MOSCOW MATHEMATICAL JOURNAL
Volume 15, Number 1, January–March 2015, Pages 31–48

ON PROJECTIONS OF SMOOTH AND NODAL PLANE CURVES

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ABSTRACT. Suppose that $C \subset \mathbb{P}^2$ is a general enough nodal plane curve of degree > 2, $\nu\colon \hat{C} \to C$ is its normalization, and $\pi\colon C' \to \mathbb{P}^1$ is a finite morphism simply ramified over the same set of points as a projection $\mathrm{pr}_p \circ \nu\colon \hat{C} \to \mathbb{P}^1$, where $p \in \mathbb{P}^2 \setminus C$ (if $\deg C = 3$, one should assume in addition that $\deg \pi \neq 4$). We prove that the morphism π is equivalent to such a projection if and only if it extends to a finite morphism $X \to (\mathbb{P}^2)^*$ ramified over C^*, where X is a smooth surface.

As a by-product, we prove the Chisini conjecture for mappings ramified over duals to general nodal curves of any degree $\geqslant 3$ except for duals to smooth cubics; this strengthens one of Victor Kulikov's results.

2010 MATH. SUBJ. CLASS. Primary: 14H50; Secondary: 14D05, 14N99.

KEY WORDS AND PHRASES. Plane algebraic curve, projection, monodromy, Picard–Lefschetz theory, Chisini conjecture.

1. Introduction

Let $C \subset \mathbb{P}^2$ be a projective curve of degree $d > 2$ such that all its singularities are nodes, and let $\nu\colon \hat{C} \to C$ be the normalization mapping. For a point $p \notin C$ consider the projection $\mathrm{pr}_p\colon C \to \mathbb{P}^1 = p^\perp$ from p; here $p^\perp \subset \mathbb{P}^2$ is the set of lines passing through p, and pr_p assigns to x the line joining p and x. The composition $\mathrm{pr}_p \circ \nu\colon \hat{C} \to \mathbb{P}^1$ will be called a generalized projection; its branch locus coincides with $p^\perp \cap C^*$, where $C^* \subset (\mathbb{P}^2)^*$ is the curve dual to C. The generalized projection has simple ramification if and only if $p^\perp$ is transversal to C^*.

Suppose that $\pi\colon C' \to p^\perp$, where C' is a smooth projective curve, is a holomorphic mapping with simple ramification and such that the branch locus of π coincides with $p^\perp \cap C^*$. We are looking for a criterion for π to be isomorphic to the generalized projection $\mathrm{pr}_p \circ \nu$.

Received April 16, 2014; in revised form October 16, 2014.

The first named author was supported by the Lady Davis foundation visiting professorship grant, HSE Scientific Foundation individual research grant 12-01-0015, the MON grant 02.740.11.0608, the Simons–IUM fellowship by the Simons foundation, by the 2013 Dobrushin professorship grant and by the RFBR grant N.Sh.5138.2014.1 ("Scientific school of V. I. Arnold"). The second named author was supported in part by AG Laboratory HSE..

Problem background. Consider first the case when C is smooth. An easy dimension count shows that if $d > 3$ then the branch locus of a projection is not arbitrary. Namely, it follows from the Riemann–Hurwitz formula that a degree d map from a curve of degree d has $d(d-1)$ critical values, provided the ramification is simple. So, its branch locus is a point of $\mathrm{Sym}^{d(d-1)}\mathbb{P}^1 = \mathbb{P}^{d(d-1)}$. The space of projective curves $C \subset \mathbb{P}^2$ of degree d has dimension $\binom{d+2}{2} - 1 = d(d+3)/2$. For a point $p \in \mathbb{P}^2$ there is a 3-dimensional group of projective automorphisms $\mathbb{P}^2 \to \mathbb{P}^2$ preserving p and all the lines containing p. So, the space of all projections has the dimension $m_d := d(d+3)/2 - 3 = (d^2 + 3d - 6)/2 \ll d(d-1)$ for large d.

Consider now small d. For $d = 1$ and $d = 2$ the situation is trivial. For $d = 3$ one has $m_3 = 6 = 3(3-1)$, and indeed, it is easy to see that any 6 pairwise distinct points $a_1, \dots, a_6 \in \mathbb{P}^1$ are branch points of a suitable generic projection of a cubic curve.

Let now $d = 4$. Here $m_4 = 11$, $4(4-1) = 12$, so branch loci of generic projections lie in a hypersurface $V \subset \mathrm{Sym}^{12}\mathbb{P}^1 = \mathbb{P}^{12}$. Take a generic point $a \in V$; then the number of pairs (C', π) such that C' is a smooth curve of genus 3 and a is the branch locus of a degree 4 mapping $\pi\colon C \to \mathbb{P}^1$, is equal to $255N$ where $N = 12(3^{10} - 1)$ (it follows from the general formula for Hurwitz numbers, see e.g. [KL07]). R. Vakil showed in [Vak01] that for $120N$ pairs the curve C' is a plane quartic and π is isomorphic to a projection, and for the remaining $135N$ pairs the curve C is hyperelliptic, hence not plane.

For $d > 4$ it is easy to derive from [EGH96, Proposition 1] that if C is a smooth plane projective curve then each degree d mapping $\pi\colon C \to \mathbb{P}^1$ is isomorphic to a projection. We do not, though, assume C' to be a plane curve. For C nodal a degree d map $\hat{C} \to \mathbb{P}^1$ does not need to be equivalent to a projection.

Main results. To make our problem more tractable we impose some generality hypotheses on the curves and mappings in question. To wit, we will assume that the mappings $\pi\colon \hat{C} \to \mathbb{P}^1$ have simple ramification (see Definition 1.5 below) and that the nodal curve $C \subset \mathbb{P}^2$ is general enough in the sense of Definition 1.6. If $\nu\colon \hat{C} \to C$ is the normalization, then for the general point $p \in \mathbb{P}^2 \setminus C$ the generalized projection $\mathrm{pr}_p \circ \nu\colon \hat{C} \to \mathbb{P}^1$ has simple ramification.

We will say that the holomorphic mappings $\varphi_1\colon C_1 \to \mathbb{P}^1$ and $\varphi_2\colon C_2 \to \mathbb{P}^1$ are *equivalent* if there exists a holomorphic isomorphism $\psi\colon C_1 \to C_2$ such that $\varphi_2 \circ \psi = \varphi_1$.

Theorem 1.1. *Suppose that $C \subset \mathbb{P}^2$ is a nodal curve of degree > 2 that is general enough in the sense of Definition 1.6. Suppose that a point $p \in \mathbb{P}^2 \setminus C$ is such that the composition $\mathrm{pr}_p \circ \nu\colon \hat{C} \to \mathbb{P}^1$, where $\nu\colon \hat{C} \to C$ is the normalization and $\mathrm{pr}_p\colon C \to p^\perp = \mathbb{P}^1$ is the projection from p, has simple ramification. Suppose that C' is a smooth projective curve and $\pi\colon C' \to p^\perp$ is a holomorphic mapping with simple ramification such that the branch locus of π coincides with $p^\perp \cap C^*$. If $\deg C = 3$, assume in addition that C has a node or $\deg \pi \neq 4$. Then the following two conditions are equivalent*:

(a) *π is equivalent to $\mathrm{pr}_p \circ \nu$: there exists an isomorphism $\varphi\colon C' \to \hat{C}$ such that $(\mathrm{pr}_p \circ \nu) \circ \varphi = \pi$.*

(b) *There exist a smooth projective surface X, a finite holomorphic mapping $f\colon X \to (\mathbb{P}^2)^*$ that is ramified exactly over C^*, and an isomorphism $\varphi\colon C' \to f^{-1}(p^\perp)$ such that $f|_{f^{-1}(p^\perp)} \circ \varphi = \pi$.*

Remark 1.2. The implication (a) $\Rightarrow$ (b) holds without the genericity hypotheses, for arbitrary nodal curves and arbitrary generalized projections. See the proof of Proposition 2.1 below.

