Complex Variables and Elliptic Equations
2011, 1–22, iFirst

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A moduli space of non-compact curves on a complex surface

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Communicated by S. Ivashkovich

(*Received 4 March 2011; final version received 23 May 2011*)

We show that under mild boundary conditions the moduli space of non-compact curves on a complex surface is (locally) an analytic subset of a ball in a Banach manifold, defined by *finitely* many holomorphic functions.

Keywords: complex variables; moduli space; Banach analytic set

AMS Subject Classifications: 32G13; 32K05; 32D15

1. Introduction and basic notation

In this article X denotes a *smooth complex surface*. A curve C in X is an effective divisor. This means that C is a locally finite formal sum $C = \sum_i m_i\, C_i$, where every C_i is a (closed) irreducible analytic set of (co)dimension 1, and m_i are positive integers. We call C_i (irreducible) the components of C and m_i the multiplicities. The set $|C| := \cup_i C_i$ is called the support of C. For an open subset $U \subset X$ we define the restriction of C to U as $C \cap U := \sum_i m_i (C_i \cap U)$.

With any component C_i we associate the ideal sheaf $\mathcal{I}_{C_i}$ whose group of sections over an open set $V \subset X$ is $\Gamma(V, \mathcal{I}_{C_i}) := \{f \in \Gamma(V, \mathcal{O}) : |_{V \cap C_i} \equiv 0\}$. This is a coherent analytic sheaf and $\mathrm{supp}\,(\mathcal{O}/\mathcal{I}_{C_i}) = C_i$. Call $\mathcal{I}_C := \prod_{i=1}^N \mathcal{I}_{C_i}^{m_i}$ the ideal sheaf of C, and $\mathcal{O}_C := \mathcal{O}_X/\mathcal{I}_C$ the structure sheaf of C. The ideal sheaf $\mathcal{I}_C$ is locally principle, i.e. has locally the form $\mathcal{I}_C|_U = f_U \cdot \mathcal{O}_X$. We call such f_U a local determining function of C in U and $C \cap U$ a divisor of f_U, $C \cap U = \mathrm{Div}(f_U)$.

The pair $(|C|, \mathcal{O}_C)$ is a complex subspace of X (in general, not reduced and reducible) which we shall also denote by C. This means that we can consider C as an analytic cycle $C = \sum_i m_i\, C_i$ and also as subspace $C = (|C|, \mathcal{O}_C)$ of X.

It is known (see, e.g. [1] or [2]) that one can associate to every curve $C = \sum_i m_i C_i$, a closed positive integer (1, 1)-current η_C such that, for any continuous 2-form φ with the compact support in X,

$$\eta_C(\varphi) = \langle \eta_C, \varphi \rangle := \sum_i m_i \int_{C_i} \varphi.$$

*Email: shevchishin@googlemail.com

ISSN 1747–6933 print/ISSN 1747–6941 online

DOI: http://dx.doi.org/10.1080/17476933.2011.592582
http://www.tandfonline.com

Moreover, C is completely determined by η_C and every closed positive integer $(1,1)$-current η in X corresponds to some curve (see [1] or [2]). Thus, we identify the set of curves in X with the space $\mathcal{PI}^{1,1}(X)$ of closed positive integer $(1,1)$-currents in X and induce the topology on a set of curves from the space $\mathcal{D}'_2(X)$ of 2-currents in X. Note that $\mathcal{PI}^{1,1}(X)$ is closed subset in the space $\mathcal{D}'_2(X)$.

The (weak) topology in the space $\mathcal{PI}^{1,1}(X)$ gives us the notion of weakly continuous family of curves in X. Namely, a family $\{C_y\}_{y\in Y}$, parameterized by a topological space Y, is called weakly continuous, iff the induced map $F\colon Y\to \mathcal{PI}^{1,1}(X)$, $F(y):=\eta_{C_y}$, is continuous. It follows from the result of Stoll [3] that this is equivalent to the following condition: There exist an open covering $\{V_\alpha\}$ of $X\times Y$ and continuous functions $f_\alpha\in C(V_\alpha,\mathbb{C})$ such that for any $y\in Y$ the restriction of f_α on $(X\times\{y\})\cap V_\alpha$ is holomorphic and generates the ideal sheaf $\mathcal{I}_{C_y}=f_\alpha\cdot\mathcal{O}_{X\times\{y\}}$ of C_y. The f_α are called local determining functions of a family $\{C_y\}$.

In particular, a sequence $\{C_\nu\}$ of curves in X converges weakly to a curve C_∞ iff for any $x\in X$ there exist a neighbourhood $V\ni x$ and a sequence of holomorphic functions $f_\nu\in\Gamma(V,\mathcal{O})$ which are determining for C_ν and which converge uniformly in V to a determining function f_∞ of C_∞.

For the definition of the category of *Banach analytic spaces*, we refer to [4, Section 3]. We note that for every Banach analytic space Y and Banach space E the sheaf $\mathcal{O}_Y(E)\colon U\subset Y\mapsto \Gamma(U,\mathcal{O}_Y(E))$ of holomorphic E-valued morphisms between open subsets $U\subset Y$ and E is a part of a definition of the structure of a Banach analytic space. In the case $E=\mathbb{C}$ we denote this sheaf by $\mathcal{O}_Y$. Any morphism $\boldsymbol{F}\colon Y\to Z$ between two Banach analytic spaces defines a continuous map $F\colon Y\to Z$ between corresponding topological spaces, and a morphism of sheaves $F^\sharp_E: F^*\mathcal{O}_Z(E)\to\mathcal{O}_Y(E)$ for any Banach space E. Here $F^*\mathcal{O}_Z(E)$ denotes the pull-back of the sheaf $\mathcal{O}_Z(E)$ w.r.t. continuous map F. Moreover, a morphism $\boldsymbol{F}\colon Y\to Z$ is defined by the data F and $F^\sharp_{(\cdot)}$.

We say that a continuous map $F\colon Y\to Z$ is *holomorphic* if it is induced by a morphism $\boldsymbol{F}\colon Y\to Z$. Note that such a morphism $\boldsymbol{F}\colon Y\to Z$ can be not unique at the sheaf level. In particular, two different morphisms $\boldsymbol{F}_1, \boldsymbol{F}_2\in \mathrm{Mor}(Y,E)=\Gamma(Y,\mathcal{O}_Y(E))$ can induce the same continuous map $F_1=F_2\colon Y\to E$. This reflects the fact that a generic Banach analytic space Y is highly non-reduced.

Definition 1 We say that a Banach analytic space Y is of finite type iff Y can covered by local charts Y_α such that any Y_α is isomorphic to a zero set of a holomorphic map $f_\alpha: B_\alpha\to\mathbb{C}^{n_\alpha}$, where B_α denotes a ball in some Banach space. In particular, we have an isomorphism $\mathcal{O}_Y|_{Y_\alpha}\cong\mathcal{O}_{B_\alpha}/(f_{\alpha,1},\dots,f_{\alpha,n_\alpha})$, where $(f_{\alpha,1},\dots,f_{\alpha,n_\alpha})$ denotes the ideal sheaf generated by the components of f_α. Such spaces are also referred to as Banach analytic spaces of finite definition or Banach analytic spaces of finite codimension.

Definition 2 A holomorphic family $\mathcal{C}=\{C_y\}_{y\in Y}$ of curves in X parameterized by a Banach analytic space Y is given by an open covering $\{V_\alpha\}$ of $X\times Y$ and holomorphic functions $f_\alpha\in\Gamma(V_\alpha,\mathcal{O}_{X\times Y})$ such that:

(i) if $V_\alpha\cap V_\beta\neq\varnothing$, then $f_\alpha=f_{\alpha\beta}\cdot f_\beta$ for some invertible $f_{\alpha\beta}\in\ \Gamma(V_\alpha\cap V_\beta,\mathcal{O}^*_{X\times Y})$;
(ii) for any $y\in Y$ the restriction of f_α on $V_\alpha\cap X\times\{y\}$ is not identically zero and is a local determining function for a curve C_y.

The functions f_α are called *local determining functions of the family* $\{C_y\}_{y\in Y}$. The collection f_α defines the sheaf of ideals $\mathcal{I}_{\mathcal{C}} \subset \mathcal{O}_{X\times Y}$ with $\mathcal{I}_{\mathcal{C}}|_{V_\alpha} := f_\alpha \cdot \mathcal{O}_{X\times Y}$. Two holomorphic families parameterized by the same Banach analytic space Y are isomorphic iff they define the same sheaf of ideals over $X \times Y$.

Now let C^* be any curve in X and $K \Subset |C^*|$ any compact subset of its support. Our main result is the following theorem.

Main Theorem *There exists an open set $U \subset X$ containing K such that the set of curves in U, which satisfy appropriate boundary conditions and which are sufficiently close to $C^* \cap U$, is a holomorphic family $\mathcal{C} = \{\mathcal{C}_t\}_{t\in\mathcal{M}}$ parameterized by a Banach analytic space $\mathcal{M}$ of finite type. Moreover, for every continuous* (*resp. holomorphic*) *family $\{C_y\}_{y\in Y}$ with $C^* = C_{y_0}$ for some $y_0 \in Y$, there exist a neighbourhood $Y_0 \subset Y$ of y_0 and a continuous map* (*resp. a morphism*) *$F\colon Y_0 \to \mathcal{M}$ such that $C_y \cap U = \mathcal{C}_{F(y)}$. Two such families $\{C'_y\}_{y\in Y}$ and $\{C''_y\}_{y\in Y}$ coincide over Y_0 iff they induce the same continuous map* (*resp. morphism*) *$F\colon Y_0 \to \mathcal{M}$.*

The theorem has several corollaries which are mainly due to the fact that Banach analytic sets of finite type have sufficiently simple structure. In particular, if X, C^* and U are as in the *Main Theorem*, and if $\{C_n\}$ is a sequence of curves in X converging to C^*, then for any $n \gg 1$ there exists a holomorphic family $\{C_\lambda\}_{\lambda\in\Delta}$ of curves in U, which is parameterized by a disk $\Delta \subset \mathbb{C}$ and contains both $C_n \cap U$ and $C^* \cap U$. This allows to obtain a generalization of the continuity principle of E. E. Levi.

The conclusion of the *Main Theorem* is obtained by an explicit construction of the space $\mathcal{M}$. The problem of deformation of a curve C leads to study of the normal sheaf $\mathcal{N}_C$ to C in X. It is defined as $\mathcal{N}_C := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$.

To obtain a parameterizing space as an analytic set in *Banach* manifold, we introduce the notion of a (Banach) *smoothness* S. This generalizes the usual smoothness classes such as k times continuous differentiability C^k, Sobolev smoothness $L^{k,p}$ or Hölder smoothness $C^{k,\alpha}$. For such a smoothness S, we define a Banach space $\Gamma_S(C, \mathcal{N}_C) \equiv \mathsf{H}^0_S(C, \mathcal{N}_C)$ of (holomorphic) sections of $\mathcal{N}_C$ which are *S-smooth up to boundary* ∂C (or simply S-smooth).

The description of a moduli space $\mathcal{M}$ in a neighbourhood of a marked point y_0 is usual for a deformation theory:

There exists a ball $B \subset \mathsf{H}^0_S(C, \mathcal{N}_C)$ and a holomorphic map $\Phi: B \to \mathsf{H}^1(C,\mathcal{N}_C)$ such that

(i) $\Phi(0) = 0$, $d\Phi(0) = 0$;
(ii) $\Phi: B \to \mathsf{H}^1(C,\mathcal{N}_C)$ is a local chart for $\mathcal{M}$.
(iii) $0 \in B$ corresponds to $y_0 \in \mathcal{M}$, parameterizing $C^* \cap U$.

The desired property of $\mathcal{M}$ is based on the fact that non-compact components of C are Stein spaces, and consequently $\mathsf{H}^1(C, \mathcal{N}_C)$ is finite-dimensional. In particular, $\mathcal{O}_{\mathcal{M}} \cong \mathcal{O}_B/(\Phi_1, \dots, \Phi_k)$, $k := \dim \mathsf{H}^1(C,\mathcal{N}_C)$. Here Φ_i denote the components of Φ and $(\Phi_1, \dots, \Phi_k)$ the ideal sheaf generated by $\Phi_1, \dots, \Phi_k$.

2. The local situation

The construction of the moduli space $\mathcal{M}$ is based on two special cases. One of them describes local deformations of curves and the other one allows to match two different local descriptions.

We first consider the local situation. For this we suppose that Σ is a smooth complex curve with a smooth nonempty boundary $\partial\Sigma$. Set $V := \Sigma \times \Delta$. Denote by $\Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$ the Banach space of n-tuples of holomorphic uniformly bounded functions on Σ. For every such $f = (f_1(z), \ldots, f_n(z)) \in \Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$ we define a *Weierstraß polynomial*

$$P_f(z, w) := w^n + \sum_{i=1}^{n} f_i(z)\, w^{n-i}, \quad z \in \Sigma, \quad w \in \Delta \tag{2.1}$$

and a curve $C_f \subset V$ to be the zero divisor of $P_f(z, w)$.

