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Weak* convergence of operator means

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Abstract. For a linear operator U with $\|U^n\| \leqslant \text{const}$ on a Banach space X we discuss conditions for the convergence of ergodic operator nets T_α corresponding to the adjoint operator U^* of U in the W*O-topology of the space $\operatorname{End} X^*$. The accumulation points of all possible nets of this kind form a compact convex set L in $\operatorname{End} X^*$, which is the kernel of the operator semigroup $G = \overline{\operatorname{co}}\,\Gamma_0$, where $\Gamma_0 = \{U_n^*, n \geqslant 0\}$. It is proved that all ergodic nets T_α weakly* converge if and only if the kernel L consists of a single element. In the case of $X = C(\Omega)$ and the shift operator U generated by a continuous transformation φ of a metrizable compactum Ω we trace the relationships among the ergodic properties of U, the structure of the operator semigroups L, G and $\Gamma = \overline{\Gamma}_0$, and the dynamical characteristics of the semi-cascade (φ, Ω). In particular, if $\operatorname{card} L = 1$, then a) for any $\omega \in \Omega$ the closure of the trajectory $\{\varphi^n\omega, n \geqslant 0\}$ contains precisely one minimal set m, and b) the restriction (φ, m) is strictly ergodic. Condition a) implies the W*O-convergence of any ergodic sequence of operators $T_n \in \operatorname{End} X^*$ under the additional assumption that the kernel of the enveloping semigroup $E(\varphi, \Omega)$ contains elements obtained from the 'basis' family of transformations $\{\varphi^n, n \geqslant 0\}$ of the compact set Ω by using some transfinite sequence of sequential passages to the limit.

Keywords: weak* ergodic theory, dynamical system, enveloping semigroup, Choquet representation.

Introduction

Let a linear operator U on a Banach space X satisfy the bounds $\|U^n\| \leqslant c$ for all integers $n \geqslant 0$. The problem concerning the convergence of the Cesàro means

$$U_n = \frac{1}{n}(I + U + \cdots + U^{n-1}), \qquad n \to \infty,$$

or other ergodic (averaging) nets $R_\alpha = R_\alpha(U)$ in appropriate topologies is treated in operator ergodic theorems. As a rule, one considers [1] the convergence of R_α in the strong, weak, or uniform topology on the space of bounded linear operators $\operatorname{End} X$. In the most natural formulation of the problem one considers the strong operator topology (the SO-topology), and the elements of an SO-ergodic net R_α are chosen in the convex hull of the set of operators $\{U^n, n \geqslant 0\}$ and satisfy the condition $R_\alpha(I - U) \xrightarrow{\text{SO}} 0$. By the Sine separation principle ([1], [2]), an arbitrary SO-ergodic operator net R_α converges strongly if and only if the invariant

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vectors of U separate the invariant vectors of the adjoint operator U^*. Various versions of this principle are also used in more general situations ([1], [3]) related to the convergence of operator means.

In the present paper we consider convergence conditions for generalized means $R_\alpha x \to z,\ z \in X^{**}$ for all $x \in X$ in the w^*-topology on the second dual space X^{**} or, equivalently, conditions for the weak* convergence (W*O-convergence) for the operators $T_\alpha = R_\alpha^*$ acting on the first dual space X^*. In particular, instead of the w^*-convergence of the means $U_n x \to z,\ z \in X^{**}$ for $x \in X$, one can speak of the W*O-convergence of the operator sequence

$$V_n = \frac{1}{n}(I + V + \cdots + V^{n-1})$$

in the space $\operatorname{End} X^*$, where $V = U^*$.

Let G_0 be the convex hull of the set of operators $\Gamma_0 = \{V^n, n \geqslant 0\}$ in $\operatorname{End} X^*$. An operator net T_α in G_0 is said to be *ergodic* if $T_\alpha(I - V) \xrightarrow{\mathrm{W^*O}} 0$. In this case $T_\alpha = R_\alpha^*$ for $R_\alpha \in \operatorname{End} X$, and the corresponding operator net R_α is also said to be *ergodic*. As we see, there is convergence with respect to the weak operator topology, $R_\alpha(I - U) \xrightarrow{\mathrm{WO}} 0$. We denote by G and Γ the operator semigroups obtained as W*O-closures of G_0 and Γ_0, respectively. The (norm) bounded subsets of $\operatorname{End} X^*$ are relatively compact in the W*O-topology, and therefore the sets G and Γ are compact. The smallest two-sided ideal (the kernel) L of the semigroup G turns out to be a non-empty convex compact subset of G consisting of all operators $Q \in G$ for which $VQ = Q$. We note that the elements of L are projections acting on the space X^*, and their norms do not exceed c.

The main idea of the present paper is to connect the ergodic properties of linear operators U and V with the algebraic and geometric structures of the semigroups L, G and Γ. Using the separation principle, one can derive from the results of Lloyd [4] that, if some SO-ergodic operator net $R_\alpha \in \operatorname{End} X$ converges strongly, then the kernel L of G consists of a single element. Developing this topic, we show (Theorem 1.3) that all ergodic operator nets $T_\alpha \in G_0$ converge (to the same projection $Q \in L$) in the W*O-topology if and only if $L = \{Q\}$. The last condition is not always satisfied; however, it follows immediately from the definition of the kernel L that every element $Q \in L$ is the limit of some operator ergodic net, that is, for any $U \in \operatorname{End} X$ with $\|U^n\| \leqslant \mathrm{const}$, there are W*O-convergent ergodic operator nets $T_\alpha \in G_0$. This shows a fundamental distinction between the strong and weak* ergodic theories, because, by the separation principle, all SO-ergodic operator nets R_α either converge simultaneously (to the same projection $P \in \operatorname{End} X$) or diverge simultaneously in the SO-topology on $\operatorname{End} X$. It is still not quite clear what the conditions might be under which a given projection $Q \in L$ is the limit of some *ergodic sequence* of operators $T_n \in G_0$ with respect to the W*O-topology.

The Iwanik criterion [5] is known as an analogue of the separation principle for the weak* convergence of the Cesàro means V_n. Useful generalizations of this criterion in the case of an arbitrary ergodic net are obtained in Theorem 1.5.

The weak* convergence of ergodic means is of interest mainly from the point of view of dynamical systems. Indeed, every continuous transformation φ of a metrizable compact set Ω generates a shift operator $U\colon x(\omega) \to x(\varphi\omega)$, $\omega \in \Omega$, on the space of continuous scalar-valued functions $X = C(\Omega)$ and, for example, the weak* convergence of the Cesàro means V_n can be reduced to the pointwise convergence of the scalar means defined by the equation

$$(U_n x)(\omega) = \frac{1}{n}\big(x(\omega) + x(\varphi\omega) + \dots + x(\varphi^{n-1}\omega)\big)$$

for any $x \in X$ and $\omega \in \Omega$. In the Krylov–Bogolyubov terminology ([6], [7]), this means that all points of Ω are quasi-regular with respect to the discrete dynamical system (a semi-cascade) φ^n, $n \geqslant 0$. The main results of the paper are related to this very case, and their statements and proofs substantially use the notion of enveloping semigroup $E(\varphi, \Omega)$ of a semi-cascade (φ, Ω); this enveloping semigroup is the closure of the family of transformations $\Phi_0 = \{\varphi^n, n \geqslant 0\}$ of the compact set Ω in the direct product topology on Ω^Ω [8].

Let $\Lambda(\Omega)$ be the family of φ-ergodic Borel probability measures on Ω. For a measure $\mu \in \Lambda(\Omega)$ we denote by $\Phi_1(\mu)$ the class of transformations $p\colon \Omega \to \Omega$ having the following property: There is a non-descending system of Borel sets $h_k \subset \Omega$ and a double sequence of positive integers $n(k,l)$ such that $\mu(h_k) > 1 - k^{-1}$ and $\lim_{l\to\infty} \varphi^{n(k,l)}\omega = p\omega$ for any $\omega \in h_k$. We continue the procedure of passing from Φ_0 to $\Phi_1(\mu)$ with the aid of transfinite induction and denote by $\Phi(\mu)$ the union of the resulting classes $\Phi_\nu(\mu)$ over all ordinals ν. Let Φ be the intersection of the classes $\Phi(\mu)$ for all $\mu \in \Lambda(\Omega)$. For any $\omega \in \Omega$ we denote by $\overline{o}(\omega)$ the closure of the trajectory $\{\varphi^n\omega, n \geqslant 0\}$ in the topology of Ω.