Remark 1.3. If C is a smooth cubic and $\deg f = 4$ then the theorem is wrong; see Remark 4.14.

Remark 1.4. The condition of smoothness of X in the theorem cannot be omitted, at least for $\deg f$ equal to 3 or 4: in Section 5 we show that for $d = 3$ and 4 there exists a general enough smooth curve $C \subset \mathbb{P}^2$ of degree d, a line $p^\perp \subset (\mathbb{P}^2)^*$ (for some $p \in \mathbb{P}^2$) transversal to C^*, a smooth projective curve C', and a morphism $\pi\colon C' \to p^\perp$ simply ramified over $p^\perp \cap C^*$ such that condition (b) of Theorem 1.1 with the smoothness omitted holds but π is not equivalent to the projection pr_p (Proposition 5.1). We do not know similar counterexamples with $d > 4$.

In other words, if we are given a simply ramified covering of $\mathbb{P}^1$ with the same branch locus as that of a (generalized) projection from a given point p, then this covering is the generalized projection if and only if it can be extended to a finite ramified covering of $(\mathbb{P}^2)^*$ branched over C^*, with a smooth surface upstairs.

Theorem 1.1 may be restated in topological terms: we will show that if $L \subset (\mathbb{P}^2)^*$ is a projective line then the mapping $\pi\colon C' \to L$ is equivalent to the generalized projection $\mathrm{pr}_p \circ \nu$ if and only if the covering $\pi^{-1}(L \setminus C^*) \to L \setminus C^*$ can be extended to a covering $X_0 \to (\mathbb{P}^2)^* \setminus C^*$ such that its fiber monodromy satisfies some extra conditions (C^* must have no bad nodes or bad cusps in the sense of Definitions 3.2 and 3.3); see Theorem 4.15.

Note that we make (almost) no assumption about genus of the curve C' or degree of the morphism f.

The (a) $\Rightarrow$ (b) implication in Theorem 1.1 is easy (see Section 2). If $\deg C \geqslant 7$, the (b) $\Rightarrow$ (a) implication follows very easily from Theorem 10 of Victor Kulikov's paper [Kul99] (see Proposition 4.2 below), but for the remaining cases $3 \leqslant \deg C \leqslant 6$, where the Chisini conjecture for coverings ramified over C^* is not completely proved, it requires more work. We give an argument that works uniformly in all cases, including the case $\deg C \geqslant 7$; our proof is quite different from Kulikov's one. As a by-product, we establish the Chisini conjecture for coverings of $\mathbb{P}^2$ ramified over curves dual to general enough nodal curves of degrees 4, 5, and 6 (and of degree 3 provided that either the curve does have a node or degree of the covering is not 4). This extends Theorem 10 from [Kul99].

The paper is organized as follows. In Section 2 we prove the easy part of Theorem 1.1. In Section 3 we study coverings of the projective plane that are ramified over a curve having only nodes and standard cusps as singularities, and in Section 4 we prove the difficult part of Theorem 1.1. Finally, in Section 5 we show that, at least if $\deg C = 3$ and $\deg C = 4$, the smoothness condition in Theorem 1.1 cannot be omitted.

The authors would like to thank Maxim Kazarian, Victor Kulikov, Alexander Kuznetsov, Sergey Lando, Sergey Natanzon, Ossip Schwarzman, Andrey Soldatenkov, and Victor Vassiliev for useful discussions. We are grateful to Igor Khavkine, Francesco Polizzi, Damian Rössler, Will Sawin, and particularly Rita Pardini for consultations at http://mathoverflow.net.

The first named author wishes to thank the Technion — Israel Institute of Technology, where the final version of the paper was written, for the warm hospitality.

Our special thanks go to Boris Shapiro, at whose instigation this research was started. Without his constant encouragement this paper would never have been written.

Notation, Conventions, and Definitions

The base field for all the algebraic varieties will be the field $\mathbb{C}$ of complex numbers.

Except for the case when we mention explicitly Zariski topology, all topological spaces will be assumed Hausdorff and locally simply connected, and all the topological terms will refer to the classical topology of complex algebraic varieties.

We will say that a singular point of a curve is a *node* if it is locally analytically isomorphic to the singularity of the plane curve defined by the equation $xy = 0$, and that a singular point is a *standard cusp* (or simply a cusp) if it is locally analytically isomorphic to the singularity of the plane curve defined by the equation $y^2+x^3 = 0$.

If $f\colon X \to Y$ is a finite holomorphic map of smooth varieties of equal dimension, then by *ramification locus* (a.k.a. the set of critical points) of f we mean the closed subset

$$R = \{x \in X\colon \text{derivative of } f \text{ is degenerate at } x\};$$

by *branch locus* (a.k.a. the set of critical values) of f we mean the subset $f(R) \subset Y$.

Let $f\colon W \to V$ be a covering of topological spaces, and take $v_0 \in V$. Then the action of the group $\pi_1(V, v_0)$ on the set $f^{-1}(v_0)$ via loop lifting will be called the *fiber monodromy* of f.

If $f\colon Y \to X$ is a finite morphism of algebraic varieties of equal dimension (i.e., a proper holomorphic map with finite fibers) and $B \subset X$ is its branch locus, then the restriction of f to $f^{-1}(X \setminus B) \subset Y$ is a finite-sheeted covering of topological spaces. Take $x_0 \in X \setminus B$. Then by fiber monodromy of f we will mean fiber monodromy of the covering $f^{-1}(X \setminus B) \to Y \setminus B$ with respect to the point x_0.

Suppose now that $X = \mathbb{P}^1$, so the branch locus $B \subset \mathbb{P}^1$ is a finite set. If $p \in B$ and γ_p is the conjugacy class in $\pi_1(\mathbb{P}^1 \setminus B)$ corresponding to a small loop around p, then the image of γ_p under the fiber monodromy homomorphism is a conjugacy class in the symmetric group S_d, where d is the number of points in $f^{-1}(x_0)$. This conjugacy class (or the corresponding partition of d) is called the *cyclic type* of the point p.

Definition 1.5. Suppose that C is a smooth projective curve. We will say that a finite morphism $f\colon C \to \mathbb{P}^1$ has *simple ramification* if each branch point $\xi \in \mathbb{P}^1$ has the cyclic type of a transposition.

If $C \subset \mathbb{P}^2$ is a nodal curve and $\nu\colon \hat{C} \to C$ is the normalization, then the generalized projection of $\hat{C}$ from a general point of $\mathbb{P}^2$ has simple ramification.

Definition 1.6. Suppose that $C \subset \mathbb{P}^2$ is an irreducible projective curve such that each singular point of C is a node. Let us say that C is *general enough* if the following conditions are satisfied.
– all the inflexion points of C are simple;
– no line is tangent to C at more than two points;
– if a line is tangent to C at an inflexion point, it is not tangent to C elsewhere.

Here, we assume that a line is tangent to C if it is either tangent to C at a smooth point or tangent to a branch of C at a node. By inflexion point we mean either a smooth point $x \in C$ for which the local intersection index of C with its tangent at x is greater than 2, or a node $x \in C$ for which the local intersection index of at least one of the two (limiting) tangent lines to branches a x with the corresponding branch is greater than 2; we say that an inflexion point is *simple* if the intersection index in question equals exactly 3.

It follows from the hypotheses on C that the dual curve $C^* \subset (\mathbb{P}^2)^*$ has no inflexion points, all the singularities of C^* are nodes and standard cusps, and no line is tangent to C^* at three points (bitangents of C^* correspond to nodes of C).

For a projective subspace $\alpha \subset \mathbb{P}^n$ we denote by $\alpha^\perp \subset (\mathbb{P}^n)^*$ the set of hyperplanes in $\mathbb{P}^n$ containing α, where $(\mathbb{P}^n)^*$ is the dual projective space. If $\dim \alpha = k$ then $\alpha^\perp \subset (\mathbb{P}^n)^*$ is a projective subspace of dimension $n-1-k$; in particular, if α is a point then $\alpha^\perp$ is a hyperplane (denoted by H_α in SGA7, see [DK73]).

