Generalized Variation of Mappings with Applications to Composition Operators and Multifunctions

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Abstract. We study (set-valued) mappings of bounded Φ-variation defined on the compact interval I and taking values in metric or normed linear spaces X. We prove a new structural theorem for these mappings and extend Medvedev's criterion from real valued functions onto mappings with values in a reflexive Banach space, which permits us to establish an explicit integral formula for the Φ-variation of a metric space valued mapping. We show that the linear span $GV_\Phi(I;X)$ of the set of all mappings of bounded Φ-variation is automatically a Banach algebra provided X is a Banach algebra. If $h : I\times X \to Y$ is a given mapping and the composition operator $\mathcal{H}$ is defined by $(\mathcal{H}f)(t) = h(t, f(t))$, where $t \in I$ and $f : I \to X$, we show that $\mathcal{H} : GV_\Phi(I;X) \to GV_\Psi(I;Y)$ is Lipschitzian if and only if $h(t,x) = h_0(t) + h_1(t)x$, $t \in I$, $x \in X$. This result is further extended to multivalued composition operators $\mathcal{H}$ with values compact convex sets. We prove that any (not necessarily convex valued) multifunction of bounded Φ-variation with respect to the Hausdorff metric, whose graph is compact, admits regular selections of bounded Φ-variation.

Mathematics Subject Classifications (1991): Primary: 26A45, 26A46; Secondary: 47H04, 47H30, 54C60, 54C65

Key words: metric space valued mappings, bounded Φ-variation, set-valued mappings, regular selections, Lipschitzian Nemytskiĭ composition operators

1. Introduction

In this paper we study mappings of bounded generalized Φ-variation defined on the closed interval $I = [a, b]$ of the real line $\mathbb{R}$ (a, $b \in \mathbb{R}$, $a < b$) and taking values in a metric space (X, d). The notion of the Φ-variation of a mapping $f \in X^I$, where X^I is the set of all mappings $f : I \to X$ from I into X, is introduced as follows. Let $\mathcal{N}$ be the set of all convex continuous functions $\Phi : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ such that Φ vanishes at zero only and $\lim_{\rho\to\infty} \Phi(\rho)/\rho = \infty$. If $\Phi \in \mathcal{N}$ and $T = \{t_i\}_{i=0}^m$ is a partition of I (i.e. $a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b$), we set

$$V_\Phi[f,T] \equiv V_{\Phi,d}[f,T] := \sum_{i=1}^m \Phi\Big(\frac{d(f(t_i), f(t_{i-1}))}{t_i - t_{i-1}}\Big)(t_i - t_{i-1}) \qquad (1.1)$$

and define the (*total*) *generalized Φ-variation of f on I* by

$$\mathsf{V}_\Phi(f) \equiv \mathsf{V}_\Phi(f, I) := \sup\{\, V_\Phi[f,T] \mid T \in \mathcal{T}(I)\,\}, \qquad (1.2)$$

where $\mathcal{T}(I)$ designates the set of all partitions of I. The set of all mappings of *bounded* (*generalized*) Φ*-variation* is denoted by

$$BV_\Phi(I;X) = \{\, f : I \to X \mid \mathsf{V}_\Phi(f) < \infty \,\}. \tag{1.3}$$

The notion of the Φ-variation of a mapping is a generalization of the classical concepts of variation introduced by C. Jordan and F. Riesz for real valued functions. If $\Phi(\rho) = \rho^q$ ($\rho \in \mathbb{R}^+$, $1 \le q < \infty$), we write $\mathsf{V}_q(f)$ and $BV_q(I;X)$ instead of (1.2) and (1.3), respectively, and say that $f \in BV_1(I;X)$ is a *mapping of bounded variation* in the sense of Jordan ([17], [39, Ch. 8, 9]) and $f \in BV_q(I;X)$ with $q > 1$ is a *mapping of bounded q-variation* in the sense of Riesz ([42], [43, Ch. 2, Sec. 3.36]).

The following three criteria for real valued functions are well known. Jordan's criterion [17]: $f \in BV_1(I;\mathbb{R})$ if and only if f is the difference of two bounded nondecreasing functions on I. Riesz's criterion [42]: if $q > 1$, then $f \in BV_q(I;\mathbb{R})$ if and only if $f : I \to \mathbb{R}$ is absolutely continuous ($f \in AC(I;\mathbb{R})$, for short) and its derivative defined almost everywhere on I is Lebesgue q-summable; moreover, $\mathsf{V}_q(f) = \int_I |f'(t)|^q dt$. Medvedev's criterion [32]: if $\Phi \in \mathcal{N}$, then $f \in BV_\Phi(I;\mathbb{R})$ if and only if $f \in AC(I;\mathbb{R})$ and $\int_I \Phi(|f'(t)|)\, dt < \infty$. Also, it is known that the space $BV_q(I;\mathbb{R})$ equipped with the norm $|f|_q = |f(a)| + (\mathsf{V}_q(f))^{1/q}$ is a Banach algebra for all $q \ge 1$ (cf. [45] if $q > 1$).

It is clearly seen that the above criteria are inapplicable for mappings from $BV_\Phi(I;X)$ if X is an arbitrary metric space. For these mappings we establish a new criterion (Theorem 2.4) which reveals their structure: a mapping $f \in X^I$ belongs to $BV_\Phi(I;X)$ if and only if f is represented as the composition $f = g \circ \varphi$, where $\varphi \in BV_\Phi(I;\mathbb{R})$ and g maps the image of φ into X and satisfies a Lipschitz condition with the Lipschitz constant not exceeding 1; moreover, the function φ can be chosen such that $\mathsf{V}_\Phi(\varphi) = \mathsf{V}_\Phi(f)$. Earlier, a structural theorem of this type was shown to be valid for mappings from $BV_1(I;X)$ [4], absolutely continuous mappings $f \in AC(I;X)$ [5] and mappings from $BV_q(I;X)$ with $q > 1$ [7] (see also [6]). Using the differentiability properties of mappings from $BV_\Phi(I;X)$ (Theorem 3.1) we extend the Riesz-Medvedev criterion onto mappings with values in a reflexive Banach space X and obtain an explicit integral formula for the Φ-variation of a mapping with values in an arbitrary metric space (Corollary 3.2). As an immediate consequence of the structural theorem, Medvedev's criterion and the fact that the space $L^1(I;\mathbb{R})$ of Lebesgue integrable functions is the union of Orlicz classes $L_\Phi(I;\mathbb{R})$ over all $\Phi \in \mathcal{N}$ (cf. [22, Sec. 8.1]), we find that $AC(I;X) = \bigcup_{\Phi \in \mathcal{N}} BV_\Phi(I;X)$.

The assumption that X is a normed linear space does not, in general, imply that the convex set $BV_\Phi(I;X)$ is a linear space. In Section 3 we define the space $GV_\Phi(I;X)$ to be the linear span of $BV_\Phi(I;X)$ and endow it with a norm. This is done in such a way that if $\Phi(\rho) = \rho^q$, $q \geq 1$, then $GV_\Phi(I;X) = BV_q(I;X)$. Having proved the counterpart of the above structural theorem for the space $GV_\Phi(I;X)$ (Lemma 3.5), we show that this space is a Banach algebra provided X is a Banach algebra (Theorem 3.6). Also, we obtain relations between spaces $GV_\Phi(I;X)$ corresponding to different functions $\Phi \in \mathcal{N}$ (Theorems 3.3 and 3.7).

In Section 4 we characterize Lipschitzian composition operators between spaces of mappings of bounded generalized Φ-variation. Let X and Y be normed linear spaces and $h : I \times X \to Y$ a given mapping of two variables. The mapping $\mathcal{H} : X^I \to Y^I$ defined by

$$(\mathcal{H}f)(t) \equiv \mathcal{H}(f)(t) := h(t, f(t)), \qquad t \in I, \quad f \in X^I, \tag{1.4}$$

is called a *composition operator* (or the *Nemytskiĭ operator of substitution*) *generated by* h. For the purpose of introduction we temporarily let $X = Y = \mathbb{R}$. Let $\mathcal{F}(I) \subset \mathbb{R}^I$ be a Banach function space with the norm $|\cdot|_\mathcal{F}$. In order to solve the functional equation $f(t) = h(t, f(t))$, $t \in I$, also written as $f = \mathcal{H}f$, with respect to the unknown function $f \in \mathcal{F}(I)$, one can try the classical Banach fixed point theorem, in which case the operator $\mathcal{H} : \mathcal{F}(I) \to \mathcal{F}(I)$ should satisfy the following Lipschitz condition:

$$|\mathcal{H}f_1 - \mathcal{H}f_2|_\mathcal{F} \leq \mu |f_1 - f_2|_\mathcal{F}, \qquad f_1,\, f_2 \in \mathcal{F}(I), \tag{1.5}$$

where μ is a constant, $0 < \mu < 1$. However, as was observed in [27] in the case of Lipschitz functions $\mathcal{F}(I) = C^{0,1}(I)$, condition (1.5) implies that the generating function h of the operator $\mathcal{H}$ has to be of the form:

$$h(t, x) = h_0(t) + h_1(t)x \quad \text{for all} \quad t \in I \quad \text{and} \quad x \in \mathbb{R}, \tag{1.6}$$

where $h_0,\, h_1 \in \mathcal{F}(I)$. Consequently, Banach's contraction principle cannot be applied directly in $\mathcal{F}(I)$ if h is a "nonlinear" function in the second variable (and hence a more powerful tool must be invoked, such as the Schauder fixed point theorem, etc.). Subsequently, this result has been extended by several authors: [19, 26, 28, 29] (for Hölder and differentiable functions and Lipschitz mappings) and [30, 31, 33, 35] (for functions of bounded variation in the sense of Jordan and Riesz and functions of bounded second p-variation). In Theorems 4.1 and 4.5 we show that for any functions $\Phi,\, \Psi \in \mathcal{N}$ the generating function h of Lipschitzian composition operator $\mathcal{H}$, which maps $GV_\Phi(I;X)$ into $GV_\Psi(I;Y)$, is (roughly) of the form (1.6) for all $(t, x) \in I \times X$. Moreover, if Φ grows at infinity "significantly slower" than Ψ, any operator $\mathcal{H}$ of this kind is constant (Theorem 4.3). These results are new even for real valued functions (Corollaries 4.4 and 4.6); they were established in [8].

In Section 5 we address set-valued mappings and multivalued composition operators. In Section 5.1 we treat the problem of the existence of selections of multifunctions (= set-valued mappings) of bounded generalized Φ-variation. Let $\mathcal{P}_{cb}(X)$ be the family of all nonempty closed bounded subsets of a metric space X, D the Hausdorff metric on $\mathcal{P}_{cb}(X)$ and $F : I \to \mathcal{P}_{cb}(X)$ a multifunction. Measurable (or Baire) selections of F were shown to exist in [23, 11] if I is a metric space, X a complete (and separable) metric space and F is continuous (or lower semicontinuous); moreover, if X is a Banach space and the values of F are convex, then F admits a continuous selection [36] (under much more general conditions on I). However, if the values of F are nonconvex (but compact), continuous selections of F may fail to exist, for example, if (a) $F : [a, b] \to \mathcal{P}_{cb}(\mathbb{R}^2)$ is continuous with respect to D [15] (or even Hölder continuous of any exponent $0 < \gamma < 1$ [9]), or (b) $F : \mathbb{R}^3 \to \mathcal{P}_{cb}(\mathbb{R}^3)$ is Lipschitzian with respect to D [15, 38]. On the other hand, the following results depend upon the domain of F being the real line $\mathbb{R} \supset I$ (the values of F are not assumed to be convex): If X is a Banach space and the graph of F is compact, selections of F of the same functional class as F, i.e. the so called *regular selections*, exist in the cases: F is Lipschitzian [16, 18, 37], absolutely continuous [47, 5], of bounded Jordan variation [4, 5] or of bounded Riesz q-variation with respect to D [7]. In Theorem 5.1 we show that any multifunction $F \in BV_\Phi(I; \mathcal{P}_{cb}(X))$ with compact graph admits regular selections $f \in BV_\Phi(I; X)$. In particular, if X is reflexive, selections f are almost everywhere strongly differentiable.

Finally, in Section 5.2 we introduce the metric space $BV_\Psi(I; \mathcal{P}_{cc}(Y))$ of multifunctions of bounded Ψ-variation taking values in $\mathcal{P}_{cc}(Y)$, the set of all nonempty compact convex subsets of a normed linear space Y. Here the metric space structure is made possible thanks to the translation invariance of the Hausdorff metric D on $\mathcal{P}_{cc}(Y)$. In Section 5.3 we extend the results of Section 4 onto multivalued Lipschitzian composition operators $\mathcal{H} : GV_\Phi(I; X) \to BV_\Psi(I; \mathcal{P}_{cc}(Y))$ generated by a multivalued mapping $h : I \times X \to \mathcal{P}_{cc}(Y)$. Recently, Lipschitzian set-valued composition operators $\mathcal{H}$ were characterized in [44] in the class of Lipschitz multifunctions and in [46] and [34] in the case of multifunctions of bounded Jordan and Riesz variation, respectively.

2. Metric Space Valued Mappings

Let (X, d) be a metric space. Recall that a mapping $f : I \to X$ is called *absolutely continuous* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any finite number of points $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b$

the condition $\sum_{i=1}^n(b_i - a_i) \le \delta(\varepsilon)$ implies $\sum_{i=1}^n d(f(b_i), f(a_i)) \le \varepsilon$. The set of all absolutely continuous mappings is denoted by $AC(I;X)$. A mapping $f \in X^I$ is said to be *Lipschitzian* (or *Lipschitz continuous*) if the (least) *Lipschitz constant* of f,

$$\mathrm{Lip}(f) := \sup\{\, d(f(t), f(s))/|t-s|\,;\, t,\, s \in I,\ t \neq s\,\},$$

is finite. We denote by $C^{0,1}(I;X)$ the set of all Lipschitzian mappings. If $(X, \|\cdot\|)$ is a normed linear space, we denote by $\|f\|_{0,1} = \|f(a)\| + \mathrm{Lip}(f)$ the norm in $C^{0,1}(I;X)$, so that $C^{0,1}(I;X)$ is a Banach space provided X is a Banach space.

Observe that any function $\Phi \in \mathcal{N}$ is strictly increasing, and so there exists the inverse function of Φ denoted by Φ^{-1}, and that the functions $\rho \mapsto \Phi(\rho)/\rho$ and $\rho \mapsto \rho\Phi^{-1}(1/\rho)$ are nondecreasing for $\rho > 0$. Note that unlike N-functions from [22, Ch. 1], functions from the set $\mathcal{N}$ are *not* subject to the condition $\Phi'(0) = \lim_{\rho\to 0}\Phi(\rho)/\rho = 0$. Also, since $\Phi \in \mathcal{N}$ is convex and continuous, we have *Jensen's inequality for sums*:

$$\Phi\biggl(\frac{\sum_{i=1}^n \alpha_i x_i}{\sum_{i=1}^n \alpha_i}\biggr) \le \frac{\sum_{i=1}^n \alpha_i \Phi(x_i)}{\sum_{i=1}^n \alpha_i}, \quad \{\alpha_i,\, x_i\}_{i=1}^n \subset \mathbb{R}^+, \quad \textstyle\sum_{i=1}^n \alpha_i > 0, \tag{2.1}$$

and (a particular case of) *Jensen's integral inequality*, which holds for functions $x : I \to \mathbb{R}^+$ such that the integrals in (2.2) are finite:

$$\Phi\biggl(\frac{1}{|I|}\int_I x(t)\,dt\biggr) \le \frac{1}{|I|}\int_I \Phi(x(t))\,dt, \quad \text{where } |I| := b - a. \tag{2.2}$$

The following lemma lists some elementary properties of mappings of bounded generalized Φ-variation.

LEMMA 2.1. *Let (X,d) be a metric space, $f : I \to X$ and $\Phi \in \mathcal{N}$.*
(a) *If J is a closed subinterval of I, then $\mathsf{V}_\Phi(f,J) \le \mathsf{V}_\Phi(f,I)$.*
(b) *If $a < t < b$, then $\mathsf{V}_\Phi(f,I) = \mathsf{V}_\Phi(f,[a,t]) + \mathsf{V}_\Phi(f,[t,b])$.*
(c) *If $f_n \in X^I$, $\Phi_n \in \mathcal{N}$ $(n \in \mathbb{N})$, $\lim_{n\to\infty} d(f_n(t), f(t)) = 0$ $(t \in I)$ and $\lim_{n\to\infty}\Phi_n(\rho) = \Phi(\rho)$ $(\rho \ge 0)$, then $\mathsf{V}_\Phi(f) \le \liminf_{n\to\infty}\mathsf{V}_{\Phi_n}(f_n)$.*
(d) *$C^{0,1}(I;X) \subset BV_\Phi(I;X) \subset AC(I;X)$ and (we set $|I| = b - a$)*

$$\mathsf{V}_\Phi(f) \le |I|\Phi(\mathrm{Lip}(f)) \qquad \text{if} \quad f \in C^{0,1}(I;X), \tag{2.3}$$

$$\mathsf{V}_1(f) \le |I|\Phi^{-1}(\mathsf{V}_\Phi(f)/|I|) \quad \text{if} \quad f \in BV_\Phi(I;X). \tag{2.4}$$

Proof. Property (a) follows immediately from (1.1) and (1.2).