Lemma 1 *Every curve $C \subset V = \Sigma \times \Delta$ satisfying condition*

$$|C| \subset \Sigma \times \Delta(r) \quad \text{for some } r < 1 \tag{2.2}$$

is a zero divisor of a uniquely defined Weierstraß polynomial $P_f(z, w) = w^n + \sum_{i=1}^{n} f_i(z)\, w^{n-i}$ with $f = (f_1, \ldots, f_n) \in \Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$.

The set $\mathcal{M}^{(n)}_{L^\infty}(V)$ of those $f \in \Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$ for which C_f satisfies condition (2.2) *is open in $\Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$. The map $\Phi : \mathcal{M}^{(n)}_{L^\infty}(V) \to \mathcal{PJ}^{(1,1)}(V)$, $\Phi(f) := \eta_{C_f}$, is continuous and injective. The topology on the image $\Phi(\mathcal{M}^{(n)}_{L^\infty}(V))$ coincides with the weak topology of $\Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$.*

Remark 1 For C as in the lemma, we shall call the corresponding degree n of P the degree of C. The weak topology of $\Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$ is understood in the sence of functional analysis (see, e.g. [5, Section 3.11]). Notice that the weak convergence $f^{(\nu)} \rightharpoonup f$ of functions in $\Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$ implies the uniform convergence on every compact $K \Subset \Sigma$ but not the uniform convergence on the whole Σ.

Proof Since the group $\mathsf{H}^2(V, \mathbb{Z})$ is trivial, any curve C in V admits a global determining function F. For C satisfying condition (2.2) the Weierstraß preparation theorem (see, e.g. [6]) insures that C is a zero-divisor of a uniquely defined Weierstraß polynomial $P_f(z, w) = w^n + \sum_{i=1}^{n} f_i(z)\, w^{n-i}$, such that $F = h \cdot P_f$ for some invertible $h \in \Gamma(V, \mathcal{O})$. We can view $f := (f_1, \ldots, f_n) \in \Gamma(\Sigma, \mathcal{O}^n)$ as a holomorphic map from Σ into the n-th symmetric power $\mathsf{Sym}^n \Delta \subset \mathsf{Sym}^n \mathbb{C} \cong \mathbb{C}^n$. Hence the f_i are necessarily uniformly bounded in Σ. Moreover, for $g \in \Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$ sufficiently close to f, the curve C_g defined by Weierstraß polynomial $P_g(z, w) := w^n + \sum_{i=1}^{n} g_i(z)\, w^{n-i}$, also satisfies condition (2.2). Thus the set $\mathcal{M}^{(n)}_{L^\infty}(V)$ is open in $\Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$.

According to the Poincaré-Lelong formula (see [1] or [6]), the map Φ is given by formula

$$f \in \Gamma_{L^\infty}(\Sigma, \mathcal{O}^n) \longrightarrow \frac{1}{i\pi}\, \partial\bar\partial \log|P_f| \in \mathcal{PJ}^{(1,1)}(V).$$

Thus $\Phi : \Gamma_{L^\infty}(\Sigma, \mathcal{O}^n) \to \mathcal{PJ}^{(1,1)}(V)$ is continuous.

Now let $f^{(\nu)} \in \mathcal{M}^{(n)}_{L^\infty}(V) \subset \Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$ be a sequence. If $\{f^{(\nu)}\}$ converges weakly to $f \in \mathcal{M}^{(n)}_{L^\infty}(V)$, then $\{C_{f^{(\nu)}}\}$ is bounded in $\mathcal{PJ}^{(1,1)}(V)$ and any Cauchy subsequence of $\{C_{f^{(\nu)}}\}$ must converge to C_f. Vice versa, let $\{C_{f^{(\nu)}}\}$ converge to a curve C satisfying condition (2.2). Since $\mathcal{M}^{(n)}_{L^\infty}(V) \subset \Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$ is a bounded subset, some subsequence of $\{f^{(\nu)}\}$ converges weakly to $f \in \Gamma_{L^\infty}(\Sigma, \mathcal{O}^n)$ such that $C_f = C$. Consequently, $f \in \mathcal{M}^{(n)}_{L^\infty}(V)$. ∎

We see that we have constructed a holomorphic family $\{C_f\}_{f \in \mathcal{M}^{(n)}_{L^\infty}(V)}$ of curves which are 'uniformly bounded' in V. Later we shall show that this family possesses the universality property. To generalize this result to other classes of boundary conditions on curves, we introduce the notion of a *smoothness*.

For $0<r<R<\infty$ denote by $A_{r,R}$ the *annulus* $\{z\in\mathbb{C}: r<|z|<R\}$. Denote also $\Delta^-(r) := \{z\in\bar{\mathbb{C}} : |z|>r\}$. Recall that any Riemann surface Σ with a complex structure which is homeomorphic to an annulus and which boundary $\partial\Sigma$ consists of two circles is in fact biholomorphic to some annulus $A_{r,R}$, $\Sigma\cong A_{r,R}$. Recall also that any holomorphic function $f\in\Gamma(A_{r,R},\mathcal{O})$ admits a unique decomposition into the sum $f=f^++f^-$ such that $f^+\in\Gamma(\Delta(R),\mathcal{O})$ and $f^-\in\Gamma(\Delta^-(r),\mathcal{O})$ with $f^-(\infty)=0$. We call it the *Laurent decomposition* of f.

Definition 3 A smoothness class S (or simply a smoothness) in Δ is defined by fixing a subalgebra $\Gamma_S(\Delta,\mathcal{O})\subset\Gamma_{L^\infty}(\Delta,\mathcal{O})$ which satisfies the following conditions:

(Si) $\Gamma_S(\Delta,\mathcal{O})$ is a Banach algebra[1] with the norm $\|\cdot\|_S$ and $\|f\|_{L^\infty(\Delta)}\leqslant C_S\|f\|_S$
(Sii) $\Gamma_S(\Delta,\mathcal{O})$ is invariant w.r.t. the action of the group $\mathbf{U}(1)$ by rotations on Δ.
(Siii) If $f\in\Gamma_S(\Delta,\mathcal{O})$, $g\in\Gamma(A_{r,R},\mathcal{O})$ with some $r<1<R$, and $fg=(fg)^++(fg)^-$ is the Laurent decomposition of the product, then $(fg)^+\in\Gamma_S(\Delta,\mathcal{O})$. Moreover, $\|(fg)^+\|_S\leqslant C(S,r,R)\cdot\|f\|_S\cdot\|g\|_{L^\infty(A_{r,R})}$ for some constant $C(S,r,R)$.
(Siv) If $f\in\Gamma_{L^\infty}(A_{r,1},\mathcal{O})$ has the Laurent decomposition $f=f^++f^-$ with $f^+\in\Gamma_S(\Delta,\mathcal{O})$ and a bounded invertible $g:=1/f\in\Gamma_{L^\infty}(A_{r,1},\mathcal{O})$ with the Laurent decomposition $g=g^++g^-$, then $g^+\in\Gamma_S(\Delta,\mathcal{O})$.

We say that $f\in\Gamma_S(\Delta,\mathcal{O})$ is S-*smooth*. Conditions (Siii) and (Siv) show that S-smoothness depends essentially only on the behaviour of f at the boundary of Δ. Obvious examples are C^k-Lipschitz–Hölder smoothness (up to the boundary) $S=C^{k,\alpha}(\bar{\Delta})$ with $k\in\mathbb{N}$ and $0\leqslant\alpha\leqslant 1$, Sobolev smoothness $S=L^{k,p}(\Delta)$ with $k\geqslant 1$, $1\leqslant p\leqslant\infty$ and $kp>2$ and also Sobolev smoothness $S=L^{k,p}(S^1)$ on boundary $S^1:=\partial\Delta$ with $k\geqslant 1$ and $1\leqslant p\leqslant\infty$. The later means that the trace $f|_{S^1}$ of $f\in\Gamma(\Delta,\mathcal{O})$ on $S^1=\partial\Delta$ is well defined and belongs to the corresponding class.

Definition 4 Let Σ be a smooth complex curve whose boundary $\partial\Sigma$ consists of finitely many components γ_i, $i=1,\dots,n$, each of which is homeomorphic to a circle S^1. The smoothness S at $\partial\Sigma$ in Σ is defined by fixing of a smoothness classes S_i in Δ and annuli $A_i\subset\Sigma$ such that one of the components of boundary ∂A_i coincides with γ_i and the other one lies in the interior of Σ. For every $i=1,\dots,n$ this induces a biholomorphic map $\varphi_i : A_i\to A_{r_i,1}$ which extends continuously up to the boundary ∂A_i and maps γ_i onto $\partial\Delta$. We say that $f\in\Gamma(\Sigma,\mathcal{O})$ is S-smooth in Σ at $\partial\Sigma$, $f\in\Gamma_S(\Sigma,\mathcal{O})$, iff for every $i=1,\dots,n$ the Laurent decomposition $\varphi_{i*}f=(\varphi_{i*}f)^++(\varphi_{i*}f)^-$ yields $(\varphi_{i*}f)^+\in\Gamma_{S_i}(\Delta,\mathcal{O})$.

LEMMA 2 *Let Σ, γ_i, $\varphi_i : A\to A_{r_i,1}$, and S_i be as above. Then $\Gamma_S(\Sigma,\mathcal{O})$ is a Banach algebra with respect to the norm $\|f\|_S := \sum_i\|(\varphi_{i*}f)^+\|_{S_i}$. Moreover,*

$$\|f\|_{L^\infty(\Sigma)}\leqslant C_S\cdot\|f\|_S \tag{2.3}$$

with a constant C_S independent of $f\in\Gamma_S(\Sigma,\mathcal{O})$. The subset

$$\Gamma_S(\Sigma,\mathcal{O})^\times := \{f\in\Gamma_S(\Sigma,\mathcal{O}) : f^{-1}\in\Gamma_{L^\infty}(\Sigma,\mathcal{O})\}$$

is open in $\Gamma_S(\Sigma,\mathcal{O})$ and the map

$$F:\Gamma_S(\Sigma,\mathcal{O})^\times\to\Gamma_{L^\infty}(\Sigma,\mathcal{O})\quad F(f):=f^{-1}$$

is $\Gamma_S(\Sigma,\mathcal{O})$-valued and holomorphic.

Proof Let $f \in \Gamma_S(\Delta, \mathcal{O})$. Set $f_i := \varphi_{i\,*}f$ and let $f_i = f_i^+ + f_i^-$ be the corresponding Laurent decompositions. Due to (Si) of the definition of smoothness, f is uniformly bounded in Σ and takes its supremum on one of the boundary component γ_j. The later means that there exists a sequence $x_\nu \in \Sigma$ such that $\lim x_\nu \in \gamma_j$ and $\lim |f(x_\nu)| = \|f\|_{L^\infty(\Sigma)}$. In particular, $\|f\|_{L^\infty(\Sigma)} = \|f_j\|_{L^\infty(A_j)}$.

For this A_j we obviously have

$$\|f_j^-\|_{L^\infty(A_j)} \leqslant \|f_j\|_{L^\infty(A_j)} + \|f_j^+\|_{L^\infty(A_j)}. \tag{2.4}$$

On the other hand, $\|f_j^-\|_{L^\infty(\gamma_j)} \leqslant \delta \cdot \|f_j^-\|_{L^\infty(A_j)}$ with $\delta := \max\{r_i\} < 1$. This is due to $f_j^-(\infty) = 0$ and the Schwarz inequality. Consequently,

$$\|f_j\|_{L^\infty(A_j)} \leqslant \delta \cdot \|f_j^-\|_{L^\infty(A_j)} + \|f_j^+\|_{L^\infty(A_j)}. \tag{2.5}$$

Comparing (2.4) and (2.5) we see that

$$\|f_j^-\|_{L^\infty(A_j)} \leqslant \frac{2}{1-\delta} \cdot \|f_j^+\|_{L^\infty(A_j)} \quad \text{and} \quad \|f_j\|_{L^\infty(A_j)} \leqslant \frac{3-\delta}{1-\delta} \cdot \|f_j^+\|_{L^\infty(A_j)}. \tag{2.6}$$

Since $\delta < 1$ is independent of f, from (Si) we obtain the estimate (2.3). Consequently, every $\|\cdot\|_S$-Cauchy sequence converges to some element in $\Gamma_S(\Sigma, \mathcal{O})$. Moreover, the map

$$\Phi : \Gamma_S(\Sigma, \mathcal{O}) \longrightarrow \prod_{i=1}^{n} \Gamma_{S_i}(\Delta, \mathcal{O}), \quad \Phi(f) := \big((\varphi_{i\,*}f)^+\big), \tag{2.7}$$

is a closed imbedding.

