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INSTANTON MODULI SPACES AND $\mathscr{W}$-ALGEBRAS

Alexander BRAVERMAN, Michael FINKELBERG & Hiraku NAKAJIMA

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INSTANTON MODULI SPACES AND $\mathscr{W}$-ALGEBRAS

by Alexander BRAVERMAN, Michael FINKELBERG & Hiraku NAKAJIMA

Abstract. —We describe the (equivariant) intersection cohomology of certain moduli spaces ("framed Uhlenbeck spaces") together with some structures on them (such as e.g., the Poincaré pairing) in terms of representation theory of some vertex operator algebras ("$\mathscr{W}$-algebras").

Résumé **(Sur la catégorie dérivée des 1-motifs.)** — Nous décrivons la cohomologie d'intersection (équivariante) de certains espaces de modules ("espaces d'Uhlenbeck encadrés") ainsi que quelques structures sur eux (comme par exemple l'accouplement de dualité de Poincaré) en termes de théorie des représentation de certaines algèbres vertex ("$\mathscr{W}$-algèbres").

CONTENTS

CHAPTER 1

INTRODUCTION

The main purpose of this paper is to describe the (equivariant) intersection cohomology of certain moduli spaces ("framed Uhlenbeck spaces") together with some structures on them (such as e.g., the Poincaré pairing) in terms of representation theory of some vertex operator algebras ("$\mathscr{W}$-algebras"). In this introduction we first briefly introduce the relevant geometric and algebraic objects (cf. Subsections 1.1 and 1.3) and then state our main result (in a somewhat weak form) in Subsection 1.4 (a more precise version is discussed in 1.9). In Subsection 1.5 we discuss the motivation for our results and relate them to some previous works. In §1.8 we mention earlier works from which we obtain strategy and techniques of the proof.

1.1. Uhlenbeck spaces

Let G be an almost simple simply-connected algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Let also $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$.

Let Bun_G^d be the moduli space of algebraic G-bundles over the projective plane $\mathbb{P}^2$ (over $\mathbb{C}$) with the instanton number d and with trivialization at the line at infinity ℓ_∞. It is a non-empty smooth quasi-affine algebraic variety of dimension $2dh^\vee$ for $d \in \mathbb{Z}_{\geq 0}$, where $h^\vee$ is the dual Coxeter number of G.

By results of Donaldson **[24]** (when G is classical) and Bando **[5]** (when G is arbitrary) Bun_G^d is homeomorphic to the moduli space of anti-self-dual connections (instantons) on S^4 modulo gauge transformations γ with $\gamma(\infty) = 1$ where the structure group is the maximal compact subgroup of G. We will use an algebro-geometric framework, as we can use various tools.

It is well-known that Bun_G^d has a natural partial compactification $\mathscr{U}_G^d$, called the Uhlenbeck space. Set-theoretically, $\mathscr{U}_G^d$ can be described as follows:

$$\mathscr{U}_G^d = \bigsqcup_{0 \leq d' \leq d} \mathrm{Bun}_G^{d'} \times S^{d-d'}(\mathbb{A}^2),$$

where $S^{d-d'}(\mathbb{A}^2)$ denotes the corresponding symmetric power of the affine plane $\mathbb{A}^2$.

The variety $\mathcal{U}_G^d$ is affine and it is always singular unless $d = 0$. It has a natural action of the group $G \times GL(2)$, where G acts by changing the trivialization at ℓ_∞ and $GL(2)$ just acts on $\mathbb{P}^2$ (preserving ℓ_∞). In what follows, it will be convenient for us to restrict ourselves to the action of $\mathbb{G} = G \times \mathbb{C}^* \times \mathbb{C}^*$ where $\mathbb{C}^* \times \mathbb{C}^*$ is the diagonal subgroup of $GL(2)$.

Remark 1.1.1. — The compactification of the moduli space of instantons on a compact C^∞ 4-manifolds, as a topological space, was introduced by Donaldson, based on the earlier fundamental work by Uhlenbeck. See [**25**, Notes to Section 4.4.1] for further historical comments. This construction works for any compact Lie group, i.e., any reductive group G, and also the case when we take the quotient only by gauge transformations γ with $\gamma(\infty) = 1$ as above.