(b) For any partition T_1 of the interval $[a,t]$ and any partition T_2 of $[t,b]$ we have $T_1 \cup T_2 \in \mathcal{T}(I)$, so that (1.1) and (1.2) yield:

$$V_\Phi[f,T_1] + V_\Phi[f,T_2] = V_\Phi[f,T_1 \cup T_2] \le \mathsf{V}_\Phi(f,I),$$

and hence $\mathsf{V}_\Phi(f,[a,t])+\mathsf{V}_\Phi(f,[t,b])\le\mathsf{V}_\Phi(f,I)$. To prove the reverse inequality, let $T=\{t_i\}_{i=0}^m\in\mathcal{T}(I)$. We may suppose that $t_{k-1}<t<t_k$ for some $k\in\{1,\dots,m\}$. Setting, for $t,\,s\in I,\,s<t$,

$$\varrho_f(t,s)=d(f(t),f(s))/(t-s),\quad U_{\Phi,f}(t,s)=(t-s)\Phi(\varrho_f(t,s)),\ (2.5)$$

using the monotonicity of Φ, the triangle inequality for d and Jensen's inequality (2.1) with $\alpha_1=t_k-t$, $\alpha_2=t-t_{k-1}$, $x_1=\varrho_f(t_k,t)$ and $x_2=\varrho_f(t,t_{k-1})$ we find that

$$U_{\Phi,f}(t_k,t_{k-1})\le U_{\Phi,f}(t_k,t)+U_{\Phi,f}(t,t_{k-1}).$$

Since $\{t_i\}_{i=0}^{k-1}\cup\{t\}$ and $\{t\}\cup\{t_i\}_{i=k}^m$ are partitions of $[a,t]$ and $[t,b]$, respectively, from (1.1) and (1.2) we have (omitting subscripts Φ, f):

$$\begin{aligned}V_\Phi[f,T]&\le\Big(\sum_{i=1}^{k-1}U(t_i,t_{i-1})+U(t,t_{k-1})\Big)+\Big(U(t_k,t)+\sum_{i=k+1}^{m}U(t_i,t_{i-1})\Big)\le\\&\le\mathsf{V}_\Phi(f,[a,t])+\mathsf{V}_\Phi(f,[t,b]),\end{aligned}$$

and the reverse inequality follows.

(c) If $T=\{t_i\}_{i=0}^m\in\mathcal{T}(I)$, the definition of $\mathsf{V}_{\Phi_n}(f_n)$ and (2.5) imply

$$\sum_{i=1}^m U_{\Phi_n,f_n}(t_i,t_{i-1})=V_{\Phi_n}[f_n,T]\le\mathsf{V}_{\Phi_n}(f_n),\qquad n\in\mathbb{N}.$$

Had we shown that $U_{\Phi_n,f_n}(t_i,t_{i-1})\to U_{\Phi,f}(t_i,t_{i-1})$ as $n\to\infty$, then taking the limit inferior in both sides of the last inequality we obtain:

$$V_\Phi[f,T]\le\liminf_{n\to\infty}\mathsf{V}_{\Phi_n}(f_n),\qquad T\in\mathcal{T}(I).$$

It suffices to prove that if $\rho_n=\varrho_{f_n}(t_i,t_{i-1})$ and $\rho=\varrho_f(t_i,t_{i-1})$, then $\Phi_n(\rho_n)\to\Phi(\rho)$ as $n\to\infty$. Let $\rho>0$ and $\varepsilon>0$. By the continuity of Φ there is $0<\delta=\delta(\varepsilon)<\rho$ such that $|\Phi(r)-\Phi(\rho)|\le\varepsilon/2$ for all $r\ge0$, $|r-\rho|\le\delta$. As $\rho_n\to\rho$ (by the continuity of d and the pointwise convergence of f_n to f) and $\Phi_n\to\Phi$ pointwise as $n\to\infty$, there exists $N(\varepsilon)\in\mathbb{N}$ such that for all $n\ge N(\varepsilon)$ we have: $\rho-\delta<\rho_n<\rho+\delta$ and

$$|\Phi_n(\rho-\delta)-\Phi(\rho-\delta)|\le\varepsilon/2,\qquad|\Phi_n(\rho+\delta)-\Phi(\rho+\delta)|\le\varepsilon/2.$$

Since Φ_n is increasing, for all $n\ge N(\varepsilon)$ it follows that

$$\begin{aligned}\Phi_n(\rho_n)<\Phi_n(\rho+\delta)\le\Phi(\rho+\delta)+(\varepsilon/2)\le\Phi(\rho)+\varepsilon,\\\Phi_n(\rho_n)>\Phi_n(\rho-\delta)\ge\Phi(\rho-\delta)-(\varepsilon/2)\ge\Phi(\rho)-\varepsilon,\end{aligned}$$

or $|\Phi_n(\rho_n)-\Phi(\rho)|<\varepsilon$. The case $\rho=0$ readily follows now with obvious modifications.

(d) In view of (1.1) and (1.2), the first inclusion and inequality (2.3) are obvious. To prove the second inclusion, let $f \in BV_\Phi(I;X)$ and $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b$. Applying Jensen's inequality for sums with $\alpha_i = b_i - a_i$ and $x_i = d(f(b_i), f(a_i))/(b_i - a_i)$ we have:

$$\Phi\biggl(\frac{\sum_{i=1}^n d(f(b_i), f(a_i))}{\sum_{i=1}^n (b_i - a_i)}\biggr) \le \frac{\mathsf{V}_\Phi(f)}{\sum_{i=1}^n (b_i - a_i)},$$

and taking the inverse function Φ^{-1} from both sides of this inequality, we get

$$\sum_{i=1}^n d(f(b_i), f(a_i)) \le \Bigl[\sum_{i=1}^n (b_i - a_i)\Bigr] \cdot \Phi^{-1}\biggl(\frac{\mathsf{V}_\Phi(f)}{\sum_{i=1}^n (b_i - a_i)}\biggr). \tag{2.6}$$

Suppose that $v := \mathsf{V}_\Phi(f) \neq 0$ (otherwise f is a constant mapping). Since $\Phi \in \mathcal{N}$,

$$\lim_{r \to 0} r\Phi^{-1}(v/r) = v \lim_{\rho \to \infty} \rho/\Phi(\rho) = 0, \tag{2.7}$$

and hence, given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $r\Phi^{-1}(v/r) \le \varepsilon$ for all $0 < r \le \delta(\varepsilon)$. Now, (2.6) implies that

$$\text{if} \quad \sum_{i=1}^n (b_i - a_i) \le \delta(\varepsilon), \quad \text{then} \quad \sum_{i=1}^n d(f(b_i), f(a_i)) \le \varepsilon.$$

Thus, $f \in AC(I;X)$ and, in particular, $f \in BV_1(I;X)$. The estimate (2.4) follows from (2.6) if we set $t_0 = a_1 = a$, $t_i = b_i = a_{i+1}$ for $i = 1, \ldots, n-1$ and $t_n = b_n = b$, and note that $\{t_i\}_{i=0}^n$ is a partition of I and $\sum_{i=1}^n (t_i - t_{i-1}) = b - a = |I|$. □

The following theorem is the counterpart of *E. Helly's selection principle* (cf. [14], [39, Ch. 8] for the case of real valued functions and [4, 5, 7] for the case of metric space valued mappings of bounded variation in the sense of Jordan and Riesz):

THEOREM 2.2. *Let X be a complete metric space, $\Phi \in \mathcal{N}$ and $\mathfrak{F}$ an infinite family of mappings from X^I such that* (a) $\sup_{f \in \mathfrak{F}} \mathsf{V}_\Phi(f) =: v$ *is finite, and* (b) *for any $t \in I$ the set $\{f(t) \mid f \in \mathfrak{F}\}$ is precompact in X. Then the family $\mathfrak{F}$ contains a sequence of mappings which converges uniformly on I to a mapping from $BV_\Phi(I;X)$.*

Proof. Let us show that the family $\mathfrak{F}$ is equicontinuous. For all $f \in \mathfrak{F}$ and all t, $s \in I$, $s < t$, by (a) and the definition of $\mathsf{V}_\Phi(f)$, we get:

$$d(f(t), f(s)) \le (t-s)\Phi^{-1}(\mathsf{V}_\Phi(f)/(t-s)) \le (t-s)\Phi^{-1}(v/(t-s)). \tag{2.8}$$

By virtue of (2.7), for any $\varepsilon > 0$ we can find $\delta(\varepsilon, v) > 0$ such that if $0 < r \le \delta(\varepsilon, v)$, then $r\Phi^{-1}(v/r) \le \varepsilon$. Now, inequality (2.8) implies $\sup_{f\in\mathfrak{F}} d(f(t), f(s)) \le \varepsilon$ for all $t, s \in I$ such that $0 < t-s \le \delta(\varepsilon, v)$, and the equicontinuity of $\mathfrak{F}$ follows. Taking into account the completeness of X and condition (b) and applying Ascoli-Arzelà's theorem (e.g., [24, Ch. 3, Sec. 3]), we can choose a sequence of mappings $\{f_n\}_{n=1}^{\infty}$ in $\mathfrak{F}$ which uniformly on I converges to a continuous mapping $f \in X^I$ as $n \to \infty$. It remains to note that, by (a) and Lemma 2.1(c), $\mathsf{V}_\Phi(f) \le v$, and so $f \in BV_\Phi(I; X)$. □

As a corollary, we establish the existence of "geodesics" of bounded Φ-variation between any two points of a compact metric space.

COROLLARY 2.3. *Suppose that X is a compact metric space, $x, y \in X$, $\Phi \in \mathcal{N}$ and there is a $f_0 \in BV_\Phi(I; X)$ such that $f_0(a) = x$ and $f_0(b) = y$. Then there exists a mapping $f \in BV_\Phi(I; X)$ of* minimal Φ-*variation such that $f(a) = x$ and $f(b) = y$.*

Proof. Set $\ell := \inf\{\mathsf{V}_\Phi(f) \mid f \in BV_\Phi(I; X),\ f(a) = x,\ f(b) = y\}$. By the assumption, $0 \le \ell < \infty$, so that there exists a sequence of mappings $\{f_n\}_{n=1}^{\infty}$ in $BV_\Phi(I; X)$ such that $f_n(a) = x$, $f_n(b) = y$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathsf{V}_\Phi(f_n) = \ell$. Since the sequence $\mathsf{V}_\Phi(f_n)$, $n \in \mathbb{N}$, is bounded and X is compact, by Theorem 2.2 there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ in $\{f_n\}_{n=1}^{\infty}$ which uniformly on I converges to a mapping $f \in BV_\Phi(I; X)$ as $k \to \infty$. Clearly, $f(a) = x$ and $f(b) = y$. The definition of ℓ and Lemma 2.1(c) imply that

$$\ell \le \mathsf{V}_\Phi(f) \le \liminf_{k\to\infty} \mathsf{V}_\Phi(f_{n_k}) = \lim_{n\to\infty} \mathsf{V}_\Phi(f_n) = \ell,$$

i.e., $\mathsf{V}_\Phi(f) = \ell$, which was to be proved. □

Another consequence of Theorem 2.2 is the existence of regular selections of set-valued mappings of bounded Φ-variation which we postpone until Section 5 (see Theorem 5.1).

Now we present a *structural theorem* which shows that any mapping from $BV_\Phi(I; X)$ can be characterized by a real valued function of bounded Φ-variation modulo a Lipschitzian mapping (similar structural theorems hold for absolutely continuous mappings [5] and mappings of bounded variation in the sense of Jordan [4], Riesz [7], Wiener [9] and Young [10]). The composition $g \circ \varphi$ of two mappings $\varphi : I \to J$ and $g : J \to X$ is defined as usual by $(g \circ \varphi)(t) = g(\varphi(t))$, $t \in I$. We have:

THEOREM 2.4. *Let X be a metric space, $\Phi \in \mathcal{N}$ and $f \in BV_1(I; X)$. Define $\varphi : I \to \mathbb{R}^+$ by $\varphi(t) = \mathsf{V}_1(f, [a, t])$, $t \in I$, and let $J = \varphi(I)$*

be the image of φ. *Then* $f \in BV_\Phi(I;X)$ *if and only if* $\varphi \in BV_\Phi(I;\mathbb{R})$ *and there exists a mapping* $g \in C^{0,1}(J;X)$ *with* $\mathrm{Lip}(g) \le 1$ *such that* $f = g \circ \varphi$ *on* I. *Moreover, in this case* $\mathsf{V}_\Phi(\varphi) = \mathsf{V}_\Phi(f)$.

Proof. Sufficiency. By Lemma 2.1(d), the function φ is absolutely continuous, and so the image $J = \varphi(I)$ is a compact interval. For any partition $T = \{t_i\}_{i=0}^m$ of the interval I we have:

$$\begin{aligned} V_\Phi[g \circ \varphi, T] &= \sum_{i=1}^m \Phi\Big(\frac{d(g(\varphi(t_i)), g(\varphi(t_{i-1})))}{t_i - t_{i-1}}\Big)(t_i - t_{i-1}) \le \\ &\le \sum_{i=1}^m \Phi\Big(\mathrm{Lip}(g)\frac{|\varphi(t_i) - \varphi(t_{i-1})|}{t_i - t_{i-1}}\Big)(t_i - t_{i-1}) \le \\ &\le \mathsf{V}_\Phi(\mathrm{Lip}(g)\varphi), \end{aligned}$$

and hence

$$\mathsf{V}_\Phi(f) = \mathsf{V}_\Phi(g \circ \varphi) \le \mathsf{V}_\Phi(\mathrm{Lip}(g)\varphi) \le \mathsf{V}_\Phi(\varphi). \tag{2.9}$$

Necessity. By virtue of Lemma 2.1(d), φ is well defined, bounded, nondecreasing and continuous on I. Hence, $J = \varphi(I) = [0, \ell]$ where $\ell = \mathsf{V}_1(f)$.

Let us show that $\varphi \in BV_\Phi(I;\mathbb{R})$. If $T = \{t_i\}_{i=0}^m \in \mathcal{T}(I)$, $I_i = [t_{i-1}, t_i]$ and $|I_i| = t_i - t_{i-1}$, $i = 1, \dots, m$, then from the additivity of Jordan's variation and inequality (2.4) we have:

$$\varphi(t_i) - \varphi(t_{i-1}) = \mathsf{V}_1(f, I_i) \le |I_i|\Phi^{-1}(\mathsf{V}_\Phi(f, I_i)/|I_i|),$$

so that the monotonicity of Φ and Lemma 2.1(b) give:

$$V_\Phi[\varphi, T] = \sum_{i=1}^m \Phi\Big(\frac{|\varphi(t_i) - \varphi(t_{i-1})|}{|I_i|}\Big)|I_i| \le \sum_{i=1}^m \mathsf{V}_\Phi(f, I_i) = \mathsf{V}_\Phi(f),$$

whence

$$\mathsf{V}_\Phi(\varphi) \le \mathsf{V}_\Phi(f). \tag{2.10}$$

Now we prove the existence of mapping g. For $\tau \in [0, \ell]$ denote by $\varphi^{-1}(\{\tau\}) = \{t \in I \mid \varphi(t) = \tau\}$ the inverse image of the one-point set $\{\tau\}$ under the function φ. We define the desired mapping $g : [0, \ell] \to X$ as follows: if $\tau \in [0, \ell]$, we set

$$g(\tau) = f(t) \quad \text{for any point} \quad t \in \varphi^{-1}(\{\tau\}). \tag{2.11}$$

This is correct, i.e., the value $f(t) \in X$ is independent of $t \in \varphi^{-1}(\{\tau\})$, since

$$d(f(t), f(s)) \le |\varphi(t) - \varphi(s)|, \qquad t, s \in I. \tag{2.12}$$

The representation $f = g \circ \varphi$ follows from (2.11), inequality $\mathrm{Lip}(g) \le 1$ is a consequence of (2.12), and equality $\mathsf{V}_\Phi(\varphi) = \mathsf{V}_\Phi(f)$ follows from (2.9) and (2.10). □

3. Normed Linear Space Valued Mappings

In this section we assume that X is a normed linear space with the norm $\|\cdot\|$. Naturally, X^I becomes a linear space with respect to pointwise operations of addition and multiplication by a scalar.