The main result of the paper (Theorem 3.2) establishes a relationship between the following statements:

A) the kernel $\operatorname{Ker} E(\varphi, \Omega)$ of the enveloping semigroup $E(\varphi, \Omega)$ contains at least one transformation $p\colon \Omega \to \Omega$ belonging to the class Φ;

A_1) the support of any ergodic measure of the semi-cascade (φ, Ω) is a minimal set;

B) for any $\omega \in \Omega$ the set $\overline{o}(\omega)$ contains a unique minimal set m, and precisely one φ-invariant Borel probability measure is concentrated on m;

B_1) for any $\omega \in \Omega$ the set $\overline{o}(\omega)$ contains a unique minimal set;

C) the kernel L of the semigroup G consists of a single element;

D) all ergodic sequences of operators $T_n \in G_0$ converge in the W*O-topology of the space $\operatorname{End} X^*$;

E) all ergodic operator nets $T_\alpha \in G_0$ converge with respect to the W*O-topology of $\operatorname{End} X^*$.

It turns out that the implications A) + B_1) $\Rightarrow$ A_1), A_1) + B) $\Rightarrow$ D), C) $\Rightarrow$ B), and C) $\Leftrightarrow$ E) hold. The equivalence of the conditions C) and E) follows from the general theorem mentioned above, Theorem 1.3. The implication A_1) + B) $\Rightarrow$ D) generalizes the well-known fact [7] that the unique ergodicity of a compact (discrete) dynamical system implies the quasi-regular property of all points of the phase space of the system. It follows from the implication C) $\Rightarrow$ B) that the condition $\operatorname{card} L = 1$ imposes severe restrictions on the dynamical system (φ, Ω).

It is also proved (Theorem 2.4) that condition A) ensures the equation $Z(\Omega) = (M(\Omega))^c$, where $M(\Omega)$ stands for the union of all minimal sets, $Z(\Omega)$ is the minimal attracting centre of the semi-cascade (φ,Ω), and $(\,\cdot\,)^c$ is the closure operation in Ω. We recall that the inclusion $(M(\Omega))^c \subset Z(\Omega)$ may be proper in the general case (see Ch. 5 in [6]).

One of the simplest possible types of behaviour of a dynamical system (φ,Ω) is related to the equality between the set $M(\Omega)$ and the family of fixed points $N(\Omega)=\{\omega\in\Omega\colon \varphi\omega=\omega\}$ for this system. Purely algebraic considerations enable one to prove (Theorem 3.5) the equivalence of the conditions $L\cap\Gamma\neq\varnothing$ and $L\cap\Gamma=\operatorname{Ker}\Gamma$, where $\operatorname{Ker}\Gamma$ stands for the kernel of the semigroup Γ. Here each of these conditions ensures the equality $M(\Omega)=N(\Omega)$.

One of the tools of our investigation in the present paper is the Choquet integral representation of operators $T\in G$ in terms of operators $P\in\Gamma$. In this connection, we describe the structure of the set of extreme points $\operatorname{ex}G$ of the convex compact set $G\subset\operatorname{End}X^*$ for a shift operator $U\colon C(\Omega)\to C(\Omega)$. It is proved that $\operatorname{ex}G=\operatorname{ex}\Gamma$. It follows from the definition of the operator semigroups G and Γ that $\operatorname{ex}G\subset\Gamma$. At the same time, by Proposition 4.4, the equality $\operatorname{ex}G=\Gamma$ turns out to be equivalent to the condition that the semigroup epimorphism $\theta\colon\Gamma\to E(\varphi,\Omega)$ whose action is defined by $P\delta(\omega)=\delta(p\omega)$ for $p=\theta P$ and $\omega\in\Omega$ is injective.

§ 1. Linear operator on an arbitrary space

In this section, we consider ergodic properties of an arbitrary linear operator U on a Banach space X. Our only assumption is that the bounds $\|U^n\|\leqslant c$ hold uniformly with respect to $n\geqslant 0$. We denote by $\operatorname{Im}(\,\cdot\,)$ and $F(\,\cdot\,)$ the image and the subspace of fixed vectors of a linear operator, and by $\operatorname{End}X$ and $\operatorname{End}X^*$ the algebras of bounded operators acting on X and on the first dual space X^*, respectively. Let $V=U^*$ be the adjoint operator on X^*; then $\|V^n\|\leqslant c$ for $n\geqslant 0$. Let us equip the algebra of operators $\operatorname{End}X^*$ with the locally convex weak* operator topology (W*O-topology) and single out the multiplicative semigroups $\Gamma_0=\{V^n, n\geqslant 0\}$ and $G_0=\operatorname{co}\Gamma_0$ in $\operatorname{End}X^*$.

We write $G=\overline{\operatorname{co}}\,\Gamma_0$ and $\Gamma=\overline{\Gamma}_0$, where the bar above stands for the W*O-closure of sets in the normed space $\operatorname{End}X^*$. Then $\Gamma\subset G$ and $\|T\|\leqslant c$ for $T\in G$. The norm bounded subsets of $\operatorname{End}X^*$ are relatively compact in the W*O-topology, and therefore the sets G and Γ are Hausdorff (non-metrizable in general), separable, and compact. The product TT_1 in $\operatorname{End}X^*$ is continuous with respect to T for any T_1; however, it is continuous with respect to T_1 only if $T=R^*$, $R\in\operatorname{End}X$. For this reason, the semigroup G and its subsemigroup Γ are non-commutative, although their elements commute with the operator V.

We note that $\operatorname{Im}T\supset F(T)\supset F(V)$ for every $T\in G$. Following [4], we consider the non-empty family

$$L=\{Q\in G\colon VQ=Q\}$$

of fixed points of the continuous map $T\to VT$ of the convex compact set G into itself. The operator V is continuous in the w^*-topology of the space X^*, and therefore the set L is closed. If $Q\in L$, then $V^nQ=Q$ for $n\geqslant 1$, and hence $TQ=Q$ for every $T\in G$; in particular, $Q^2=Q$. The convex compact set L consists of projections $Q\in\operatorname{End}X^*$, $\|Q\|\leqslant c$. One can give an alternative definition: $L=\{T\in G\colon \operatorname{Im}T=T(V)\}$. This implies that $L=\{0\}$ if $F(V)=\{0\}$.

We note moreover that $L \cap G_0 = \varnothing$ if linear independence of the system of operators $\{U^n, n \geqslant 0\}$ on $\operatorname{End} X$ is assumed.

To describe the algebraic structure of L, and also to prove some assertions in § 3, we need some fundamentals of the general theory of semigroups (see, for example, [9] and [10]). Namely, if a semigroup S has a minimal left ideal, then it also has a kernel $\operatorname{Ker} S$, which is the non-empty intersection of all two-sided ideals. Here $\operatorname{Ker} S$ coincides with the union of all minimal left ideals, and every left ideal contains a minimal one. If a left ideal $I \subset S$ is minimal and $u \in S$, then the left ideal Iu is also minimal, and all minimal left ideals of S can be obtained in this way. Everything presented here holds for right ideals, subject to straightforward modifications.

Lemma 1.1. *The set L is the kernel of G consisting of the one-element left ideals. Moreover, L is a unique minimal right ideal of the semigroup G.*

Proof. It is clear that L is a right ideal of G. If $Q \in L$, then $TQ = Q$ for every $T \in G$. Thus, $\{Q\}$ is a one-element minimal left ideal of G, and therefore L is a two-sided ideal. The semigroup G has a kernel $\operatorname{Ker} G$, which is the union of all minimal left ideals, and this kernel contains the ideal L, and thus coincides with L. Further, $QQ_1 = Q_1$ for any projections $Q, Q_1 \in L$, and hence the right ideal L is principal and generated by any of its elements. Thus, L is a minimal right ideal. Since $\operatorname{Ker} G$ is the union of all minimal right ideals, it follows that L is a unique ideal of this kind in the semigroup G, which completes the proof of the lemma.

We note that every one-sided ideal of the semigroup G contains a minimal one, and hence every right ideal contains the kernel L, whereas every left ideal has a non-empty intersection with L.

For an arbitrary operator net T_α in G we denote by $\operatorname{cl}(T_\alpha)$ the set of generalized limit points (accumulation points) of the net, which is non-empty because G is compact. A net of this kind is said to be *ergodic* if $T_\alpha \in G_0$ for any α and

$$T_\alpha(I - V) \xrightarrow{\mathrm{W^*O}} 0. \tag{1.1}$$

Since $T_\alpha \in G_0$, it follows that $T_\alpha \in R_\alpha^*$, where $R_\alpha \in \operatorname{End} X$. In this case the operator net R_α is also said to be ergodic. The sequence of Cesàro means V_n is ergodic, because $V_n(I - V) = n^{-1}(I - V^n)$ and $\|V_n(I - V)\| \leqslant (1 + c)n^{-1}$. If an ergodic net T_α converges to the operator Q in the W*O-topology, then $Q \in L$, and it follows from the w^*-convergence $T_\alpha y \to y_0$ for $y \in X^*$ that $Vy_0 = y_0$.

We claim that the condition $\operatorname{card} L = 1$ is equivalent to the simultaneous convergence of all ergodic nets of operators $T_\alpha \in G_0$ in the W*O-topology of the space $\operatorname{End} X^*$.

Lemma 1.2. *For an arbitrary net of operators $T_\alpha \in G$ the inclusion $\operatorname{cl}(T_\alpha) \subset L$ is equivalent to* (1.1).