If $X \subset \mathbb{P}^n$ is a projective variety, then by $X^* \subset (\mathbb{P}^n)^*$ we will denote its projective dual, i.e., the closure of the set of hyperplanes tangent to X at smooth points.

We also will be using the notation $\alpha^\perp$ and X^* when $\alpha,\, X \subset (\mathbb{P}^n)^*$, where the canonical isomorphism $((\mathbb{P}^n)^*)^* = \mathbb{P}^n$ is assumed.

If $X \subset \mathbb{P}^2$ is a projective curve, we say that a line $L \subset \mathbb{P}^2$ is transversal to X if L does not pass through singular points of X and is not tangent to X at any non-singular point.

If D_1 and D_2 are Cartier divisors on a projective surface, then $(D_1,\, D_2)$ stands for their intersection index.

2. Proof of the (a) ⇒ (b) Implication in Theorem 1.1

This part of Theorem 1.1 follows immediately from

Proposition 2.1. *Suppose that $C \subset \mathbb{P}^2$ is a nodal curve and $\nu\colon \hat{C} \to C$ is its normalization. Then there exist a smooth projective surface X_C and a finite regular mapping $f_C\colon X_C \to (\mathbb{P}^2)^*$ having the dual curve $C^* \subset (\mathbb{P}^2)^*$ as its branch locus and such that for any point $p \in \mathbb{P}^2 \setminus C$ there is an isomorphism $\varphi\colon \hat{C} \to f_C^{-1}(p^\perp)$ such that $f_C \circ \varphi = \mathrm{pr}_p \circ \nu$ (as usual, $\mathrm{pr}_p\colon C \to p^\perp$ is the projection from p).*

Proof. Put

$$X_C = \{(x,\, t) \in \hat{C} \times (\mathbb{P}^2)^* \colon \nu(x) \in t^\perp\}. \tag{1}$$

The surface X_C, being the projectivization of the vector bundle $\nu^*\mathcal{T}_{\mathbb{P}^2}(-1)$, is smooth.

Define the mapping $f_C\colon X_C \to (\mathbb{P}^2)^*$ by the formula

$$f(x, t) = t. \tag{2}$$

For $t \in (\mathbb{P}^2)^*$,

$$f_C^{-1}(t) = \{(x, t)\colon \nu(x) \in t^\perp\};$$

f is a finite morphism of degree d since for the general $t \in (\mathbb{P}^2)^*$ the line $t^\perp$ intersects C at d smooth points. Its branch locus is the set of t such that $t^\perp$ is either tangent to C at a smooth point or tangent to a branch of C at a node, so the branch locus of f coincides with C^*.

If $p \in \mathbb{P}^2 \setminus C$ then

$$f_C^{-1}(p^\perp) = \{(x, t) \in \hat{C} \times (\mathbb{P}^2)^*\colon \nu(x),\, p \in t^\perp\}.$$

It is clear that the mapping $\varphi\colon \hat{C} \to f^{-1}(p^\perp)$ sending x to the pair $(x, \overline{px})$ is the required isomorphism. □

3. Plastering of Generic Coverings over Complements to Curves

In this section we prepare for the proof of the implication (b) ⇒ (a) in Theorem 1.1.

We start with a simple general fact. Suppose that $D \subset \mathbb{P}^2$ is an arbitrary projective curve, X_0 is a smooth affine surface and $f_0\colon X_0 \to \mathbb{P}^2 \setminus D$ is a finite (topological) covering.

Proposition 3.1. *Suppose that, in the above setting, there exists a line $L_0 \subset \mathbb{P}^2$ transversal to D such that the induced covering $f_0^{-1}(L_0) \to L_0 \setminus D$ has simple ramification. Then for any line $L \subset \mathbb{P}^2$ transversal to D the induced covering $f_0^{-1}(L) \to L \setminus D$ also has simple ramification.*

Following [Kul99], we will say that coverings satisfying hypotheses of the proposition are *generic*.

Proof of the proposition. The set

$$U = \{t \in (\mathbb{P}^2)^*\colon t^\perp \text{ is transversal to } D\}$$

is arcwise connected; let $\gamma\colon [0, 1] \to U$ be a path such that $\gamma(0) = L_0$, $\gamma(1) = L$. Let also

$$\mathcal{X} = \{(x, t) \in (\mathbb{P}^2 \setminus D) \times U\colon x \in t^\perp\};$$

the natural projection $p\colon \mathcal{X} \to U$ is a locally trivial fiber bundle with the fiber homeomorphic to $\mathbb{P}^1$ punctured at $\deg D$ points. The pullback covering $p^*f_0\colon \tilde{\mathcal{X}} \to \mathcal{X}$ restricted to $p^{-1}(t) \subset \mathcal{X}$ is isomorphic to the covering $f_0^{-1}(t^\perp) \to t^\perp$. The pullback bundle $\gamma^*p\colon \overline{X} \to [0, 1]$ is trivial, so the proposition follows. □

If $f_0\colon X_0 \to \mathbb{P}^2 \setminus D$ is a covering, then X_0 carries a unique structure of complex variety for which the mapping f is a finite unramified morphism. Suppose now that the covering is generic and all the singularities of the curve D are nodes or standard cusps. It follows from a theorem of Grauert and Remmert [Gro71b, Exposé XII, Théorème 5.4] that in this situation f_0 extends uniquely to a finite mapping $f\colon X \to \mathbb{P}^2$ with normal X. A well-known GAGA-style result [Gro71b, Exposé 12, Corollaire 4.6] implies that the resulting complex space X will be a projective surface. We are going now to describe explicitly the singularities of surface X as well as the structure of the mapping f near ramification locus.

In the sequel, Δ will denote the unit disk $\{z\colon |z| < 1\} \subset \mathbb{C}$, and Δ^* will stand for the punctured disk $\Delta \setminus \{0\}$.

Suppose that q is a node of D; choose a neighborhood $U \ni q$ and an isomorphism $U \to \Delta^2$ such that $D \cap U$ is mapped to the set $\{(x, y) \in D^2\colon xy = 0\}$. Fix a point $q_0 \in U \setminus D$. The covering $f_0^{-1}(U \setminus D) \to U \setminus D$ induces a fiber monodromy action of the group $\pi_1(U \setminus D, q_0) = \mathbb{Z} \oplus \mathbb{Z}$ on the set $f^{-1}(q_0)$.

The two generators of $\pi_1(U \setminus D, q_0)$ are the small loops around two branches of D at q; since f_0 is a generic covering, Proposition 3.1 shows that the generators act on $f^{-1}(q_0)$ by transpositions. These transpositions commute, so they can be either disjoint or equal.

Definition 3.2. In the above setting, we will say that a node $q \in D$ is *good* (with respect to the covering f_0) if the two generators of $\pi_1(U \setminus D, q_0)$ act by disjoint transpositions.

We will say that a node $q \in D$ is *bad* (with respect to f_0) if they act by equal transpositions.

Suppose now that q is a (standard) cusp of the curve D. Choose a neighborhood $U \ni q$ and an isomorphism $U \to \Delta^2$ such that $D \cap U$ is mapped to the set $\{(x, y) \in D^2\colon x^3 + y^2 = 0\}$. Fix a point $q_0 \in U \setminus D$. The covering $X \setminus f^{-1}(D) \to \mathbb{P}^2 \setminus D$ induces a fiber monodromy action of the group $\pi_1(U \setminus D, q_0)$ on the set $f^{-1}(q_0)$. It is well known that $\pi_1(U \setminus D, q_0)$ is isomorphic to the Artin braid group on 3 strings $B_3 = \langle u, v\colon uvu = vuv\rangle$, where the generators u and v correspond to the two loops around two intersection points $\ell \cap D$, where ℓ is a line close to q.