Take another $g \in \Gamma_S(\Sigma, \mathcal{O})$ and set $g_i := \varphi_{i*}g$ with the corresponding Laurent decompositions $g_i = g_i^+ + g_i^-$. Then

$$(\varphi_{i\,*}(f \cdot g))^+ = f_i^+ g_i^+ + (f_i^- g_i^+)^+ + (f_i^+ g_i^-)^+.$$

For any $i = 1, \ldots, n$ we obviously have

$$\|f_i^-\|_{L^\infty(\Delta^-(r_i))} \leqslant c(r_i) \cdot \|f_i\|_{L^\infty(A_{r_i,1})} \leqslant c_S \|f\|_S$$

and the same estimates for g. Due to (Siii), we obtain

$$\|f \cdot g\|_S \leqslant c'_S \cdot \|f\|_S \cdot \|g\|_S.$$

Condition (Siv) implies that $\Gamma_S(\Sigma, \mathcal{O})^\times$ consists of those $f \in \Gamma_S(\Sigma, \mathcal{O})$ which are invertible in $\Gamma_S(\Sigma, \mathcal{O})$. Thus, the last statement of the lemma is a standard fact of the theory of commutative Banach algebras (see, e.g. [7]). ■

Definition 5 Let Σ and S be as above and $V = \Sigma \times \Delta$. A function $F(z, w) \in \Gamma(V, \mathcal{O})$ is called S-smooth, $F \in \Gamma_S(V, \mathcal{O})$, iff $\mathrm{F}(z, a) \in \Gamma_S(\Sigma, \mathcal{O})$ for every $a \in \Delta$ and the induced map $F : \Delta \to \Gamma_S(\Sigma, \mathcal{O})$ is holomorphic and bounded. A curve $C \subset V$ is called S-smooth iff C satisfies condition (2.2) and $C = \mathsf{Div}(F)$ for some $F \in \Gamma_S(V, \mathcal{O})$, i.e. C is defined by such an F. Let $\mathcal{F}_S^{(n)}$ denote the set of those $F \in \Gamma_S(V, \mathcal{O})$ such that $\inf\{|F(z,a)| : z \in \Sigma,\ a \in A_{r,1}\} > 0$ for some $r < 1$ and for which the curve $C_F := \mathsf{Div}(F)$ has degree n.

Lemma 3 *A set $\Gamma_S(V, \mathcal{O})$ of S-smooth functions is a Banach space with a norm*

$$\|F\|_S := \sup\{\|F(z,a)\|_S \,:\, a \in \Delta\}.$$

Every S-smooth curve $C \subset V$ of degree n is represented by a unique Weierstraß polynomial $P = P_f(z,w) = w^n + \sum_{i=1}^n f_i(z) w^{n-i}$ with $f = (f_1, \dots, f_n) \in \Gamma_S(\Sigma, \mathcal{O}^n)$. The set $\mathcal{F}_S^{(n)}$ is open in $\Gamma_S(V, \mathcal{O})$ and the map $\Psi : \mathcal{F}_S^{(n)} \to \Gamma_S(\Sigma, \mathcal{O}^n)$, $\Psi : F \mapsto f$, is holomorphic.

Proof The part of the lemma concerning $\Gamma_S(V, \mathcal{O})$ is obvious. Let F lie in $\mathcal{F}_S^{(n)} \subset \Gamma_S(V, \mathcal{O})$ and $C_F \subset \Sigma \times \Delta$ be the corresponding curve. For $k \in \mathbb{N}$ and $z \in \Sigma$ we set

$$f_k(z) := \frac{1}{2\pi i} \int_{|w|=r} \frac{w^k}{F(z,w)} \frac{\partial F}{\partial w}(z,w) \mathrm{d}w,$$

where $r < 1$ is chosen sufficiently close to 1. Then f_0 is constant and equals the degree n of C, whereas f_i, $i = 1, \dots, n$, are the coefficient of the Weierstraß polynomial P_f of C. Since the operations in the definition of f_k – taking inverse, differentiating, integrating – are holomorphic, f_k depends holomorphically on $F \in \Gamma_S(V, \mathcal{O}^n)$. ■

Let $\mathcal{M}_S^{(n)}(V)$ be the image $\Psi(\mathcal{F}_S^{(n)})$. One can regard an embedding $\mathcal{M}_S^{(n)}(V) \subset \Gamma_S(\Sigma, \mathcal{O}^n)$ as a local chart of a moduli space of curves on a complex surface with an appropriate smoothness condition. To be able to patch such local models together we need an invariant description of $\mathcal{M}_S^{(n)}(V)$.

Take $F \in \mathcal{F}_S^{(n)}$, set $C := \mathsf{Div}(F)$, $f := \Psi(F)$, and consider the tangent map

$$d\Psi_F : T_F \mathcal{F}_S^{(n)} = \Gamma_S(V, \mathcal{O}) \longrightarrow T_C \mathcal{M}_S^{(n)}(V) = \Gamma_S(\Sigma, \mathcal{O}^n).$$

Lemma 4 *Let $\pi : |C| \to \Sigma$ be the natural projection and $\pi_* \mathcal{O}_C$ a push-forward with respect to π. The map $d\Psi_F$ induces on $\mathcal{O}_\Sigma^n$ a structure of a $\pi_*\mathcal{O}_C$-module which is independent of the choice of F with $\mathsf{Div}(F) = C$. With respect to this structure, there exists a $\pi_*\mathcal{O}_C$-isomorphism θ_C between $\mathcal{O}_\Sigma^n$* and *the push-forward $\pi_*\mathcal{N}_C$ of the normal sheaf $\mathcal{N}_C = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$.*

The isomorphism θ_C admits the following characterization: Let $F_\lambda(z,w)$, $\lambda \in \Delta(\rho)$, be a holomorphic family of functions in $\mathcal{F}_S^{(n)}$ such that $C = \mathsf{Div}(F_0)$. Furthermore, let $\varphi := \theta_C(d\Psi_{F_0}(F_0')) \in \Gamma(C, \mathcal{N}_C)$ where $F_0' := \frac{\partial F_\lambda}{\partial \lambda}|_{\lambda=0}$. Then

$$\varphi : [F_0]_{\mathcal{I}_C^2} \in \Gamma(C, \mathcal{I}_C/\mathcal{I}_C^2) \mapsto [F_0']_{\mathcal{I}_C} \in \Gamma(C, \mathcal{O}_V/\mathcal{I}_C). \tag{2.8}$$

Proof For convenience, we slightly modify the definition of the map Ψ. For $F \in \mathcal{F}_S^{(n)} \subset \Gamma_S(V, \mathcal{O})$ and $f = (f_1, \dots, f_n) = \Psi(F) \in \Gamma_S(\Sigma, \mathcal{O}^n)$ we set $\Psi^{WP}(F) := w^n + \sum_{i=1}^n f_i(z) w^{n-i} \in \Gamma_S(V, \mathcal{O})$, where WP stands for 'Weierstraß polynomial'.

Set $P := \Psi^{WP}(F)$ and $g := P^{-1}F$. Then g is holomorphic and bounded in V, $g(z,a) \in \Gamma_S(\Sigma, \mathcal{O})$ for $a \in A_{r,1}$, and the induced map $g : A_{r,1} \to \Gamma_S(\Sigma, \mathcal{O})$ is holomorphic. Considering the Cauchy representation for g,

$$g(z,w) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{g(z,\zeta)}{w - \zeta} \, \mathrm{d}\zeta,$$

we see that $g \in \Gamma_S(V, \mathcal{O})$. By the same argumentation we have $g^{-1} \in \Gamma_S(V, \mathcal{O})$. Moreover, the map $F \in \mathcal{F}_S^{(n)} \longmapsto F/\Psi^{WP}(F) \in \Gamma_S(V, \mathcal{O})$ is holomorphic.

Now let $F_\lambda(z, w)$, $\lambda \in \Delta(\rho)$, be a holomorphic family of functions in $\Gamma_S(V, \mathcal{O})$ with $F_0 = F$. Put $P_\lambda := \Psi^{WP}(F_\lambda)$ and $g_\lambda := F_\lambda/P_\lambda$. Then P_λ and g_λ depend holomorphically in λ. Differentiating the identity $F_\lambda = g_\lambda \cdot P_\lambda$ with respect to λ in $\lambda = 0$, we obtain

$$F_0' = g_0' \cdot P_0 + g_0 \cdot P_0'$$

with $g_0 = g$ and $P_0 = P$. Thus the tangent map

$$d\Psi_F^{WP} : T_F \mathcal{F}_S^{(n)} = \Gamma_S(V, \mathcal{O}) \to \Gamma_S(V, \mathcal{O})$$

is given by the formula $d\Psi_F^{WP}(F') = \{\frac{g^{-1}F'}{P}\}$ the Weierstraß remainder of a division of $g^{-1}\, F'$ by P. It is a unique polynomial $R_{f'} = \sum_{i=1}^n f_i'(z) w^{n-i}$ of degree $< n$ such that $R_{f'} \equiv g^{-1}\, F' (\mathrm{mod}\, F)$. This implies that $d\Psi_F$ yields an isomorphism of Banach spaces

$$\psi_F : \Gamma_S(V, \mathcal{O}) / \big(F \cdot \Gamma_S(V, \mathcal{O})\big) \cong \Gamma_S(\Sigma, \mathcal{O}^n),$$
$$\psi_F : [H]_F \longmapsto h = (h_1, \ldots, h_n) \quad \text{with } g^{-1} H \equiv R_h (\mathrm{mod}\, F).$$

Due to its definition, ψ_F is essentially local and induces the isomorphism of $\mathcal{O}_\Sigma$-modules

$$\psi_F : \pi_* \big(\mathcal{O}_V/(F(z, w) \cdot \mathcal{O}_V)\big) \cong \mathcal{O}_\Sigma[w]/(P(z, w) \cdot \mathcal{O}_\Sigma[w]) \cong \mathcal{O}_\Sigma^n.$$

Since $\mathcal{O}_V/P(z,w) \cdot \mathcal{O}_V = \mathcal{O}_C$, ψ_F defines on $\mathcal{O}_\Sigma^n$ a structure of a free $\pi_*\mathcal{O}_C$-module of rank 1. If $\widetilde{F} \in \Gamma_S(V, \mathcal{O})$ is another function with $\mathrm{Div}(\widetilde{F}) = C$, then $\widetilde{F} = h \cdot F$ for some invertible $h \in \Gamma_S(V, \mathcal{O})$. Then, by the definition, $\psi_{\widetilde{F}}(H) = \psi_F(h^{-1}H)$. Consequently, the induced structure of the $\pi_*\mathcal{O}_C$-module on $\mathcal{O}_\Sigma^n$ is independent of the choice of F.

Using ψ_F, we define a $\pi_*\mathcal{O}_C$-homomorphism $\theta_F : \mathcal{O}_\Sigma^n \to \pi_*\mathcal{N}_C$. For a local section $f = (f_1, \ldots, f_n)$ of $\mathcal{O}_\Sigma^n$ over an open set $\Omega \subset \Sigma$ we take a holomorphic function $H \in \Gamma(\Omega \times \Delta, \mathcal{O})$ such that $H \equiv g \cdot R_f\, (\mathrm{mod}\, F)$ in $\Omega \times \Delta$, where $g = P^{-1}F$ is as above. Since $C = \mathrm{Div}(F)$, the sheaf $\mathcal{I}_C/\mathcal{I}_C^2$ is free $\mathcal{O}_C$-module of rank 1 with generator $[F]_{\mathcal{I}_C^2}$. Using $\mathcal{N}_C = \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_V/\mathcal{O}_C)$, we define

$$\theta_F(f) \in \Gamma(\Omega, \pi_*\mathcal{N}_C) = \Gamma\big((\Omega \times \Delta) \cap C, \mathcal{N}_C\big), \quad \theta_F(f) : [F]_{\mathcal{I}_C^2} \longmapsto [H]_{\mathcal{I}_C}.$$

If $\widetilde{H} \in \Gamma(\Omega \times \Delta, \mathcal{O})$ is another holomorphic function with $\widetilde{H} \equiv g \cdot R_f\, (\mathrm{mod}\, F)$, then $[\widetilde{H}]_{\mathcal{I}_C} = [H]_{\mathcal{I}_C}$. This shows that the definition of $\theta_F(f)$ is independent of the choice of H.

Similarly, if $\widetilde{F} \in \Gamma_S(V, \mathcal{O})$ is another defining function for C, $C = \mathrm{Div}(\widetilde{F})$, then $\widetilde{F} = h \cdot F$ with $h \in \Gamma_S(V, \mathcal{O})$ invertible. In this case $P^{-1}\widetilde{F} = h \cdot g$ and hence $\theta_{\tilde{F}}(f) : [\widetilde{F}]_{\mathcal{I}_C^2} \mapsto [h \cdot H]_{\mathcal{I}_C}$. This means that $\theta_{\tilde{F}}(f) = \theta_F(f)$ as sections of $\mathcal{N}_C$. Thus the definition of $\theta_C := \theta_F$ is independent of the choice of defining function F for the curve C.

