leq r+1$, hence \mathcal{DY}^ε is generated by $\mathbf{a}_k(u)$ and $\mathbf{x}_l^+(u)$. On the other hand, the quotient of $\mathbb{C}[\mathbf{a}_{k,r}]_{r=1}^{\infty} \cdot \mathcal{DY}^+$ by relation (17) is $\mathbb{C}[\mathbf{a}_{k,r}]_{r=1}^{\infty} \otimes \mathcal{DY}^+$ as a filtered vector space. The same argumentation holds for \mathcal{Y}^ε . The assertion follows. \square

4.2 The Yangian of $\widehat{\mathfrak{sp}}_N$

Let $(c_{kl})_{k,l \in I}$ stand for the symmetrized Cartan matrix of $\widehat{\mathfrak{sp}}_N$. That is, $c_{kk} = 4$ for $k = 0$ or $k = N/2$, $c_{kk} = 2$ for $0 < k < N/2$, $c_{kl} = 0$ for $|k-l| > 1$, $c_{kl} = -1$ for $0 < k, l < N/2$ and $l = k \pm 1$ and $c_{kl} = -2$ otherwise.

As for the finite case, we will consider the ‘‘affine Borel Yangian’’. This is an associative algebra $\widehat{\mathcal{Y}}^\varepsilon$ generated by the series

$$\mathbf{x}_k^+(u) := 1 + \sum_{r=0}^{\infty} \mathbf{x}_{k,r} u^{-r-1}, \quad (18)$$

$$\mathbf{A}_k(u) := u^{d_k} + \sum_{r=0}^{\infty} \mathbf{A}_{k,r} u^{d_k-r-1}, \quad (19)$$

with $k \in \mathbb{Z}$, subject to the relations

$$\mathbf{A}_k(u)\mathbf{A}_l(v) = \mathbf{A}_l(v)\mathbf{A}_k(u), \quad (20)$$

$$\mathbf{x}_k^\pm(u)\mathbf{x}_l^\pm(v)(2u - 2v \mp c_{kl}) = \mathbf{x}_l^\pm(v)\mathbf{x}_k^\pm(u)(2u - 2v \pm c_{kl}), \quad (21)$$

where (c_{kl}) stands for the symmetrized Cartan matrix of \widetilde{C}_n , and

$$\mathbf{A}_k(u)\mathbf{x}_l^+(v) \frac{2u - 2v + \frac{1}{2}c_{kl}\delta_{kl}}{2u - 2v - \frac{1}{2}c_{kl}\delta_{kl}} = \mathbf{x}_l^+(v)\mathbf{A}_k(u), \quad (22)$$

in the sense that the negative Fourier components of the LHS and RHS are equal, and the Serre relations (9) and (10).

4.3 Symplectic Yangians and symplectic zastava spaces

THEOREM 4.3. *The algebra $\mathcal{Y}_{\underline{d}}^\varepsilon$ is a quotient of the Borel Yangian $\widehat{\mathcal{Y}}^\varepsilon$ of $\widehat{\mathfrak{sp}}_N$ by some ideal containing $\mathbf{A}_{k,r} = 0$ for $r > d_k$.*

Proof. For $k \in I$, $l \in I_0$, we introduce the following generating series in $\mathcal{Y}_{\underline{d}}^\varepsilon$:

$$a_k(u) := 1 - d_k u^{-1} - \sum_{r=1}^{\infty} a_{k,r} u^{-r-1}, \quad b_l(u) := \sum_{s=0}^{\infty} b_{l,s} u^{-s-1}. \quad (23)$$

For $l \in I_1$, $i = 1, 2, \dots, d_l$, we introduce the following generating series in $U(\mathfrak{a}_{\underline{d}}^\varepsilon)$ (warning: these are not in $\mathcal{Y}_{\underline{d}}^\varepsilon$):

$$b_l^{(i)}(u) := \sum_{s=0}^{\infty} b_{l,s}^{(i)} u^{-s-1}, \quad b_{l,s}^{(i)} := (A_l^s p_l)^{(i)}, \quad (24)$$

the i -th coordinate of the vector $A_l^s p_l$ in the orthonormal basis of V_l . We then have the following result (see [FR14]).

LEMMA 4.4. *The following relations hold:*

$$(u - v)[b_k(u), b_k(v)] = (b_k(u)b_k(v) + b_k(v)b_k(u)) \text{ for } k \in I_0, \quad (25)$$

$$2(u - v)[b_k(u), b_l(v + d_l)] = -(b_k(u)b_l(v + d_l) + b_l(v + d_l)b_k(u)) \text{ for } k, l \in I_0, l = k + 1, \quad (26)$$

$$(u - v)[a_k(u), b_l(v)] = -\frac{\delta_{kl}}{u - v} b_l(v) a_k(u) \text{ for } k \in I, l \in I_0. \quad (27)$$

$$2(u-v)[b_k^{(i)}(u), b_k^{(j)}(v)] = c_{kl}(b_k^{(i)}(u)b_k^{(j)}(v) + b_k^{(j)}(v)b_k^{(i)}(u)) \text{ for } k \in I_1, i, j = 1, \dots, d_l, \quad (28)$$

$$2(u-v)[b_k^{(i)}(u), b_l(v+d_l)] = c_{kl}(b_k^{(i)}(u)b_l(v+d_l) + b_l(v+d_l)b_k^{(i)}(u)) \text{ for } k \in I_1, l = k+1, i = 1, \dots, d_l, \quad (29)$$

$$2(u-v)[b_k(u), b_l^{(i)}(v+d_l)] = c_{kl}(b_k(u)b_l^{(i)}(v+d_l) + b_l^{(i)}(v+d_l)b_k(u)) \text{ for } l \in I_1, l = k+1, i = 1, \dots, d_l, \quad (30)$$

$$(u-v)[a_k(u), b_l^{(i)}(v)] = -\frac{\delta_{kl}}{u-v}b_l^{(i)}(v)a_k(u) \text{ for } k \in I, l \in I_1, i = 1, \dots, d_l. \quad (31)$$