Since f_0 is a generic covering, the elements u and v of $\pi_1(U \setminus D, q_0)$ act on $f^{-1}(q_0)$ by transpositions. These transpositions (call them U and V) satisfy the relations $UVU = VUV$, so they are either non-commuting or equal.

Definition 3.3. In the above setting, we will say that the cusp $q \in D$ is *good* (with respect to the covering f_0) if the transpositions U and V are non-commuting.

We will say that the cusp $q \in D$ is *bad* (with respect to f_0) if $U = V$.

Proposition 3.4 (cf. [Kul99, Section 1.3])**.** *The local behavior of the mapping f near points of the preimage $f^{-1}(q)$, $q \in D$, is as follows:*

(1) *If q is a smooth point of D then $f^{-1}(q)$ consists of $d - 1$ points, and X smooth at all of them. At exactly one of them f is ramified, and near this point f is locally isomorphic to $f(x, y) = (x, y^2)$.*
(2) *If q is a good node then $f^{-1}(q)$ consists of $d - 2$ points, and X is smooth at all of them. At exactly two of them f is ramified, with the same local behavior as in Case 1.*
(3) *If q is a bad node then $f^{-1}(q)$ consists of $d - 1$ points. All of them except one are smooth. The remaining point is singular, locally isomorphic to $\{(x, y, z) \in \Delta^3\colon z^2 = xy\}$ (Du Val's A_1); the mapping f in the same coordinates is $f(x, y, z) = (x, y)$.*

(4) *If q is a good cusp then $f^{-1}(q)$ consists of $d-2$ points, and X is smooth at all of them. At exactly one of them f is ramified, and the pair (X, f) near this point is locally isomorphic to $X = \{(x, y, z) \in \Delta^3 \colon z^3+3xz+2y=0\}$, $f(x, y, z) = (x, y)$.*
(5) *If q is a bad cusp then $f^{-1}(q)$ consists of $d-1$ points. All of them except one are smooth, the remaining point is singular, locally isomorphic to $\{(x, y, z) \in \Delta^3 \colon z^2 = x^3+y^2\}$ (Du Val's A_2); the mapping f in the same coordinates is $f(x, y, z) = (x, y)$.*

Proof. A direct computation shows that the branch loci B of the local models described above are as follows: $B = \{(x, 0) \colon x \in \Delta\}$ for case 1, $B = \{(x, y) \in \Delta^2 \colon xy = 0\}$ for the cases 2 and 3, and $B = \{(x, y) \in \Delta^2 \colon x^3 + y^2 = 0\}$ for the cases 4 and 5. Thus, in all the cases the branch loci are locally biholomorphic to the curve D in a neighbourhood of q.

By the uniqueness statement of the Grauert–Remmert theorem it is enough to check that the monodromy action of $\pi_1(\mathbb{P}^2 \setminus D)$ on the fiber $f_0^{-1}(q) \subset X$ is isomorphic to the action of $\pi_1(\Delta^2 \setminus B)$ on the fiber over 0 in the local model. If $a \in f^{-1}(q)$ is a smooth point such that f has the normal form $f(z) = z^k$ in it, then the monodromy action is a cycle of length k; this proves the statement in cases 1 and 2. In case 3 the branch locus is a union of two lines; the group $\pi_1(\Delta^2 \setminus B)$ is generated by two loops circling around these lines; apparently, they both act by the same transposition of the two preimages of the origin.

To cover cases 4 and 5 notice that $\pi_1(\Delta^2 \setminus B) = B_3$ in both of them. Any homomorphism $B_3 \to S_3$ maps the standard generators of the group into permutations U and V satisfying $UVU = VUV$; it is easy to see that such U and V must be transpositions, either equal or non-commuting. So, there exist only two actions, up to conjugation, of B_3 on a set of three elements; this proves the statement in cases 4 and 5. □

Suppose that $C \subset \mathbb{P}^2$ is a nodal projective curve that is general enough in the sense of Definition 1.6 and that $L \subset (\mathbb{P}^2)^*$ is a line transversal to C^*.

Suppose in addition that $\pi \colon C' \to L$, where C' is a smooth projective curve, is a holomorphic mapping with simple ramification and with branch locus $L \cap C^*$. Choose a point $t_0 \in L \setminus C^*$.

Corollary 3.5. *If fiber monodromy homomorphism $\pi_1(L \setminus C^*, t_0) \to \mathrm{Perm}(\pi^{-1}(t_0))$ factors through the homomorphism $\pi_1(L \setminus C^*, t_0) \to \pi_1(\mathbb{P}^2 \setminus C^*, t_0)$ induced by the embedding $L \hookrightarrow (\mathbb{P}^2)^*$, then there exists a unique pair (X, f), where X is a normal projective surface and $f \colon X \to (\mathbb{P}^2)^*$ is a finite holomorphic mapping such that $f|_{f^{-1}(L)} = \pi$.*

Over each node of C^*, there is at most one singular point of X, and it must be a Du Val A_1-singularity. Over each cusp of C^*, there is at most one singular point of X, and it must be a Du Val A_2-singularity. The other points of X are smooth.

Proof. Since the monodromy $\pi_1(L \setminus C^*, t_0) \to \mathrm{Perm}(\pi^{-1}(t_0))$ factors through the homomorphism $\pi_1(L \setminus C^*, t_0) \to \pi_1(\mathbb{P}^2 \setminus C^*, t_0)$, there exists a topological covering $X_0 \to (\mathbb{P}^2)^* \setminus C^*$ of degree $\deg \pi$ extending the covering $\pi^{-1}(L \setminus C^*) \to L \setminus C^*$. The restriction of f_0 to the line transversal to C^* has simple ramification, so f_0 is a generic covering by Proposition 3.1. The rest follows from Proposition 3.4 with $D = C^*$. ☐

4. Smooth Surfaces Simply Ramified over Duals to General Nodal Curves

In this section we prove the (b) $\Rightarrow$ (a) implication in Theorem 1.1.

Throughout this section we will be working in the following setting. $C \subset \mathbb{P}^2$ is a nodal curve that is general enough in the sense of Definition 1.6, $C^* \subset (\mathbb{P}^2)^*$ is the dual curve, X is a smooth projective surface, $f\colon X \to (\mathbb{P}^2)^*$ is a finite holomorphic mapping with simple ramification and branch locus C^*. Recall that the smooth surface X_C and the mapping $f_C\colon X_C \to (\mathbb{P}^2)^*$ were defined by equations (1) and (2), respectively.

For the point $p \in \mathbb{P}^2$ denote by $\mathrm{pr}_p\colon C \to p^\perp = \mathbb{P}^1$ the projection of C from p. Also denote by $\nu\colon \hat{C} \to C$ the normalization mapping.

Proposition 4.1. *Suppose that $\Phi\colon X \to X_C$ is an isomorphism such that $f_C \circ \Phi = f$. Let p be such that the composition $\mathrm{pr}_p\colon C \to p^\perp$ has simple ramification. Then for $C' := f^{-1}(p^\perp)$ there exists an isomorphism $\varphi\colon \hat{C} \to C'$ such that $f \circ \varphi = \mathrm{pr}_p \circ \nu$.*

Proof. The required isomorphism φ is just the restriction of Φ^{-1} to $f_C^{-1}(p^\perp)$. ☐

Proposition 4.2. *The* (b) $\Rightarrow$ (a) *implication in Theorem* 1.1 *holds in each of the following cases.*

(1) $\deg C \geqslant 7$;
(2) $\deg C = 6$ *and C has at least one node*;
(3) $\deg C = 5$ *and C has at least three nodes.*

Proof. It follows imeediately from [Kul99, Theorem 10] and Plücker formulas for the degree of the dual to a plane curve that if C satisfies the hypothesis of the proposition, then a finite mapping $f\colon X \to (\mathbb{P}^2)^*$ with simple ramification over C^* is unique. Thus, the mapping $f\colon X \to (\mathbb{P}^2)^*$ is equivalent to $f_C\colon X_C \to (\mathbb{P}^2)^*$, and Proposition 4.1 applies. ☐

The rest of this section is devoted to the proof of the (b) $\Rightarrow$ (a) implication in Theorem 1.1 in the remaining cases. This proof is independent of the results of [Kul99] and works for all the cases in Theorem 1.1, including those covered by Proposition 4.2.