Proof. If $\operatorname{cl}(T_\alpha) \subset L$ and (1.1) fails to hold, then for some neighbourhood D of zero in $\operatorname{End} X^*$ and for any index α there is an index $\beta = \beta(\alpha)$, $\beta \geqslant \alpha$, such that $T_\beta(I - V) \notin D$. Let $Q \in \operatorname{cl}(T_\beta)$; then $Q \in \operatorname{cl}(T_\alpha) \subset L$ and $Q(I - V) \notin D$. Thus, $Q(I - V) \neq 0$, which contradicts the relation $Q \in L$.

Conversely, $TV = VT$ for $T \in G$, and the function $T \to T(I-V)$ is a continuous map from the compact set G to $\operatorname{End} X^*$, and therefore $\operatorname{cl}(T_\alpha(I-V)) \supset (I-V)\operatorname{cl}(T_\alpha)$. If condition (1.1) holds, then $\operatorname{cl}(T_\alpha(I-V)) = \{0\}$, and thus $(I-V)\operatorname{cl}(T_\alpha) = \{0\}$, $\operatorname{cl}(T_\alpha) \subset L$, which completes the proof of the lemma.

Theorem 1.3. *All ergodic nets of operators $T_\alpha \in G_0$ converge in the* W*O*-topology if and only if the kernel L of the semigroup G consists of a unique element Q. In this case,* $\lim_\alpha T_\alpha = Q$.

Proof. If the kernel L is a one-element set, then, by Lemma 1.2, an arbitrary ergodic net of operators $T_\alpha \in G_0$ has a unique accumulation point, namely, the point $Q \in G$, and hence $T_\alpha \to Q$.

Suppose now that the kernel L contains two different elements, Q and Q'. Since $L \subset \overline{G}_0$, it follows that $T_\alpha \to Q$ and $T'_\alpha \to Q'$, where T_α and T'_α are ergodic nets in G_0 defined by the system of W*O-neighbourhoods of zero in the space $\operatorname{End} X^*$. In more detail, let $\alpha = (B, B^*, k)$, where k are positive integers and B and B^* are arbitrary finite sets in X and X^*, respectively. We introduce a partial order on the set of indices α, namely, $\alpha_1 \geqslant \alpha$ if $B_1 \supset B$, $B_1^* \supset B^*$, and $k_1 \geqslant k$. For any value of α we choose operators T_α and T'_α in G_0 in such a way that

$$\big|(x, (T_\alpha - Q)y)\big| < k^{-1}, \qquad \big|(x, (T'_\alpha - Q')y)\big| < k^{-1}$$

for the elements $x \in B$ and $y \in B^*$. We write $T''_\alpha = T_\alpha$ or $T''_\alpha = T'_\alpha$, depending on whether k is even or odd. As we can see, the operator net T''_α is ergodic, but has no limit in the W*O-topology. This completes the proof of the theorem.

Let us now discuss necessary and sufficient conditions for the convergence of an ergodic net $T_\alpha = R^*_\alpha$ in the W*O-topology on the operator space $\operatorname{End} X^*$ that generalize the well-known Iwanik criterion [5] for the Cesàro means. Let $\operatorname{End}(X, X^{**})$ be the Banach space of bounded linear operators acting from X into X^{**}. We denote the w^*-topologies in X^* and X^{**} by τ and σ, respectively. It follows from the identity $(R_\alpha x, y) = (x, T_\alpha y)$ for $x \in X$ and $y \in X^*$ that the weak* convergence $T_\alpha \to T$ in $\operatorname{End} X^*$ is equivalent to the σ-convergence $R_\alpha x \to Rx$ on X, where $R \in \operatorname{End}(X, X^{**})$. It is useful to note that the relationships $(Rx, y) = (x, Ty)$, $R = T^*\big|_X$, and $T = R^*\big|_{X^*}$ define a natural linear isometry of the spaces $\operatorname{End}(X, X^{**})$ and $\operatorname{End} X^*$.

Let $(\,\cdot\,)^\sigma$ denote the closure of sets in the σ-topology. We write

$$Y = \{y \in X^* \colon T_\alpha y \xrightarrow{\tau} 0\}, \qquad X_0 = \{x \in X \colon R_\alpha x \xrightarrow{\sigma} z, z \in X^{**}\}.$$

Then $z = R_0 x$, where $R_0 \in \operatorname{End}(X_0, X^{**})$. Moreover, we write $X_1 = \operatorname{Im} R_0$ and note that $Y \cap F(V) = \{0\}$. Since $R_\alpha x = x$ and $R_0 x = x$ for $x \in F(U)$, we have $F(U) \subset X_1$. For a set $(\,\cdot\,)$ in X^{**} or X^* we denote by $(\,\cdot\,)^\perp$ its annihilator (orthogonal complement) in X^* or X^{**}, respectively. It follows from the ergodic property (1.1) that $T_\alpha y \xrightarrow{\tau} 0$ on $\operatorname{Im}(I-V) \subset X^*$ and, since $(R_\alpha x, y) = (x, T_\alpha y)$ for every $x \in X$ and $y \subset X^*$, we also have $R_\alpha x \xrightarrow{\sigma} 0$ on $\operatorname{Im}(I-U) \subset X$. Therefore, $X_0 \supset \operatorname{Im}(I-U)$, and thus $X_0^\perp \subset \operatorname{Im}(I-U)^\perp = F(V)$.

Lemma 1.4. *For every ergodic net of operators* $T_\alpha \in G_0$
a) $X_1^\sigma \subset Y^\perp \subset F(V^*)$ *and* $X_1^\perp \supset Y$;
b) $X_0^\perp = X_1^\perp \cap F(V)$.

Proof. We note that $Y \supset \operatorname{Im}(I-V)$ and $Y^\perp \subset \operatorname{Im}(I-V)^\perp = F(V^*)$, where $V^* = U^{**}$. Let $x \in X_0,\ y \in X^*$ and $z = R_0 x$. Then $z \in X_1$ and

$$(z,y) = \lim_\alpha (R_\alpha x, y) = \lim_\alpha (x, T_\alpha y). \tag{1.2}$$

As we can see, $(z,y) = 0$ for every $y \in Y$, and therefore $X_1 \subset Y^\perp$ and $X_1^\sigma \subset Y^\perp$. The relations $(X_1^\sigma)^\perp = X_1^\perp$ and $Y^{\perp\perp} = Y$ imply the embedding $X_1^\perp \supset Y$. Hence, part a) of the lemma is established. Furthermore, the equalities $T_\alpha y = y$ hold for the functionals $y \in F(V)$. Hence we find that $(R_\alpha x, y) = (R_0 x, y) = (x,y)$ for any $x \in X_0$. This means that the conditions $y \in X_0^\perp$ and $y \in X_1^\perp$ are equivalent for any such y. However, $X_0^\perp \subset F(V)$, and therefore $X_0^\perp = X_1^\perp \cap F(V)$. This proves the lemma.

The condition that an ergodic net T_α is W*O-convergent is equivalent to either of the following two relations: $X_0 = X$ or $X^* = F(V) \oplus Y$. We say that X_1 *separates* $F(V)$ if for any distinct vectors $y_1, y_2 \in F(V)$ there is a vector $x \in X_1$ such that $(x, y_1) \neq (x, y_2)$.

Theorem 1.5. *The following three conditions are pairwise equivalent for any ergodic operator net* $T_\alpha \in G_0$:
a) X_1 *separates* $F(V)$;
b) $X_0 = X$;
c) $X_1^\perp = Y$.

Proof. The subspace X_1 separates $F(V)$ if and only if $X_1^\perp \cap F(V) = \{0\}$. By Lemma 1.4 b), it follows from condition a) that $X_0^\perp = \{0\}$, that is, $X_0 = X$. Suppose now that $X_0 = X$. If $x \in X_0,\ z = R_0 x$ and $y \in X_1^\perp$, then $(z,y) = 0$, and it follows from (1.2) that $T_\alpha y \xrightarrow{\tau} \overline{y}$ and the vector $\overline{y}$ is orthogonal to X. Thus, $\overline{y} = 0$ for any $y \in X_1^\perp$, or, in other words, $X_1^\perp \subset Y$. The converse embedding always holds by Lemma 1.4, a), and the implication b) $\Rightarrow$ c) is established. Finally, we obtain the implication c) $\Rightarrow$ a), because $Y \cap F(V) = \{0\}$. This completes the proof of the theorem.

In the case of Cesàro means V_n, the equivalence of conditions a) and b) of Theorem 1.5 was established in [5]. We note that condition c) of Theorem 1.5 is equivalent to the relation $X_1^\sigma = Y^\perp$.

Remark 1.6. The subspaces X_0 and Y and the linear subspace X_1 depend on the choice of the ergodic operator net T_α. If all nets of this kind converge, then, by Theorem 1.3, the kernel L consists of the single element Q, and the subspace Y coinciding with the kernel of the projection Q does not depend on the choice of the net T_α. Here the linear subspaces $X_1 \subset X^{**}$ can be different for different nets T_α, although they have the same σ-closures, because, in this case, $X_1^\sigma = Y^\perp$.