It is well known (see [2, Ch. 1, Sec. 2.1], [20, 21]) that a Lipschitzian mapping $f : I \to X$ with values in an arbitrary Banach space X need not be differentiable (strongly or weakly) at points of I. The same references show that for any mapping $f \in AC(I; X)$, where X is a *reflexive* Banach space, the strong derivative $f'(t) \in X$ (i.e. the derivative with respect to the norm $\|\cdot\|$) exists for almost all $t \in I$ and f can be represented as the Bochner integral of its derivative. Under these circumstances in the following theorem we derive an explicit formula for the Φ-variation of a mapping.

THEOREM 3.1. *Let X be a reflexive Banach space, $\Phi \in \mathcal{N}$ and f be in $BV_\Phi(I; X)$. Then f is almost everywhere on I strongly differentiable, its derivative f' is strongly measurable and Bochner integrable, f is represented as $f(t) = f(a) + \int_a^t f'(\tau)\, d\tau$ for all $t \in I$ and*

$$\mathsf{V}_\Phi(f) = \int_I \Phi(\|f'(t)\|)\, dt. \tag{3.1}$$

Proof. 1. First we prove the following auxiliary inequality:

$$\int_a^{b-s} \Phi(\|f(t+s) - f(t)\|/s)\, dt \le \mathsf{V}_\Phi(f), \qquad 0 < s < b - a. \tag{3.2}$$

By Lemma 2.1(a) the function $t \mapsto \mathsf{V}_\Phi(f, [a, t])$ is nondecreasing on I and hence it is Riemann integrable. Let $0 < s < b-a$. By Lemma 2.1(d), $f \in AC(I; X)$, so that the function $[a, b - s] \ni t \mapsto \|f(t + s) - f(t)\|$ is continuous. From the definition of the Φ-variation and Lemma 2.1(b) for all $t \in [a, b - s]$ we have:

$$\Phi(\|f(t+s) - f(t)\|/s) \le \frac{1}{s}\,(\mathsf{V}_\Phi(f, [a, t+s]) - \mathsf{V}_\Phi(f, [a, t])).$$

It remains to integrate this inequality over the interval $[a, b - s]$ with respect to t and change variables appropriately:

$$\int_a^{b-s} \Phi(\|f(t+s) - f(t)\|/s)\, dt \le \frac{1}{s} \int_{b-s}^{b} \mathsf{V}_\Phi(f, [a, t])\, dt \le \mathsf{V}_\Phi(f, I).$$

2. Since f is absolutely continuous, all assertions of the theorem, except (3.1), follow, e.g., from [2, Ch. 1, Thm. 2.1]. To obtain (3.1), note that

$$\|f'(t)\| \leq \liminf_{s\to 0} \|(f(t+s)-f(t))/s\| \quad \text{for almost all} \quad t \in I,$$

and so inequality (3.2) and Fatou's lemma imply that

$$\int_I \Phi(\|f'(t)\|)\,dt \leq \liminf_{s\to 0} \int_a^{b-s} \Phi(\|f(t+s)-f(t)\|/s)\,dt \leq \mathsf{V}_\Phi(f).$$

The reverse inequality follows from the integral representation of f and Jensen's integral inequality (2.2): if $T = \{t_i\}_{i=0}^m \in \mathcal{T}(I)$, $I_i = [t_{i-1}, t_i]$ and $|I_i| = t_i - t_{i-1}$, then

$$\begin{aligned} V_\Phi[f,T] &\leq \sum_{i=1}^m \Phi\Big(\frac{1}{|I_i|}\int_{I_i} \|f'(t)\|\,dt\Big)|I_i| \leq \\ &\leq \sum_{i=1}^m \int_{I_i} \Phi(\|f'(t)\|)\,dt = \int_I \Phi(\|f'(t)\|)\,dt, \end{aligned}$$

and so $\mathsf{V}_\Phi(f) \leq \int_I \Phi(\|f'(t)\|)\,dt$, which completes the proof. □

COROLLARY 3.2. *Let $f : I \to X$ and $\Phi \in \mathcal{N}$.*

(a) *If X is a reflexive Banach space, then f is in $BV_\Phi(I;X)$ if and only if $f \in AC(I;X)$ and $\int_I \Phi(\|f'(t)\|)\,dt < \infty$.*

(b) *If X is an arbitrary metric space and $\varphi(t) = \mathsf{V}_1(f,[a,t])$, $t \in I$, then $f \in BV_\Phi(I;X)$ if and only if $\varphi \in BV_\Phi(I;\mathbb{R})$ if and only if $\varphi \in AC(I;\mathbb{R})$ and $\int_I \Phi(|\varphi'(t)|)\,dt < \infty$. Moreover,*

$$\mathsf{V}_\Phi(f) = \mathsf{V}_\Phi(\varphi) = \int_I \Phi(|\varphi'(t)|)\,dt = \int_I \Phi\Big(\big|\tfrac{d}{dt}\mathsf{V}_1(f,[a,t])\big|\Big)\,dt.$$

(c) *If X is a normed linear space and $f \in C^1(I;X)$ (a continuously differentiable mapping), then $\mathsf{V}_\Phi(f) = \int_I \Phi(\|f'(t)\|)\,dt$.*

Proof. (a) follows immediately from Theorem 3.1. Item (b) is a consequence of (a), the reflexivity of $\mathbb{R}$ and Theorem 2.4. The formula in (c) follows from (b), since $\varphi(t) = \mathsf{V}_1(f,[a,t]) = \int_a^t \|f'(\tau)\|\,d\tau$ for $t \in I$. □

As $\Phi \in \mathcal{N}$ is a convex function, the set $BV_\Phi(I;X)$ is convex and the mapping $f \mapsto \mathsf{V}_\Phi(f)$ is a convex functional on it, i.e.

$$\mathsf{V}_\Phi(\theta f + (1-\theta)g) \leq \theta\,\mathsf{V}_\Phi(f) + (1-\theta)\,\mathsf{V}_\Phi(g) \qquad (3.3)$$

for all f, $g \in BV_\Phi(I;X)$ and $\theta \in [0,1]$. However, the set $BV_\Phi(I;X)$ need not be a linear space in general. Using the technique from [22, Ch. 2] one can show that if X is a Banach space, then $BV_\Phi(I;X)$ is a linear space if and only if Φ *satisfies the* Δ_2*-condition*, i.e., there exist constants $\rho_0 \ge 0$ and $C > 0$ such that $\Phi(2\rho) \le C\Phi(\rho)$ for all $\rho \ge \rho_0$.

We define the space $GV_\Phi(I;X)$ with $\Phi \in \mathcal{N}$ as the linear span of $BV_\Phi(I;X)$ and call it the *space* of mappings of bounded (generalized) Φ-variation. It is clear that a mapping $f : I \to X$ belongs to $GV_\Phi(I;X)$ if and only if there exists a constant $r > 0$, depending on f, such that $f/r \in BV_\Phi(I;X)$. Note that

$$BV_\Phi(I;X) \subset GV_\Phi(I;X) \subset AC(I;X), \qquad \Phi \in \mathcal{N}.$$

Moreover, for any normed linear space X, we have:

$$AC(I;X) = \bigcup_{\Phi\in\mathcal{N}} GV_\Phi(I;X). \tag{3.4}$$

In fact, if $f \in AC(I;X)$, the function $t \mapsto \varphi(t) = \mathsf{V}_1(f,[a,t])$ is in $AC(I;\mathbb{R})$ and so its derivative φ' is Lebesgue integrable on I. It follows from [22, Sec. 8.1] that there exists a function $\Phi \in \mathcal{N}$ such that $\int_I \Phi(|\varphi'(t)|)\,dt < \infty$. By Corollary 3.2(b) we conclude that $f \in BV_\Phi(I;X)$, which proves the inclusion $\subset$ in (3.4).

Now we study the inclusion relations between spaces $GV_\Phi(I;X)$ corresponding to different functions $\Phi \in \mathcal{N}$. They resemble the relations between Orlicz spaces (cf. [22, Thm. 13.1]). Let us recall a few definitions [22, Secs. 3, 13]. Let Φ, $\Psi \in \mathcal{N}$. We say that Ψ *precedes* Φ (*at infinity*) and write $\Psi \preccurlyeq \Phi$ if there exist constants $\rho_0 \ge 0$ and $C > 0$ such that $\Psi(\rho) \le \Phi(C\rho)$ for all $\rho \ge \rho_0$. For instance, if $\Phi(\rho) = \rho^p$, $\Psi(\rho) = \rho^q$, where p, $q \ge 1$, then $\Psi \preccurlyeq \Phi$ if and only if $q \le p$. Two functions Φ and Ψ are said to be *equivalent* (*at infinity*), denoted by $\Phi \sim \Psi$, if $\Phi \preccurlyeq \Psi$ and $\Psi \preccurlyeq \Phi$. Clearly, $\Phi \sim \Psi$ if and only if there exist constants $\rho_0 \ge 0$ and C_1, $C_2 > 0$ such that $\Phi(C_1\rho) \le \Psi(\rho) \le \Phi(C_2\rho)$ for all $\rho \ge \rho_0$. For example, if the limit of $\Phi(\rho)/\Psi(\rho)$ as $\rho \to \infty$ is finite > 0, then $\Phi \sim \Psi$.

THEOREM 3.3. *Let X be a normed linear space and Φ, $\Psi \in \mathcal{N}$. If $\Psi \preccurlyeq \Phi$, then $GV_\Phi(I;X) \subset GV_\Psi(I;X)$. If X is a Banach space and $GV_\Phi(I;X) \subset GV_\Psi(I;X)$, then $\Psi \preccurlyeq \Phi$. Consequently, spaces $GV_\Phi(I;X)$ and $GV_\Psi(I;X)$ consist of the same mappings if and only if $\Phi \sim \Psi$.*

Proof. Suppose that $\Psi \preccurlyeq \Phi$. Then there exist constants $\rho_0 \ge 0$ and $C > 0$ such that $\Psi(\rho) \le \Phi(C\rho)$ for all $\rho \ge \rho_0$. If $f \in GV_\Phi(I;X)$, then $\mathsf{V}_\Phi(f/r) < \infty$ for some $r > 0$, and so for any partition T of the interval I, we have:

$$V_\Psi[f/(rC), T] \le \Psi(\rho_0)|I| + \mathsf{V}_\Phi(f/r).$$

It follows that $\mathsf{V}_\Psi(f/(rC)) < \infty$ and therefore $f \in GV_\Psi(I; X)$.

Let X be a Banach space, and assume that condition $\Psi \preccurlyeq \Phi$ does not hold. There exists an increasing sequence $\{\rho_n\}_{n=1}^\infty$ of positive numbers such that $\lim_{n\to\infty} \rho_n = \infty$ and $\Psi(\rho_n) > \Phi(n2^n\rho_n)$ for all $n \in \mathbb{N}$. From $\Phi(\theta\rho) \le \theta\Phi(\rho)$ with $\theta = 1/2^n$ and $\rho = n2^n\rho_n$ we obtain $\Phi(n2^n\rho_n) \ge 2^n\Phi(n\rho_n)$, so that

$$\Psi(\rho_n) > 2^n\Phi(n\rho_n), \qquad n \in \mathbb{N}. \tag{3.5}$$

Let $\{a_n\}_{n=0}^\infty \subset I$ be an increasing sequence such that $a_0 = a$ and

$$a_n - a_{n-1} = 2^{-n}|I|\Phi(\rho_1)/\Phi(n\rho_n), \qquad n \in \mathbb{N}.$$

For $t \in I$ define $\chi(t) = n\rho_n$ if $a_{n-1} \le t < a_n$, $n \in \mathbb{N}$, and $\chi(t) = 0$ otherwise, and define $f : I \to X$ by setting $f(t) = \left(\int_a^t \chi(\tau)\, d\tau\right)x_0$, $t \in I$, where $x_0 \in X$, $\|x_0\| = 1$. We are going to show that $f \in GV_\Phi(I; X)$ and, at the same time, $f \notin GV_\Psi(I; X)$. In fact,

$$\begin{aligned}\mathsf{V}_\Phi(f) &= \sum_{n=1}^\infty \Phi\left(\frac{\|f(a_n) - f(a_{n-1})\|}{a_n - a_{n-1}}\right)(a_n - a_{n-1}) = \\ &= \sum_{n=1}^\infty \Phi(n\rho_n)(a_n - a_{n-1}) = |I|\Phi(\rho_1).\end{aligned}$$

It follows that $f \in BV_\Phi(I; X)$. Taking into account (3.5), for any $r \ge 1$ and any $m \in \mathbb{N}$ such that $m \ge r$, we have:

$$\begin{aligned}\mathsf{V}_\Psi(f/r) &\ge \sum_{n=m}^{2m} \Psi\left(\frac{\|f(a_n) - f(a_{n-1})\|}{r(a_n - a_{n-1})}\right)(a_n - a_{n-1}) \ge \\ &\ge \sum_{n=m}^{2m} \Psi(\rho_n)(a_n - a_{n-1}) \ge m|I|\Phi(\rho_1).\end{aligned}$$

Therefore $\mathsf{V}_\Psi(f/r) = \infty$ for all $r \ge 1$. □

On the linear space $GV_\Phi(I; X)$ we define the following nonnegative Luxemburg type functional (cf. [25], [22, Chap. 2, Sec. 9.7]):

$$p_\Phi(f) := \inf\{\, r > 0 \mid \mathsf{V}_\Phi(f/r) \le 1\,\}, \quad f \in GV_\Phi(I; X), \quad \Phi \in \mathcal{N}. \tag{3.6}$$

Since the mapping p_Φ is the Minkowski functional of the convex set $E_\Phi = \{f \in BV_\Phi(I;X) \mid \mathsf{V}_\Phi(f) \le 1\}$, the core of E_Φ contains the zero mapping and $\lambda E_\Phi \subset E_\Phi$ (see (3.3)) for all scalars λ such that $|\lambda| < 1$, p_Φ is a *seminorm* on $GV_\Phi(I;X)$ (cf. [12, Ch. 1, Sec. 3, Lemma 2]). For instance, if $\Phi(\rho) = \rho^q$, $q \ge 1$, then $p_\Phi(f) = (\mathsf{V}_q(f))^{1/q}$, $f \in BV_q(I;X)$. Also, we will need the following properties of p_Φ:

LEMMA 3.4. *Let* $\Phi \in \mathcal{N}$ *and* $f \in GV_\Phi(I;X)$. *We have*:
(a) *if* $t,\ s \in I$, $t \ne s$, *then* $\|f(t) - f(s)\| \le |t-s|\Phi^{-1}(1/|t-s|)p_\Phi(f)$;
(b) *if* $p_\Phi(f) > 0$, *then* $\mathsf{V}_\Phi(f/p_\Phi(f)) \le 1$;
(c) *if* $r > 0$, *then* $\mathsf{V}_\Phi(f/r) \le 1$ *if and only if* $p_\Phi(f) \le r$;
(d) *if* $r > 0$ *and* $\mathsf{V}_\Phi(f/r) = 1$, *then* $p_\Phi(f) = r$ (*but not vice versa in general*);
(e) *if the sequence* $\{f_n\}_{n=1}^\infty \subset GV_\Phi(I;X)$ *converges to* $f \in X^I$ *pointwise on* I *as* $n \to \infty$, *then* $p_\Phi(f) \le \limsup_{n\to\infty} p_\Phi(f_n)$;
(f) *the following inequalities hold*:

$$\Phi^{-1}(1/|I|)p_\Phi(f) \le \mathrm{Lip}(f) \qquad \textit{if} \quad f \in C^{0,1}(I;X), \tag{3.7}$$

$$\mathsf{V}_1(f) \le |I|\Phi^{-1}(1/|I|)p_\Phi(f) \qquad \textit{if} \quad f \in GV_\Phi(I;X). \tag{3.8}$$

Proof. (a) If $t,\ s \in I$, $s < t$, then according to (1.1), (1.2) and (3.6) we have:

$$\Phi\Big(\frac{\|f(t)-f(s)\|}{(t-s)\,r}\Big)(t-s) \le \mathsf{V}_\Phi(f/r) \le 1 \quad \text{for all} \quad r > p_\Phi(f),$$

so taking the inverse function Φ^{-1} we arrive at

$$\|f(t)-f(s)\| \le (t-s)\Phi^{-1}(1/(t-s))r, \qquad r > p_\Phi(f).$$

(b) The definition of $p_\Phi(f)$ implies $\mathsf{V}_\Phi(f/r) \le 1$ for all $r > p_\Phi(f)$. Choose a sequence $r_n > p_\Phi(f)$, $n \in \mathbb{N}$, which converges to $p_\Phi(f)$ as $n \to \infty$. Then f/r_n converges to $f/p_\Phi(f)$ uniformly on I, so that Lemma 2.1(c) yields:

$$\mathsf{V}_\Phi(f/p_\Phi(f)) \le \liminf_{n\to\infty} \mathsf{V}_\Phi(f/r_n) \le 1.$$

It follows that $p_\Phi(f) \in \{r > 0 \mid \mathsf{V}_\Phi(f/r) \le 1\} =: \Lambda$ and $p_\Phi(f) = \min \Lambda$.