Now let $F_\lambda(z,w) \in \Gamma_S(V, \mathcal{O})$, $\lambda \in \Delta(\rho)$, be a holomorphic family of functions such that $F_0 = F$ and $F'_0 = H$. The relation (2.8) for $\varphi := \theta_C(d\Psi_F(H))$ follows immediately from the construction of Ψ_F and θ_C. ■

Definition 6 Note that the constructed isomorphism θ_C induces the bijection $\theta_C : \Gamma(C, \mathcal{N}_C) \cong \Gamma(\Sigma, \mathcal{O}_\Sigma^n)$. A section φ of $\mathcal{N}_C$ is called S-smooth, $\varphi \in \Gamma_S(C, \mathcal{N}_C)$, iff $\theta(\varphi) \in \Gamma_S(\Sigma, \mathcal{O}_\Sigma^n)$.

As we have already noted, the map $\Psi : \mathcal{M}_S^{(n)}(V) \to \Gamma_S(\Sigma, \mathcal{O}_\Sigma^n)$ is a chart in a moduli space of curves in V with an appropriate smoothness condition at boundary. However, the map Ψ depends on the choice of local (holomorphic) coordinates (z, w) in V. In particular, if in the construction of Ψ we replace w by some other coordinate function $\tilde{w} = \tilde{w}(w)$, then the map Ψ as well as a $\pi_*\mathcal{O}_C$-module structure on $\mathcal{O}_\Sigma^n$ will change. However, relation (2.8) remains valid, since it is independent of the choice of coordinates in V. This leads us to the following corollary.

Corollary 1 *The tangent space $T_C\mathcal{M}_S^{(n)}(V)$ is canonically isomorphic to $\Gamma_S(C, \mathcal{N}_C)$.*

3. Curves in the 'distorted cylinder'

To be able to patch local descriptions, we consider the following special situation. In $\mathbb{C}^2$ with standard coordinates (z, w) we consider an annulus $A_{r,R} := \{(z,w) : w = 0, r < |z| < R\}$. We assume that in some neighbourhood U of the closure $\bar{A}_{r,R}$ we are given two holomorphic functions z_1 and z_2 which coincide with z along $A_{r,R}$. Without loss of generality, we may also assume that the both pairs (z_1, w) and (z_2, w) are coordinates in U so that we can express $z_1 = z_1(z_2, w)$ and $z_2 = z_2(z_1, w)$.

For $\rho > 0$ let

$$W_{r,R,\rho} := \{x \in U \, : \, |z_1(x)| > r, |z_2(x)| < R, |w(x)| < \rho\}.$$

We shall always suppose that $\rho > 0$ is chosen sufficiently small such that $W_{r,R,\rho} \Subset U$ and that the sets

$$\begin{aligned} \partial_- W_{r,R,\rho} &:= \{x \in U \, : \, |z_1(x)| = r, |w(x)| \leqslant \rho\} \\ \partial_+ W_{r,R,\rho} &:= \{x \in U \, : \, |z_2(x)| = R, |w(x)| \leqslant \rho\} \end{aligned}$$

are disjoint. One can regard the set $W_{r,R,\rho}$ as a distorted cylinder with the non-parallel lower side $\partial_- W_{r,R,\rho}$ and upper side $\partial_+ W_{r,R,\rho}$. Note also that there exist real numbers $r < r' < R' < R$ such that both sets

$$\begin{aligned} V^-_{r,r'} &:= \{x \in U : r < |z_1(x)| < r', |w(x)| < \rho\} \\ V^+_{R',R} &:= \{x \in U : R' < |z_2(x)| < R, |w(x)| < \rho\} \end{aligned}$$

are products of an annulus and a disk. This allows us to make the following definition.

Definition 7 A smoothness S in $W_{r,R,\rho}$ is defined by fixing smoothness classes S^- and S^+ in Δ. A holomorphic function F in $W_{r,R,\rho}$ is S-smooth, $F \in \Gamma_S(W_{r,R,\rho}, \mathcal{O})$, iff $F|_{V^-_{rr'}} \in \Gamma_{S^-}(V^-_{r,r'}, \mathcal{O})$ and $F|_{V^+_{R',R}} \in \Gamma_{S^+}(V^+_{R',R}, \mathcal{O})$. Note that S also defines a smoothness in $A_{r,R}$: It is suffucient to fix annuli $A^- = \{r < |z| < r'\}$, $A^+ = \{R' < |z| < R\}$ and smoothness classes S^+ and S^-. We shall also denote this smoothness by S. Thus we obtain a continuous projection map $F \in \Gamma_S(W_{r,R,\rho}, \mathcal{O}) \mapsto F|_{A_{r,R}} \in \Gamma_S(A_{r,R}, \mathcal{O})$.

A curve $C \subset W_{r,R,\rho}$ is S-smooth iff $C \subset W_{r,R,\rho'}$ for some $\rho' < \rho$ and $C = \mathsf{Div}(F)$ for some $F \in \Gamma_S(W_{r,R,\rho}, \mathcal{O})$. The degree of an S-smooth curve $C \subset W_{r,R,\rho}$ is an integer $\deg C := \int_\gamma \, d\log F$, where $F \in \Gamma_S(W_{r,R,\rho}, \mathcal{O})$ is any function defining C and γ is a simple smooth loop (i.e. a closed real curve) in $\{x \in W_{r,R,\rho} : |w(x)| = \rho\}$ with

$\int_\gamma d\log w = 1$. It is obvious that $\deg C$ is a positive interger independent of the choice of F and γ. The set of S-smooth curves of degree n in $W_{r,R,\rho}$ will be denoted by $\mathcal{M}_S^{(n)}(W_{r,R,\rho})$. Note that for every S-smooth curve $C \subset W_{r,R,\rho}$ the Weierstraß polynomials P^+ and P^- of $C \cap V^-_{r,r'}$ and $C \cap V^+_{R',R}$ are uniquely defined. This yields an injective map

$$\kappa^{(n)} : \mathcal{M}_S^{(n)}(W_{r,R,\rho}) \to \Gamma_{S^-}(A_{r,r'}, \mathcal{O}^n) \times \Gamma_{S^+}(A_{R',R}, \mathcal{O}^n). \tag{3.1}$$

A family $\{C_y\}_{y\in Y}$ of S-smooth curves of a degree n in $W_{r,R,\rho}$ parameterized by a topological space Y is called *continuous iff* the induced map

$$\Psi : Y \to \Gamma_{S^-}(A_{r,r'}, \mathcal{O}^n) \times \Gamma_{S^+}(A_{R',R}, \mathcal{O}^n), \quad \Psi(y) := \kappa^{(n)}(C_y),$$

is continuous.

We say that $\{C_y\}_{y\in Y}$ is a *holomorphic family of S-smooth curves*, if it is a continuous family of S-smooth curves, Y has a structure of a Banach analytic space, and both restricted families $C_y \cap V^-_{r,r'}$ and $C_y \cap V^+_{R',R}$ are given by holomorphic morphisms $\psi^-_Y : Y \to \Gamma_{S^-}(V^-_{r,r'}, \mathcal{O})$ and $\psi^+_Y : Y \to \Gamma_{S^+}(V^+_{R',R}, \mathcal{O})$. Note that $\psi^\pm_Y$ induce local determining functions $F^\pm(z,w,y) := \psi^\pm_Y(y)(z,w)$ on $V^-_{r,r'} \times Y$ and $V^+_{R',R} \times Y$, respectively.

To generalise the results of Section 2 for curves in $W_{r,R,\rho}$, one must find an appropriate analogue of a Weierstraß polynomial for $W_{r,R,\rho}$.

Definition 8 Let the components of $f = (f_1, \ldots, f_n) \in \Gamma_S(A_{r,R}, \mathcal{O}^n)$ have a Laurent decomposition $f_i(z) = f_i^+(z) + f_i^-(z)$. A (distorted) Weierstraß polynomial $\widetilde{P}_f(z_1, w)$ in $W_{r,R,\rho}$ of degree n with coefficients $(f_1, \ldots, f_n)$ is defined as

$$\widetilde{P}_f(z_1, z_2, w) := w^n + \sum_{i=1}^n \left(f_i^-(z_1) + f_i^+(z_2)\right), w^{n-i}. \tag{3.2}$$

One can expect that there is one-to-one correspondence between S-smooth curves in $W_{r,R,\rho}$ of degree n and distorted Weierstraß polynomials $\widetilde{P}_f(z_1, z_2, w)$ of the same degree n with S-smooth coefficients. Since a difference of z_1 and z_2 introduces a 'non-linearity', one can hope to obtain the corresponding relationship only in some neighbourhood of a trivial case $\widetilde{P}_0 = w^n$ and $C_0 = n \cdot A_{r,R}$, when the coefficients $f = (f_1, \ldots, f_n)$ of $\widetilde{P}_f$ are sufficiently small with respect to the norm in $\Gamma_S(A_{r,R}, \mathcal{O}^n)$. The corresponding condition on a curve is that $\|\kappa^{(n)}(C)\|_S$ (with κ^n from (3.1)) should be small. Here $\|\cdot\|_S$ denotes the norm in $\Gamma_{S^-}(A_{r,r'}, \mathcal{O}^n) \oplus \Gamma_{S^+}(A_{R,R'}, \mathcal{O}^n)$.

LEMMA 5 *Let z_1, z_2, w, $r < r' < R' < R$, ρ, and S have the same meaning as above. There exists $\varepsilon > 0$ such that every S-smooth curve C of degree n in $W_{r,R,\rho}$, satisfying*

$$\|\kappa^{(n)}(C)\|_S \leqslant \varepsilon \tag{3.3}$$

is a zero divisor of a uniquely defined distorted Weierstraß polynomial $\widetilde{P}_f(z_1, z_2, w)$.

If Y is a topological (resp. Banach analytic) space and $\{C_y\}_{y\in Y}$ is a continuous (resp. holomorphic) family of curves satisfying (3.3), then the induced map $\psi_Y : Y \to \Gamma_S(A_{r,R}, \mathcal{O}^n)$ with $\mathsf{Div}(\widetilde{P}_{\psi_Y(y)}) = C_y$ is continuous (resp. holomorphic).

Proof Without loss of generality, we may assume that $\varepsilon > 0$ is chosen sufficiently small so that $C \subset W_{r,R,\rho'}$ for any given $\rho' < \rho$ also small enough.

Furthermore, we may also assume that

$$\begin{aligned} r &< |z_1(z_2,w)| < R', \quad \text{for any } (z_2,w) \text{ with } |z_2| = r' \text{ and } |w| \leqslant \rho', \\ r' &< |z_2(z_1,w)| < R, \quad \text{for any } (z_1,w) \text{ with } |z_1| = R' \text{ and } |w| \leqslant \rho'. \end{aligned} \tag{3.4}$$

Then $W_{r,R,\rho'}$ is a union of $V_1 := V^-_{r,R'} = \{r < |z_1| < R', |w| < \rho'\}$ and $V_2 := V^+_{r',R} = \{r' < |z_2| < R, |w| < \rho'\}$. Since $V_i = A_i \times \Delta(\rho')$ with $A_1 := \{r < |z| < R'\}$ and $A_2 := \{r' < |z| < R\}$, we can apply the result of Lemma 1.

Let B be a sufficiently small ball in $\Gamma_S(A_{r,R}, \mathcal{O}^n)$, $f = (f_1, \ldots, f_1) \in B$, and $\widetilde{P}_f$ a corresponding distorted Weierstraß polynomial. Then the zero divisor $C_f := \mathsf{Div}(\widetilde{P}_f)$ is an S-smooth curve of degree n lying in $W_{r,R,\rho}$. Moreover, both curves $C_f \cap V_i$, $i = 1, 2$, are S_i-smooth and of degree n in V_1 and V_2, respectively. Here we set $S_1 := S^-$ and $S_2 := S^+$. Thus there exist uniquely defined Weierstraß polynomials P_1 and P_2 in V_1 and V_2, respectively, such that $C_f \cap V_i = \mathsf{Div}(P_i)$. This defines the maps $\varphi_i : B \to \Gamma_{S_i}(A_i, \mathcal{O}^n)$, $i = 1, 2$.

Formula (2.8) provides that the derivation of φ_i at $f \equiv 0 \in B$ is simply the restriction map $\Gamma_S(A_{r,R}, \mathcal{O}^n) \to \Gamma_{S_i}(A_i, \mathcal{O}^n)$. Set $Y := \Gamma_{S_1}(A_1, \mathcal{O}^n) \oplus \Gamma_{S_2}(A_2, \mathcal{O}^n)$ and $\varphi = (\varphi_1, \varphi_2) : B \to Y$ so that $\varphi(f) = \kappa^{(n)}(\mathsf{Div}(\widetilde{P}_f))$. Since the differential $\mathrm{d}\varphi(0)$ of φ at $f \equiv 0 \in B$ consists of the pair of restrictions, $\mathrm{d}\varphi(0)$ is an injection with a closed image. This implies the injectivity of φ in some smaller ball $B(0, \varepsilon) \subset \Gamma_S(A_{r,R}, \mathcal{O}^n)$.

We state our conclusion in the following way: there exists an $\varepsilon > 0$ such that two distorted Weierstraß polynomials $\widetilde{P}_f$ and $\widetilde{P}_g$ in $W_{r,R,\rho}$ of given degree n with $\|f\|_S \leqslant \varepsilon$ and $\|g\|_S \leqslant \varepsilon$ coincide provided they define the same curve $C \subset W_{r,R,\rho}$.