Proof. This follows from Propositions 3.24 and 3.32 of [FR14]. \square

LEMMA 4.5. *We have*

$$[b_{k,r_2}, [b_{k,r_1}, b_{l,s}]] + [b_{k,r_1}, [b_{k,r_2}, b_{l,s}]] = 0 \text{ for } k, l \in I_0, |k-l| = 1, \quad (32)$$

$$[b_{k,r_2}, [b_{k,r_1}, b_{l,s}^{(i)}]] + [b_{k,r_1}, [b_{k,r_2}, b_{l,s}^{(i)}]] = 0 \text{ for } k \in I, l \in I_1, |k-l| = 1, i = 1, \dots, d_l, \quad (33)$$

$$[b_{k,r_2}^{(i)}, [b_{k,r_1}^{(j)}, b_{l,s}]] + [b_{k,r_1}^{(j)}, [b_{k,r_2}^{(i)}, b_{l,s}]] = 0 \text{ for } k \in I_1, l \in I, |k-l| = 1, i, j = 1, \dots, d_k. \quad (34)$$

Proof. This follows from Proposition 3.32 of [FR14]. \square

For $l \in I$, let $D_l(u)$ be the (unique) solution of the functional equation

$$a_l(u) := D_l(u - \frac{1}{2})D_l(u + \frac{1}{2})^{-1}. \quad (35)$$

We have

$$D_l(u)b_k(v) \frac{2u-2v+\delta_{kl}}{2u-2v-\delta_{kl}} = b_k(v)D_l(u) \text{ for } k \in I_0$$

and

$$D_l(u)b_k^{(i)}(v) \frac{2u-2v+\delta_{kl}}{2u-2v-\delta_{kl}} = b_k^{(i)}(v)D_k(u) \text{ for } k \in I_1.$$

Set $\widetilde{b}_k^{(i)}(u) := D_k(u - \frac{1}{2})^{-1}b_k^{(i)}(u)$. From Lemma 4.4, we have

$$\widetilde{b}_k^{(i)}(u)\widetilde{b}_k^{(j)}(v) = \widetilde{b}_k^{(j)}(v)\widetilde{b}_k^{(i)}(u).$$

Note that the rest of the relations for $b_k^{(i)}$ from Lemma 4.4 also hold for $\widetilde{b}_k^{(i)}$.

For $l \in I_1, i, j = 1, \dots, d_l$, set

$$\widetilde{b}_l^{(ij)}(u) := D(u - \frac{1}{2})\widetilde{b}_l^{(i)}(u)\widetilde{b}_l^{(j)}(u+1). \quad (36)$$

Note that

$$\widetilde{b}_l^{(ii)}(u)\widetilde{b}_l^{(jj)}(v) \frac{u-v+2}{u-v-2} = \widetilde{b}_l^{(jj)}(v)\widetilde{b}_l^{(ii)}(u) \quad (37)$$

and

$$\widetilde{b}_l^{(ii)}(u)b_k(v)\frac{u-v+1}{u-v-1} = b_k(v)\widetilde{b}_l^{(ii)}(u) \tag{38}$$

for $|k-l|=1$.

For $l \in I_1$, set $\widetilde{b}_l(u) = \sum_{i=1}^{d_l} \widetilde{b}_l^{(ii)}(u)$. By Proposition 3.6, we see that the algebra $\mathcal{Y}_d^\varepsilon$ is generated by (the Fourier coefficients of) $D_l(u)$, $b_k(u)$ for $k \in I_0$ and $\widetilde{b}_k(u)$ for $k \in I_1$. Now the theorem reduces to the following result.

LEMMA 4.6. *There is a homomorphism $\varphi_d : \widehat{\mathcal{Y}}^\varepsilon \rightarrow \mathcal{Y}_d^\varepsilon$ sending $\mathbf{A}_k(u)$ to $D_k(u + \sum_{m=1}^k d_m)$ and $\mathbf{x}_l^+(u)$ to $b_l(u + \sum_{m=1}^l d_m)$ for $l \in I_0$ and to $\widetilde{b}_l(u + \sum_{m=1}^l d_m)$ for $l \in I_1$.*

Proof. We need to prove the quadratic relations and the Serre relations for the elements $D_l(u)$, $b_k(u)$. The quadratic relations follow from the relations (37) and (38). The proof of the Serre relations is entirely similar to that of Proposition 3.32 from [FR14]. \square

According to the Newton identity (see Theorem 7.1.3 of [Mol07]), we have

$$a_l(u) = \frac{C_l(-u + d_l)}{C_l(-u + d_l - 1)}, \tag{39}$$

where $C_l(u)$ is the Capelli determinant. This means that $D(u) = C(-u + d_l - \frac{1}{2})$. In particular, $D_{l,r} = 0$ for $r > d_l$. \square

CONJECTURE 4.7. $\mathcal{Y}_d^\varepsilon = \widehat{\mathcal{Y}}^\varepsilon / \{\mathbf{A}_{k,r} \mid r > d_k\}$.

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