(c) If $\mathsf{V}_\Phi(f/r) \le 1$, definition (3.6) implies $p_\Phi(f) \le r$. If $p_\Phi(f) = r$, then $\mathsf{V}_\Phi(f/r) \le 1$ by (b). Let us show that

$$\text{if } p_\Phi(f) < r, \text{ then } \mathsf{V}_\Phi(f/r) < 1. \tag{3.9}$$

By (a), if $p_\Phi(f) = 0$, then f is a constant mapping and $\mathsf{V}_\Phi(f/r) = 0$, so assume that $p_\Phi(f) > 0$. From (3.3) and item (b) we have:

$$\mathsf{V}_\Phi(f/r) \le (p_\Phi(f)/r)\,\mathsf{V}_\Phi(f/p_\Phi(f)) \le p_\Phi(f)/r < 1.$$

(d) Let $\mathsf{V}_\Phi(f/r) = 1$. By (c), if $p_\Phi(f) > r$, then $\mathsf{V}_\Phi(f/r) > 1$, which is impossible. Taking into account (3.9) we conclude that $p_\Phi(f) = r$.

(e) Set $\alpha = \limsup_{n\to\infty} p_\Phi(f_n)$. If $\alpha = \infty$, the inequality in (e) is obvious. Suppose now that α is finite. Then there exists $n_0 \in \mathbb{N}$ such that $\alpha_n := \sup_{k\geq n} p_\Phi(f_k)$ is finite for all $n \geq n_0$. Let $\varepsilon > 0$. Since $\alpha_n+\varepsilon > p_\Phi(f_n)$, by the definition of $p_\Phi(f_n)$ we have: $\mathsf{V}_\Phi(f_n/(\alpha_n+\varepsilon)) \leq 1$, $n \geq n_0$. The pointwise convergence of f_n to f and the convergence of α_n to α imply that $f_n/(\alpha_n + \varepsilon)$ converges to $f/(\alpha + \varepsilon)$ pointwise on I as $n \to \infty$. Applying Lemma 2.1(c) we find that

$$\mathsf{V}_\Phi(f/(\alpha+\varepsilon)) \leq \liminf_{n\to\infty} \mathsf{V}_\Phi(f_n/(\alpha_n+\varepsilon)) \leq 1.$$

From the definition of $p_\Phi(f)$ it follows that $p_\Phi(f) \leq \alpha + \varepsilon$ for all $\varepsilon > 0$.

(f) Set $r = \mathrm{Lip}(f)/\Phi^{-1}(1/|I|)$. If $\mathrm{Lip}(f) = 0$, then $p_\Phi(f) = 0$. Let $\mathrm{Lip}(f) > 0$. Using inequality (2.3) we have:

$$\mathsf{V}_\Phi(f/r) \leq |I|\Phi(\mathrm{Lip}(f/r)) = |I|\Phi(\mathrm{Lip}(f)/r) = 1,$$

so (c) yields $p_\Phi(f) \leq r$, which proves (3.7).

Put $r = \mathsf{V}_1(f)/(|I|\Phi^{-1}(1/|I|))$. We can assume that $r > 0$. Applying inequality (2.4) we get:

$$\mathsf{V}_\Phi(f/r) \geq |I|\Phi(\mathsf{V}_1(f/r)/|I|) = |I|\Phi(\mathsf{V}_1(f)/(r|I|)) = 1.$$

Now (3.9) gives $p_\Phi(f) \geq r$, which yields (3.8). □

Now let us show that mappings from $GV_\Phi(I;X)$ have the same structure as mappings from $BV_\Phi(I;X)$ (cf. Theorem 2.4).

LEMMA 3.5. *Let X be a normed linear space, $\Phi \in \mathcal{N}$ and f be in $BV_1(I;X)$. Set $\varphi_f(t) = \mathsf{V}_1(f,[a,t])$, $t \in I$, and $J = \varphi_f(I)$. Then f belongs to $GV_\Phi(I;X)$ if and only if $\varphi_f \in GV_\Phi(I;\mathbb{R})$ and there exists a mapping $g \in C^{0,1}(J;X)$ with $\mathrm{Lip}(g) \leq 1$ such that $f = g \circ \varphi_f$ on I. Moreover, $p_\Phi(\varphi_f) = p_\Phi(f)$.*

Proof. Sufficiency. Since $\varphi_f \in GV_\Phi(I;\mathbb{R})$, there exists $r > 0$ such that φ_f/r is in $BV_\Phi(I;\mathbb{R})$, so if $f = g \circ \varphi_f$, where $\mathrm{Lip}(g) \leq 1$, then by (2.9) we have:

$$\mathsf{V}_\Phi(f/r) \leq \mathsf{V}_\Phi(\mathrm{Lip}(g/r)\varphi_f) \leq \mathsf{V}_\Phi(\varphi_f/r),$$

and hence $f \in GV_\Phi(I;X)$.

Necessity. From the definition of φ_f we have: $\varphi_{f/r} = \varphi_f/r$, $r > 0$. Since $f \in GV_\Phi(I;X)$, then $f/r \in BV_\Phi(I;X)$ for some $r > 0$, and so Theorem 2.4 implies $\varphi_{f/r} \in BV_\Phi(I;\mathbb{R})$ and there exists a mapping $\widetilde{g}$

in $C^{0,1}(J/r;X)$ with $\mathrm{Lip}(\widetilde{g})\le 1$ such that $f/r=\widetilde{g}\circ\varphi_{f/r}$ on I. Setting $g(s)=r\widetilde{g}(s/r)$, $s\in J$, we find that $g\in C^{0,1}(J;X)$, $\mathrm{Lip}(g)\le 1$ and $f=g\circ\varphi_f$ on I, where $\varphi_f\in GV_\Phi(I;\mathbb{R})$.

Finally, since, by Theorem 2.4, $\mathsf{V}_\Phi(\varphi_f/r)=\mathsf{V}_\Phi(\varphi_{f/r})=\mathsf{V}_\Phi(f/r)$ for all $r>0$, we conclude that $p_\Phi(\varphi_f)=p_\Phi(f)$. □

Remark 3.1. Theorem 2.2 holds if we replace the set $BV_\Phi(I;X)$ by $GV_\Phi(I;X)$, where X is a Banach space, and condition (a) there—by: the family $\{p_\Phi(f)\mid f\in\mathfrak{F}\}$ is uniformly bounded. To see this, it suffices to apply Lemma 3.4(a) instead of inequality (2.8) and Lemma 3.4(e) in place of Lemma 2.1(c).

We define the *norm* $\|\cdot\|_\Phi$ on the space $GV_\Phi(I;X)$ as follows:

$$\|f\|_\Phi:=\|f(a)\|+p_\Phi(f),\qquad f\in GV_\Phi(I;X),\quad \Phi\in\mathcal{N}. \tag{3.10}$$

In particular, $\|f\|_\Phi=\|f\|_q:=\|f(a)\|+(\mathsf{V}_q(f))^{1/q}$ if $\Phi(\rho)=\rho^q$ $(q\ge 1)$.

To formulate the next theorem, we introduce some terminology. Let X, Y and Z be normed linear spaces over the same field and the norms denoted by the same symbol $\|\cdot\|$ (which won't lead to ambiguities). We say that the triple (X,Y,Z) is *multiplicative* if there exists a bilinear mapping $M:X\times Y\to Z$ such that $\|M(x,y)\|\le\|x\|\cdot\|y\|$ for all $x\in X$ and $y\in Y$. The mapping M is called the *product mapping* and the value $M(x,y)\in Z$ is written simply as xy. If $f\in X^I$ and $g\in Y^I$, we define the product $fg\in Z^I$ by $(fg)(t):=f(t)g(t)$, $t\in I$.

THEOREM 3.6. *Let (X,Y,Z) be a multiplicative triple of normed linear spaces and $\Phi\in\mathcal{N}$. If $f\in GV_\Phi(I;X)$ and $g\in GV_\Phi(I;Y)$, then their product fg belongs to $GV_\Phi(I;Z)$ and the following inequality holds:*

$$\|fg\|_\Phi\le\gamma\|f\|_\Phi\|g\|_\Phi, \tag{3.11}$$

where $\gamma\equiv\gamma(\Phi,|I|):=\max\{1,\ 2|I|\Phi^{-1}(1/|I|)\}$.

If X is a Banach space, then $GV_\Phi(I;X)$ is also a Banach space.

Consequently, if X is a normed (respectively, Banach) algebra, then $GV_\Phi(I;X)$ is also a normed (respectively, Banach) algebra.

Proof. 1. Let us prove the following inequality:

$$p_\Phi(fg)\le p_\Phi(f)\|g\|_u+\|f\|_u p_\Phi(g), \tag{3.12}$$

where $\|f\|_u=\sup_{t\in I}\|f(t)\|$ and $\|g\|_u=\sup_{t\in I}\|g(t)\|$. Using the definition of the function φ_f from Lemma 3.5, estimating the sums for the Jordan variation and using the triangle inequality, we have:

$$\varphi_{fg}(t)=\mathsf{V}_1(fg,[a,t])\le\varphi_f(t)\|g\|_u+\|f\|_u\varphi_g(t),\qquad t\in I. \tag{3.13}$$

Applying Lemma 3.5 we get (3.12):

$$\begin{aligned} p_\Phi(fg) &= p_\Phi(\varphi_{fg}) \le p_\Phi(\varphi_f \|g\|_u + \|f\|_u \varphi_g) \le \\ &\le p_\Phi(\varphi_f)\|g\|_u + \|f\|_u p_\Phi(\varphi_g) = p_\Phi(f)\|g\|_u + \|f\|_u p_\Phi(g). \end{aligned}$$

To obtain inequality (3.11), observe that, by virtue of (3.8),

$$\|f\|_u \le \|f\|_1 = \|f(a)\| + \mathsf{V}_1(f) \le \|f(a)\| + |I|\Phi^{-1}(1/|I|)p_\Phi(f), \quad (3.14)$$

and so it remains to take into account (3.10) and (3.12).

2. If X is a Banach space, let us prove that $GV_\Phi(I;X)$ is complete. Suppose that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $GV_\Phi(I;X)$, i.e.

$$\|f_n - f_m\|_\Phi = \|f_n(a) - f_m(a)\| + p_\Phi(f_n - f_m) \to 0 \quad \text{as} \quad n, m \to \infty.$$

By Lemma 3.4(a) it follows that the sequence $\{f_n(t)\}_{n=1}^\infty$ is Cauchy in X for all $t \in I$ and therefore, by the completeness of X, there exists a mapping $f \in X^I$ such that f_n converges to f pointwise as $n \to \infty$. Since $f_n - f_m$ converges to $f_n - f$ pointwise as $m \to \infty$, Lemma 3.4(e) yields:

$$\|f_n - f\|_\Phi \le \limsup_{m\to\infty} \|f_n - f_m\|_\Phi = \lim_{m\to\infty} \|f_n - f_m\|_\Phi \in \mathbb{R}^+, \qquad n \in \mathbb{N}.$$

Taking into account that $\{f_n\}_{n=1}^\infty$ is Cauchy in $GV_\Phi(I;X)$ we have:

$$\limsup_{n\to\infty} \|f_n - f\|_\Phi \le \lim_{n\to\infty} \lim_{m\to\infty} \|f_n - f_m\|_\Phi = 0.$$

Hence $\|f_n - f\|_\Phi \to 0$ as $n \to \infty$. It follows that there exists $n_0 \in \mathbb{N}$ such that $\|f_{n_0} - f\|_\Phi \le 1$ from which

$$\|f\|_\Phi \le \|f - f_{n_0}\|_\Phi + \|f_{n_0}\|_\Phi \le 1 + \|f_{n_0}\|_\Phi < \infty.$$

Therefore $f \in GV_\Phi(I;X)$, and this completes the proof. □

Remark 3.2. If X is a normed or Banach algebra, the norm (3.10) can always be replaced by an equivalent norm $|\cdot|_\Phi$ such that

$$|fg|_\Phi \le |f|_\Phi |g|_\Phi \quad \text{for all} \quad f, g \in GV_\Phi(I;X). \qquad (3.15)$$

This is a consequence of the following (general) observation. For any f from $GV_\Phi(I;X)$ consider the linear continuous operator M_f from $GV_\Phi(I;X)$ into itself defined by $M_f(g) = fg$ whenever $g \in GV_\Phi(I;X)$. The operator norm $|f|_\Phi = \|M_f\| := \sup\{\|fg\|_\Phi\,;\ \|g\|_\Phi = 1\}$ is the desired norm on $GV_\Phi(I;X)$ satisfying (3.15) and $|f|_\Phi \le \|f\|_\Phi \le \gamma|f|_\Phi$ for all $f \in GV_\Phi(I;X)$.

We finish this section with the following theorem.

THEOREM 3.7. (a) *Let* (X, Y, Z) *be a multiplicative triple of Banach spaces,* Φ_1*,* Φ_2*,* $\Psi \in \mathcal{N}$*, and suppose that the product* fg *belongs to* $GV_\Psi(I; Z)$ *provided* $f \in GV_{\Phi_1}(I; X)$ *and* $g \in GV_{\Phi_2}(I; Y)$*. Then there exists a constant* κ_0 *such that* $\|fg\|_\Psi \le \kappa_0 \|f\|_{\Phi_1} \|g\|_{\Phi_2}$*.*

(b) *Let* X *be a Banach space,* Φ*,* $\Psi \in \mathcal{N}$ *and* $\Psi \preccurlyeq \Phi$ *(so that* $GV_\Phi(I; X) \subset GV_\Psi(I; X)$*). Then there exists a constant* $\kappa > 0$ *such that*

$$\|f\|_\Psi \le \kappa \|f\|_\Phi, \qquad f \in GV_\Phi(I; X). \tag{3.16}$$

(*Conversely, it is clear that if* (3.16) *holds, then* $\Psi \preccurlyeq \Phi$*.*)

(c) *Let* X *be a Banach space and* Φ*,* $\Psi \in \mathcal{N}$*. Then the spaces* $GV_\Phi(I; X)$ *and* $GV_\Psi(I; X)$ *are equal if and only if* $\Phi \sim \Psi$ *if and only if the norms* $\|\cdot\|_\Phi$ *and* $\|\cdot\|_\Psi$ *are equivalent.*

Proof. (a) For $f \in GV_{\Phi_1}(I; X)$ and $g \in GV_{\Phi_2}(I; Y)$ set $M(f, g) = fg$. By the assumption, $M(f, g) \in GV_\Psi(I; Z)$.

1. The linear operator $M(\,\cdot\,, g) : GV_{\Phi_1}(I; X) \to GV_\Psi(I; Z)$ is closed for any $g \in GV_{\Phi_2}(I; Y)$. To see this, let f_n, $f \in GV_{\Phi_1}(I; X)$, $n \in \mathbb{N}$, $w \in GV_\Psi(I; Z)$, $\|f_n - f\|_{\Phi_1} \to 0$ and $\|M(f_n, g) - w\|_\Psi \to 0$ as $n \to \infty$. Then, by virtue of (3.14), f_n converges to f uniformly on I, so that $M(f_n, g) = f_n g$ converges uniformly to fg; similarly, $f_n g = M(f_n, g)$ converges uniformly to w. Thus, $w = fg = M(f, g)$, and so $M(\,\cdot\,, g)$ is closed. By the closed graph theorem $M(\,\cdot\,, g)$ is continuous and hence there exists a constant $\kappa(g) > 0$ such that

$$\|fg\|_\Psi = \|M(f, g)\|_\Psi \le \kappa(g) \|f\|_{\Phi_1}, \qquad f \in GV_{\Phi_1}(I; X).$$

2. Set $B_1 = \{f \in GV_{\Phi_1}(I; X)\,;\, \|f\|_{\Phi_1} \le 1\}$. The last estimate shows that the family $\mathfrak{B} = \{M(f, \cdot\,) \mid f \in B_1\}$ of linear continuous operators, which map $GV_{\Phi_2}(I; Y)$ into $GV_\Psi(I; Z)$, is pointwise bounded:

$$\|fg\|_\Psi \le \kappa(g) \|f\|_{\Phi_1} \le \kappa(g), \qquad f \in B_1.$$

By the uniform boundedness principle the family $\mathfrak{B}$ is uniformly bounded, i.e. there exists a positive constant κ_0 such that $\|fg\|_\Psi \le \kappa_0 \|g\|_{\Phi_2}$ for all $f \in B_1$ and $g \in GV_{\Phi_2}(I; Y)$. It remains to note that

$$\|fg\|_\Psi = \|(f/\|f\|_{\Phi_1})g\|_\Psi \|f\|_{\Phi_1} \le \kappa_0 \|f\|_{\Phi_1} \|g\|_{\Phi_2}$$

for all $0 \ne f \in GV_{\Phi_1}(I; X)$ and $g \in GV_{\Phi_2}(I; Y)$.