Now let $C \subset W_{r,R,\rho}$ be a curve which satisfies the hypotheses of the lemma. In particular, $C \cap V_1$ is S^--smooth and $C \cap V_1 = \mathsf{Div}(P)$ for a uniquely defined Weierstraß polynomial $P = w^n + \sum_{i=1}^n g_i(z_1)\, w^{n-i}$. Further, from $C \subset W_{r,R,\rho'}$ we obtain $\|g_k\|_{L^\infty(A_{r,R'})} \leqslant c \cdot \rho'$ where the constant c is independent of the curve C. This yields

$$\|g_k\|_{\Gamma_{S^-}(A_{r,r'}, \mathcal{O})} \leqslant c' \cdot \rho'. \tag{3.5}$$

Consider the restriction of P to the set

$$W_{r,r',\rho} = \{x \in U : |z_1(x)| > r, |z_2(x)| < r', |w(x)| < \rho.$$

Note that every S-smooth function F in $W_{r,r',\rho}$ is uniquely represented in the form

$$F = \sum_{i=1}^n (f_i^+(z_1) + f_i^-(z_2))\, w^{n-i} + w^n(1 + Q) \tag{3.6}$$

with $Q \in \Gamma_S(W_{r,r',\rho}, \mathcal{O})$ and $f = (f_1, \ldots, f_n) \in \Gamma_S(A_{r,r'}, \mathcal{O}^n)$. Here $f_i(z) = f_i^+(z) + f_i^-(z)$ denotes the Laurent decomposition of the components of $f(z)$. The corresponding f_i are obtained inductively by the formula

$$f_{n-k}(z) := \frac{F - \sum_{i=n-k+1}^n (f_i^+(z_1) + f_i^-(z_2))\, w^{n-i}}{w^k}\Big|_{A_{r,r'}}, \quad k = 0, \ldots, n-1, \tag{3.7}$$

so that

$$1 + Q := \frac{F - \sum_{i=1}^n (f_i^+(z_1) + f_i^-(z_2))\, w^{n-i}}{w^n}. \tag{3.8}$$

Let us denote P by $\widetilde{P}_0$. Define inductively $\widetilde{P}_{k+1} := (1 - Q_k)^{-1}\widetilde{P}_k$, where Q_k is determined by the relation

$$\widetilde{P}_k = \sum_{i=1}^{n}(f^+_{k,i}(z_1) + f^-_{k,i}(z_2))\, w^{n-i} + w^n(1 + Q_k), \tag{3.9}$$

and $f_{k,i}(z) = f^+_{k,i}(z) + f^-_{k,i}(z)$ is the Laurent decomposition. We shall represent $\widetilde{P}_k$ in the form $\widetilde{P}_k = w^n + R_k + w^nQ_k$ with $R_k(z_1, z_2, w) := \sum_{i=1}^{n}(f^+_{k,i}(z_1) + f^-_{k,i}(z_2))\, w^{n-i}$. The estimates (2.5) on the coefficients g_i of $\widetilde{P}_0$ and the recursive formulas for $f_{0,i}(z)$ and Q_0 provide the estimate

$$\|R_0\| + \|Q_0\| \leqslant c'' \cdot \rho',$$

where $\|\cdot\|$ denotes the norm in $\Gamma_S(W_{r,r',\rho}, \mathcal{O})$ and the constant c'' independent of the choice of a curve C. In the same way one obtains the estimate

$$\|R_{k+1}\| \leqslant (1 + c'''\, \|Q_k\|)\, \|R_k\| \quad \text{and} \quad \|Q_{k+1}\| \leqslant c'''\, (\|Q_k\| + \|R_k\|)\, \|Q_k\|,$$

where the constant c''' is independent of C and ρ'. Since Q_0 and R_0 are small enough, the iteration converges to $\widetilde{P}_\infty = w^n + R_\infty$ which is of the desired form.

To show the existence of a distorted Weierstraß-type polynomial $\widetilde{P}$ in the entire set $W_{r,R,\rho}$, we additionally fix real numbers r'' and R'' with the property $r' < r'' < R'' < R'$. Then there exists a $\rho'' > 0$ such that

$$\begin{aligned} r' &< |z_2(z_1, w)| < R'', \quad \text{for any } (z_1, w) \text{ with } |z_1| = r'' \text{ and } |w| \leqslant \rho'', \\ r'' &< |z_1(z_2, w)| < R', \quad \text{for any } (z_2, w) \text{ with } |z_2| = R'' \text{ and } |w| \leqslant \rho''. \end{aligned} \tag{3.10}$$

We may assume that $\varepsilon > 0$ was fixed so small that every curve C satisfying the hypotheses of the lemma lies in $W_{r,R,\rho''}$.

Let C be such a curve. The above procedure allows us to construct the corresponding distorted Weierstraß-type polynomials $\widetilde{P}^-$ in the set $W_{r,R'',\rho''}$ and $\widetilde{P}^+$ in the set $W_{r'',R,\rho''}$. Due to condition (2.10), the intersection $W_{r,R'',\rho''} \cap W_{r'',R,\rho''}$ is also a distorted cylinder $W_{r'',R'',\rho''}$. Thus $\widetilde{P}^-$ and $\widetilde{P}^+$ coincide and define the desired distorted Weierstraß-type polynomial $\widetilde{P}$ in the whole set $W_{r,R,\rho}$.

Let Y be a topological space and $\{C_y\}_{y\in Y}$ a continuous family of curves satisfying condition (2.3). Furthermore, let $\psi_Y \colon Y \to \Gamma_S(A_{r,R}, \mathcal{O}^n)$ be an induced map. The explicit construction provides that ψ_Y is continuous.

Suppose also that Y is a Banach analytic space and $\{C_y\}_{y\in Y}$ is a holomorphic family of S-smooth curves. Assume additionally that $z_1 \equiv z_2$, i.e. $W_{r,R,\rho}$ is a usual (not distorted) cylinder $A_{r,R} \times \Delta(\rho)$. Let $\psi^-_Y : Y \to \Gamma_{S^-}(A_{r,r'} \times \Delta(\rho), \mathcal{O})$ and $\psi^+_Y : Y \to \Gamma_{S^+}(A_{R',R} \times \Delta(\rho), \mathcal{O})$ be holomorphic maps, inducing corresponding local determining functions $F^{\pm}(z,w,y)$ for the family $\{C_y\}$ in $A_{r,r'} \times \Delta(\rho) \times Y$ and $A_{R',R} \times \Delta(\rho) \times Y$, respectively (Definition 7).

Lemma 3 implies that there exist holomorphic maps $\varphi^-_i : Y \to \Gamma_{S^-}(A_{r,r'}, \mathcal{O})$ such that the Weierstraß polynomial $w^n + \sum_{i=1}^{n} f^-_i(z, y)w^{n-i}$ with coefficients $f^-_i(z, y) := \varphi^-_i(y)(z)$ is a local determining function for the family $\{C_y\}$ in $A_{r,r'} \times \Delta(\rho) \times Y$. Indeed, the map $\varphi^- := (\varphi^-_1, \ldots, \varphi^-_n)$ is obtained as a composition of ψ^-_Y with the map Ψ from Lemma 3. Repeating the same argumentation for ψ^+_Y, we obtain a holomorphic map $\varphi^+ = (\varphi^+_1, \ldots, \varphi^+_n) : Y \to \Gamma_{S^+}(A_{R',R}, \mathcal{O}^n)$ with similar properties.

The condition that both $w^n + \sum_{i=1}^n f_i^{\pm}(z,y)w^{n-i}$ are local determining functions for the same holomorphic family $\{C_y\}$ means that the map $(\varphi^-, \varphi^+) : Y \to \Gamma_{S^-}(A_{r,r'}, \mathcal{O}^n) \times \Gamma_{S^+}(A_{R',R}, \mathcal{O}^n)$ takes values in the subset consisting of the tuples $(f^+, f^-) = ((f_1^+, \dots, f_n^+), (f_1^-, \dots, f_n^-)) \in \Gamma_{S^-}(A_{r,r'}, \mathcal{O}^n) \times \Gamma_{S^+}(A_{R',R}, \mathcal{O}^n)$ which are restrictions onto $A_{r,r'}$ and $A_{R',R}$ of some holomorphic function $f = (f_1, \dots, f_n) \in \Gamma_S(A_{r,R}, \mathcal{O}^n)$. This implies that (φ^-, φ^+) takes value in $\Gamma_S(A_{r,R}, \mathcal{O}^n) \subset \Gamma_{S^-}(A_{r,r'}, \mathcal{O}^n) \oplus \Gamma_{S^+}(A_{R',R}, \mathcal{O}^n)$. Thus any holomorphic family $\{C_y\}_{y\in Y}$ in $A_{r,R} \times \Delta(\rho)$ of curves satisfying condition $\mathsf{supp}(C_y) \subset A_{r,R} \times \Delta(\rho')$ with some $\rho' < \rho$ of given degree n is defined by a holomorphic map $\varphi_Y : Y \to \Gamma_S(A_{r,R}, \mathcal{O}^n)$.

Now let us return to $W_{r,R,\rho}$ of the general type satisfying the conditions of the lemma. Note that all the above constructions of the proof respect holomorphic structure; in particular, they can be interpreted as holomorphic maps between corresponding Banach manifolds. This implies that the statement of the lemma about *holomorphic* families $\{C_y\}_{y\in Y}$ of S-smooth curves is valid. ■

It follows from the above that a small ball in $\Gamma_S(A_{r,R}, \mathcal{O}^n)$ is a local chart for the space $\mathcal{M}_S^{(n)}(W_{r,R,\rho})$. In this situation an invariant description is also possible.

LEMMA 6 *The tangent space $T_C\mathcal{M}_S^{(n)}(W_{r,R,\rho})$ at $C = n\cdot A_{r,R}$ is canonically isomorphic to $\Gamma_S(A_{r,R}, \mathcal{N}_C)$. Formula* (1.8) *also remains valid.*

Proof is identical to that for Lemma 4.

4. Globalization

Let U be an open set in a smooth complex surface X and $\mathcal{PT}^{(1,1)}(U)$ be the set of all curves in U. One can regard $\mathcal{PT}^{(1,1)}(U)$ as the base of the 'universal' (weakly continuous) family of curves in U. However, the weak topology of currents in $\mathcal{PT}^{(1,1)}(U)$ is not convenient to deal with. As we have seen in the previous sections, it is more useful to describe (a family of) curves by appropriate determining functions. Here we shall show that every continuous family of curves in U can be locally represented as a continuous deformation of determining functions.

It is enough to consider the situation when $U = \Sigma \times \Delta$ with Σ a smooth complex curve. Let z be a (local) coordinate on Σ and w a standard one on Δ. Fix a relatively compact subcurve $\Sigma' \Subset \Sigma$ with a smooth boundary and a smoothness S in Σ'. Thus for every neighbourhood Ω of $\bar{\Sigma}'$ in Σ the restriction map $\Gamma(\Omega, \mathcal{O}) \to \Gamma(\Sigma', \mathcal{O})$ takes values in $\Gamma_S(\Sigma', \mathcal{O})$ and is continuous w.r.t. usual Fréchet topology in $\Gamma(\Omega, \mathcal{O})$.

LEMMA 7 *Let $0 < r < R < 1$ and let C_0 be a curve in U whose restriction $C_0 \cap V_R$, $V_R := \Sigma' \times \Delta(R)$, is S-smooth and lies in $V_r = \Sigma' \times \Delta(r)$. Suppose also that C_0 does not contain components of the form $\{z\} \times \Delta$ with $z \in \bar{\Sigma}'$. Then there exists a neighbourhood $\mathcal{U}^{(n)}$ of C_0 in $\mathcal{PT}^{(1,1)}(U)$ with the following properties*:

(i) *For every $C \in \mathcal{U}^{(n)}$ the restriction $C \cap V_r$ is S-smooth in V_r and has degree* $\mathsf{deg}(C \cap V) = \mathsf{deg}(C_0 \cap V) =: n$.

(ii) *The induced map $\kappa : \mathcal{U}^{(n)} \to \mathcal{M}_S^{(n)}(V_r)$ is continuous with respect to the weak topology in $\mathcal{PT}^{(1,1)}(U)$.*

(iii) *If $\{C_y\}_{y\in Y}$ is a holomorphic family of curves in U with $C_y \in \{\mathcal{U}\}^{(n)}$ for all $y \in Y$, then the induced map $\varphi_Y : Y \to \mathcal{M}_S^{(n)}(V_r)$ is holomorphic.*

Proof Define

$$\mathcal{U} := \{C \in \mathcal{PJ}^{(1,1)}(U) : |C| \cap (\bar{\Sigma}' \times (\bar{\Delta}(R) \backslash \Delta(r)) = \varnothing\}.$$

Since the weak convergence $C_i \to C$ of currents in $\mathcal{PJ}^{(1,1)}(U)$ implies the Hausdorf convergence of supports $|C_i| \to |C|$, the set $\mathcal{U}$ is open in $\mathcal{PJ}^{(1,1)}(U)$.