A continuous action $P \to VP$ generates a semi-cascade (V, Γ) on the W*O-compact set $\Gamma \subset \operatorname{End} X^*$. Let us use the following notation: $A(\Gamma)$ is the compact (in the w^*-topology of the space $C^*(\Gamma)$) convex set of Borel probability measures on Γ, $\mathrm{Ai}(\Gamma)$ is the closed convex subset of V-invariant measures in $A(\Gamma)$, and $\Lambda(\Gamma)$ is the subset of ergodic measures in $\mathrm{Ai}(\Gamma)$. By one of the equivalent definitions in Ch. 10 of [11], the set of ergodic measures coincides with the family of extreme points $\operatorname{ex}\mathrm{Ai}(\Gamma)$.

Since $G = \overline{\mathrm{co}}\,\Gamma_0$ and $\Gamma = \overline{\Gamma}_0$, it follows that $\operatorname{ex} G \subset \Gamma$ by Ch. 1 in [11] and, since the difference $G \setminus \Gamma$ is a Baire set, it follows that (see Ch. 4 of [11]) corresponding to every operator $T \subset G$ there is a measure $\lambda \in A(\Gamma)$ realizing the Choquet representation

$$T = \int_\Gamma P\lambda(dP). \tag{1.3}$$

Integration must be treated here in the following sense:

$$(x, Ty) = \int_\Gamma (x, Py)\lambda(dP)$$

for any $x \in X$ and $y \in X^*$. A measure λ is said to be *representing*, and the notation $T = T_\lambda$ will sometimes be used. Formula (1.3) defines a continuous surjection $A(\Gamma) \to G$; however, a representing measure need not be unique.

Lemma 1.7. *If* $\lambda \in \mathrm{Ai}(\Gamma)$, *then* $T_\lambda \in L$.

Proof. It follows from the V-invariance of the measure λ that

$$VT_\lambda = \int_\Gamma VP\lambda(dP) = \int_\Gamma P\lambda(dP) = T_\lambda, \qquad T_\lambda \in L.$$

§ 2. A shift on $C(\Omega)$: preliminaries

From now on throughout the paper, we consider ergodic properties of the shift operator U on the space of continuous scalar-valued functions $X = C(\Omega)$ on a metrizable compactum Ω. Thus, $(Ux)(\omega) = x(\varphi\omega)$ for $x \in C(\Omega)$ and $\omega \in \Omega$, where φ is a continuous (not necessarily invertible) transformation of Ω. We denote by $A(\Omega)$ and $K(\Omega)$ the sets of Borel probability measures and Dirac measures on Ω, respectively. Let $\Lambda(\Omega)$ be the class of ergodic measures for the semi-cascade (φ, Ω), that is, the φ-invariant measures $\lambda \in A(\Omega)$ for which $\lambda(h)$ is equal to 0 or 1 for any Borel set $h \subset \Omega$ with the property $\varphi^{-1}h = h$. Moreover, we denote by $M(\Omega)$ and $Z(\Omega)$ the union of all minimal sets and minimal attracting centres of the semi-cascade (φ, Ω), respectively. We recall that $Z(\Omega)$ coincides with the closure of the union of the supports of all ergodic measures. The set $A(\Omega)$ is convex, compact, and metrizable with respect to the w^*-topology of the dual space X^*, whereas $K(\Omega)$ is a closed subset of $A(\Omega)$. As is well known, $(M(\Omega))^c \subset Z(\Omega)$, where $(\,\cdot\,)^c$ stands for the closure operation in Ω, and this embedding may be proper.

Let Σ_μ be the sigma-algebra of sets $h \subset \Omega$ that are measurable with respect to the Lebesgue extension of a Borel measure $\mu \in A(\Omega)$. Then Σ_μ contains the sigma-algebra Σ_b of Borel subsets of Ω. A map $p\colon \Omega \to \Omega$ is said to be *μ-measurable* if $p^{-1}\Sigma_b \subset \Sigma_\mu$. We denote the family of such maps by Π_μ and write

$$\Pi_A = \bigcap_{\mu \in A(\Omega)} \Pi_\mu.$$

The class Π_A consists of all universally measurable transformations of the compactum Ω.

An important role in the forthcoming considerations is played by the notion of *enveloping semigroup* of a dynamical system. The enveloping semigroup $E(\varphi, \Omega)$ of a semi-cascade (φ, Ω) is the closure of the family $\Phi_0 = \{\varphi^n, n \geqslant 0\}$ in the topology t of pointwise convergence in the space Π of all possible maps $p\colon \Omega \to \Omega$ [8]. This topology is separable and coincides with the direct product topology of Ω^Ω. By the Tikhonov theorem, Π is compact, and hence the set $E(\varphi, \Omega)$ is also compact (but non-metrizable in general). Generally speaking, the semigroup $E(\varphi, \Omega)$ is non-commutative, and its centre contains the subsemigroup Φ_0. Every enveloping semigroup is separable (as a topological space) and has non-empty kernel.

The adjoint operator $V = U^*$ acting on the space of measures $X^* = C^*(\Omega)$ preserves the cone of positive Borel measures on Ω, and thus every operator $T \in G$ has the same property (as a W*O-limit of convex combinations of the powers V^n). If $\mu \in A(\Omega)$, then $(1, V^n\mu) = (U^n 1, \mu) = (1, \mu) = 1$ for $n \geqslant 0$, and hence $(1, T\mu) = 1$ and $T\colon A(\Omega) \to A(\Omega)$ for every $T \in G$. In particular, $P\colon A(\Omega) \to A(\Omega)$ for $P \in \Gamma$.

Thus, V generates a semi-cascade (V, A) on the compactum $A = A(\Omega)$, and the operators $P \in \Gamma$ act on $A(\Omega)$ as elements of the enveloping semigroup $E(V, A)$ of this semi-cascade. This semigroup is the closure of the sequence of continuous transformations $\{V^n, n \geqslant 0\}$ of the compactum $A(\Omega)$ in the direct product topology of A^A. Moreover, the topology of w^*-convergence on the measures $\mu \in A(\Omega)$ coincides for the operators in $\operatorname{End} X^*$ with the (formally stronger) W*O-topology. Thus, the operator semigroup Γ can be identified with the enveloping semigroup $E(V, A)$.

Further, $V\delta(\omega) = \delta(\varphi\omega)$ for the points $\omega \subset \Omega$, and the set of Dirac measures $K = K(\Omega)$ is closed in $A(\Omega)$. Therefore, $VK \subset K$ and $PK \subset K$ for all operators $P \in \Gamma$. The correspondence $V \to \varphi$ generates a continuous algebraic homomorphism $\theta\colon P \to p$ of the semigroup Γ into the semigroup $E(\varphi, \Omega)$ whose action is defined by

$$P\delta(\omega) = \delta(p\omega), \qquad \omega \in \Omega. \tag{2.1}$$

Since Γ is compact, $\varphi^n = \theta(V^n)$, and the family of transformations $\Phi_0 = \{\varphi^n, n \geqslant 0\}$ is dense in $E(\varphi, \Omega)$, it follows that $\theta(\Gamma) = E(\varphi, \Omega)$, and θ is an epimorphism.

Let us try to characterize (at least partially) the class of maps $p \in \Pi$ that satisfy the properties which hold for the 'basis' transformations $\varphi^n \in \Phi_0$:

i) $p \in \Pi_A$;

ii) $\mu(p^{-1}h) = \mu(h)$ for $\mu \in \Lambda(\Omega)$ and $h \in \Sigma_b$.

For a measure $\mu \in A(\Omega)$ and a set $B \subset \Pi$ we define a class $\mathcal{F}_\mu(B)$ of maps p $\Omega \to \Omega$ as follows: $p \in \mathcal{F}_\mu(B)$ if there are a non-decreasing (depending on B) countable system of sets $h_k \in \Sigma_b$ and a double sequence of elements $p_{kl} \in B$ such that $\mu(h_k) > 1 - k^{-1}$ and $\lim_{l\to\infty} p_{kl}\omega = p\omega$ for any $\omega \in h_k$. Here $\mu(H) = 1$, where $H = \bigcup_{k=1}^\infty h_k$, and $B \subset \mathcal{F}_\mu(B)$. We denote the class of scalar-valued Borel (or μ-measurable) functions on a metrizable compact set Ω by X_b (or by X_μ, respectively).