(b) It suffices to set $\Phi_1 = \Phi$, $g \equiv 1$ and $\kappa = \kappa(1)$ in step 1 in (a).

(c) is a consequence of Theorem 3.3 and item (b). □

4. Lipschitzian Composition Operators

In this section we characterize Lipschitzian composition operators between spaces of mappings of bounded generalized Φ-variation. Let X and Y be two normed linear spaces with the norms $\|\cdot\|$. We denote by $L(X;Y)$ the space of all linear continuous operators from X into Y equipped with the standard norm.

THEOREM 4.1. *Assume that the composition operator $\mathcal{H} : X^I \to Y^I$ is generated by a mapping $h : I \times X \to Y$ according to formula* (1.4). *Let Φ, $\Psi \in \mathcal{N}$, and let $\mathcal{F}(I;X)$ designate either $GV_\Phi(I;X)$ or $C^{0,1}(I;X)$.*

If $\mathcal{H}$ maps $\mathcal{F}(I;X)$ into $GV_\Psi(I;Y)$ and is Lipschitzian, then there exists a function $\mu_0 : I \to \mathbb{R}^+$ such that

$$\|h(t,x_1) - h(t,x_2)\| \le \mu_0(t)\|x_1 - x_2\|, \quad t \in I, \quad x_1,\, x_2 \in X, \quad (4.1)$$

and there exist mappings $h_0 \in GV_\Psi(I;Y)$ and $h_1 \in L(X;Y)^I$ such that the mapping $t \mapsto h_1(t)x$ belongs to $GV_\Psi(I;Y)$ for all $x \in X$ and

$$h(t,x) = h_0(t) + h_1(t)x, \qquad t \in I, \quad x \in X. \qquad (4.2)$$

Conversely, if $h_0 \in GV_\Psi(I;Y)$, $h_1 \in GV_\Psi(I;L(X;Y))$ and h is of the form (4.2)*, and, moreover, if $\Psi \preccurlyeq \Phi$ and X is a Banach space in the case $\mathcal{F}(I;X) = GV_\Phi(I;X)$, then $\mathcal{H}$ maps $\mathcal{F}(I;X)$ into $GV_\Psi(I;Y)$ and is Lipschitzian.*

Proof. For α, $\beta \in \mathbb{R}$, $\alpha < \beta$, we define functions $\eta_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ by

$$\eta_{\alpha,\beta}(t) = \begin{cases} 0 & \text{if} \quad t \le \alpha, \\ \dfrac{t-\alpha}{\beta-\alpha} & \text{if} \quad \alpha \le t \le \beta, \\ 1 & \text{if} \quad \beta \le t. \end{cases} \qquad (4.3)$$

1. First, we obtain an auxiliary inequality which will be used later several times. Since $\mathcal{H} : \mathcal{F}(I;X) \to GV_\Psi(I;Y)$ is Lipschitzian, there exists a constant $\mu > 0$ such that $\|\mathcal{H}f_1 - \mathcal{H}f_2\|_\Psi \le \mu\|f_1 - f_2\|_\mathcal{F}$ for all f_1, $f_2 \in \mathcal{F}(I;X)$, where $\|\cdot\|_\mathcal{F}$ is the norm in $\mathcal{F}(I;X)$. The definition of the norm $\|\cdot\|_\Psi$ implies, in particular, $p_\Psi(\mathcal{H}f_1 - \mathcal{H}f_2) \le \mu\|f_1 - f_2\|_\mathcal{F}$. By Lemma 3.4(c), if $\|f_1 - f_2\|_\mathcal{F} > 0$, the last inequality is equivalent to

$$\mathsf{V}_\Psi\left(\frac{\mathcal{H}f_1 - \mathcal{H}f_2}{\mu\|f_1 - f_2\|_\mathcal{F}}\right) \le 1.$$

From definitions of $\mathsf{V}_\Psi(\cdot)$ and $\mathcal{H}$, for any α, $\beta \in I$, $\alpha < \beta$, it follows that

$$\Psi\left(\frac{\|h(\beta,f_1(\beta)) - h(\beta,f_2(\beta)) - h(\alpha,f_1(\alpha)) + h(\alpha,f_2(\alpha))\|}{\mu\|f_1 - f_2\|_\mathcal{F}(\beta-\alpha)}\right)(\beta-\alpha) \le 1,$$

or, by taking the inverse function Ψ^{-1} from both sides, that

$$\|h(\beta, f_1(\beta)) - h(\beta, f_2(\beta)) - h(\alpha, f_1(\alpha)) + h(\alpha, f_2(\alpha))\| \le \\ \le \mu\|f_1 - f_2\|_{\mathcal{F}}(\beta-\alpha)\Psi^{-1}(1/(\beta-\alpha)). \quad (4.4)$$

2. Let us prove (4.1). First we prove it for $a < t \le b$. Let $x_1, x_2 \in X$. Consider two Lipschitzian mappings $f_j : I \to X$ defined by

$$f_j(t) = \eta_{\alpha,\beta}(t)x_j, \qquad t \in I, \quad j = 1, 2, \quad (4.5)$$

so that $f_j(\beta)=x_j$, $f_j(\alpha)=0$, $j=1, 2$. Let us compute $\|f_1 - f_2\|_{\mathcal{F}}$.

If $\mathcal{F}(I; X) = GV_\Phi(I; X)$, then choosing $r > 0$ such that

$$\mathsf{V}_\Phi((f_1 - f_2)/r) = \Phi\Big(\frac{\|x_1 - x_2\|}{(\beta-\alpha)\,r}\Big)(\beta - \alpha) = 1,$$

by Lemma 3.4(d) we get:

$$\|f_1 - f_2\|_\Phi = p_\Phi(f_1 - f_2) = r = \frac{\|x_1 - x_2\|}{(\beta-\alpha)\Phi^{-1}(1/(\beta-\alpha))}\,. \quad (4.6)$$

Substituting mappings (4.5) into (4.4), for all $a \le \alpha < \beta \le b$ and all $x_1, x_2 \in X$ we have:

$$\|h(\beta, x_1) - h(\beta, x_2)\| \le \mu\frac{\Psi^{-1}(1/(\beta-\alpha))}{\Phi^{-1}(1/(\beta-\alpha))}\|x_1 - x_2\|. \quad (4.7)$$

In particular, if $a < t \le b$, then setting $\alpha = a$ and $\beta = t$ in (4.7) we obtain (4.1). If $\mathcal{F}(I; X) = C^{0,1}(I; X)$, it is easy to see that

$$\|f_1 - f_2\|_{0,1} = \mathrm{Lip}(f_1 - f_2) = \|x_1 - x_2\|/(\beta - \alpha), \quad (4.8)$$

and hence, if $a < t \le b$, $\alpha = a$ and $\beta = t$, (4.4) yields

$$\|h(t, x_1) - h(t, x_2)\| \le \mu\Psi^{-1}(1/(t - a))\|x_1 - x_2\|,$$

which proves (4.1) in the case when $a < t \le b$.

To show that (4.1) is valid at $t = a$, consider mappings

$$f_j(t) = (1 - \eta_{\alpha,\beta}(t))x_j, \qquad t \in I, \quad j = 1, 2. \quad (4.9)$$

Note that $f_j(\beta) = 0$ and $f_j(\alpha) = x_j$, $j = 1, 2$. As above, we have:

$$\|f_1 - f_2\|_{\mathcal{F}} = (1 + 1/\omega_{\mathcal{F}}(\beta - \alpha))\|x_1 - x_2\|, \quad (4.10)$$

where $\omega_{\mathcal{F}}(r)=r\Phi^{-1}(1/r)$ if $\mathcal{F}=GV_\Phi$, and $\omega_{\mathcal{F}}(r)=r$ if $\mathcal{F}=C^{0,1}$, $r>0$. Substituting mappings (4.9) into (4.4) and setting $\alpha = a$, $\beta = b$, we obtain (4.1) at $t = a$.

3. Now we show that (4.2) holds. Define mappings $f_j : I \to X$ by

$$f_j(t) = \eta_{\alpha,\beta}(t)x_1 + (2-j)x_2, \qquad t \in I, \quad j = 1, 2. \tag{4.11}$$

and observe that $f_1(\beta) = x_1 + x_2$, $f_2(\beta) = x_1$, $f_1(\alpha) = x_2$, $f_2(\alpha) = 0$, and that $f_1 - f_2 \equiv x_2$, and so $\|f_1 - f_2\|_\Phi = \|f_1 - f_2\|_{0,1} = \|x_2\|$. Hence, inequality (4.4) provides the estimate:

$$\begin{aligned} &\|h(\beta, x_1 + x_2) - h(\beta, x_1) - h(\alpha, x_2) + h(\alpha, 0)\| \leq \\ &\qquad \leq \mu\|x_2\|(\beta - \alpha)\Psi^{-1}(1/(\beta - \alpha)). \end{aligned} \tag{4.12}$$

Since $\mathcal{H}$ maps $\mathcal{F}(I;X)$ into $GV_\Psi(I;Y)$ and constant mappings belong to $\mathcal{F}(I;X)$, for all $x \in X$ the mapping $h(\,\cdot\,,x) = \mathcal{H}\,x$ is in $GV_\Psi(I;Y)$, and so it is (absolutely) continuous by (3.4). Noting that (cf. (2.7))

$$\lim_{\beta-\alpha\to 0}(\beta - \alpha)\Psi^{-1}(1/(\beta - \alpha)) = 0, \tag{4.13}$$

and letting $\beta - \alpha$ tend to zero in (4.12) in such a way that α, $\beta \in I$, $\alpha < \beta$, and $[\alpha, \beta] \ni t$, for all $t \in I$ and x_1, $x_2 \in X$ we get:

$$h(t, x_1 + x_2) - h(t, x_1) - h(t, x_2) + h(t, 0) = 0. \tag{4.14}$$

Now, for a fixed $t \in I$ define the operator $S_t : X \to Y$ by

$$S_t(x) = h(t, x) - h(t, 0), \qquad x \in X.$$

Inequality (4.1) shows that S_t is continuous (even Lipschitzian), and since (4.14) can be rewritten in the form

$$S_t(x_1 + x_2) = S_t(x_1) + S_t(x_2), \qquad x_1,\, x_2 \in X,$$

S_t is an additive operator. Consequently, $S_t \in L(X;Y)$. Defining mappings $h_0 : I \to Y$ and $h_1 : I \to L(X;Y)$ by $h_0(t) = h(t, 0)$ and $h_1(t)x = S_t(x)$, respectively, where $t \in I$ and $x \in X$, we obtain (4.2). Taking into account that $h_0 = \mathcal{H}(0)$ and $h_1(\cdot)x = \mathcal{H}(x) - \mathcal{H}(0)$ we conclude that h_0 and $h_1(\cdot)x$ belong to $GV_\Psi(I;Y)$ for all $x \in X$.

4. Let us prove the reciprocal assertion. Let $\mathcal{F}(I;X) = GV_\Phi(I;X)$ and $\Psi \preccurlyeq \Phi$. In view of (4.2) and (1.4), the composition operator $\mathcal{H}$ is given by

$$(\mathcal{H}f)(t) = h_0(t) + h_1(t)f(t), \qquad t \in I, \quad f \in \mathcal{F}(I;X). \tag{4.15}$$

Theorem 3.3 implies $GV_\Phi(I;X) \subset GV_\Psi(I;X)$, and since the triple $(L(X;Y), X, Y)$ is multiplicative, by Theorem 3.6 the mapping $h_1 f$

is in $GV_\Psi(I;Y)$, so that $\mathcal{H}f \in GV_\Psi(I;Y)$ for all $f \in GV_\Phi(I;X)$, i.e. $\mathcal{H}$ maps $GV_\Phi(I;X)$ into $GV_\Psi(I;Y)$. From (3.11) and (3.16) we have:

$$\|\mathcal{H}f_1 - \mathcal{H}f_2\|_\Psi = \|h_1(f_1 - f_2)\|_\Psi \le \gamma(\Psi, |I|)\|h_1\|_\Psi\|f_1 - f_2\|_\Psi \le \\ \le \mu\|f_1 - f_2\|_\Phi, \qquad f_1,\, f_2 \in GV_\Phi(I;X), \tag{4.16}$$

where $\mu = \gamma(\Psi, |I|)\kappa\|h_1\|_\Psi$. Hence, $\mathcal{H}$ is a Lipschitzian operator.

Since $\mathcal{F}(I;X) = C^{0,1}(I;X) \subset GV_\Psi(I;X)$, (4.15) implies that $\mathcal{H}$ maps $C^{0,1}(I;X)$ into $GV_\Psi(I;Y)$. Estimates (3.7) and (4.16) prove that $\mathcal{H}$ is Lipschitzian. □

Theorem 4.1 shows that Lipschitzian composition operators between certain spaces of mappings of bounded generalized variation are generated by mappings of the form (4.2). Now we treat the case when these operators are automatically constant. This happens when the domain of $\mathcal{H}$ is "significantly larger" than the range of $\mathcal{H}$ (cf. Theorem 4.3).

We say that $\Phi \in \mathcal{N}$ *grows* (*at infinity*) *significantly slower than* $\Psi \in \mathcal{N}$ (in symbols, $\Phi \lhd \Psi$) if $\lim_{\rho\to\infty} \Phi(C\rho)/\Psi(\rho) = 0$ for all $C > 0$, or, equivalently, if

$$\forall\, \varepsilon > 0 \;\; \exists\, \rho_0(\varepsilon) > 0 \;\text{ such that }\; \Phi(\rho) \le \Psi(\varepsilon\rho) \;\; \forall\, \rho \ge \rho_0(\varepsilon). \tag{4.17}$$

For instance, if $\Phi(\rho) = \rho^p$, $\Psi(\rho) = \rho^q$, p, $q \ge 1$, then $\Phi \lhd \Psi$ if and only if $p < q$. Note that if $\Phi \lhd \Psi$, then $\Phi \preccurlyeq \Psi$. We have the following characterization of relation $\lhd$ between arbitrary functions Φ, $\Psi \in \mathcal{N}$:

LEMMA 4.2. $\Phi \lhd \Psi$ *if and only if* $\lim_{r\to\infty} \Psi^{-1}(r)/\Phi^{-1}(r) = 0$.

Proof. Necessity. By (4.17), for any $\varepsilon > 0$ there exists $\rho_0 = \rho_0(\varepsilon) > 0$ such that $\Phi(\rho) \le \Psi(\varepsilon\rho) =: \Psi_\varepsilon(\rho)$ for all $\rho \ge \rho_0$. It follows that

$$\Psi_\varepsilon^{-1}(r) \le \Phi^{-1}(r) \quad \text{for all} \quad r \ge \Psi_\varepsilon(\rho_0); \tag{4.18}$$

indeed, for any $r \ge \Psi_\varepsilon(\rho_0)$ there exist unique $\rho_1 \ge \rho_0$ and $\rho_2 \ge \rho_0$ satisfying $\Psi_\varepsilon(\rho_1) = r$, $\Phi(\rho_2) = r$ and $\rho_1 \le \rho_2$. Since $\Psi_\varepsilon^{-1}(r) = \Psi^{-1}(r)/\varepsilon$, (4.18) implies $\Psi^{-1}(r)/\Phi^{-1}(r) \le \varepsilon$ for all $r \ge \Psi(\varepsilon\rho_0(\varepsilon))$, and so $\Psi^{-1}(r)/\Phi^{-1}(r) \to 0$ as $r \to \infty$.

Sufficiency. Given $\varepsilon > 0$, there exists $r_0 = r_0(\varepsilon) > 0$ such that $\Psi^{-1}(r)/\Phi^{-1}(r) \le \varepsilon$ or $\Psi^{-1}(r)/\varepsilon \le \Phi^{-1}(r)$ for all $r \ge r_0$. Hence, as in (4.18), we have: $\Phi(\rho) \le \Psi(\varepsilon\rho)$ for all $\rho \ge \rho_0(\varepsilon) := \Phi^{-1}(r_0(\varepsilon))$. □

THEOREM 4.3. *Let X, Y, Φ, Ψ, $\mathcal{H}$ and h be as in Theorem* 4.1. *If*

(a) *$\mathcal{H}$ maps $GV_\Phi(I;X)$ with $\Phi \lhd \Psi$, $AC(I;X)$ or $BV_1(I;X)$ into $GV_\Psi(I;Y)$; or*

(b) *$\mathcal{H}$ maps $GV_\Phi(I;X)$, $AC(I;X)$ or $BV_1(I;X)$ into $C^{0,1}(I;Y)$; and $\mathcal{H}$ is Lipschitzian, then there exists a mapping h_0 in* (a) *$GV_\Psi(I;Y)$, or* (b) *$C^{0,1}(I;Y)$, respectively, such that $h(t,x) = h_0(t)$, $(t,x) \in I \times X$ (i.e. any Lipschitzian composition operator of this kind is constant).*

Proof. (a) Suppose that $\Phi \lhd \Psi$ and $\mathcal{H} : GV_\Phi(I;X) \to GV_\Psi(I;Y)$ is Lipschitzian. As we have already seen in step 3 of the proof of Theorem 4.1, the mapping $h(\,\cdot\,,x)$ is continuous on I for all $x \in X$. If $a < t \leq b$, then setting $\beta = t$, $x_1 = x$ and $x_2 = 0$ in (4.7) and letting α go to $t - 0$ in (4.7), by Lemma 4.2 we obtain: $h(t,x) = h(t,0)$ for all $a < t \leq b$ and $x \in X$. From the continuity of $h(\,\cdot\,,x)$ we infer that $h(\,\cdot\,,x) = h(\,\cdot\,,0) =: h_0$ on I for all $x \in X$.