Proposition 4.3. *Suppose that $L \subset (\mathbb{P}^2)^*$ is a line such that the point $L^\perp$ does not lie on C. Then the curve $f^{-1}(L) \subset X$ is smooth.*

Proof. By virtue of projective duality one has $C = (C^*)^*$. Hence, for a line $L \subset (\mathbb{P}^2)^*$, one has $L^\perp \in C$ if and only if L either is tangent to C^* at a smooth point, or is tangent to a branch of C^* at a node (we call such L a tangent to C at the node), or is the (limiting) tangent to C at a cusp. In the third case $L^\perp$ is an inflexion point of C, in the second case it lies on a double tangent to C. Now the result follows immediately from Proposition 3.4, where one puts $D = C^*$. ☐

Proposition 4.4. *Suppose that a line $L \subset (\mathbb{P}^2)^*$ is tangent to C^* (at a smooth point, a node, or a cusp). Then the curve $f^{-1}(L)$ has singular points only over the points of tangency; moreover, there is exactly one singular point of $f^{-1}(L)$ over each point of tangency, and this singular point is a node.*

Proof. Since the curve $(C^*)^* = C$ has no cusps, the curve C^* has no inflexion points; now everything follows from Proposition 3.4. □

Let the line $L \subset (\mathbb{P}^2)^*$, where $L^\perp \notin C$, vary. Proposition 4.3 shows that the curve $f^{-1}(L)$ is smooth for such L, so variation of L induces a monodromy action of $\pi_1(\mathbb{P}^2 \setminus C)$ on $H^1(f^{-1}(L), \mathbb{Z})$. More formally, consider the incidence variety

$$I = \{(p, x) \in \mathbb{P}^2 \times X \colon f(x) \in p^\perp\} \tag{3}$$

and denote by $\mathrm{pr}_1 \colon I \to \mathbb{P}^2$ and $\mathrm{pr}_2 \colon I \to X$ the projections. For each $p \in \mathbb{P}^2 \setminus C$ the fiber $\mathrm{pr}_1^{-1}(p)$ is isomorphic to $f^{-1}(p^\perp)$. It follows from Proposition 4.3 that the derivative of pr_1 has maximal rank everywhere on $\mathrm{pr}_1^{-1}(\mathbb{P}^2 \setminus C)$, whence pr_1 restricts to a locally trivial fibration over the preimage of $\mathbb{P}^2 \setminus C$. So, for a point $q \in \mathbb{P}^2 \setminus C$ the fundamental group $\pi_1(\mathbb{P}^2 \setminus C, q)$ acts on $H^1(f^{-1}(q^\perp), \mathbb{Z})$; this is the monodromy action in question.

Proposition 4.5. *The image of the monodromy action of $\pi_1(\mathbb{P}^2 \setminus C, q)$ on the first cohomology $H^1(f^{-1}(q^\perp), \mathbb{Z})$ is cyclic of order dividing $d = \deg C$.*

Proof. Since C is a nodal projective curve, the group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian [Del81], whence it is isomorphic to $H_1(\mathbb{P}^2 \setminus C, \mathbb{Z})$; the latter is obviously cyclic of order d. □

Recall the main result of local Picard–Lefschetz theory for Lefschetz pencils with one-dimensional fiber ([DK73, Exposé XIV, 3.2.5] or [Lam81, Sections 5 and 6]).

Proposition 4.6. *Suppose that $\mathcal{X}$ is a 2-dimensional complex manifold and $p \colon \mathcal{X} \to \Delta$ is a proper surjective holomorphic mapping with the following properties.*

(i) *Over Δ^*, the mapping p has no critical points.*

(ii) *In $p^{-1}(0)$, the mapping p has only one critical point w and the Hessian of p at w is non-degenerate.*

(iii) *All fibers of p are connected.*

Fix a point $z_0 \in \Delta^*$ and put $C = p^{-1}(z_0)$ (in view of (i)*, C is a compact Riemann surface). Then*

(a) *The curve $C_0 = p^{-1}(0)$ is smooth everywhere except for a node at w.*

(b) *The curve C contains an embedded circle S such that C_0 is homeomorphic to the quotient space C/S.*

(c) *The monodromy operator on $H^1(C, \mathbb{Z})$ corresponding to the generator of $\pi_1(\Delta^*)$ is defined by the formula*

$$x \mapsto x - (x, c)c, \tag{4}$$

where $c \in H^1(C, \mathbb{Z})$ is the Poincaré dual of the fundamental class of S.

(The circle S, as well as its Poincaré dual cohomology class, is called *vanishing cycle*.)

Put $C_0 = f^{-1}(\ell)$, where $\ell \subset (\mathbb{P}^2)^*$ is a general line, and denote the genus of C_0 by g.

Proposition 4.7. *Suppose that $L \subset (\mathbb{P}^2)^*$ is a line such that $L^\perp \in C$. Then the curve $f^{-1}(L)$ is connected, and L can be tangent to C^* only at one or two points.*

Proof. The connectedness of $f^{-1}(L)$ (for arbitrary L) follows from [FH79, Proposition 1]. A line L cannot be tangent to C^* at more than two points since all the singularities of the curve $C = (C^*)^*$ are nodes. □

Proposition 4.8. *If a line $L \subset (\mathbb{P}^2)^*$ is tangent to C^* at one point, then the curve $f^{-1}(L)$ is homeomorphic to the quotient space C_0/S, where $S \subset C_0$ is homeomorphic to the circle and S is homologous to zero in C_0.*

Proof. Take a point $t \in L \setminus C^*$, so that $L^\perp \in t^\perp \subset \mathbb{P}^2$, and $t^\perp$ is transversal to C. Then consider the incidence variety

$$\tilde{X} = \{(x, a) \in X \times t^\perp : f(x) \in a^\perp\} \tag{5}$$

(the surface $\tilde{X}$ is just the blow-up of X at $f^{-1}(t)$). The existence of S follows now from Proposition 4.6 applied to the natural projection $\tilde{X} \to t^\perp$ restricted on the preimage of a small disk $\Delta \subset t^\perp$ centered at $L^\perp$.

We are left only to check that the vanishing cycle is zero-homologous. Choosing $z_0 \in \Delta \setminus \{L^\perp\}$ and putting $C_0 = f^{-1}(L)$, observe that $\Delta \setminus L^\perp \subset \mathbb{P}^2 \setminus C$, so the monodromy representation

$$\pi_1(\Delta \setminus \{L^\perp\}) \to \mathrm{Aut}(H^1(C_0, \mathbb{Z})) \tag{6}$$

factors through the monodromy representation

$$\pi_1(\mathbb{P}^2 \setminus C) \to \mathrm{Aut}(H^1(C_0, \mathbb{Z})). \tag{7}$$

The image of the homomorphism (7) is finite by Proposition 4.5, whence the image of the generator of $\pi_1(\Delta \setminus \{L^\perp\})$ under the homomorphism (6) has finite order k. The kth power of the monodromy operator (4) is

$$x \mapsto x - k \cdot (x, c)c,$$

which is identity if and only if $(c, x) = 0$ for all $x \in H^1(C_0, \mathbb{Z})$. So, $c = 0$ and the curve S is homologous to zero. □

Proposition 4.9. *Suppose that* $\deg C > 2$ *and $L \subset (\mathbb{P}^2)^*$ is a line tangent to C^* at exactly one point q. Then only the following two cases are possible*:

(a) *$f^{-1}(L) = Y_1 \cup Y_2$, where Y_1 and Y_2 are smooth projective curves intersecting transversally at a point lying over q, where the restriction $f|_{Y_1} : Y_1 \to L$ is an isomorphism, the restriction $f|_{Y_2} : Y_2 \to L$ has degree > 1, and $(Y_1, Y_1) = 0$;*

(b) *C is a smooth cubic curve and f is equivalent to a projection of the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^4$ (in particular, $\deg f = 4$). In this case both Y_1 and Y_2 is a smooth rational curve, and each of the restrictios $f|_{Y_1} : Y_j \to L$ has degree 2.*

Proof. Propositions 4.8 and 4.7 show that $f^{-1}(L) = Y$ is a connected curve homeomorphic to the quotient space Y'/S, where Y' is a sphere with handles, and $S \subset Y'$ is a zero-homologous circle. Then Y'/S is homeomorphic to the wedge sum of two spheres with handles, so the curve Y has exactly two components.