Lemma 2.1. *If $\mu \in A(\Omega)$, then the operation $B \to \mathcal{F}_\mu(B)$ preserves the property $B \subset \Pi_\mu$. If $\mu \in \Lambda(\Omega)$ and $\mu(q^{-1}h) = \mu(h)$ for $q \in B$ and $h \in \Sigma_b$, then the elements $p \in \mathcal{F}_\mu(B)$ have the same property.*

Proof. Let us show first that the transformations $p \in \mathcal{F}_\mu(B)$ are measurable with respect to μ if this holds for every $q \in B$. Let $x \in C(\Omega)$ and $g(\omega) = x(p\omega)$, and let $\{h_k\}$ be an increasing system of sets corresponding to p. We denote by $\chi_k(\omega)$ and $\overline{\chi}_k(\omega)$ the indicators of the sets h_k and H. If $p_{kl} \in B$ and $x_{kl}(\omega) = x(p_{kl}\omega)$, then $x_{kl} \in X_\mu$ and $x_{kl}(\omega) \xrightarrow{l} g(\omega)$ on h_k, that is, $x_{kl}(\omega)\chi_k(\omega) \xrightarrow{l} g(\omega)\chi_k(\omega)$ on Ω. Thus, $g\chi_k \in X_\mu$ and, since $g(\omega)\chi_k(\omega) \xrightarrow{k} g(\omega)\overline{\chi}(\omega)$ for any $\omega \in H$, it follows that $g\overline{\chi} \in X_\mu$ as well. However, $\mu(H) = \mu(\Omega)$, and hence $g(\omega)\overline{\chi}(\omega) = g(\omega)$ μ-almost everywhere.

Thus, $x(p\omega) \in X_\mu$ for any continuous function $x(\omega)$. By the theory of Baire classes (see [12], § 39), the functions $\psi \in X_b$ can be obtained from continuous ones by a transfinite sequence of sequential pointwise passages to the limit. Thus, all functions of the form $\psi(p\omega)$, where $\psi \in X_b$, turn out to be μ-measurable. In particular, if $\chi(\omega)$ is the indicator of an arbitrary Borel set $h \subset \Omega$, then $\chi(p\omega) \in X_\mu$ and $p^{-1}h \in \Sigma_\mu$. This implies that $p \in \Pi_\mu$.

Further, let $\mu \in \Lambda(\Omega)$, $h \in \Sigma_b$ and $p \in \mathcal{F}_\mu(B)$. Let also $f(\omega) = \chi(p\omega)$, where $\chi(\omega)$ stands for the indicator of the set h. By the assumption of the lemma, $\mu(p_{kl}^{-1}h) = \mu(h)$ for $p_{kl} \in B$, or, in other words, $(\chi, \mu) = (\chi_{kl}, \mu)$ if we set in addition $\chi_{kl}(\omega) = \chi(p_{kl}\omega)$. Since $\mu(h_k) > 1 - k^{-1}$, it follows that

$$\left|(\chi, \mu) - \int_{h_k} \chi_{kl}(\omega)\mu(d\omega)\right| < \frac{1}{k}\,.$$

Since $\chi_{kl}(\omega) \to f(\omega)$ on h_k as $l \to \infty$, we obtain the following bound by Lebesgue's theorem:

$$\left|(\chi, \mu) - \int_{h_k} f(\omega)\mu(d\omega)\right| \leqslant \frac{1}{k}\,.$$

Passing here to the limit as $k \to \infty$, we arrive at the equation

$$\int_\Omega \chi(\omega)\mu(d\omega) = \int_H f(\omega)\mu(d\omega).$$

Since $\mu(H) = \mu(\Omega)$, it follows that $(\chi, \mu) = (f, \mu)$ or $\mu(h) = \mu(p^{-1}h)$. This completes the proof of the lemma.

Suppose now that $\mu \in \Lambda(\Omega)$ and $\Phi_1(\mu) = \mathcal{F}_\mu(\Phi_0)$. Let us construct a non-decreasing transfinite sequence of classes of transformations $\Phi_\nu(\mu)$ by setting $\Phi_\nu(\mu) = \bigcup_{\eta<\nu} \Phi_\eta(\mu)$ or $\Phi_\nu(\mu) = \mathcal{F}_\mu(\Phi_{\nu-1}(\mu))$, depending on whether the number ν is a limit or non-limit ordinal. We denote the union of $\Phi_\nu(\mu)$ over all ordinals ν by $\Phi(\mu)$. By Lemma 2.1, the elements $p \in \Phi_\nu(\mu)$ inherit the properties of the elements $q \in \Phi_{\nu-1}(\mu)$ defined by $q \subset \Pi_\mu$ and $\mu(q^{-1}h) = \mu(h)$ for $h \in \Sigma_b$. Hence, using transfinite induction, we see that all $p \in \Phi(\mu)$ have the same properties.

We write

$$\Phi = \bigcap_{\mu \in \Lambda(\Omega)} \Phi(\mu);$$

then the following proposition is in fact proved.

Proposition 2.2. *The transformations $p \in \Phi$ satisfy conditions* i), ii).

In the general case, Φ is not a part of $E(\varphi,\Omega)$. On the other hand, Φ contains the sequential closure $E_0(\varphi,\Omega)$ of the family of transformations Φ_0 with respect to the t-topology of Π. In other words, $E_0(\varphi,\Omega)$ can be obtained from the family Φ_0 by a transfinite sequence of sequential passages to the t-limit. Thus, $E_0(\varphi,\Omega)\subset E(\varphi,\Omega)$, and $E_0(\varphi,\Omega)$ consists of Borel transformations of the compactum Ω. The condition that some transformation $p\in\operatorname{Ker}E(\varphi,\Omega)$, where $\operatorname{Ker}E(\varphi,\Omega)$ stands for the kernel of $E(\varphi,\Omega)$, belongs to the class Φ turns out to impose certain restrictions on the original dynamical system. In this connection, we note an elementary general property of the enveloping semigroups in a formulation useful for subsequent references. Here and below, we denote by $M(A)$ the union of the minimal sets of the semi-cascade (V,A) on the compactum $A=A(\Omega)$.

Lemma 2.3. *If* $p\in\operatorname{Ker}E(\varphi,\Omega)$, *then* $p\Omega\subset M(\Omega)$. *But if* $P\in\operatorname{Ker}\Gamma$, *then* $PA\subset M(A)$.

Proof. As is well known (see Proposition 3.5 in [8]), every element $p\in\operatorname{Ker}E(\varphi,\Omega)$ can be represented in the form $p=\pi p$ for some idempotent $\pi\in\operatorname{Ker}E(\varphi,\Omega)$. The embedding $\pi\Omega\subset M(\Omega)$, and thus also the embedding $p\Omega\subset M(\Omega)$, follows from Proposition 3.7 in [8]. The other part of the lemma is similar to the first one, because $\Gamma\simeq E(V,A)$.

We note that the book [8] deals with invertible dynamical systems. Nevertheless, it can readily be verified directly that the results from [8] cited here and below remain valid in the non-invertible case.

Theorem 2.4. *If the kernel* $\operatorname{Ker}E(\varphi,\Omega)$ *contains an element* $p\in\Phi$, *then* $Z(\Omega)=(M(\Omega))^c$.

Proof. Since $p\in\operatorname{Ker}E(\varphi,\Omega)$, we see that Lemma 2.3 ensures the inclusion $p\Omega\subset M(\Omega)$. The closed set $M^c=(M(\Omega))^c$ belongs to the sigma-algebra Σ_b, and

$$\mu(M^c)=\mu(p^{-1}M^c)=\mu(\Omega)=1$$

for any measure $\mu\in\Lambda(\Omega)$ by Lemma 2.1. Thus, $\mu(M^c)=1$ and $\operatorname{supp}\mu\subset M^c$. Since

$$Z(\Omega)=\biggl(\bigcup_{\mu\in\Lambda(\Omega)}\operatorname{supp}\mu\biggr)^c,$$

it follows that $Z(\Omega)\subset(M(\Omega))^c$. The converse inclusion is always true (see, for example, Ch. 5, § 7 in [6]). This completes the proof of the theorem.

If the embedding $(M(\Omega))^c\subset Z(\Omega)$ is proper, then $\operatorname{Ker}E(\varphi,\Omega)\cap\Phi=\varnothing$.

§ 3. Shift on $C(\Omega)$: main results

As above, suppose that $X=C(\Omega)$ and U is a shift operator on X generated by a continuous transformation φ of the compactum Ω.

For a point $\omega\in\Omega$ we denote by $\overline{o}(\omega)$ the closure in Ω of the trajectory $\{\varphi^n\omega, n\geqslant 0\}$ and note that this set is semi-invariant, that is, $\varphi\overline{o}(\omega)\subset\overline{o}(\omega)$. As is well known, every closed semi-invariant set of a compact dynamical system contains a minimal set m with the same properties. Here $\varphi m = m$ and every trajectory is dense in m. We note that $p\overline{o}(\omega) \subset \overline{o}(\omega)$ for every transformation $p \in E(\varphi, \Omega)$.

Let us establish a relationship between the minimal sets of the semi-cascade (φ, Ω) and the minimal left ideals of the enveloping semigroup $E = E(\varphi, \Omega)$. By Ch. 2 in [8], every ideal of this kind is a minimal set of the discrete dynamical system (φ, Ω) generated by a continuous transformation $p \to \varphi p$ of the compactum $E(\varphi, \Omega)$, and therefore this set is closed in the Tikhonov topology of $E(\varphi, \Omega)$.