Assume that $\mathcal{H}$ maps $AC(I;X)$ or $BV_1(I;X)$ into $GV_\Psi(I;Y)$. For functions (4.5) we have $\|f_1 - f_2\|_1 = \|x_1 - x_2\|$ and hence the counterpart of (4.7) is the inequality:

$$\|h(\beta,x_1) - h(\beta,x_2)\| \leq \mu(\beta-\alpha)\Psi^{-1}(1/(\beta-\alpha))\|x_1 - x_2\|,$$

which holds for all $a \leq \alpha < \beta \leq b$ and x_1, $x_2 \in X$. From (4.13) it follows that $h(\,\cdot\,,x) = h(\,\cdot\,,0) \in GV_\Psi(I;Y)$ for all $x \in X$.

(b) If $\mathcal{H} : GV_\Phi(I;X) \to C^{0,1}(I;Y)$ is Lipschitzian, then the inequality $\|\mathcal{H}f_1 - \mathcal{H}f_2\|_{0,1} \leq \mu\|f_1 - f_2\|_\Phi$ implies, in particular,

$$\frac{\|h(\beta,f_1(\beta)) - h(\beta,f_2(\beta)) - h(\alpha,f_1(\alpha)) + h(\alpha,f_2(\alpha))\|}{\beta-\alpha} \leq \mu\|f_1 - f_2\|_\Phi, \tag{4.19}$$

for all α, $\beta \in I$, $\alpha < \beta$. Substituting mappings (4.5) into this inequality we get

$$\|h(\beta,x_1) - h(\beta,x_2)\| \leq \mu\|x_1 - x_2\|/\Phi^{-1}(1/(\beta-\alpha)), \quad x_1,\, x_2 \in X.$$

Setting $x_2 = 0$, it remains to note that $\Phi^{-1}(1/(\beta-\alpha)) \to \infty$ as $\beta - \alpha \to 0$.

In the case when $\mathcal{H}$ maps $AC(I;X)$ or $BV_1(I;X)$ into $C^{0,1}(I;Y)$ it suffices to replace the norm $\|f_1 - f_2\|_\Phi$ in (4.19) with the norm $\|f_1 - f_2\|_1 = \|x_1 - x_2\|$. □

In the following corollary we set $GV_\Phi(I) = GV_\Phi(I;\mathbb{R})$, $\Phi \in \mathcal{N}$.

COROLLARY 4.4. *Let $\mathcal{H} : \mathbb{R}^I \to \mathbb{R}^I$ be the composition operator generated by a function $h : I \times \mathbb{R} \to \mathbb{R}$ and Φ, $\Psi \in \mathcal{N}$. If $\Psi \preccurlyeq \Phi$, then $\mathcal{H}$ maps $GV_\Phi(I)$ into $GV_\Psi(I)$ and is Lipschitzian if and only if $\mathcal{H}$ is of the form $\mathcal{H}f = h_0 + h_1 f$, $f \in GV_\Phi(I)$, where h_0, $h_1 \in GV_\Psi(I)$. Moreover, if $\Phi \lhd \Psi$, then $\mathcal{H}$ maps $GV_\Phi(I)$ into $GV_\Psi(I)$ and is Lipschitzian if and only if $\mathcal{H}$ is a constant operator.*

Remark 4.1. Lipschitzian composition operators from $C^{0,1}(I;X)$ into $C^{0,1}(I;Y)$ were characterized in [27, 29] (with I a convex subset of a normed linear space). Corollary 4.4 is in coherence with the results from [33] ($\Phi(\rho)=\Psi(\rho)=\rho^q$, $q>1$) and [35] ($\Phi(\rho)=\rho^p$, $\Psi(\rho)=\rho^q$, $p>1$, $q\geq 1$).

Now we consider composition operators $\mathcal{H}$ with values in the space $BV_1(I;Y)$ where Y is a Banach space. If $h:I\times X\to Y$ is a mapping such that $h(\,\cdot\,,x)\in BV_1(I;Y)$ for all $x\in X$, we define the left regularization $h^*:I\times X\to Y$ of h (with respect to the first variable) as follows: for any $x\in X$ we set

$$\begin{aligned} &h^*(t,x):=\lim_{s\to t-0}h(s,x) \text{ if } a<t\leq b, \text{ and}\\ &h^*(a,x):=\lim_{t\to a+0}h^*(t,x). \end{aligned} \tag{4.20}$$

Then for all $x\in X$ the function $h^*(\,\cdot\,,x)$ is in $BV_1(I;Y)$ and is continuous from the left on $(a,b]$.

THEOREM 4.5. *Let X be a normed linear space, Y a Banach space and $\Phi\in\mathcal{N}$. Suppose that a mapping $h:I\times X\to Y$ generates the composition operator $\mathcal{H}$ via formula* (1.4)*. Let $\mathcal{F}(I;X)$ be either $GV_\Phi(I;X)$ or $C^{0,1}(I;X)$.*

If $\mathcal{H}$ maps $\mathcal{F}(I;X)$ into $BV_1(I;Y)$ and is Lipschitzian, then there exists a function $\mu_0:I\to\mathbb{R}^+$ such that

$$\|h^*(t,x_1)-h^*(t,x_2)\|\leq\mu_0(t)\|x_1-x_2\|,\quad t\in I,\quad x_1,\,x_2\in X, \tag{4.21}$$

and there exist a mapping $h_0\in BV_1(I;Y)$, left continuous on $(a,b]$, and a mapping $h_1\in L(X;Y)^I$ such that $h_1(\cdot)x\in BV_1(I;Y)$ and $h_1(\cdot)x$ is left continuous on $(a,b]$ for all $x\in X$ and

$$h^*(t,x)=h_0(t)+h_1(t)x,\qquad t\in I,\quad x\in X. \tag{4.22}$$

Conversely, if $h_0\in BV_1(I;Y)$, $h_1\in BV_1(I;L(X;Y))$ (*where Y is not necessarily complete*) *and the mapping h is of the form* (4.2)*, then $\mathcal{H}$ maps $\mathcal{F}(I;X)$ into $BV_1(I;Y)$ and is Lipschitzian.*

Proof. 1. We begin by proving (4.21). Since $\mathcal{H}:\mathcal{F}(I;X)\to BV_1(I;Y)$ is Lipschitzian, there exists a positive constant μ such that

$$\|\mathcal{H}f_1-\mathcal{H}f_2\|_1\leq\mu\|f_1-f_2\|_{\mathcal{F}}\quad\text{for all}\quad f_1,\,f_2\in\mathcal{F}(I;X), \tag{4.23}$$

where $\|\cdot\|_{\mathcal{F}}$ denotes the norm in $\mathcal{F}(I;X)$. Definitions of $\|\cdot\|_1$, $\mathsf{V}_1(\cdot)$ and $\mathcal{H}$ imply, in particular, that if $\alpha,\,\beta\in I$, $\alpha<\beta$, then

$$\|h(\beta,f_1(\beta))-h(\beta,f_2(\beta))-h(\alpha,f_1(\alpha))+h(\alpha,f_2(\alpha))\|\leq\mu\|f_1-f_2\|_{\mathcal{F}}. \tag{4.24}$$

Let $a < t \le b$. If $a \le \alpha < \beta \le b$ and $x_1,\ x_2 \in X$, substituting mappings f_j from (4.5) into (4.24), taking into account (4.6) and (4.8) and setting $\alpha = a$ and $\beta = t$ we find that

$$\|h(t,x_1) - h(t,x_2)\| \le \mu\|x_1 - x_2\|/\omega_{\mathcal{F}}(t-a),$$

where $\omega_{\mathcal{F}}(r)$ is defined for $r > 0$ as in (4.10). Since $h(\,\cdot\,,x) \in BV_1(I;Y)$ for all $x \in X$ and $\omega_{\mathcal{F}}$ is continuous, it follows, by taking the left limits (in t), that

$$\|h^*(t,x_1) - h^*(t,x_2)\| \le \mu\|x_1 - x_2\|/\omega_{\mathcal{F}}(t-a), \quad a < t \le b.$$

To prove that (4.21) holds at $t = a$, we substitute mappings (4.9) into (4.24), which, after setting $\beta = b$, yields by virtue of (4.10):

$$\|h(\alpha,x_1) - h(\alpha,x_2)\| \le \mu(1 + 1/\omega_{\mathcal{F}}(b-\alpha))\|x_1 - x_2\|, \quad a \le \alpha < b,$$

so that letting $\alpha \to t - 0$ for $t > a$ and then $t \to a + 0$ we obtain:

$$\|h^*(a,x_1) - h^*(a,x_2)\| \le \mu(1 + 1/\omega_{\mathcal{F}}(b-a))\|x_1 - x_2\|.$$

2. In order to prove (4.22), let $a < t \le b$, $n \in \mathbb{N}$ and let $a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_n < \beta_n < t$. Using (4.23) we have, in particular,

$$\sum_{i=1}^{n} \|h(\beta_i, f_1(\beta_i)) - h(\beta_i, f_2(\beta_i)) - h(\alpha_i, f_1(\alpha_i)) + h(\alpha_i, f_2(\alpha_i))\| \le \\ \le \mu\|f_1 - f_2\|_{\mathcal{F}}. \tag{4.25}$$

Setting

$$\eta_n(t) = \begin{cases} 0 & \text{if} \quad a \le t \le \alpha_1, \\ \eta_{\alpha_i,\beta_i}(t) & \text{if} \quad \alpha_i \le t \le \beta_i,\ i = 1,\ldots,n, \\ 1 - \eta_{\beta_i,\alpha_{i+1}}(t) & \text{if} \quad \beta_i \le t \le \alpha_{i+1},\ i = 1,\ldots,n-1, \\ 1 & \text{if} \quad \beta_n \le t \le b, \end{cases}$$

where $\eta_{\alpha,\beta}$ is defined in (4.3), consider mappings $f_j\colon I \to X$ defined by:

$$f_j(t) = \eta_n(t)x_1 + (2-j)x_2, \quad t \in I,\ x_1,\ x_2 \in X,\ j = 1,\ 2. \tag{4.26}$$

Observe that $f_1(\beta_i) = x_1 + x_2$, $f_2(\beta_i) = x_1$, $f_1(\alpha_i) = x_2$, $f_2(\alpha_i) = 0$, $i = 1,\ldots,n$, and $f_1 - f_2 \equiv x_2$, and so $\|f_1 - f_2\|_{\mathcal{F}} = \|x_2\|$. Hence, from (4.25) and (4.26) we get:

$$\sum_{i=1}^{n} \|h(\beta_i, x_1 + x_2) - h(\beta_i, x_1) - h(\alpha_i, x_2) + h(\alpha_i, 0)\| \le \mu\|x_2\|. \tag{4.27}$$

Since $\mathcal{H} : \mathcal{F}(I;X) \to BV_1(I;Y)$, $h(\,\cdot\,,x) \in BV_1(I;Y)$ for all $x \in X$, and therefore $h^*(\,\cdot\,,x) \in BV_1(I;Y)$ and is left continuous on $(a,b]$ for all $x \in X$. Letting α_1 go to $t-0$ in (4.27) we have:

$$\|h^*(t,x_1+x_2) - h^*(t,x_1) - h^*(t,x_2) + h^*(t,0)\| \leq \mu\|x_2\|/n.$$

Passing to the limit as $n \to \infty$, for any $a < t \leq b$, x_1, $x_2 \in X$ we find:

$$h^*(t,x_1+x_2) - h^*(t,x_1) - h^*(t,x_2) + h^*(t,0) = 0.$$

Taking into account (4.21) and arguments following (4.14) we infer that for any $t \in (a,b]$ there exist $h_0(t) \in Y$ and $h_1(t) \in L(X;Y)$ such that

$$h^*(t,x) = h_0(t) + h_1(t)x, \qquad a < t \leq b, \quad x \in X.$$

Setting $h_0(a) := h^*(a,0)$ and $h_1(a)x := h^*(a,x) - h^*(a,0)$, $x \in X$, we obtain equality (4.22). Finally, since $h_0(\cdot) = h^*(\,\cdot\,,0)$ and $h_1(\cdot)x = h^*(\,\cdot\,,x) - h^*(\,\cdot\,,0)$ on I, mappings h_0 and $h_1(\cdot)x$ are in $BV_1(I;Y)$ and are left continuous on $(a,b]$.

3. To prove the last part of the theorem, it suffices to note that $\mathcal{F}(I;X)$ is contained in $BV_1(I;X)$, that $(L(X;Y),X,Y)$ is a multiplicative triple and that (cf. (3.13)) if $h_1 \in BV_1(I;L(X;Y))$ and $f \in BV_1(I;X)$, then $\|h_1 f\|_1 \leq 2\|h_1\|_1\|f\|_1$. □

COROLLARY 4.6. *Suppose that the composition operator* $\mathcal{H} : \mathbb{R}^I \to \mathbb{R}^I$ *is generated by a function* $h : I \times \mathbb{R} \to \mathbb{R}$ *such that* $h^*(t,x) = h(t,x)$, $(t,x) \in I \times \mathbb{R}$. *Let* $\Phi \in \mathcal{N}$ *and* $\mathcal{F}(I)$ *be either* $GV_\Phi(I)$ *or* $C^{0,1}(I)$. *Then* $\mathcal{H} : \mathcal{F}(I) \to BV_1(I)$ *is Lipschitzian if and only if there exist two functions* h_0, $h_1 \in BV_1(I)$, *left continuous on* $(a,b]$, *such that* h *is of the form* $h(t,x) = h_0(t) + h_1(t)x$ *for all* $t \in I$ *and* $x \in \mathbb{R}$.

Remark 4.2. Lipschitzian composition operators from $BV_1(I)$ into itself were described in [31]. If $\mathcal{F}(I) = BV_p(I)$, $p > 1$, in Corollary 4.6, we get a result from [35]. The mapping $h^*(t,x)$ in (4.22) cannot in general be replaced by $h(t,x)$, because $h(t,x)$ need not be linear in the variable x: the corresponding example is given in [31, p. 157]. Clearly, a theorem similar to Theorem 4.5 holds for the right regularization of $h(\,\cdot\,,x)$.

5. Applications to Multifunctions

5.1. Existence of Regular Selections

Let $\mathcal{P}(X)$ be the family of all *nonempty* subsets of the metric space (X,d) and $\mathcal{P}_{cb}(X)$ be the family of all nonempty *closed* and *bounded* subsets there.

The *Hausdorff metric* on $\mathcal{P}_{cb}(X)$ is defined by

$$D(A,B)=\max\{\mathrm{e}(A,B),\,\mathrm{e}(B,A)\},\qquad A,\,B\in\mathcal{P}_{cb}(X),\tag{5.1}$$

where

$$\mathrm{e}(A,B)=\sup_{x\in A}\mathrm{dist}(x,B)\quad\text{and}\quad\mathrm{dist}(x,B)=\inf_{y\in B}d(x,y).\tag{5.2}$$

A *multifunction* (or a *set-valued mapping*) *from* I *into* X is a mapping $F:I\to\mathcal{P}(X)$, so that $F(t)\subset X$ for every $t\in I$. The set $\mathrm{Gr}(F)=\{(t,x)\in I\times X\mid x\in F(t)\}$ is called the *graph of* F and the set $\mathrm{Ra}(F)=\bigcup_{t\in I}F(t)$ is called the *range of* F. For the detailed exposition of the theory of set-valued mappings and properties of the Hausdorff metric see [1, Ch. 1, Secs. 1–5] and [3, Ch. 2, Sec. 1].

A mapping $f\in X^I$ is said to be a *selection* of the multifunction $F\in\mathcal{P}(X)^I$ if $f(t)\in F(t)$ for all $t\in I$; the selection f of F is called *regular* if f is of the same functional class as F (this is to be made precise below, cf. Theorem 5.1).