Since the divisor $Y = Y_1 + Y_2$ is the inverse image of the line $L \subset (\mathbb{P}^2)^*$ and $f(Y_1) = f(Y_2) = L$ (which follows from the finiteness of the morphism f), one has, for $j = 1$ or 2,

$$(Y_j, Y_1 + Y_2) = \deg(f|_{Y_j}) > 0. \tag{8}$$

The curves Y_1 and Y_2 intersect transversally at one point, whence $(Y_1, Y_2) = 1$ and therefore

$$(Y_1, Y_1) \geqslant 0, \quad (Y_2, Y_2) \geqslant 0. \tag{9}$$

Denote by $V \subset H^2(X, \mathbb{R})$ the subspace generated by fundamental classes of Y_1 and Y_2. Consider two cases.

Case 1: $\dim V = 1$. In this case the classes of Y_1 and Y_2 are proportional: $Y_2 = rY_1$.

Observe that the number r must be positive because $(Y_1, f^{-1}L) > 0$, $(Y_2, f^{-1}L) > 0$ for a general line $L \subset (\mathbb{P}^2)^*$. Since

$$1 = (Y_1, Y_2) = r(Y_1, Y_1) = r^{-1}(Y_2, Y_2),$$

and both (Y_1, Y_1) and (Y_2, Y_2) are integers, one has $r = 1$, so that $Y_1 = Y_2$ in cohomology, and $(Y_1, Y_1) = (Y_2, Y_2) = 1$. Since homology classes of C and C' are equal and $C + C' = f^{-1}(L)$ is an ample divisor, the divisors C and C' are also ample. Now $\deg f = (Y_1 + Y_2, Y_1 + Y_2) = 4$. Since fundamental classes of the curves Y_1 and Y_2 coincide, their genera are the same; denote this number by g.

Lemma 4.10. *The curves Y_1 and $f^{-1}(L)$ have the same genus.*

Proof. Observe that the surface $\tilde{X}$ defined by the equation (5) with the natural projection $q\colon \tilde{X} \to t^{\perp}$ is a Lefschetz pencil. Now a standard argument (see, for example, [DK73, Exposé XVIII, Theorems 5.6.8 and 5.6.2]) shows that image of the natural injection $H^1(X, \mathbb{C}) \hookrightarrow H^1(f^{-1}(L), \mathbb{C})$, where the line $L \subset (\mathbb{P}^2)^*$ is transversal to C^* and passes through t, coincides with the invariant subspace

$$H^1(f^{-1}(L))^{\pi_1(L \setminus C^*)}.$$

Put $f^{-1}(L) = H$; since the monodromy group is generated by the operators (6) and all the vanishing cycles c are zero in view of Proposition 4.8, we conclude that the restriction $H^1(X, \mathbb{C}) \hookrightarrow H^1(H, \mathbb{C})$ is an isomorphism, whence the homomorphism $H^1(X, \mathcal{O}_X) \to H^1(H, \mathcal{O}_H)$ from the exact sequence

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$$

is also an isomorphism; the Kodaira vanishing theorem and Serre's duality show then that this is equivalent to the equation $\dim |K_X| = \dim |K_X + H|$; since $\dim |H| > 0$, this is equivalent to the relation $|K_X + H| = \varnothing$ (cf. [Lvo94, Proposition 6.1]).

Since $H = Y_1 + Y_2$ and Y_2 is an effective divisor, it follows now that the linear system $|K_X + Y_1|$ is also empty; since the divisor Y_1 is ample, the above argument in the reverse order shows that $H^1(X, \mathcal{O}_X) \cong H^1(Y_1, \mathcal{O}_{Y_1})$. Thus, genera of the curves $f^{-1}(L) = H$ and Y_1 are both equal to $\dim H^1(X, \mathcal{O}_X)$. □

If a line $L' \subset (\mathbb{P}^2)^*$ is transversal to C^*, then, according to Proposition 4.6b, $Y_1 \cup Y_2$ is homeomorphic to $Y' = f^{-1}(L')$ with a circle contracted to a point. Hence, the genus of Y' equals $2g$. Since $2g = g$ by Lemma 4.10, we infer that $g = 0$. Thus, Y_1 and Y_2 are smooth rational ample curves with self-intersection indices 1 on the smooth projective surface X. It is well known (see, for example, [Lvo14, Proposition 2.3]) that if a smooth rational curve Y on a smooth projective surface X is ample and $(Y, Y) = 1$, then there exists an isomorphism from X to $\mathbb{P}^2$ mapping Y to a line. Thus, $X \cong \mathbb{P}^2$ and $\mathcal{O}_X(f^{-1}(L)) \cong \mathcal{O}_{\mathbb{P}^2}(2)$, so the mapping $f\colon X \to \mathbb{P}^2$ is isomorphic to a projection $\mathrm{pr}_\Lambda\colon v_2(\mathbb{P}^2) \to \mathbb{P}^2$, where $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ is the quadratic Veronese surface and $\Lambda \subset \mathbb{P}^5$ is a 2-plane disjoint from $v_2(\mathbb{P}^2)$. Since the curve C^* is the branch locus of the projection pr_Λ, its dual curve $C \subset \mathbb{P}^2$ coincides with the intersection $(v_2(\mathbb{P}^2))^* \cap \Lambda^\perp$ (cf. the proof of Proposition 5.1 below), which is clearly a smooth plane cubic since $(v_2(\mathbb{P}^2))^*$ is the variety of degenerate symmetric 3×3-matrices.

Case 2: $\dim V = 2$. By the Hodge index theorem,

$$\begin{vmatrix} (Y_1, Y_1) & (Y_1, Y_2) \\ (Y_1, Y_2) & (Y_2, Y_2) \end{vmatrix} < 0,$$

whence $(Y_1, Y_1)(Y_2, Y_2) < 1$. So at least one of the factors must be zero; assume without loss of generality that $(Y_1, Y_1) = 0$, then

$$\deg(f|_{Y_1}) = (Y_1, Y_1 + Y_2) = 1$$

Thus, the restriction of f to Y_1 is an isomorphism from Y_1 to the line L. Let us prove that degree of the restriction $f|_{Y_2}\colon Y_2 \to f(Y_2)$ is greater than one. Indeed, L is tangent to C^* at only one point, so Y has only two components. Thus, if Y_2 maps with degree 1 onto L, then

$$(Y_2, Y_2) = (Y_2, Y_1 + Y_2) - (Y_2, Y_1) = 1 - 1 = 0$$

and $\deg f = (Y_1 + Y_1, Y_2 + Y_2) = 2(Y_1, Y_2) = 2$, so all nodes and/or cusps of C^* must be bad in the sense of Definition 3.3, which contradicts the smoothness of X in view of Proposition 3.4. Thus, C^* must be smooth. Since its dual curve $(C^*)^* = C$ has no cusps, the smooth curve C^* has no inflexion points. It is well known (and follows, for example, from the Plücker formulas [BK86, Section 9.1, Theorem 1]) that the only smooth plane curve without inflexions is the conic, whence C is a conic, which contradicts the hypothesis. □

Remark 4.11. The trick with Hodge index theorem is essentially contained in Van de Ven's paper [VdV79] (see the proof of Theorem I). A similar argument allows one to give a proof of the main result of the paper [Zak73] that is valid in arbitrary characteristic (see [Lvo14]).