Let $\{C_i\}$ be a sequence in $\mathcal{U}$ which converges to $C \in \mathcal{U}$ with $\mathsf{deg}(C_i \cap V_r) = n$. Then Lemma 1 implies that $\mathsf{deg}(C \cap V_r) = n$. Thus $\mathcal{U}$ is a disjoint union of components $\mathcal{U}^{(n)} := \{C \in \mathcal{U} : \mathsf{deg}(C \cap V_r) = n\}$ which are open in $\mathcal{PJ}^{(1,1)}(U)$. Take any $C \in \mathcal{U}^{(n)}$. Since U is Stein and $\mathsf{H}^2(U, \mathbb{Z}) = 0$, there exists $F \in \Gamma(\mathrm{U}, \mathcal{O})$ such that $C = \mathsf{Div}(F)$. But then $F|_{V_r} \in \Gamma_S(V_r, \mathcal{O})$ and this proves the S-smoothness of $C \cap V_r$.

Since $\mathcal{PJ}^{(1,1)}(U)$ is a subset of a space of distributions, its topology is sequential. This means that a set $A \subset \mathcal{PJ}^{(1,1)}(U)$ is closed iff for any sequence $\{C_i\} \subset A$ which converges to $C = \lim C_i \in \mathcal{PJ}^{(1,1)}(U)$, the limit point C belongs to A. In particular, a map $\kappa : \mathcal{U}^{(n)} \to \mathcal{M}_S^{(n)}(V_r)$ is continuous iff the image of every convergent sequence is also a convergent sequence.

So let $\{C_i\}$ be a sequence in $\mathcal{U}^{(n)}$ converging to $C \in \mathcal{U}^{(n)}$. Then there exists a neighbourhood Ω of $\bar{\Sigma}'$ in Σ such that $|C_i| \cap (\Omega \times (\bar{\Delta}(R) \backslash \Delta(r))) = \varnothing$ for every $i \gg 1$. It follows that every restricted curve $C_i \cap \Omega \times \Delta(r)$ is a zero divisor of a uniquely defined Weierstraß polynomial P_{f_i} of degree n with coefficients $f_i = (f_{i\,1}, \ldots, f_{i\,,n}) \in \Gamma_{L^\infty}(\Omega, \mathcal{O}^n)$. Moreover, the coefficients f_i are L^∞-bounded uniformly in i. Consequently, f_i weakly converge in Ω to the coefficients $g = (g_1, \ldots, g_n) \in \Gamma_{L^\infty}(\Omega, \mathcal{O}^n)$ of the Weierstraß polynomial P_g of the curve $C \cap \Omega \times \Delta(r)$.

By the hypotheses of the lemma, the restrictions of f_i onto Σ' are S-smooth and converge to $g|_{\Sigma'}$ with respect to the norm topology in $\Gamma_S(\Sigma', \mathcal{O}^n)$. This shows that the map $\kappa : \mathcal{U}^{(n)} \to \mathcal{M}^{(n)}(V_r)$ is continuous.

Now let $\{C_y\}_{y \in Y}$ be a holomorphic family of curves in U with all $C_y \in \mathcal{U}^{(n)}$. Fix some $y_0 \in Y$. Then there exist a neighbourhood Y_0 of $y_0 \in Y$ and a neighbourhood Ω of $\bar{\Sigma}'$ in Σ such that $|C_y| \cap (\Omega \times (\bar{\Delta}(R) \backslash \Delta(r))) = \varnothing$ for every $y \in Y_0$. Take $z^* \in \bar{\Omega}$ and $y^* \in Y_0$, and consider the set $(\{z^*\} \times \Delta(R)) \cap |C_{y^*}|$. By the construction, it consists of finitely many points $x_1, \ldots, x_k$. For every x_i an appropriate multiplicity m_i is defined such that $\sum_{i=1}^k m_i = n$.

By the definition of a holomorphic family of curves, in some neighbourhood $W_i \subset U \times Y_0$ of every $(x_i,\ y^*)$ a holomorphic function $F_i(z, w;\ y) \in \Gamma(W_i,\ \mathcal{O}_{U \times Y})$ is defined such that $\mathsf{Div}(F_i(\cdot;\ y)) = C_y \cap W_i$. As in the proof of Lemma 3, we can construct local determining functions $P_i(z, w; y) \in \Gamma(W_i,\ \mathcal{O}_{U \times Y})$ for $C_y \cap W_i$ which are polynomial in w, $P_i(z, w; y) = w^{m_i} + \sum_{j=1}^{m_i} f_{ij}(z, y)\, , w^{m_i - j}$. The product $P(z, w; y) := \prod_{i=1}^k P_i(z, w; y)$ is the Weierstraß polynomial of C_y. This shows that in a neighbourhood of $(z^*, y^*) \in \Sigma \times Y$ the coefficients $f(z; y) = (f_1(z; y), \ldots, f_n(z; y)) \in \Gamma_{L^\infty}(\Omega, \mathcal{O}^n)$ depend holomorphically (i.e. analytically) on both variables z and y. It follows that the induced map $\varphi_Y : Y \to \Gamma_{L^\infty}(\Omega, \mathcal{O}^n)$, sending $y \in Y$ into the coefficients of the Weierstraß polynomial of $C_y \cap (\Omega \times \Delta(r))$ is holomorphic. To finish the proof, we apply the restriction map $\Gamma_{L^\infty}(\Omega, \mathcal{O}^n) \to \Gamma_S(\Sigma', \mathcal{O}^n)$. ■

Remark 2 In fact, we have constructed a morphism $\varphi_Y : Y \to \Gamma_S(\Sigma',\ \mathcal{O}^n)$.

Definition 9 Let U be an open set in a smooth complex surface X and C a curve in U. Suppose that there exists a finite collection $\{U_i\}_{i=1}^N$ of open subsets of U which satisfies the following properties:

(i) Every U_i is a product $U_i = \Sigma_i \times \Delta$ with Σ_i being an annulus $A_{r_i,1}$. Moreover, in some neighbourhood $\widetilde{U}_i$ of the closure $\bar{U}_i$ there exists a holomorphic function z_i whose restriction on Σ_i coincides with the standard coordinate z on $A_{r_i,1} = \{z : r_i < |z| < 1\}$.
(ii) $U \cap \widetilde{U}_i = \{x \in U : |z_i(x)| < 1\}$.
(iii) For every U_i there is a fixed smoothness S_i such that $C \cap U_i$ is a zero divisor of a Weierstraß polynomial P_{g_i} of a degree n_i with S_i-smooth coefficients $g_i = (g_{i\,1}, \ldots, g_{in_i}) \in \Gamma_{S_i}(\Sigma_i, \mathcal{O}^{n_i})$.
(iv) Distinct U_i are disjoint and $|C| \setminus \cup_{i=1}^{N} U_i$ is compact in U.

Then we say that $S := \{(U_i, z_i, S_i)\}$ is a *smoothness in* U and C is an S-smooth curve in U. A family $\{C_y\}_{y \in Y}$ of S-smooth curves in U is called *continuous* (resp. holomorphic) iff Y is a topological (resp. complex) space, $\{C_y\}_{y \in Y}$ is a continuous family of curves in U and every restricted family $\{C_y \cap U_i\}_{y \in Y}$ is induced by a continuous (resp. holomorphic) map $F_i : Y \to \Gamma_{S_i}(\Sigma_i, \mathcal{O}^{n_i})$.

A section f of the structure sheaf $\mathcal{O}_C$ (resp. the normal sheaf $\mathcal{N}_C$) is called S-smooth, $f \in \Gamma_S(C, \mathcal{O}_C)$ (resp. $f \in \Gamma_S(C, \mathcal{N}_C)$), iff for every U_i the restriction $f|_{U_i}$ is S_i-smooth. An S-smooth curve C is called *extendible* iff there exists an (abstract) holomorphic curve $\widetilde{C}$ (i.e. a complex analytic space of pure dimension 1) and an open embedding $C \hookrightarrow \widetilde{C}$, such that $|C|$ is relatively compact in $|\widetilde{C}|$, $|C| \Subset |\widetilde{C}|$, $\mathcal{O}_{\tilde{C}}|_C = \mathcal{O}_C$, and such that the restriction map $\Gamma(C, \mathcal{O}_{\widetilde{C}}) \longrightarrow \Gamma(C, \mathcal{O}_C)$ takes values in $\Gamma_S(C, \mathcal{O}_C)$.

Theorem 1 *Let X be a smooth complex surface, $U \subset X$ an open subset, S a smoothness in U, and C an S-smooth curve in U. Suppose that C is extendible. Then there exists a ball $B \subset \Gamma_S\ (C, \mathcal{N}_C)$ and a holomorphic map $\Phi : B \to \mathsf{H}^1(C, \mathcal{N}_C)$ with $\Phi(0) = 0$ and $d\Phi(0) = 0$ such that the set $Z := \Phi^{-1}(0)$ is*

(a) *a Banach analytic set of finite codimension in B and*
(b) *the base of a holomorphic family $\mathcal{C} = \{\mathcal{C}_z\}$ of S-smooth curves in U with $\mathcal{C}_0 = C$ which possesses the following universality property*:
(c) *For every continuous (resp. holomorphic) family $\{C_y\}_{y \in Y}$ of S-smooth curves in U with $C_{y_0} = C$ there exists a neighbourhood Y' of y_0 in Y and a continuous (resp. holomorphic) map $\Psi_Y : Y' \to Z$ with $\Psi_Y(y_0) = 0$ and $\mathcal{C}_{\Psi_Y(y)} = C_y$.*

Denote by $\mathcal{M}_S(U)$ the set of S-smooth curves in U. Due to Definition 9 this is a subset of $\prod_i \Gamma_{S_i}(\Sigma_i, \mathcal{O}^{n_i})$ with the induced topology. Thus Theorem 1 provides that in a neighbourhood of an extendible curve $\mathcal{M}_S(U)$ has the natural structure of a Banach analytic space of finite type and that Z is a local chart for $\mathcal{M}_S(U)$ at C. We call $\mathcal{M}_S(U)$ the moduli space of S-smooth curves in U.

Proof Let $\{U_i\}$ be as in Definition 9. We construct a special covering $\{V_i\}$ of $|C|$ in U which satisfy the following conditions:

(i′) Every V_i is biholomorphic to $\Sigma_i \times \Delta$ for some smooth complex curve Σ_i with a boundary $\partial\Sigma_i$ consisting of finitely many smooth circles γ_{ij}, $\partial\Sigma_i = \sqcup_j \gamma_{ij}$.
(ii′) If $i \leqslant N$, then $V_i = A_{r'_i,1} \times \Delta \subset U_i$ for some $r_i \leqslant r'_i < 1$.
(iii′) With respect to the isomorphism $V_i \cong \Sigma_i \times \Delta$, the restricted curve $C \cap V_i$ is a divisor of a Weierstraß polynomial P_i. Moreover, for every $i > N$ there is fixed a smoothness S_i on Σ_i and the coefficients of P_i are S_i-smooth.

(iv′) An intersection $V_i \cap V_j$ is either empty or is biholomorphic to a distorted cylinder $W_{ij} := W_{r_{ij}, R_{ij}, \rho_{ij}}$ with corresponding holomorphic coordinates z'_{ij}, z''_{ij}, and w_{ij}. In the latter case $|C| \cap V_i \cap V_j$ is a (non-empty) annulus $\Sigma_{ij} = A_{r_{ij}, R_{ij}} = \mathsf{Div}(w_{ij})$ and $C_{ij} := C \cap V_i \cap V_j = n_{ij} \cdot A_{r_{ij}, R_{ij}}$.
(v′) If γ_{ij} is a boundary component of Σ_i with $i > N$, then $\gamma_{ij} \subset V_j$.

The construction of $\{V_i\}$ can be realized as follows: First, for every $i \leqslant N$ we find r'_i with $r_i \leqslant r'_i < 1$ such that $|C|$ is a smooth analytic set in a neighbourhood of $|C| \cap (\{x \in \Sigma_i : |z(x)| = r'_i\} \times \Delta)$. Next, we consider the singular points of $|C| \backslash (\cup_{i=1}^N V_i)$ and find an appropriate neighbourhood V_i, $i = N+1, \ldots, N_1$, of every such a point, so that V_i and V_j are disjoint for $1 \leqslant i, j \leqslant N_1$. Then the set $|C| \backslash (\cup_{i=1}^{N_1} V_i)$ can be covered by finitely many smooth complex non-closed curves C'_k with a smooth boundary which we enumerate by $k = N_1 + 1, \ldots, N_2$.

For any C'_k we fix a neighbourhood V'_k of a closure $\bar{C}'_k$ such that $|C| \cap V'_k$ is also smooth with a smooth boundary and $\mathsf{H}^2(V'_k, \mathbb{Z}) = 0$. In particular, the (holomorphic) line bundle $L_{C \cap V'_k}$, corresponding to a divisor $|C| \cap V'_k$, is topologically trivial. Due to a result of Siu [8], the set $|C| \cap V'_k$ admits a Stein neighbourhood $V''_k \subset V'_k$. The condition of topological triviality of $L_{C \cap V'_k}$ provides the existence of a holomorphic function $w_k \in \Gamma(V''_k, \mathcal{O})$ such that $|C| \cap V''_k = \mathsf{Div}(w_k)$.