Using the metric space $(\mathcal{P}_{cb}(X),D)$ in place of (X,d) we introduce the notions of Lipschitzian multifunctions $F\in C^{0,1}(I;\mathcal{P}_{cb}(X))$, absolutely continuous multifunctions $F\in AC(I;\mathcal{P}_{cb}(X))$ and multifunctions of bounded generalized Φ-variation $F\in BV_\Phi(I;\mathcal{P}_{cb}(X))$, $\Phi\in\mathcal{N}$. For instance, $F\in BV_\Phi(I;\mathcal{P}_{cb}(X))$ if

$$\mathsf{V}_\Phi(F)\equiv\mathsf{V}_{\Phi,D}(F,I):=\sup\{V_{\Phi,D}[F,T]\mid T\in\mathcal{T}(I)\}<\infty,$$

where for a partition $T=\{t_i\}_{i=0}^m\in\mathcal{T}(I)$ we set

$$V_{\Phi,D}[F,T]=\sum_{i=1}^m\Phi\Big(\frac{D(F(t_i),F(t_{i-1}))}{t_i-t_{i-1}}\Big)(t_i-t_{i-1}).$$

The main result of this section is the following

THEOREM 5.1. *Let $F\in BV_\Phi(I;\mathcal{P}_{cb}(X))$ be a multifunction with compact graph where X is a Banach space and $\Phi\in\mathcal{N}$. Then F admits a regular selection f, i.e. $f\in BV_\Phi(I;X)$, $f(t)\in F(t)$ for all $t\in I$, $\mathsf{V}_\Phi(f)\le\mathsf{V}_\Phi(F)$ and $\mathsf{V}_1(f)\le\mathsf{V}_1(F)$. Moreover, for any finite number of points $t_i\in I$ and $x_i\in F(t_i)$, $i=0,1,\dots,m$, a regular selection f of F can be chosen such that $f(t_i)=x_i$ for all $i=0,1,\dots,m$.*

Proof. We shall prove the theorem for $m=0$, the general case will follow immediately. For each $n\in\mathbb{N}$ let $T_n=\{t_i^n\}_{i=0}^n$ be a partition of the interval $I=[a,b]$ with the properties:

1) $t_0\in T_n$, i.e., $t_0=t_{k(n)}^n$ for some $k(n)\in\{0,\,1,\,\dots,\,n\}$, and

2) $\max_{1\le i\le n}(t_i^n - t_{i-1}^n) \to 0$ as $n\to\infty$.

We define the elements $x_i^n \in F(t_i^n)$ inductively as follows. To start with, let $n\in\mathbb{N}$ and let $a < t_0 < b$. (Below $\|\cdot\|$ denotes the norm in X.)

(a) Set $x_{k(n)}^n = x_0$.

(b) If $i \in \{1, \ldots, k(n)\}$ and if $x_i^n \in F(t_i^n)$ is already chosen, pick $x_{i-1}^n \in F(t_{i-1}^n)$ such that $\|x_i^n - x_{i-1}^n\| = \operatorname{dist}(x_i^n, F(t_{i-1}^n))$.

c) If $i \in \{k(n)+1, \ldots, n\}$ and if $x_{i-1}^n \in F(t_{i-1}^n)$ is already chosen, pick $x_i^n \in F(t_i^n)$ such that $\|x_{i-1}^n - x_i^n\| = \operatorname{dist}(x_{i-1}^n, F(t_i^n))$.

Now, if $t_0 = a$, so that $k(n) = 0$, then we use only (a) and (c) to define x_i^n, and if $t_0 = b$, so that $k(n) = n$, we define x_i^n by (a) and (b).

We define a sequence of mappings $f_n : I \to X$, $n\in\mathbb{N}$, as follows:

$$f_n(t) = x_{i-1}^n + \frac{t - t_{i-1}^n}{t_i^n - t_{i-1}^n}(x_i^n - x_{i-1}^n), \quad t\in[t_{i-1}^n, t_i^n], \quad i = 1, \ldots, n. \tag{5.3}$$

Note at once that $f_n(t_{i-1}^n) = x_{i-1}^n$, $f_n(t_i^n) = x_i^n$ and $f_n(t_0) = x_0$ for all $n\in\mathbb{N}$, and that, by virtue of (b), (c), (5.1) and (5.2),

$$\|x_i^n - x_{i-1}^n\| \le D(F(t_i^n), F(t_{i-1}^n)), \quad n\in\mathbb{N}, \quad i = 1, \ldots, n. \tag{5.4}$$

Using Lemma 2.1(b) and inequality (5.4), for all $n\in\mathbb{N}$ we find that

$$\begin{aligned} \mathsf{V}_\Phi(f_n) &= \sum_{i=1}^n \mathsf{V}_\Phi(f_n, [t_{i-1}^n, t_i^n]) = \sum_{i=1}^n \Phi\biggl(\frac{\|x_i^n - x_{i-1}^n\|}{t_i^n - t_{i-1}^n}\biggr)(t_i^n - t_{i-1}^n) \le \\ &\le \sum_{i=1}^n \Phi\biggl(\frac{D(F(t_i^n), F(t_{i-1}^n))}{t_i^n - t_{i-1}^n}\biggr)(t_i^n - t_{i-1}^n) \le \mathsf{V}_\Phi(F). \end{aligned} \tag{5.5}$$

Since F is of bounded Jordan variation by Lemma 2.1(d), the calculations in (5.5) with $\Phi(\rho) = \rho$ also give $\mathsf{V}_1(f_n) \le \mathsf{V}_1(F)$ for all $n\in\mathbb{N}$.

It is seen from (5.3) that all the images $f_n(I)$ are contained in the closed convex hull $\overline{\mathrm{co}}\mathrm{Ra}(F)$ of the range $\mathrm{Ra}(F)$, and since the graph $\mathrm{Gr}(F)$ is compact in $I\times X$, $\mathrm{Ra}(F)$ is compact in X and, hence, by Mazur's theorem [13, Ch. 5, Sec. 2], $\overline{\mathrm{co}}\mathrm{Ra}(F)$ is also compact in X. By Theorem 2.2 there exists a subsequence in $\{f_n\}_{n=1}^\infty$ (which will be denoted by the same symbol) which converges uniformly on I to a mapping $f\in BV_\Phi(I;X)$ as $n\to\infty$. Clearly, $f(t_0) = x_0$. Lemma 2.1(c) and inequality (5.5) imply $\mathsf{V}_\Phi(f) \le \mathsf{V}_\Phi(F)$ and $\mathsf{V}_1(f) \le \mathsf{V}_1(F)$.

It remains to show that f is a selection of F. Fixing $t\in I$, for every $n\in\mathbb{N}$ there is a number $i(n)\in\{1,\ldots,n\}$ such that $t_{i(n)-1}^n \le t \le t_{i(n)}^n$, and hence, by condition 2) above, the sequences $t_{i(n)-1}^n$ and $t_{i(n)}^n$ converge to t as $n \to \infty$. As the graph of F is compact and $x^n_{i(n)} \in F(t^n_{i(n)})$ by the construction, i.e., $(t^n_{i(n)}, x^n_{i(n)}) \in \mathrm{Gr}(F)$, then there exists a subsequence in $\{(t^n_{i(n)}, x^n_{i(n)})\}_{n=1}^{\infty}$ (denoted by the same symbol) which converges in $I \times X$ to a point $(\tau, x) \in \mathrm{Gr}(F)$ as $n \to \infty$. But $t^n_{i(n)} \to t$ as $n \to \infty$, so that $\tau = t$, and hence, $(t, x) \in \mathrm{Gr}(F)$ or $x \in F(t)$. From the convergence of the subsequence it follows also that $f_n(t^n_{i(n)}) = x^n_{i(n)} \to x$, and the continuity of the mapping f yields that $f(t^n_{i(n)}) \to f(t)$ in X as $n \to \infty$. Since the sequence f_n converges uniformly to f, we have $f_n(t^n_{i(n)}) - f(t^n_{i(n)}) \to 0$ as $n \to \infty$. We conclude that $x = f(t)$, and so $f(t) \in F(t)$. Since $t \in I$ is arbitrary, the proof is complete. □

5.2. The Metric Space $BV_\Phi(I; \mathcal{P}_{cc}(X))$

The material of this section is preparatory. It will be used in Section 5.3.

Let $(X, \|\cdot\|)$ be a normed linear space and $\mathcal{P}_{cc}(X)$ be the family of all nonempty *compact convex* subsets of X. The Hausdorff metric D is translation invariant on $\mathcal{P}_{cc}(X)$ in the sense that (see [41, Lemma 3]):

$$D(A, B) = D(A + Q, B + Q) \tag{5.6}$$

for all A, $B \in \mathcal{P}_{cc}(X)$ and bounded $Q \in \mathcal{P}(X)$.

We are going to endow the set $BV_\Phi(I; \mathcal{P}_{cc}(X))$ with a metric. To do this, let F_1, $F_2 \in BV_\Phi(I; \mathcal{P}_{cc}(X))$. For t, $s \in I$, $s < t$, and $r > 0$ we set

$$U_r^{F_1,F_2}(t, s) = \Phi\Big(\frac{D(F_1(t) + F_2(s), F_2(t) + F_1(s))}{(t - s)\, r}\Big)(t - s), \tag{5.7}$$

and

$$W_r(F_1, F_2) = \sup\Big\{\sum_{i=1}^{m} U_r^{F_1,F_2}(t_i, t_{i-1}) \,\Big|\, T = \{t_i\}_{i=0}^{m} \in \mathcal{T}(I)\Big\} \tag{5.8}$$

($m \in \mathbb{N}$ is arbitrary). We define a *semimetric* on $BV_\Phi(I; \mathcal{P}_{cc}(X))$ by

$$\Delta(F_1, F_2) \equiv \Delta_\Phi(F_1, F_2) = \inf\{r > 0 \mid W_r(F_1, F_2) \le 1\}, \tag{5.9}$$

and we define the *metric* D_Φ on $BV_\Phi(I; \mathcal{P}_{cc}(X))$ by

$$D_\Phi(F_1, F_2) = D(F_1(a), F_2(a)) + \Delta(F_1, F_2). \tag{5.10}$$

Below we verify that the above definitions are correct. First, let us show that the value (5.9) is finite. Observe that

$$D(A, B) \le D(A+P, B+Q) + D(P, Q), \quad A,\ B,\ P,\ Q \in \mathcal{P}_{cc}(X), \tag{5.11}$$

since equality (5.6) and the triangle inequality for D imply

$$D(A,B) = D(A{+}P,B{+}P) \le D(A{+}P,B{+}Q){+}D(B{+}P,B{+}Q) = \\ = D(A{+}P,B{+}Q) + D(P,Q).$$

Now, if t, $s \in I$, then (5.11) and (5.6) yield

$$D(F_1(t)+F_2(s),F_2(t)+F_1(s)) \le D(F_1(t),F_1(s)) + D(F_2(t),F_2(s)),$$

and it follows from (5.7), (5.8), the monotonicity and convexity of Φ that

$$W_r(F_1,F_2) \le \frac{1}{r}(\mathsf{V}_\Phi(F_1) + \mathsf{V}_\Phi(F_2)), \qquad r \ge 2.$$

Hence, $\lim_{r\to\infty} W_r(F_1,F_2) = 0$, and the value $\Delta(F_1,F_2)$ is finite.

To prove that D_Φ is a metric, we need the counterpart of Lemma 3.4, which will be useful in Section 5.3 as well.

LEMMA 5.2. *Let* $\Phi \in \mathcal{N}$ *and* F_1, $F_2 \in BV_\Phi(I;\mathcal{P}_{cc}(X))$. *We have*:
(a) *if* t, $s \in I$ *and* $t \ne s$, *then*

$$|D(F_1(t),F_2(t)){-}D(F_1(s),F_2(s))| \le D(F_1(t){+}F_2(s),F_2(t){+}F_1(s)) \le \\ \le |t-s|\Phi^{-1}(1/|t-s|)\Delta(F_1,F_2);$$

(b) *if* $\Delta(F_1,F_2) > 0$, *then* $W_{\Delta(F_1,F_2)}(F_1,F_2) \le 1$;
(c) *if* $r > 0$, *then* $W_r(F_1,F_2) \le 1$ *if and only if* $\Delta(F_1,F_2) \le r$;
(d) *if* $r > 0$ *and* $W_r(F_1,F_2) = 1$, *then* $\Delta(F_1,F_2) = r$;
(e) *if* $F_j^n \in BV_\Phi(I;\mathcal{P}_{cc}(X))$ $(n \in \mathbb{N})$ *and* $D(F_j^n(t),F_j(t)) \to 0$ *as* $n \to \infty$ $(t \in I)$, $j = 1$, 2, *then* $\Delta(F_1,F_2) \le \limsup_{n\to\infty} \Delta(F_1^n,F_2^n)$.

Proof. (a) If t, $s \in I$, $t \ne s$, then using (5.7)–(5.9) we have:

$$\Phi\Big(\frac{D(F_1(t)+F_2(s),F_2(t)+F_1(s))}{|t-s|\,r}\Big)|t-s| \le W_r(F_1,F_2) \le 1$$

for all $r > \Delta(F_1,F_2)$, whence

$$D(F_1(t)+F_2(s),F_2(t)+F_1(s)) \le |t-s|\Phi^{-1}(1/|t-s|)\Delta(F_1,F_2).$$

Since, by virtue of (5.11),

$$D(F_1(t),F_2(t)) \le D(F_1(t)+F_2(s),F_2(t)+F_1(s)) + D(F_1(s),F_2(s)),$$

it suffices to interchange the variables t and s.

(b) First, let us show that if conditions of Lemma 5.2(e) hold and $\lim_{n\to\infty} r_n = r$ where $r_n > 0$ and $r > 0$, then

$$W_r(F_1, F_2) \le \liminf_{n\to\infty} W_{r_n}(F_1^n, F_2^n). \tag{5.12}$$

Observe that if $t,\, s \in I$, then

$$\lim_{n\to\infty} D(F_1^n(t)+F_2^n(s), F_2^n(t)+F_1^n(s)) = D(F_1(t)+F_2(s), F_2(t)+F_1(s)), \tag{5.13}$$

since the triangle inequality for D and (5.11) imply that the absolute value of the difference between the left and right hand sides of (5.13) is not greater than

$$D(F_1^n(t)+F_2^n(s), F_1(t)+F_2(s)) + D(F_2^n(t)+F_1^n(s), F_2(t)+F_1(s)) \le$$

$$\le D(F_1^n(t),F_1(t))+D(F_2^n(s),F_2(s))+D(F_2^n(t),F_2(t))+D(F_1^n(s),F_1(s)),$$

and the latter expression tends to zero as $n \to \infty$. Now, if $T = \{t_i\}_{i=0}^m$ is a partition of I, by (5.8) we have:

$$\sum_{i=1}^m U_{r_n}^{F_1^n,F_2^n}(t_i, t_{i-1}) \le W_{r_n}(F_1^n, F_2^n), \qquad n \in \mathbb{N}.$$

Taking into account (5.7), (5.8), the continuity of Φ and (5.13) we get:

$$\sum_{i=1}^m U_r^{F_1,F_2}(t_i, t_{i-1}) \le \liminf_{n\to\infty} W_{r_n}(F_1^n, F_2^n), \quad T = \{t_i\}_{i=0}^m \in \mathcal{T}(I),$$

from which (5.12) follows.

To prove (b), let $r_n > \Delta(F_1, F_2) =: r$ ($n \in \mathbb{N}$) be such that $r_n \to r$ as $n \to \infty$. Since, by (5.9), $W_{r_n}(F_1, F_2) \le 1$, inequality (5.12) yields: $W_r(F_1, F_2) \le 1$.

(c) It suffices to show only that if $\Delta(F_1, F_2) < r$, then $W_r(F_1, F_2) < 1$. In fact, if $\Delta(F_1, F_2) = 0$, then $W_r(F_1, F_2) = 0$ by Lemma 5.2(a), (5.7) and (5.8). If $\Delta(F_1, F_2) > 0$, then by the convexity of Φ and (b) we have:

$$W_r(F_1, F_2) \le (\Delta(F_1, F_2)/r) W_{\Delta(F_1,F_2)}(F_1, F_2) \le \Delta(F_1, F_2)/r < 1.$$

(d) is an easy consequence of (c).