Proposition 4.12. *Suppose that $C \subset \mathbb{P}^2$ is a smooth or nodal curve of degree > 2 that is general enough in the sense of Definition 1.6. Let also X be a smooth projective surface and $f\colon X \to (\mathbb{P}^2)^*$ be a finite morphism with branch locus C^* and such that the ramification is simple. If $\deg C = 3$, assume in addition that C has a node or $\deg f \neq 4$. Then there exists an isomorphism $\Phi\colon X \to X_C$ such that $f_C \circ \Phi = f$, where X_C and f_C are defined by equations (1) and (2), respectively.*

We know two proofs of this proposition, one "topological" and one algebraic geometric. For the exposition in this paper, we have chosen the latter since its rigorous version appears to be shorter.

Proof. As before, denote the normalization of C by $\nu\colon \hat{C} \to C$. Let $\gamma\colon \hat{C} \to C^*$ be the Gauss mapping, which attaches to a point $x \in C$ the tangent to C^* at $\nu(x)$, where by tangent we mean the limiting tangent if $\nu(x)$ is a cusp of C^* and the limiting tangent at the branch corresponding to x if $\nu(x)$ is a node of C^*.

If follows from projective duality that the definition of the surface X_C (see (1)) may be rewritten as

$$X_C = \{(x, t) \in \hat{C} \times (\mathbb{P}^2)^* \colon t \in \gamma(x)^{\perp}\}.$$

Now put

$$\mathcal{Z} = \{(x, y) \in \hat{C} \times X \colon f(y) \in \gamma(x)^{\perp}\}.$$

(i.e. $\mathcal{Z}$ is the fiber product of $\hat{C}$ and X over $(\mathbb{P}^2)^*$).

Lemma 4.13. *There exists a component $\mathcal{Z}_1 \subset \mathcal{Z}$ consisting of components of fibers that have intersection index* 1 *with $D = f^{-1}(L)$ (or, equivalently, project isomorphically onto tangents to C^*; in the proof of Proposition 4.9 these components were called Y_1).*

Proof of the lemma. We will be using the language of schemes.

Denote the natural projection $\mathcal{Z} \to \hat{C}$ by π_1 and the morphism $(x, y) \mapsto f(y)$ by $\pi_2\colon \mathcal{Z} \to (\mathbb{P}^2)^*$. Pick a line $L \subset (\mathbb{P}^2)^*$ transversal to C^* and denote by $D \subset \mathcal{Z}$ the closed subscheme $\pi_2^{-1}L$. Proposition 4.9 shows that for almost all (in the sense of Zariski topology) closed points $x \in \hat{C}$, the fiber $\pi_1^{-1}(x)$ consists of two components Y_1 and Y_2 and $D \cap \pi_1^{-1}(x)$ is a reduced scheme of length $\deg C^*$; moreover, one of the points of $D \cap \pi_1^{-1}(x)$ lies in Y_1 and the remaining $\deg C^* - 1$ lie in Y_2. Let K stand for the function field of $\hat{C}$ and $\bar{K}$ for its algebraic closure; put $\eta = \operatorname{Spec} K$, $\bar{\eta} = \operatorname{Spec} \bar{K}$, and let $\mathcal{Z}_\eta$ and $\mathcal{Z}_{\bar{\eta}}$ be the generic and generic geometric fibers of π_1 respectively. Then it is apparently evident that $\mathcal{Z}_{\bar{\eta}}$ consists of two components and the pullback of D is also a reduced scheme of length $\deg C^*$ such that one of its points lies on one of the components of $\mathcal{Z}_{\bar{\eta}}$, while the remaining $\deg C^* - 1$ points lie on the other component. To prove this assertion rigorously, we invoke EGA4.

To wit, consider the coherent sheaf $\mathcal{F} = \mathcal{O}_{\mathcal{Z}} \oplus \mathcal{O}_D$ on $\mathcal{Z}$. The primary type ("type primaire") [Gro66a, Remarque 9.8.9] (roughly speaking, it is the collection of lengths of fibers of $\mathcal{F}$ at the generic points of components of its support and at the generic points of various intersections of the closures of these points) of pullbacks of $\mathcal{F}$ to fibers over almost all (in the sense of Zariski topology) closed points is the same; since the set of points with the property "the pullback of $\mathcal{F}$ to the geometric fiber over the point in question has a given *type primaire*" is constructible due to [Gro66a, 9.8.9.1], it follows that *type primaire* of the pullback of $\mathcal{F}$ to $\mathcal{Z}_{\bar{\eta}}$ is the same as that of its pullbacks to almost all fibers over closed points, whence the assertion.

Now observe that the Galois group $G = \operatorname{Gal}(\overline{K}/K)$ acts on the set of the two components of $\mathcal{Z}_{\bar{\eta}}$. Since the pullback of D to $\mathcal{Z}_{\bar{\eta}}$ is K-rational, it is G-invariant. On the other hand, one of these component contains only one point of D, while the other contains $\deg C^* - 1 > 1$ such points (if $\deg C^* - 1 = 1$, then C^* and C are conics, which contradicts the hypothesis). Thus, both these components are G-invariant, whence $\mathcal{Z}_\eta$ contains a component containing only one point from the pullback of D. This proves the lemma. □

Returning to the proof of the proposition, observe that self-intersection index of almost all of the fibers of the projection $\mathcal{Z}_1 \to C^*$ as curves on X is zero due to Proposition 4.9, so they are disjoint. Thus, the natural mapping $\Phi_1 \colon \mathcal{Z}_1 \to X$ (induced by projection to the second factor) is generically one to one; since X is smooth, Zariski's main theorem implies that Φ_1 is an isomorphism. On the other hand, if we define the morphism $\Phi_2 \colon \mathcal{Z}_1 \to X_C$ by the formula $(x, y) \mapsto (x, f(y))$, then it is easy to see that Φ_2 is also generically bijective, whence isomorphic. Now we may put $\Phi = \Phi_2 \circ \Phi_1^{-1}$. □

Now the (b) ⇒ (a) implication in Theorem 1.1 follows immediately from Propositions 4.12 and 4.1.

Remark 4.14. If $\deg C = 3$ and $\deg f = 4$, Theorem 1.1 does not hold. Indeed, if $f \colon v_2(\mathbb{P}^2) \to \mathbb{P}^2$ is a general projection of the quadratic Veronese surface, then the branch locus of f is the curve $C^* \subset \mathbb{P}^2$ that is dual to a smooth cubic C. If $L \subset \mathbb{P}^2$ is a general line, then the restriction $\pi = f|_{f^{-1}(L)} \colon f^{-1}(L) \to L$ is a generic covering of degree 4 ramified over $L \cap C^*$, that is, over the branch locus of the projection $\mathrm{pr}_{L^\perp} \colon C \to L$. By the very construction, the mapping π can be extended to the mapping $f \colon v_2(\mathbb{P}^2) \to \mathbb{P}^2$ ramified over C^*, but $\deg f = 4 \neq 3$, so f is not equivalent to a projection of the plane cubic.