Lemma 3.1. *For any $\omega \in \Omega$ and for an arbitrary minimal left ideal $I \subset E(\varphi, \Omega)$ the set $I\omega \subset \overline{o}(\omega)$ is minimal. Conversely, for any minimal set $m \subset \overline{o}(\omega)$ there is a minimal left ideal $I \subset E(\varphi, \Omega)$ for which $I\omega = m$.*

Proof. The left ideal I is minimal, and $\varphi p = p\varphi$ for every $p \in E(\varphi, \Omega)$. Therefore, $I\varphi = \varphi I = I$. For a fixed $\omega \in \Omega$ the function $\gamma\colon p \to p\omega$ acts continuously from $E(\varphi, \Omega)$ to Ω, and thus the set $I\omega$ is closed and $\varphi I\omega = I\omega$, so $I\omega$ contains a minimal set m of the semi-cascade (φ, Ω). The function $\gamma = \gamma_\omega$ maps the closed ideal I onto $I\omega$, and thus the subset $\gamma^{-1}m \cap I$ of the semigroup $E(\varphi, \Omega)$ is closed and semi-invariant with respect to the semi-cascade (φ, E). But then we have $\gamma^{-1}m = I$ and $m = I\omega$.

Suppose now that $m \subset \overline{o}(\omega)$ and $\xi \in m$; then $\varphi^{n(k)}\omega \to \xi$ for some sequence $n(k)$, $k \geqslant 1$, of positive integers. The sequence of transformations $\varphi^{n(k)}$ has an accumulation point q in the compact Hausdorff space $E(\varphi, \Omega)$. Here we have $q\omega = \xi$, and

$$I_{\omega,m} = \{p \in E(\varphi, \Omega) \colon p\omega \in m\}$$

is a non-empty (because it contains q) left ideal of $E(\varphi, \Omega)$. This ideal necessarily contains some minimal left ideal I, and $I\omega \subset m$. Since $\varphi I\omega = I\omega$ and the set m is minimal, it follows that $I\omega = m$, and this completes the proof of the lemma.

As was already noted in § 2, the operator semigroup Γ can be interpreted as the enveloping semigroup $E(V, A)$ of the semi-cascade (V, A) on the metrizable compactum $A = A(\Omega)$. According to the general principles [8], the minimal left ideals $J \subset \Gamma$ are precisely the minimal sets of the dynamical system (V, Γ) on the compact set Γ. We note that the continuous algebraic epimorphism $\Gamma \xrightarrow{\theta} E(\varphi, \Omega)$ acting according to (2.1) preserves the classes of left, right, and two-sided ideals of the semigroups Γ and $E(\varphi, \Omega)$. The same holds for the operation of passing to the full pre-image under θ^{-1}.

Let us now state the main result of the present paper. We note that the support $\operatorname{supp} \mu$ of an ergodic measure $\mu \in \Lambda(\Omega)$ is a closed semi-invariant set. Consider the following statements:

A) the kernel $\operatorname{Ker} E(\varphi, \Omega)$ of the enveloping semigroup $E(\varphi, \Omega)$ has non-empty intersection with the class of transformations $\Phi \subset \Pi$;

A_1) the support of every ergodic measure $\mu \in \Lambda(\Omega)$ is a minimal set;

B) for any $\omega \in \Omega$ the closure of the trajectory $\overline{o}(\omega)$ contains a unique minimal set m, and precisely one φ-invariant probability measure is concentrated on m;

B_1) for any $\omega \in \Omega$ the closure of the trajectory $\overline{o}(\omega)$ contains a unique minimal set;

C) the kernel L of the semigroup G consists of a single element;

D) all ergodic operator sequences $T_n \in G_0$ converge in the W*O-topology of $\operatorname{End} X^*$;

E) all ergodic operator nets $T_\alpha \in G_0$ converge with respect to the W*O-topology of $\operatorname{End} X^*$.

It is clear that B) ⇒ B$_1$) and E) ⇒ D). In the case under consideration with $X = C(\Omega)$, the W*O-convergence of the ergodic operator net $T_\alpha \in R_\alpha^*$ in $\operatorname{End} X^*$ implies the convergence of the scalar nets $(R_\alpha x)(\omega)$ for any $x \in X$ and $\omega \in \Omega$. The converse assertion follows for ergodic sequences T_n from Lebesgue's theorem on passing to the limit under the integral sign, whereas it fails to hold in the general case.

Theorem 3.2. *The following implications hold*: A) + B$_1$) ⇒ A$_1$), A$_1$) + B) ⇒ D), C) ⇒ B), *and* C) ⇔ E).

The combined conditions A$_1$) + B) mean that for any $\omega \in \Omega$ the dynamical system $(\varphi, \overline{o}(\omega))$ is uniquely ergodic, that is, has a unique invariant Borel probability measure. The equivalence of conditions C) and E) is a special case of Theorem 1.3. The implication A$_1$) + B) ⇒ D) slightly generalizes the well-known statement (see Theorem 5.2 in [7]) according to which the unique ergodicity of a semi-cascade $(\varphi, \overline{o}(\omega))$ implies the quasi-regularity of all points of the set $\overline{o}(\omega)$.

Proof of Theorem 3.2. *The implication* A) + B$_1$) ⇒ A$_1$). The support $s_\mu = \operatorname{supp}\mu$ of any ergodic measure $\mu \in \Lambda(\Omega)$ is topologically transitive (see § 4.1 of [13]), that is, $\overline{o}(\omega) = s_\mu$ for some $\omega \in s_\mu$. In view of assumption B$_1$), this implies that s_μ contains a unique minimal set m. Since $\theta\Gamma = E(\varphi, \Omega)$, where θ is the epimorphism (2.1), for any $p \in \operatorname{Ker} E(\varphi, \Omega) \cap \Phi$ there is an element $P \in \theta^{-1}p$ in the operator semigroup Γ. By Lemma 2.3, we have $ps_\mu \subset s_\mu \cap M(\Omega)$; however, $s_\mu \cap M(\Omega) = m$, and therefore $ps_\mu \subset m$ and $p^{-1}m \supset s_\mu$. Since $p \in \Phi$, Proposition 2.2 ensures that

$$p^{-1}m \subset \Sigma_\mu, \qquad \mu(m) = \mu(p^{-1}m) = \mu(s_\mu)$$

for the closed set $m \subset s_\mu$. It follows now from the definition of the support of a measure that $m = \operatorname{supp}\mu$, as was to be proved.

The implication A$_1$) + B) ⇒ D). Let R_n be an arbitrary ergodic operator sequence in $\operatorname{End} X$ and let $\omega \in \Omega$. By the assumption of the theorem, the dynamical system $(\varphi, \overline{o}(\omega))$ has a unique invariant (probability) measure μ. Since the constant functions are φ-invariant on the compactum $\overline{o}(\omega)$ and $(1, \mu) \neq (2, \mu)$, it follows from Theorem 1.5 that for every continuous function $x \in X$ the scalar sequence $(R_n x)(\xi)$ converges for any $\xi \in \overline{o}(\omega)$. This implies that the function sequence $R_n x$ converges pointwise on Ω, and hence the ergodic operator sequence $T_n = R_n^*$ converges in the W*O-topology of the operator space $\operatorname{End} X^*$. Thus, every ergodic operator sequence $T_n \in G_0$ converges weakly*, which completes the proof of the implication in question.

The implication C) ⇒ B). By Theorem 1.3, under condition C), the Cesàro means $U_n x$ converge pointwise for an arbitrary continuous function $x \in X$, and therefore (see Theorem 5.4 in [7]) precisely one invariant Borel probability measure is concentrated on every minimal set $m \subset \Omega$. Let us now prove that the closure of every trajectory of the dynamical system (φ, Ω) contains a unique minimal set. If $\omega \in \Omega$

and $m \subset \bar{o}(\omega)$, then, by Lemma 3.1, the enveloping semigroup $E(\varphi, \Omega)$ contains a minimal left ideal I such that $I\omega = m$. Suppose that there is an ω for which $\bar{o}(\omega)$ contains two distinct minimal sets, m_1 and m_2. Then $I_k\omega = m_k$ for $k = 1, 2$ and for some minimal left ideals $I_k \subset E(\varphi, \Omega)$. If $\theta\colon \Gamma \to E(\varphi, \Omega)$ is the canonical semigroup epimorphism of $\Gamma \simeq E(V, A)$ onto $E(\varphi, \Omega)$, then the full pre-images $\theta^{-1}I_1$ and $\theta^{-1}I_2$ are left ideals in Γ, which necessarily contain some minimal left ideals J_1 and J_2. Here the left ideals θJ_k are contained in I_k, and thus $\theta J_k = I_k$, $k = 1, 2$. Since the ideals J_k are minimal sets of the semi-cascade (V, Γ), there are ergodic measures $\lambda_k \in \Lambda(\Gamma)$ concentrated on J_k. We write

$$Q_k = \int_\Gamma P\lambda_k(dP);$$

then, in view of the equalities $P\delta(\omega) = \delta(p\omega)$ and $p = \theta P$, we have

$$Q_k\delta(\omega) = \int_\Gamma \delta(p\omega)\lambda_k(dP).$$

Since the measures λ_k are invariant with respect to (V, Γ), it follows from Lemma 1.7 that $Q_k \in L$. Since $\operatorname{supp}\lambda_k = J_k$ and $\theta J_k\omega = m_k$, we have $p\omega \in m_k$ for any operator $P \subset J_k$ and $p = \theta P$. The integral in the Choquet representation (1.3) is treated in the sense of W*O-convergence, and therefore the measures $Q_k\delta(\omega)$ are concentrated on the sets m_k. Thus, $Q_1 \neq Q_2$, and the contradiction thus obtained accomplishes the proof of the implication C) ⇒ B). This completes the proof of the theorem.