(e) We may assume that the values $\alpha = \limsup_{n\to\infty} \Delta(F_1^n, F_2^n)$ and $\alpha_n = \sup_{k\ge n} \Delta(F_1^k, F_2^k)$, $n \in \mathbb{N}$, are finite. Since for any $\varepsilon > 0$ we have $\alpha_n + \varepsilon > \Delta(F_1^n, F_2^n)$, (5.9) implies $W_{\alpha_n+\varepsilon}(F_1^n, F_2^n) \le 1$, and so (5.12) gives: $W_{\alpha+\varepsilon}(F_1, F_2) \le 1$. Hence, $\Delta(F_1, F_2) \le \alpha + \varepsilon$ for all $\varepsilon > 0$. □

LEMMA 5.3. *Let X be a normed linear space and $\Phi \in \mathcal{N}$. Then the mapping D_Φ defined by* (5.7)–(5.10) *is a metric on $BV_\Phi(I; \mathcal{P}_{cc}(X))$.*

Proof. Let $F_j \in BV_\Phi(I;\mathcal{P}_{cc}(X))$, $j = 1,2,3$. If $D_\Phi(F_1,F_2) = 0$, it follows from (5.10) and Lemma 5.2(a) that

$$D(F_1(t),F_2(t)) = D(F_1(a),F_2(a)) = 0, \qquad t \in I, \quad t > a,$$

and hence, $F_1(t) = F_2(t)$ for all $t \in I$.

It is clear that D_Φ is symmetric: $D_\Phi(F_1,F_2) = D_\Phi(F_2,F_1)$.

To prove the triangle inequality for D_Φ, it suffices to show that

$$\Delta(F_1,F_2) \le \Delta(F_1,F_3) + \Delta(F_2,F_3).$$

Note that, thanks to (5.11) and (5.6), the following inequality holds:

$$D(F_1(t)+F_2(s),F_2(t)+F_1(s)) \le D(F_1(t)+F_3(s),F_3(t)+F_1(s))+ \\ +D(F_2(t)+F_3(s),F_3(t)+F_2(s)), \qquad t,\,s \in I. \tag{5.14}$$

Assume that $\Delta(F_1,F_3) = 0$. By Lemma 5.2(a), for all $t,\,s \in I$ we have:

$$D(F_1(t)+F_3(s),F_3(t)+F_1(s)) = 0,$$

and hence, (5.14), (5.7) and (5.8) imply $W_r(F_1,F_2) \le W_r(F_2,F_3)$ for all $r > 0$, so that $\Delta(F_1,F_2) \le \Delta(F_2,F_3)$. An analogous argument applies in the case when $\Delta(F_2,F_3) = 0$, so that $\Delta(F_1,F_2) \le \Delta(F_1,F_3)$.

Now, let $r_1 = \Delta(F_1,F_3) > 0$ and $r_2 = \Delta(F_2,F_3) > 0$. By Lemma 5.2(b) we have: $W_{r_1}(F_1,F_3) \le 1$ and $W_{r_2}(F_2,F_3) \le 1$. Taking into account (5.14), (5.7), (5.8), the monotonicity and convexity of Φ, we get:

$$W_{r_1+r_2}(F_1,F_2) \le \frac{r_1}{r_1+r_2}W_{r_1}(F_1,F_3) + \frac{r_2}{r_1+r_2}W_{r_2}(F_2,F_3) \le 1,$$

which proves that $\Delta(F_1,F_2) \le r_1 + r_2 = \Delta(F_1,F_3) + \Delta(F_2,F_3)$. □

5.3. Multivalued Composition Operators

The aim of this section is to obtain multivalued versions of Theorems 4.1, 4.3 and 4.5 for mappings of bounded generalized variation.

We assume that $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ are normed linear spaces and that $K \subset X$ is a *convex cone*, i.e. $K + K \subset K$ and $rK \subset K$, $r \in \mathbb{R}^+$. A multivalued operator $S : K \to \mathcal{P}_{cc}(Y)$ is said to be *linear* if

$$S(x+x') = S(x) + S(x'), \quad S(rx) = rS(x), \quad \forall\, x,\, x' \in K, \forall\, r \in \mathbb{R}^+.$$

Denote by $L(K;\mathcal{P}_{cc}(Y))$ the set of all linear continuous multivalued operators from K into $\mathcal{P}_{cc}(Y)$. In what follows D designates the Hausdorff metric on $\mathcal{P}_{cc}(Y)$.

THEOREM 5.4. *Suppose that* $\mathcal{H} : K^I \to \mathcal{P}_{cc}(Y)^I$ *is the composition operator generated by a multivalued mapping* $H : I \times K \to \mathcal{P}_{cc}(Y)$ *via*:

$$(\mathcal{H}f)(t) \equiv \mathcal{H}(f)(t) := H(t, f(t)), \qquad t \in I, \quad f \in K^I. \tag{5.15}$$

Let Φ, $\Psi \in \mathcal{N}$. *If* $\mathcal{H}$ *maps* $GV_\Phi(I;K)$ *or* $C^{0,1}(I;K)$ *into* $BV_\Psi(I;\mathcal{P}_{cc}(Y))$ *and is Lipschitzian, then there exists a function* $\mu_0 : I \to \mathbb{R}^+$ *such that*

$$D(H(t,x_1), H(t,x_2)) \le \mu_0(t)\|x_1 - x_2\|, \quad t \in I, \quad x_1,\, x_2 \in K, \tag{5.16}$$

and there exist two multivalued mappings $H_0 \in BV_\Psi(I;\mathcal{P}_{cc}(Y))$ *and* $H_1 \in L(K;\mathcal{P}_{cc}(Y))^I$ *such that* $H(t,x) = H_0(t) + H_1(t)(x)$, $t \in I$, $x \in K$.

Moreover, if $\mathcal{H} : GV_\Phi(I;K) \to BV_\Psi(I;\mathcal{P}_{cc}(Y))$ *is Lipschitzian and* $\Phi \lhd \Psi$, *then* $H(t,x) = H(t,0)$, $(t,x) \in I \times K$ (*and so* $\mathcal{H}$ *is constant*).

Proof. Denote by $\mathcal{F}(I;K)$ either $GV_\Phi(I;K)$ or $C^{0,1}(I;K)$ and the generic norm there—by $\|\cdot\|_{\mathcal{F}}$. Since $\mathcal{H} : \mathcal{F}(I;K) \to BV_\Psi(I;\mathcal{P}_{cc}(Y))$ is Lipschitzian, there exists a constant $\mu > 0$ such that

$$D_\Psi(\mathcal{H}f_1, \mathcal{H}f_2) \le \mu\|f_1 - f_2\|_{\mathcal{F}}, \qquad f_1,\, f_2 \in \mathcal{F}(I;K),$$

so that the definition of D_Ψ (see (5.9) and (5.10)) implies, in particular,

$$\Delta_\Psi(\mathcal{H}f_1, \mathcal{H}f_2) \le \mu\|f_1 - f_2\|_{\mathcal{F}}.$$

If $\|f_1 - f_2\|_{\mathcal{F}} > 0$, by Lemma 5.2(c) this inequality is equivalent to

$$W_{\mu\|f_1 - f_2\|_{\mathcal{F}}}(\mathcal{H}f_1, \mathcal{H}f_2) \le 1,$$

and so definitions (5.7) and (5.8) yield: if α, $\beta \in I$ and $\alpha < \beta$, then

$$\Psi\Big(\frac{D((\mathcal{H}f_1)(\beta) + (\mathcal{H}f_2)(\alpha), (\mathcal{H}f_2)(\beta) + (\mathcal{H}f_1)(\alpha))}{(\beta - \alpha)\,\mu\,\|f_1 - f_2\|_{\mathcal{F}}}\Big)(\beta - \alpha) \le 1.$$

Applying Ψ^{-1} and taking into account (5.15) we arrive at the inequality which is the counterpart of (4.4):

$$D\Big(H(\beta, f_1(\beta)) + H(\alpha, f_2(\alpha)), H(\beta, f_2(\beta)) + H(\alpha, f_1(\alpha))\Big) \le \\ \le \mu\|f_1 - f_2\|_{\mathcal{F}}(\beta - \alpha)\Psi^{-1}(1/(\beta - \alpha)). \tag{5.17}$$

To prove (5.16), we follow the arguments in step 2 of the proof of Theorem 4.1 with $x_j \in K$ in (4.5) and (4.9), use (5.17) instead of (4.4) and take into account the translation property (5.6) of D. Consequently, setting $\omega_\Phi(r) = r\Phi^{-1}(1/r)$ and $\omega_{0,1}(r) = r$, $\Phi \in \mathcal{N}$, $r > 0$, we obtain inequality (5.16) with

$$\mu_0(t) = \begin{cases} \mu\,\omega_\Psi(b-a)(1+1/\omega_{\mathcal{F}}(b-a)) & \text{if} \quad t = a, \\ \mu\,\omega_\Psi(t-a)/\omega_{\mathcal{F}}(t-a) & \text{if} \quad a < t \le b, \end{cases}$$

where $\omega_{\mathcal{F}} = \omega_\Phi$ if $\mathcal{F} = GV_\Phi$ and $\omega_{\mathcal{F}} = \omega_{0,1}$ if $\mathcal{F} = C^{0,1}$.

For $a \le \alpha < \beta \le b$ and $x_1, x_2 \in K$ define Lipschitzian mappings $f_j : I \to K$ by

$$f_j(t) = \frac{1}{2}\Big(\eta_{\alpha,\beta}(t)(x_1 - x_2) + x_j + x_2\Big), \qquad t \in I, \quad j = 1, 2,$$

where $\eta_{\alpha,\beta}$ is defined in (4.3). Substituting them into (5.17) we have:

$$D\bigg(H(\beta, x_1) + H(\alpha, x_2), H\Big(\beta, \frac{x_1+x_2}{2}\Big) + H\Big(\alpha, \frac{x_1+x_2}{2}\Big)\bigg) \le \\ \le \mu\,\|(x_1 - x_2)/2\|\,\omega_\Psi(\beta - \alpha). \tag{5.18}$$

Since $\mathcal{H}$ maps $\mathcal{F}(I;K)$ into $BV_\Psi(I;\mathcal{P}_{cc}(Y))$ and constant mappings belong to $\mathcal{F}(I;K)$, $H(\,\cdot\,,x) = \mathcal{H}(x) \in BV_\Psi(I;\mathcal{P}_{cc}(Y))$ for all $x \in K$, so that, by Lemma 2.1(d), $H(\,\cdot\,,x)$ is absolutely continuous. Letting $\beta - \alpha \to 0$ in (5.18) in such a way that $[\alpha,\beta] \ni t$, where $t \in I$, we get, by virtue of (4.13),

$$D\bigg(H(t, x_1) + H(t, x_2), H\Big(t, \frac{x_1+x_2}{2}\Big) + H\Big(t, \frac{x_1+x_2}{2}\Big)\bigg) = 0,$$

and so, since D is a metric and H takes convex values,

$$H(t,x_1) + H(t,x_2) = H\Big(t, \frac{x_1+x_2}{2}\Big) + H\Big(t, \frac{x_1+x_2}{2}\Big) = 2H\Big(t, \frac{x_1+x_2}{2}\Big).$$

Hence, for every $t \in I$ the multivalued operator $H(t,\,\cdot\,) : K \to \mathcal{P}_{cc}(Y)$ satisfies the Jensen equation:

$$\frac{1}{2}\Big(H(t,x_1) + H(t,x_2)\Big) = H\Big(t, \frac{x_1+x_2}{2}\Big), \qquad x_1, x_2 \in K.$$

It follows (cf. [40, Thm. 5.6]) that for every $t \in I$ there exists a set $H_0(t) \in \mathcal{P}_{cc}(Y)$ and an additive multivalued operator $H_1(t)(\cdot) : K \to \mathcal{P}_{cc}(Y)$, i.e. $H_1(t)(x_1 + x_2) = H_1(t)(x_1) + H_1(t)(x_2)$ for all $x_1, x_2 \in K$, such that

$$H(t,x) = H_0(t) + H_1(t)(x), \qquad x \in K. \tag{5.19}$$

In view of (5.16), the operator $H_1(t)(\cdot)$ is continuous, and since it is additive, it is also linear (cf. [40, Thm. 5.3]), i.e. H_1 maps I into the space $L(K;\mathcal{P}_{cc}(Y))$. Hence, $H_1(t)(0) = \{0\}$ for all $t \in I$ and, therefore, (5.19) implies $H(t,0) = H_0(t)$ for all $t \in I$. Consequently, $H_0 \in BV_\Psi(I;\mathcal{P}_{cc}(Y))$.

If $\Phi \lhd \Psi$, note that the counterpart of (4.7) is the inequality:

$$D(H(\beta,x_1),H(\beta,x_2)) \le \mu\|x_1-x_2\|\omega_\Psi(\beta-\alpha)/\omega_\Phi(\beta-\alpha), \quad (5.20)$$

which holds for all $a \le \alpha < \beta \le b$ and $x_1,\ x_2 \in K$. Setting $\beta = t$ for $a < t \le b$, $x_1 = x \in K$ and $x_2 = 0$ and passing to the limit as $\alpha \to t-0$ in (5.20) and applying Lemma 4.2 we get: $D(H(t,x),H(t,0)) = 0$. Hence, $H(t,x) = H(t,0)$ for all $a < t \le b$ and $x \in K$, and it suffices to take into account the continuity of $H(\,\cdot\,,x)$. □

Remark 5.1. Observe that a theorem similar to Theorem 4.3 is valid for multivalued composition operators. We omit the details.

Remark 5.2. If K is a linear subspace of X, then, due to its additivity, the operator $H_1(t)(\cdot)$ in (5.19) is single-valued for all $t \in I$: in fact, if $x \in K$, then $(-x) \in K$, and so $H_1(t)(x) + H_1(t)(-x) = \{0\}$.

Finally, we extend Theorem 4.5 onto multivalued composition operators. Let Y be a Banach space. The set $BV_1(I;\mathcal{P}_{cc}(Y))$ is a metric space with the metric D_1 defined by (5.7)–(5.10) with $\Phi(\rho) = \rho$. Suppose that a multivalued mapping $H : I \times K \to \mathcal{P}_{cc}(Y)$ is such that $H(\,\cdot\,,x) \in BV_1(I;\mathcal{P}_{cc}(Y))$ for all $x \in K$. Since Y is complete, $(\mathcal{P}_{cc}(Y),D)$ is a *complete* metric space (cf. [3, Thm. II-14]), so that any mapping from $BV_1(I;\mathcal{P}_{cc}(Y))$ has one-sided limits at each point of I. The left regularization $H^* : I \times K \to \mathcal{P}_{cc}(Y)$ of H is defined by

$$H^*(t,x) = \lim_{s\to t-0} H(s,x),\ a < t \le b, \quad H^*(a,x) = \lim_{t\to a+0} H^*(t,x),$$

for all $x \in K$, where the limits are taken with respect to the Hausdorff metric D on $\mathcal{P}_{cc}(Y)$. Then $H^*(\,\cdot\,,x) \in BV_1(I;\mathcal{P}_{cc}(Y))$ is left continuous on $(a,b]$ for all $x \in K$.

THEOREM 5.5. *Let X be a normed linear space, $K \subset X$ a convex cone, Y a Banach space and $\Phi \in \mathcal{N}$. Suppose that $\mathcal{H} : K^I \to \mathcal{P}_{cc}(Y)^I$ is the composition operator generated by a multivalued mapping $H : I\times K \to \mathcal{P}_{cc}(Y)$ according to* (5.15). *If $\mathcal{H}$ maps $GV_\Phi(I;K)$ or $C^{0,1}(I;K)$ into $(BV_1(I;\mathcal{P}_{cc}(Y)),D_1)$ and is Lipschitzian, then there exists a function $\mu_0 : I \to \mathbb{R}^+$ such that*

$$D(H^*(t,x_1),H^*(t,x_2)) \le \mu_0(t)\|x_1-x_2\|, \quad t \in I, \quad x_1,\ x_2 \in K,$$

and there exist a multivalued mapping $H_0 \in BV_1(I;\mathcal{P}_{cc}(Y))$, left continuous on $(a,b]$, and a mapping $H_1 : I \to L(K;\mathcal{P}_{cc}(Y))$ such that

$$H^*(t,x) = H_0(t) + H_1(t)(x), \qquad t \in (a,b], \quad x \in K.$$

The proof of this theorem can be easily compiled from the proofs of Theorems 4.5 and 5.4 with $\Psi(\rho) = \rho$, and so we omit it. The only new ingredient consists of replacing mappings f_j from (4.26) by Lipschitzian mappings $f_j : I \to K$ defined by

$$f_j(t) = \frac{1}{2}\Big(\eta_n(t)(x_1 - x_2) + x_j + x_2\Big), \quad t \in I,\ x_j \in K,\ j = 1, 2,$$

and taking into account the (continuity) equality similar to (5.13) while passing to the limit as $\alpha_1 \to t-0$ (cf. step 2 in the proof of Theorem 4.5).

Acknowledgements

This work was done in part when the author was a visiting scientist at the Moscow State (Lomonosov) University, Moscow, Russia, September-December 1998. The author is grateful to M. G. Eroshenkov, O. G. Smolyanov, P. L. Ul'yanov and S. A. Vakhrameev for their kind hospitality and support. The author is also indebted to the unknown reviewer for valuable critical comments which led to complete revision of the initial version of this paper.

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