We may assume that $|C| \cap V''_k$ is biholomorphic to a subdomain of the complex plane $\mathbb{C}$. Let z_k be a holomorphic function on $|C| \cap V''_k$ which corresponds to a standard coordinate on $\mathbb{C}$. Since V''_k is Stein, we can extend z_k to a holomorphic function in V''_k. Now one can see that, choosing appropriate $\Sigma_k \subset |C| \cap V''_k$, $k = N_1 + 1, \ldots, N_2$, and setting $V_k := \{x \in V''_k : z_k(x) \in \Sigma_k | w_k(x)| < r_k\}$, it is possible to obtain the desired covering $\{V_i\}$ with $i = 1, \ldots, N_2$.

Due to the construction of $V_i = \Sigma_i \times \Delta$, the boundary components γ_{ij} of Σ_i are naturally separated into two groups which consist respectively of 'inner' components lying in U and 'outer' components lying on ∂U. It is easy to see that the property of a curve C to be S-smooth in U is independent of the choice of inner smoothness classes S_{ij} which correspond to inner components γ_{ij}. Thus, without loss of generality we may assume that all inner smoothnesses classes S_{ij} are Hilbert, i.e. the corresponding spaces $\Gamma_{S_i}(\Delta, \mathcal{O})$ are Hilbert spaces. For example, one can take all S_i to be some Sobolev smoothness class $L^{k,2}$.

For any index pair (i,j) with nonempty $W_{ij} = V_i \cap V_j$ we denote by n_{ij} the multiplicity of $C_{ij} = C \cap W_{ij}$. Note that for such (i,j) the smoothnesses S_i and S_j in $V_i = \Sigma_i \times \Delta$ and $V_j = \Sigma_j \times \Delta$ induce the smoothness S_{ij} on Σ_{ij} with the continuous projections $\Gamma_{S_i}(V_i, \mathcal{O}) \to \Gamma_{S_{ij}}(W_{ij}, \mathcal{O})$ and $\Gamma_{S_j}(V_j, \mathcal{O}) \to \Gamma_{S_{ij}}(W_{ij}, \mathcal{O})$. We fix sufficiently small balls $B_{ij} \subset \Gamma_{S_{ij}}(\Sigma_{ij}, \mathcal{O}^{n_{ij}})$ which parameterize S_{ij}-smooth curves in W_{ij} which are sufficiently close to $C \cap W_{ij}$.

Now fix some V_i. Then the restricted curve $C \cap V_i$ is a zero divisor of a uniquely defined Weierstraß polynomial P_i of the degree n_i with S_i-smooth coefficients $g_i = (g_{i\,1}, \ldots) \in \Gamma_{S_i}(\Sigma_i, \mathcal{O}^{n_i})$. Fix a sufficiently small ball $B_i := B(g_i, a_i) \subset \Gamma_{S_i}(\Sigma_i, \mathcal{O}^{n_i})$ centred at g_i. If the radius a_i of B_i is chosen sufficiently small, then, for every j such that $W_{ij} \neq \varnothing$ and for every $f \in B_i$, the restricted curve $\mathsf{Div}(P_f) \cap W_{ij}$ is a zero divisor of a uniquely defined distorted Weierstraß polynomial $\widetilde{P}_g$ of the degree n_{ij} with S_{ij}-smooth coefficients $g \in \Gamma_{S_{ij}}(\Sigma_{ij}, \mathcal{O}^{n_{ij}})$.

This defines a map $\varphi_{ij} : B_i \to B_{ij}$ which is holomorphic (Lemma 5). We may assume that the image of φ_{ij} lies in the ball $1/2B_{ij}$. Consider the holomorphic map

$$\widetilde{\Phi} : \prod_i B_i \longrightarrow \prod_{i<j} B_{ij}, \quad (\widetilde{\Phi}(f))_{ij} := \varphi_{ij}(f_i) - \varphi_{ji}(f_j).$$

The product $\prod_i B_i$ parameterized the space of tuples (C_i), where every $C_i := \mathsf{Div}(f_i)$ is an S_i-smooth curve in V_i which is sufficiently close to $C \cap V_i$. Two such 'local deformations' C_i and C_j coincide exactly when $\varphi_{ij}(f_i) = \varphi_{ji}(f_j)$. It follows that the analytic set $\widetilde{Z} := \widetilde{\Phi}^{-1}(0) \subset \prod_i B_i$ satisfies property (ii) of the theorem.

Due to Lemmas 4, 6 and Definition 6, the tangent space to $\prod_i B_i$ at $g = (g_i)$ is isomorphic to $\sum_i \Gamma_{S_i}(\Sigma_i, \mathcal{O}^{n_i}) = \sum_i \Gamma_{S_i}(C \cap V_i, \mathcal{N}_C)$, whereas the tangent space to $\prod_{i<j} B_{ij}$ at 0 is isomorphic to $\sum_{i<j} \Gamma_{S_{ij}}(\Sigma_{ij}, \mathcal{O}^{n_{ij}}) = \sum_{i<j} \Gamma_{S_{ij}}(C \cap V_{ij}, \mathcal{N}_C)$. Formula (1.8) implies that the differential $d\widetilde{\Phi}_g$ of $\widetilde{\Phi}$ at g coincides with the Čecch coboundary operator

$$d\widetilde{\Phi}_g = \delta_S : \sum_i \Gamma_{S_i}(C \cap V_i, \mathcal{N}_C) \longrightarrow \sum_{i<j} \Gamma_{S_{ij}}(C \cap V_{ij}, \mathcal{N}_C), \quad (\delta(f_i))_{ij} = f_i|_{W_{ij}} - f_j|_{W_{ij}}.$$

The key point of the proof is that for an extendible curve C the operator δ has a closed image and splits. To show this we fix some $i \leqslant N$ so that $V_i = \Sigma_i \times \Delta$ touches the boundary ∂U. Let γ_{i1} be a boundary component of Σ_i lying on ∂U and S_{i1} the corresponding smoothness class. Then there exists curves $C'_i \subset C''_i \subset \widetilde{C}$ such that:

(a) $|C'_i| \cap |C| = |C''_i| \cap |C| = |C_i|$ so that both C'_i and C''_i are extensions of C_i 'outwards' from $|C|$.
(b) $|C'_i|$ is relatively compact in $|\widetilde{C}|$ and the 'outer' part of the (topological) boundary of $|C'_i|$ lying outside $|C|$ is smooth and consists of finitely many circles γ'_{ij} which lie in $|C''_i|$.
(c) If $f \in \Gamma(C'_i, \mathcal{O}_{\tilde{C}})$, then $f|_{C'_i}$ is S_{i1}-smooth at the 'outer' part of the boundary of $|C_i|$ which lie on ∂U.

We repeat this construction for every $1 \leqslant i \leqslant N$ and set $C'_i = C''_i = C_i$ and so on for $i > N$. Set $C' := C \cup_i C'_i$ and $C'' := C \cup_i C''_i$. These are complex curves. The boundary $\partial C'$ of C' consists of smooth circles γ'_{ij}. Since the restriction $\mathcal{N}_C|_{C_i}$ is trivial, we can extend $\mathcal{N}_C$ to a rank 1 locally free $\mathcal{O}_{C''}$-module $\mathcal{N}_{C''}$ with $\mathcal{N}_{C''}|_{C''_i}$ trivial. For any component γ'_{ij} of the 'outer' part of the boundary of $|C'_i|$ we fix a Hilbert smoothness class S'_{ij}. This defines the Hilbert space $\Gamma_{S'_i}(C'_i, \mathcal{N}_{C'})$ and a (continuous) restriction map $\Gamma_{S'_i}(C'_i, \mathcal{N}_{C'}) \to \Gamma_{S_i}(C_i, \mathcal{N}_C)$.

Consider the induced Čech coboundary operators

$$\delta' : \sum_i \Gamma_{S'_i}(C'_i, \mathcal{N}_{C'}) \longrightarrow \sum_{i<j} \Gamma_{S_{ij}}(C_{ij}, \mathcal{N}_C), \quad (\delta'(f_i))_{ij} := f_i|_{C_{ij}} - f_j|_{C_{ij}}, \tag{4.1}$$

and

$$\delta'' : \sum_i \Gamma(C''_i, \mathcal{N}_{C''}) \longrightarrow \sum_{i<j} \Gamma(C_{ij}, \mathcal{N}_C), \quad (\delta''(f_i))_{ij} := f_i|_{C_{ij}} - f_j|_{C_{ij}}. \tag{4.2}$$

By the construction, all C''_i are Stein spaces. Thus (4.2) is an acyclic Čech resolvent for $\mathcal{N}_{C''}$. Consequently, $\mathsf{Ker}(\delta'') = \mathsf{H}^0(C'', \mathcal{N}_{C''}) = \Gamma(C'', \mathcal{N}_{C''})$ and

$\mathsf{Coker}(\delta'') = \mathsf{H}^1(C'', \mathcal{N}_{C''})$. We note the canonical isomorphisms

$$\mathsf{H}^1(C'', \mathcal{N}_{C''}) \cong \mathsf{H}^1(C', \mathcal{N}_{C'}) \cong \mathsf{H}^1(C, \mathcal{N}_C).$$

These are finite-dimensional spaces. Denote by p the composition

$$\sum_{i<j} \Gamma_{S_{ij}}(C_{ij}, \mathcal{N}_C) \hookrightarrow \sum_{i<j} \Gamma(C_{ij}, \mathcal{N}_C) \twoheadrightarrow \mathsf{H}^1(C'', \mathcal{N}_{C''})$$

and set $T := \mathsf{Ker}(p)$.

First, we note that p is a surjection onto $\mathsf{H}^1(C'', \mathcal{N}_{C''})$. For this observe that one can find an acyclic Čech resolvent

$$\hat{\delta} : \sum_i \Gamma(\widehat{C}_i, \mathcal{N}_{C''}) \longrightarrow \sum_{i<j} \Gamma(\widehat{C}_{ij}, \mathcal{N}_C), \quad (\hat{\delta}(f_i))_{ij} := f_i|_{\widehat{C}_{ij}} - f_j|_{\widehat{C}_{ij}}, \tag{4.3}$$

for $\mathcal{N}_{C''}$ with $C_{ij} \Subset \widehat{C}_{ij}$. Then every $[h] \in \mathsf{H}^1(C'', \mathcal{N}_{C''})$ can be represented by $h = (h_{ij})$ with $h_{ij} \in \Gamma(\widehat{C}_{ij}, \mathcal{N}_C)$ so that the restriction gives $h \in \sum_{i<j} \Gamma_{S_{ij}}(C_{ij}, \mathcal{N}_C)$.

Now take $h = (h_{ij}) \in T$. Since $p(h) = 0$, there exists $f'' = (f_i'') \in \sum_i \Gamma(C_i'', \mathcal{N}_{C''})$ such that $h = \delta''(f)$. Let $f_i' \in \Gamma(C_i', \mathcal{N}_{C'})$ denote the restriction of f_i'' onto C_i'.

Now in fact $f_i' \in \Gamma_{S_i'}(C_i', \mathcal{N}_{C'})$. The corresponding smoothness of f_i' at the outer component γ_{ij}' of the boundary $\partial C_i'$ follows from the fact that f_i' is holomorphic in a neighbourhood of γ_{ij}'. Similarly, if γ_{ij} is an inner component of the boundary $\partial C_i'$, then γ_{ij} lies in some V_j. In this case f_j' is holomorphic in a neighbourhood of γ_{ij} and $f_i' = h_{ij} + f_j'$. Since h_{ij} is S_{ij}-smooth at γ_{ij}, the same holds for f_i'.

This implies that the image of δ' is T and is of finite codimension. Consequently, T is a closed subspace of $\sum_{i<j} \Gamma_{S_{ij}}(C_{ij}, \mathcal{N}_C)$. Since all the smoothnesses S_i are Hilbert, $\sum_i \Gamma_{S_i'}(C_i', \mathcal{N}_{C'})$ is a Hilbert space and $\mathsf{Ker}(\delta')$ admits a complement. Therefore there exists a splitting operator $\sigma' : \Gamma_{S_{ij}}(C_{ij}, \mathcal{N}_C) \to \sum_i \Gamma_{S_i'}(C_i', \mathcal{N}_{C'})$ such that for every $h \in T$ holds $\delta'(\sigma'(h)) = h$.

Let $\sigma : \Gamma_{S_{ij}}(C_{ij}, \mathcal{N}_C) \to \sum_i \Gamma_{S_i'}(C_i', \mathcal{N}_{C'})$ denote the composition of σ' with the restriction map $\sum_i \Gamma_{S_i'}(C_i', \mathcal{N}_{C'}) \to \sum_i \Gamma_{S_i}(C_i, \mathcal{N}_C)$. Then again $\delta(\sigma(h)) = h$.