In fact, the proof of the implication A) + B_1) ⇒ A_1) in Theorem 3.2 uses only the property ii) of the transformation $p \in \operatorname{Ker} E(\varphi, \Omega)$, because the μ-measurability of the set $p^{-1}m$ follows simply from the relations $p^{-1}m \supset s_\mu$ and $\mu(s_\mu) = 1$. The same remark also holds in connection with the proof of Theorem 2.4. Both statements remain valid under the assumption that $\operatorname{Ker} E(\varphi, \Omega)$ contains a transformation $p\colon \Omega \to \Omega$ with the following property: for any ergodic measure $\mu \in \Lambda(\Omega)$ the equality $\mu(p^{-1}h) = \mu(h)$ holds when the set h is closed and $p^{-1}h \subset \Sigma_\mu$.

When applying the theorem to the W*O-convergence of Cesàro means V_n, we obtain the following corollary.

Corollary 3.3. *If* $\operatorname{Ker} E(\varphi, \Omega) \cap \Phi \neq \varnothing$, *then condition* B) *implies the quasi-regularity of all points* $\omega \in \Omega$.

Indeed, Theorem 3.2 yields the implication A) + B) ⇒ D), that is, in the case in question, all ergodic operator sequences T_n in $\operatorname{End} X^*$, including the means V_n, weakly* converge.

Oxtoby's result cited above (Theorem 5.4 in [7]) can be stated in the following sharpened form: *if some ergodic operator sequence* $T_n \in G_0$ *converges with respect to the* W*O-*topology, then precisely one invariant Borel probability measure is concentrated on any given minimal set* $m \subset \Omega$.

This implies the following unexpected conclusion: if more than one ergodic measure is concentrated on some minimal set of the semi-cascade (φ, Ω), then all ergodic operator sequences $T_n \in G_0$ diverge in the W*O-topology, although there are certainly convergent ergodic operator nets $T_\alpha \in G_0$. In other words, in this case, the kernel L of the semigroup G has no common elements with the family of sequential weak* limits of elements of the convex set $G_0 \subset \operatorname{End} X^*$ of operators.

Let us determine conditions ensuring that the kernel L of the semigroup G has non-empty intersection with the subsemigroup Γ. We note that Γ contains the identity element, namely, the identity operator on X^*. We also recall that the kernel of every enveloping semigroup is the non-empty union of all minimal left ideals of the semigroup.

Lemma 3.4. *The function* $\theta\colon \Gamma \to E(\varphi,\Omega)$ *maps* $\operatorname{Ker}\Gamma$ *onto* $\operatorname{Ker} E(\varphi,\Omega)$.

Proof. The epimorphism θ (as well as the passage to the full pre-image, θ^{-1}) preserves the classes of one-sided ideals in the semigroups Γ and $E = E(\varphi,\Omega)$. Since $\theta(\operatorname{Ker}\Gamma)$ is a two-sided ideal, it follows that $\theta(\operatorname{Ker}\Gamma) \supset \operatorname{Ker} E$. Let us establish the opposite inclusion. Let $p \in \theta(\operatorname{Ker}\Gamma)$; then $p = \theta P$ for some element $P \in \operatorname{Ker}\Gamma$. If I is a minimal left ideal in $E(\varphi,\Omega)$ and $J = \theta^{-1}I$, then J is a left ideal in Γ. Since $\theta J = I$ and $\theta P = p$, it follows that $\theta JP = Ip$ and $JP \subset \theta^{-1}(Ip)$. Further, $P \in \operatorname{Ker}\Gamma$ and Γ is a semigroup with unity, and hence the principal ideal ΓP is minimal and contains P. It follows from $JP \subset \Gamma P$ that $JP = \Gamma P$ and $P \in JP$. Thus, $P \subset \theta^{-1}(Ip)$, and therefore $p \in Ip$. The minimal left ideal Ip is contained in the kernel of the semigroup $E(\varphi,\Omega)$, and thus $p \in \operatorname{Ker} E$, which completes the proof.

Let $N(\Omega)$ and $N(A)$ be the sets of fixed points of the semi-cascades (φ,Ω) and (V,A) on the compacta Ω and $A = A(\Omega)$, respectively. As above, we denote by $M(\Omega)$ and $M(A)$ the unions of all minimal sets of the corresponding dynamical systems.

Theorem 3.5. *The following three conditions are pairwise equivalent*:

a) $L \cap \Gamma \neq \varnothing$;

b) $L \cap \Gamma = \operatorname{Ker}\Gamma$;

c) $M(A) = N(A)$.

Moreover, each of the conditions a)–c) *implies that* $M(\Omega) = N(\Omega)$.

Proof. The implication a) $\Rightarrow$ b). Since $L = \operatorname{Ker} G$ and Γ is a subsemigroup of G, it follows from condition a) that $L \cap \Gamma$ is a non-empty two-sided ideal of the semigroup Γ. By Lemma 1.1, L consists of the one-element minimal left ideals of G. Therefore, $L \cap \Gamma \subset \operatorname{Ker}\Gamma$, and thus $L \cap \Gamma = \operatorname{Ker}\Gamma$.

The implication b) $\Rightarrow$ c). It is clear that $M(A) \supset N(A)$. Since $\Gamma \simeq E(V,A)$, it follows from [8], p. 20, that for an arbitrary element (a measure) $\mu \in M(A)$ there is an operator $Q \in \operatorname{Ker}\Gamma$ for which $\mu \in QA$. Under assumption b) we have $Q \in L$ and $VQ = Q$, and hence $V\mu = \mu$ for any $\mu \in M(A)$, and $M(A) = N(A)$.

The implication c) $\Rightarrow$ a). By Lemma 2.3, for any element $Q \in \operatorname{Ker}\Gamma$ we have $QA \subset M(A)$. Thus, assumption c) leads to the equality $VQ = Q$, and so to the embedding $L \cap \Gamma \supset \operatorname{Ker}\Gamma$.

Finally, using the above result from [8] again, we see that for every point $\omega \in M(\Omega)$ there is an element $\pi \in \operatorname{Ker} E(\varphi,\Omega)$ such that $\omega \in \pi\Omega$. By Lemma 3.4 we have $\pi = \theta Q$, where Q is an operator in $\operatorname{Ker}\Gamma$. If $L \cap \Gamma = \operatorname{Ker}\Gamma$, then $VQ = Q$. Since

θ is a semigroup homomorphism and $\theta V = \varphi$, it follows that $\theta VQ = \theta Q = \varphi\theta Q$, and thus $\varphi\pi = \pi$. If $\omega = \pi\xi$ for some $\xi \in \Omega$, then $\varphi\omega = \omega$ and $\omega \in N(\Omega)$. As we can see, condition b) implies that $M(\Omega) \subset N(\Omega)$. The converse embedding is trivial, and thus the proof of the theorem is complete.

§ 4. Shift on $C(\Omega)$: the structure of ex G

Let us proceed to the discussion of the structure of the set $\operatorname{ex} G$ of extreme points of the convex set $G \subset \operatorname{End} X^*$ in the above situation in which $X = C(\Omega)$ and the shift operator $U \in \operatorname{End} X$ corresponds to a continuous transformation φ of the compactum Ω.

The relation $\operatorname{ex} G \subset \Gamma$ was already noted above. Moreover, we claim that the injectivity of the epimorphism $\theta\colon \Gamma \to E(\varphi, \Omega)$ defined by (2.1) is equivalent to the equality $\operatorname{ex} G = \Gamma$. We recall that $K = K(\Omega)$ denotes the family of Dirac measures on Ω, and $A(\Omega)$ the compact convex set of all Borel probability measures on Ω. As is well known (Ch. 1 in [11]), $K(\Omega) = \operatorname{ex} A(\Omega)$.

Lemma 4.1. *The conditions $T \in \Gamma$ and $TK \subset K$ are equivalent for any $T \in G$.*

Proof. The semigroup Γ is the closure of the countable operator family $\{V^n, n \geqslant 0\}$ with respect to the W*O-topology of $\operatorname{End} X^*$. The set $K(\Omega)$ of point measures is closed in $A(\Omega)$, and $V^n\delta(\omega) = \delta(\varphi^n\omega)$ for any $\omega \in \Omega$. Therefore, $TK \subset K$ for $T \in \Gamma$.