The following corollary of Theorem 1.1 is essentially its "topological" reformulation:

Corollary 4.15. *Suppose that $C \subset \mathbb{P}^2$ is a nodal curve of degree > 2 that is general enough in the sense of Definition 1.6. Suppose that a point $p \in \mathbb{P}^2 \setminus C$ is such that the composition $\mathrm{pr}_p \circ \nu \colon \hat{C} \to \mathbb{P}^1$, where $\nu \colon \hat{C} \to C$ is the normalization and pr_p is the projection from p, has simple ramification. Denote by $L \subset (\mathbb{P}^2)^*$ the line in the dual plane corresponding to the point $p \in \mathbb{P}^2$, and suppose that C' is a smooth projective curve and $\pi \colon C' \to L$ is a holomorphic mapping with simple ramification and such that the branch locus of π coincides with $L \cap C^*$. If $\deg C = 3$, assume in addition that C has a node or $\deg f \neq 4$. Then the following two conditions are equivalent.*

(a) *There exists an isomorphism $\varphi \colon C' \to \hat{C}$ such that $(\mathrm{pr}_p \circ \nu) \circ \varphi = \pi$.*

(b) *The covering $\pi^{-1}(L \setminus C^*) \to L \setminus C^*$ can be extended to a covering $X_0 \to (\mathbb{P}^2)^* \setminus C^*$ with respect to which all nodes and cusps of the curve C^* are good in the sense of Definitions 3.2 and 3.3.*

Proof. Immediate from Theorem 1.1 and Proposition 3.4. □

Our proof of Theorem 1.1 implies the following corollary:

Corollary 4.16. *Suppose that $C \subset \mathbb{P}^2$ is a nodal curve that is general enough in the sense of Definition 1.6. If C is not a smooth cubic, then any two generic coverings $X \to (\mathbb{P}^2)^*$, ramified over C^* and with smooth X, are equivalent.*

If $\deg C \geqslant 7$, or $\deg C = 6$ and C has at least one node, or $\deg C = 5$ and C has at least three nodes, then this corollary is implied by [Kul99, Theorem 10].

It is well known (see, for example, [Kul99]) that this assertion is wrong if C is a smooth cubic. A relevant counterexample is essentially contained in our proof of Proposition 4.9 (Case 1).

5. The Smoothness Condition is (Sometimes) Essential

In this section we show that the smoothness condition in Theorem 1.1 is not trivial, at least for the case of smooth curves of degree $d = 3$ or 4 and mappings of degree d.

To wit, we prove the following.

Proposition 5.1. *Suppose that $d = 3$ or 4. Then there exist a smooth curve $C \subset \mathbb{P}^2$, $\deg C = d$, a line $p^\perp \subset (\mathbb{P}^2)^*$ (where $p \in \mathbb{P}^2$ is a point) transversal to C, a smooth projective curve C', and a holomorphic mapping $\pi\colon C' \to p^\perp$ with simple ramification such that $\deg \pi = d$ and π is ramified exactly over $C^* \cap p^\perp$ with the following properties.*

(1) For a point $t_0 \in p^\perp \setminus C^*$, the fiber monodromy $\pi_1(p^\perp \setminus C^*, t_0) \to \mathrm{Perm}(\pi^{-1}(t_0))$ can be factored through the homomorphism $\pi_1(p^\perp \setminus C^*, t_0) \to \pi_1((\mathbb{P}^2)^* \setminus C^*, t_0)$ induced by the embedding $p^\perp \setminus C^* \hookrightarrow (\mathbb{P}^2)^* \setminus C^*$.

(2) The mapping $\pi\colon C' \to p^\perp$ is not *equivalent to the projection $\mathrm{pr}_p\colon C \to p^\perp$.*

We will need the following lemma.

Lemma 5.2. *If $d = 3$ or 4, then there exists a surface with isolated singularities $X \subset \mathbb{P}^3$, $\deg X = d$, such that its dual $X^* \subset (\mathbb{P}^3)^*$ is isomorphic to X and the general hyperplane section of X is general enough in the sense of Definition 1.6.*

Proof of the proposition modulo Lemma 5.2. Let $X \subset \mathbb{P}^3$ be a surface from Lemma 5.2. For a general point $r \in \mathbb{P}^3$, the projection $\mathrm{pr}_r\colon X \to \mathbb{P}^2$, where $\mathbb{P}^2$ is a projective plane (incidentally, $\mathbb{P} = (r^\perp)^*$), is simply ramified. If $B \subset \mathbb{P}^2$ is the branch locus of pr_r, then a line $\ell \subset \mathbb{P}^2$ is tangent to B if and only if the plane Π_r spanned by r and ℓ is tangent to X. Thus, the dual curve $B^* \subset (\mathbb{P}^2)^*$ is projectively isomorphic to $X^* \cap r^\perp = C$; if p is general, Lemma 5.2 asserts that C is a general enough smooth plane curve.

Take $p \in \mathbb{P}^3$ general and let $C' = X \cap \Pi_p$; denote the restriction of pr_p to C' by $\pi\colon C' \to L$. We see (taking into account that $B = C^*$) that π extends to a finite morphism $\mathrm{pr}_p\colon X \to (\mathbb{P}^2)^*$ ramified over C^*, where the surface X is singular. Thus, π satisfies Condition (b) of Theorem 1.1 with the smoothness omitted. On the other hand, π cannot be equivalent to the projection pr_r. Indeed, if such an equivalence existed, then by Proposition 2.1 the restriction of π to the preimage of the complement of the branch locus could be extended to a finite morphism $X_C \to \mathbb{P}^2$ ramified over C^* with smooth X_C. However, if such an extension to a finite mapping $X \to (\mathbb{P}^2)^*$ ramified over C^* and with normal (in particular, smooth) X exists, this extension is unique by the theorem of Grauert and Remmert [Gro71b, Exposé XII, Théorème 5.4] we cited before. Since we already have such an extension with singular X, we arrive at the desired contradiction. $\square$

It remains to prove Lemma 5.2. If $d = 3$, we let X be the projective closure of surface in $\mathbb{A}^3$ defined by the equation $x_1x_2x_3 = 1$ (this surface was considered in [ACT02]). A direct computation shows that X contains only three singular points (of the type A_2) and that X is projectively isomorphic to X^*. Since any smooth plane cubic is automatically general enough in the sense of Definition 1.6, the surface X is as required.

To treat the case $d = 4$ we need to recall the definition and some properties of Kummer surfaces.

Suppose that J is a principally polarized Abelian variety of dimension 2 that is not the product of two elliptic curves and $\Theta \subset J$ is a theta divisor of polarization. Then the complete linear system $|2\Theta|$ is base point free and defines a finite morphism $J \to \mathbb{P}^3$; the image of this morphism is a surface of degree 4 with exactly 16 singular points of the type A_1; the surfaces obtained by this construction are called Kummer surfaces (see, for example, [Dol12, Section 10.3]). Any Kummer surface $X \subset \mathbb{P}^3$ is projectively isomorphic to its dual $X^* \subset (\mathbb{P}^3)^*$ (see [Dol12, Theorem 10.3.19]). Thus, to finish the proof of Lemma 5.2, it remains to show that there exists a Kummer surface such that its general plane section is general enough in the sense of Definition 1.6; since such sections are plane quartics, it suffices to establish that a general plane section has only simple inflexion points.

The following lemma and its proof were pointed out to us by Rita Pardini.

Lemma 5.3 (R. Pardini)**.** *Each non-hyperelliptic curve of genus* 3 (*smooth and projective*) *is isomorphic to a plane section of an appropriate Kummer surface in* $\mathbb{P}^3$.

Proof of the lemma. Suppose that C is such a curve, and let $\sigma\colon C_1 \to C$ be an unramified covering of degree 2. Denote by (J, Θ) the Prym variety corresponding to this covering [BL04, Section 12.2]. According to [BL04, Proposition 12.5a], the Abel–Prym map embeds C_1 in J as an element of the linear system 2Θ. Multiplication by -1 on J restricts on C_1 to the involution induced by σ, hence the image of C_1 via the map given by $|2\Theta|$ is a plane section of the Kummer surface associated to (J, Θ), and this section is isomorphic to C. □

Now if we pick a smooth plane curve $C_0 \subset \mathbb{P}^2$ of degree 4 such that all its Weierstrass points have index one, or, equivalently, all its inflexion points are simple, Lemma 5.3 shows that C_0 is a plane section of a Kummer surface $X \subset \mathbb{P}^3$; since the property "all the Weierstrass points have index one" is open, a generic plane section of X has only simple inflexion points, which completes the proof of Lemma 5.2.

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