Recall that $g = (g_i) \in \prod_i B_i$ is a tuple of coefficients Weierstraß polynomials parameterizing our curve C and δ is the differential $\mathrm{d}\widetilde{\Phi}_g$. Denote by $\widetilde{\Phi}_T$ the composition of $\widetilde{\Phi}$ with the orthogonal projection on T and by Φ the composition of $\widetilde{\Phi}$ with the projection onto $\mathsf{Coker}\,\mathrm{d}\widetilde{\Phi}_g = \mathsf{H}^1(C, \mathcal{N}_C)$.

The implicit function theorem implies that the set $Z_1 := \widetilde{\Phi}_T^{-1}(0)$ is a complex Banach submanifold of $\prod_i B_i$ containing g with the tangent space $T_g Z_1$ canonically isomorphic to $\mathsf{Ker}\,(d\widetilde{\Phi}_g) = \mathsf{H}^0_S(C, \mathcal{N}_C)$. One can easily see that the set $Z_2 = \widetilde{\Phi}^{-1}(0)$ is an analytic subset of Z_1 defined by the equation $Z_2 = \{y \in Z_1 : \Phi(y) = 0\}$. Take a neighbourhood B of g in Z_1 biholomorphic to a small ball in $\mathsf{H}^0_S(C, \mathcal{N}_C)$ and set $Z := Z_2 \cap B$. Then B, Φ and Z satisfy the condition of the theorem. ■

5. Applications and related questions

Proof of the Main Theorem Let X be a smooth complex surface, C^* a curve in X and $K \Subset |C^*|$ some compact subset. Repeating the constructions of the proof of Theorem 1, one can find an open neighbourhood $U \subset X$ of K and appropriate collections $\{U_i\}$ and $\{V_i\}$ of open sets in U such that $C^* \cap U$ is S-smooth with respect to an appropriate smoothness S in U. Furthermore, $\{U_i\}$ and $\{V_i\}$ can be chosen to satisfy the conditions (i)–(iv) of the Definition 9 and (i′)–(v′) of the proof of Theorem 1. Then the statement of the Main Theorem immediately follows Theorem 1. ■

It is important to underline the fact that Banach analytic sets of finite type, in contrast to general Banach analytic sets, have a simple structure [9, Chapter II, Section 3]). Namely, let B be a ball in a Banach space E and let Z be an analytic subset of B, which is defined by finitely many holomorphic function and contains $0 \in E$. Then in a neighbourhood $V \subset B$ of 0 the set Z has finitely many components, $Z \cap V = \cup_{i=1}^{N} Z_i$, each of them is irreducible at 0 and is defined by finitely many holomorphic functions too. Further, for every such component Z_i of Z at 0 there exist a closed (Banach) subspace $E_i \subset E$ of finite codimension and a (linear) projection $\pi_i : E \to E_i$ such that in some smaller neighbourhood $V_i \subset V$ of 0 the restricted projection $\pi_i : Z_i \cap V_i \to \pi_i(V_i)$ is a proper branched analytic covering with $\pi_i^{-1}(0) \cap Z_i = \{0\}$.

As a corollary of the Main Theorem we obtain the following statements.

Proposition 1 *Let X, C^* and U be as in the Main Theorem. Suppose that $\{C_n\}$ is a sequence of curves in X converging (weakly) to C^*. Then for any n which is sufficiently big, there exists a holomorphic family $\{C_\lambda\}_{\lambda\in\Delta}$ of curves in U which is parameterized by a disk $\Delta \subset \mathbb{C}$ and contains both $C_n \cap U$ and $C^* \cap U$.*

Proof Denote $C := C^* \cap U$. Let $B \subset \Gamma_S(C, \mathcal{N}_C)$ be a small ball and $Z := \Phi^{-1}(0) \subset B$ a local chart for $\mathcal{M}_S(U)$. Let $Z = \cup Z_i$ be the decomposition of Z into components such that every Z_i is a proper branched analytic covering of a ball B_i in an appropriate Banach subspace $E_i \subset \Gamma_S(C,\mathcal{N}_C)$ with respect to a projection π_i: $\Gamma_S(C,\mathcal{N}_C) \to E_i$. Then for $n \gg 1$ a curve $C_n \cap U$ is parameterized by a uniquely defined $a_n \in Z$, in particular, a_n lies in some Z_i.

Set $a'_n := \pi_i(a_n)$ and let L_n be a complex line in E_i through a'_n and 0. Then $\pi_i^{-1}(L_n) \cap Z_i$ is a complex curve which consists of finitely many irreducible components, each of which contains 0. Consequently, there exists a holomorphic map $f_n : \Delta \to \Gamma_S(C,\mathcal{N}_C)$ such that $f_n(\lambda) \in Z_i$, $f_n(0) = 0$ and $f_n(\lambda_n) = a_n$ for some $\lambda_n \in \Delta$. The map f_n defines the desired one parameter family $\{C_\lambda\}_{\lambda\in\Delta}$ connecting $C^* \cap U$ and $C_n \cap U$. ■

As an application, the Main Theorem and Proposition 1 yields a generalization of Levi's continuity principle. For a precise statement we need the notion of a meromorphic hull of a domain W in a complex manifold X. Recall that this is a maximal Riemann domain $(\widehat{W}, \pi)$ over X containing W (i.e. $\pi : \widehat{W} \to X$ is a locally biholomorphic map and there is given an inclusion $i : W \to \widehat{W}$ with $\pi \circ i = \mathsf{Id}_W$), such that every meromorphic function f on W extends to a function $\hat{f}$ on $\widehat{W}$. We refer to [10,11] for details.

Theorem 2 (Continuity principle) *Let X be a complex surface, $W \subset X$ a domain, $\widehat{W}$ its meromorphic hull, and $\widehat{C}_n$ a sequence of curves in $\widehat{W}$ without multiple components. Suppose that there exists a domain $U \Subset X$, such that the projected curves $C_n := \pi(\widehat{C}_n)$ are close in U and (weakly) converge to a curve C_∞. Suppose also the boundary of C_∞ is not empty and lies in W. Then the sequence $\widehat{C}_n$ converge to a curve $\widehat{C}_\infty$ with $\pi(\widehat{C}_\infty) = C_\infty$.*

Remark 3 Theorem 2 has the following meaning. If f is a meromorphic function in W which extends meromorphically to a neighbourhood of every C_n (possibly as a several sheeted function), then it can be extended in a neighbourhood of $|C_\infty|$. Thus, we obtain the generalization of the classical result of E.E. Levi which deals with the case where C_∞ and every C_n are disks.

Proof Set $K_0 := |C_\infty| \backslash W$. Let K be the union of K_0 with those connected components of $|C_\infty| \cap W$ which are relatively compact in $|C_\infty|$. Then K is compact. According to the Main Theorem, we can find an open set $U_1 \subset U$ and a smoothness S in U_1 such that $K \subset U_1$ and $C_\infty \cap U_1$ is S-smooth.

Due to Proposition 1, for any $n \gg 1$ there exists a holomorphic family $\{Z_\lambda\}_{\lambda \in \Delta}$ of curves in U_1 which contains C_n and C_∞. Let λ_0 and λ_∞ be the corresponding parameter values. If n was chosen big enough, the boundary of every Z_λ lies in W. In an obvious way, this family defines a complex space $Z \subset U_1 \times \Delta$ such that every Z_λ is identified with $Z \cap U_1 \times \{\lambda\}$.

Let f be a meromorphic function in W and $\hat{f}$ its extension on $\widehat{W}$. Using the technique of [10,11] for meromorphic extension along a holomorphic family of curves, one can show that the restriction of f onto Z_{λ_0} extends to a holomorphic function F on the entire space Z such that F coincides with f in a neighbourhood of boundary of every Z_λ. Since f can be any meromorphic function in W, this means that the family $\{Z_\lambda\}_{\lambda \in \Delta}$ can be lifted to a family $\{\widehat{Z}_\lambda\}_{\lambda \in \Delta}$ of curves in $\widehat{W}$ such that $\pi(\widehat{Z}_\lambda) = \widehat{Z}_\lambda$ and $\widehat{Z}_\lambda \cap W = Z_\lambda \cap W$. For further details, see [10,11].

Now it is not difficult to show that the desired curve $\widehat{C}_\infty \subset \widehat{W}$ can be constructed as $\widehat{Z}_{\lambda_\infty} \cup (C_\infty \cap W)$. ■

Trying to generalize the results of this article, one must overcome the following difficulties:

(1) Considering deformation problem, a non-compact complex subspace Z in a complex manifold X such that $\mathsf{dim}_{\mathbb{C}}\, Z > 1$, one confronts with the fact that non-compact components of Z can be non-Stein. Thus, an appropriate cohomology group $H^1(Z, \mathcal{N}_Z)$ can be infinite-dimensional and even worse non-separated topological vector space.

(2) We illustrate the problem appeared by deformation of non-compact cycles by the following example. Let $n \geqslant 2$ and $k \geqslant 2$ be integers. Consider disc $\Delta_0 := \Delta \times \{0\} \subset \Delta \times \mathbb{C}^n$ and cycle $Z := k \cdot \Delta_0$ in $X := \Delta \times \mathbb{C}^n$. The problem of deformation of Z leads to consideration of the space $\mathsf{Hom}(\Delta, \mathsf{Sym}^k\, \mathbb{C}^n)$ of holomorphic maps between Δ and k-th symmetric power of $\mathbb{C}^n$. The space $\mathsf{Sym}^k \mathbb{C}^n$ is naturally realized as an analytic subset of some $\mathbb{C}^N$. This gives a natural inclusion $\mathsf{Hom}(\Delta, \mathsf{Sym}^k\, \mathbb{C}^n) \subset \mathsf{Hom}(\Delta, \mathbb{C}^N)$. Fixing some smoothness class S, e.g. $S = L^\infty$, we obtain the set $\mathsf{Hom}_S(\Delta, \mathsf{Sym}^k\, \mathbb{C}^n)$ of S-smooth maps as an *analytic* subset of the Banach space $\mathsf{Hom}_S(\Delta, \mathbb{C}^N)$. Thus $\mathsf{Hom}_S(\Delta, \mathsf{Sym}^k\, \mathbb{C}^n)$ is equipped with the natural structure of Banach analytic space. However, $\mathsf{Hom}_S(\Delta, \mathsf{Sym}^k\, \mathbb{C}^n)$ is has infinite codimension in $\mathsf{Hom}_S(\Delta, \mathbb{C}^N)$. Moreover, there are infinitely many irreducible components of $\mathsf{Hom}_S(\Delta, \mathsf{Sym}^k\, \mathbb{C}^n)$ in $f \equiv 0$ (zero map).

(3) One can obtain a statement similar to Theorem 1 considering the deformation of stable maps from non-compact nodal curves to a given smooth complex manifold X of arbitrary dimension. Recall, that an abstract

nodal curve C is a complex space of dimension one whose singularities are only ordinary double points. Such curves are also called semi-stable. A holomorphic map $f: C \to X$ is called stable, if there are only finitely many biholomorphisms $g: C \to C$ with the property $f \circ g = f$. In [10] the deformation problem of the stable maps (C,f) is considered under the following additional assumptions.

($*$) Curves C have finitely many irreducible components, each of them being of finite genus and bordered by finitely many smooth circles, and maps $f: C \to X$ are $L^{1,p}$-smooth up to boundary ∂C.

The set of such pairs (C,f) is equipped with *Gromov topology*. The following result is proved.

For given (C_0,f_0) there exist Banach analytic spaces $\mathcal{M}$, $\mathcal{C}$ of finite type and holomorphic maps $\pi: \mathcal{C} \to \mathcal{M}$, $F: \mathcal{C} \to X$, such that:

(i) for any $\lambda \in \mathcal{M}$ the fibre $C_\lambda := \pi^{-1}(\lambda)$ is a nodal curve and $f_\lambda := F|_{C_\lambda} : C_\lambda \to X$ is a stable map with the property ($*$);
(ii) for any stable map (C,f) with the property ($*$) sufficiently close to (C_0,f_0) w.r.t. Gromov topology there exist a $\lambda \in \mathcal{M}$ and biholomorphism $\varphi: C \to C_\lambda$ such that $f = f_\lambda \circ \varphi$.

Acknowledgements

I express my gratitude to S. Ivashkovich for the constant support during preparation of this article and explaining possible applications. I also want to thank D. Barlet, H. Flenner, S. Kosarew, G. Schumacher, and B. Siebert for numerous helpful discussions on various questions of the deformation theory. I am grateful to the referee for reading this article attentively and pointing out some misprints and minor mistakes.

Note

1. This means that $\|fg\|_S \leqslant c_S\|f\|_S\|g\|_S$, where c_S is a constant independent of $f, g \in \Gamma_S(\Delta, \mathcal{O})$ and possibly greater than 1. This can always be corrected by introducing a new norm $\|f\|_S^* \sup\{\frac{\|fg\|_S}{\|g\|_S} : g \neq 0 \in \Gamma_S(\Delta, \mathcal{O})\}$ which is equivalent to $\|\cdot\|$ and for which $\|fg\|_S^* \leqslant \|f\|_S^*\|g\|_S^*$.

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