Suppose now that $T \in G$ and $TK \subset K$. Let $\lambda \in A(\Gamma)$ be some representing measure for the operator T in the integral formula (1.3); in this case,

$$T\delta(\omega) = \int_\Gamma \delta(p\omega)\lambda(dP) = \delta(\xi)$$

for a given point $\omega \in \Omega$, where $p = \theta P$ and $\xi \in \Omega$. We write

$$\mathrm{H}(\omega, T) = \{P \in \Gamma\colon P\delta(\omega) = T\delta(\omega)\},$$

or, in another notation, $\mathrm{H}(\omega, T) = \{P \in \Gamma\colon p\omega = \xi\}$. As we can see, $\mathrm{H}(\omega, T)$ is a closed convex subset of the compactum $\Gamma \subset G$. We claim that $\operatorname{supp}\lambda \subset \mathrm{H}(\omega, T)$. The Borel measure $\lambda \in A(\Gamma)$ is regular, and thus, if $\operatorname{supp}\lambda \not\subset \mathrm{H}(\omega, T)$, then there is a closed set $s \subset \operatorname{supp}\lambda$ disjoint from $\mathrm{H}(\omega, T)$ and such that $\lambda(s) > 0$. Taking a function $g \in C(\Omega)$ such that $0 \leqslant g \leqslant 1$, $g(\xi) = 0$ and $g = 1$ on a closed set $(\theta s)\omega$ (not containing the point ξ), we arrive at the mutually excluding relationships $(g, T\delta(\omega)) = 0$ and $(g, T\delta(\omega)) \geqslant \lambda(s)$. The point $\omega \in \Omega$ has been chosen in an arbitrary way, and therefore $\operatorname{supp}\lambda \subset \mathrm{H}(T)$ if we set

$$\mathrm{H}(T) = \bigcap_{\omega\in\Omega} \mathrm{H}(\omega, T)$$

or, in another notation,

$$\mathrm{H}(T) = \bigl\{P \in \Gamma\colon P\big|_K = T\big|_K\bigr\}.$$

The operator set $\mathrm{H}(T) \subset \Gamma$ is closed and convex and, in fact,

$$T = \int_{\mathrm{H}(T)} P\lambda(dP). \tag{4.1}$$

This implies that $T \in \mathrm{H}(T)$, and therefore $T \in \Gamma$, which completes the proof of the lemma.

Thus, the integral representation (1.3) for the operators $T \in \Gamma$ can be represented in the form (4.1). If $\mathrm{H}(T) = \{T\}$, then we definitely have $\lambda = \delta(T)$.

We note (by the example of subsets of G) a general property of the extreme points: if $B \subset G$, then $\operatorname{ex} B \supset B \cap \operatorname{ex} G$.

Lemma 4.2. *The conditions* $P \in \operatorname{ex} G$ *and* $P \in \operatorname{ex} \mathrm{H}(P)$ *are equivalent.*

Proof. For an operator $P \in \operatorname{ex} G$ we can see that $P \in \operatorname{ex} \mathrm{H}(P)$ in view of the relations $P \in \Gamma$, $P \in \mathrm{H}(P)$, and $\operatorname{ex} \mathrm{H}(P) \supset \mathrm{H}(P) \cap \operatorname{ex} G$. Let us establish the converse implication. Let $P \in \operatorname{ex} \mathrm{H}(P)$ and $p = \theta P$. If $2P = T_1 + T_2$, where $T_i \in G$ for $i = 1, 2$, then $2\delta(p\omega) = T_1\delta(\omega) + T_2\delta(\omega)$ for every $\omega \in \Omega$. In view of the equality $\operatorname{ex} A(\Omega) = K(\Omega)$, this implies that $T_i\delta(\omega) = \delta(p\omega)$. It follows from Lemma 4.1 that $T_i \in \Gamma$ and, since $T_i\big|_K = P\big|_K$, we have $T_i \in \mathrm{H}(P)$. This gives $T_i = P$ and $P \in \operatorname{ex} G$ and completes the proof of the lemma.

It follows from the inclusions $\Gamma \subset G$ and $\operatorname{ex} G \subset \Gamma$ that $\operatorname{ex} \Gamma \supset \Gamma \cap \operatorname{ex} G$ and $\operatorname{ex} \Gamma \supset \operatorname{ex} G$. On the other hand, for an arbitrary element $P \in \Gamma$ we have $P \in \mathrm{H}(P) \subset \Gamma$ and $\operatorname{ex} \mathrm{H}(P) \supset \mathrm{H}(P) \cap \operatorname{ex} \Gamma$. Thus, if $P \in \operatorname{ex} \Gamma$, then $P \in \operatorname{ex} \mathrm{H}(P)$. By Lemma 4.2, we have $\operatorname{ex} \mathrm{H}(P) \subset \operatorname{ex} G$, and therefore $P \in \operatorname{ex} G$. Thus, $\operatorname{ex} \Gamma \subset \operatorname{ex} G$, and the following statement is valid.

Corollary 4.3. $\operatorname{ex} G = \operatorname{ex} \Gamma$.

Lemma 4.2 enables us to describe the extreme points of G as follows:

$$\operatorname{ex} G = \bigcup_{P \in \Gamma} \operatorname{ex} \mathrm{H}(P). \tag{4.2}$$

We can now formulate the main assertion of the present section.

Proposition 4.4. *The equality* $\operatorname{ex} G = \Gamma$ *is equivalent to the injectivity of the epimorphism* $\theta\colon \Gamma \to E(\varphi, \Omega)$.

Proof. Let $P \in \Gamma$. It follows from the injectivity of θ that $\mathrm{H}(P) = \{P\}$, and $P \in \operatorname{ex} G$ by (4.2). Thus, $\Gamma \subset \operatorname{ex} G$ and, since the converse inclusion is always true, it follows that $\operatorname{ex} G = \Gamma$. Conversely, if we originally have $\operatorname{ex} G = \Gamma$, then every element $P \in \Gamma$ is an extreme point of the convex set $\mathrm{H}(P) \subset \Gamma$. Therefore, $\mathrm{H}(P)$ consists entirely of extreme points, and thus $\mathrm{H}(P) =\{\mathrm{P}\}$, and θ is injective. This completes the proof of the proposition.

§ 5. Some appendices

An important role in the above considerations was played by the condition

$$\operatorname{Ker} E(\varphi, \Omega) \cap \Phi \neq \varnothing, \tag{5.1}$$

which means that the kernel of the enveloping semigroup $E(\varphi, \Omega)$ contains at least one 'sufficiently regular' map $p\colon \Omega \to \Omega$. It is desirable to reformulate this condition in more constructive terms, even at the expense of making it stronger. From this point of view, several properties of enveloping semigroups of dynamical systems that are described in the survey [14] are of interest. Consider two conditions:

a) the compact space $E(\varphi, \Omega)$ is metrizable;

b) every closed φ-invariant set $\Theta \subset \Omega$ contains a trajectory which is Lyapunov stable with respect to the semi-cascade (φ, Θ).

As above, we consider one-sided trajectories of discrete dynamical systems. Slightly modifying the terminology used by the author, one can formulate the following statement, using the results of [14].

Lemma 5.1. *Conditions* a) *and* b) *are equivalent, and each of them implies that the compactum* $E(\varphi, \Omega)$ *coincides with the set of all possible sequential pointwise limits of elements of the basis family of transformations* $\Phi_0 = \{\varphi^n, n \geqslant 0\}$.

Since the class of transformations Φ contains the sequential closure of the family Φ_0, each of the conditions a) or b) ensures (even excessively) the validity of (5.1). Moreover ([14], [15]), these conditions also ensure that the epimorphism $\theta\colon \Gamma \to E(\varphi, \Omega)$ given by the rule (2.1) is injective, and thus, by Proposition 4.4, ensure also that $\operatorname{ex} G = \Gamma$. We note that it was in [15] that the operator semigroup $\Gamma \subset \operatorname{End}(C(\Omega))^*$ and its relationship with the enveloping semigroup $E(\varphi, \Omega)$ were considered for the first time. There is a known example ([14], p. 2356) of a minimal distal cascade on the two-dimensional torus for which θ is not injective.

The implication A) + B_1) ⇒ A_1) of Theorem 3.2, together with Lemma 5.1, leads to the following result.

Theorem 5.2. *Suppose that condition* b) *holds and the closure of every trajectory of the semi-cascade* (φ, Ω) *contains precisely one minimal set. Then all* φ*-ergodic measures* $\mu \in \Lambda(\Omega)$ *are concentrated on minimal sets.*

In [14] and in the papers cited therein one can find some other conditions equivalent to assumption b), which characterizes in a certain sense the 'non-chaotic' dynamical systems.

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