A construction as an affine variety was given in [**21**], which is one of our main references. See Remark 1.5.2 for comments in type A.

1.2. Main geometric object

The main object of our study on the geometric side is the $\mathbb{G}$-equivariant intersection cohomology $\mathrm{IH}^*_{\mathbb{G}}(\mathcal{U}_G^d)$. By the definition, it is endowed with the following structures:

1) It is a module over $H^*_{\mathbb{G}}(\mathrm{pt})$. The latter algebra can be canonically identified with the algebra of polynomial functions on $\mathfrak{h} \times \mathbb{C}^2$ which are invariant under W, where W is the Weyl group of G. In what follows we shall denote this ring by $\mathbf{A}_G$; let also $\mathbf{F}_G$ denote its field of fractions. We shall typically denote an element of $\mathfrak{h} \times \mathbb{C}^2$ by $(\boldsymbol{a}, \varepsilon_1, \varepsilon_2)$.

2) There exists a natural symmetric (Poincaré) pairing $\mathrm{IH}^*_{\mathbb{G}}(\mathcal{U}_G^d) \underset{\mathbf{A}_G}{\otimes} \mathrm{IH}^*_{\mathbb{G}}(\mathcal{U}_G^d) \to \mathbf{F}_G$ (this follows from the fact that $(\mathcal{U}_G^d)^{T \times \mathbb{C}^2}$ consists of one point).

3) For every $d \geq 0$ we have a canonical unit cohomology class $|1^d\rangle \in \mathrm{IH}^*_{\mathbb{G}}(\mathcal{U}_G^d)$.

The main purpose of this paper is to describe the above structures in terms of representation theory. To formulate our results, we need to introduce the main algebraic player – the $\mathcal{W}$-algebra.

1.3. Main algebraic object: $\mathcal{W}$-algebras

In this subsection we recall some basic facts and constructions from the theory of $\mathcal{W}$-algebras (cf. [**30**] and references therein). First, we need to recall the notion of Kostant-Whittaker reduction for finite-dimensional Lie algebras.

Let $\mathfrak{g}$ be as before a simple Lie algebra over $\mathbb{C}$ with the universal enveloping algebra $U(\mathfrak{g})$. Let us choose a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ for $\mathfrak{g}$. Let $\chi \colon \mathfrak{n}_+ \to \mathbb{C}$ be a non-degenerate character of $\mathfrak{n}_+$, i.e., a Lie algebra homomorphism such that $\chi|_{\mathfrak{n}_{+,i}} \neq 0$ for every vertex i of the Dynkin diagram of $\mathfrak{g}$ (here $\mathfrak{n}_{+,i}$ denotes the corresponding simple root subspace). Then we can define the finite $\mathcal{W}$-algebra of $\mathfrak{g}$ (to be denoted by $\mathcal{W}_{\text{fin}}(\mathfrak{g})$) as the quantum Hamiltonian reduction of $U(\mathfrak{g})$ with respect to $(\mathfrak{n}_+, \chi)$. In other words, we have

$$\mathcal{W}_{\text{fin}}(\mathfrak{g}) = \operatorname{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \underset{U(\mathfrak{n}_+)}{\otimes} \mathbb{C}_\chi, U(\mathfrak{g}) \underset{U(\mathfrak{n}_+)}{\otimes} \mathbb{C}_\chi).$$

A well-known result of Kostant [**41**, Theorem 2.4.2] asserts that

(1f) $\mathcal{W}_{\text{fin}}(\mathfrak{g})$ is naturally isomorphic to the center $\mathcal{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$.

In particular, we have

(2f) The algebra $\mathcal{W}_{\text{fin}}(\mathfrak{g})$ has a natural embedding into $S(\mathfrak{h})$, whose image coincides with the algebra $S(\mathfrak{h})^W$.

(3f) The algebra $\mathcal{W}_{\text{fin}}(\mathfrak{g})$ is a polynomial algebra in some variables $F^{(1)}, ..., F^{(\ell)}$, where $\ell = \operatorname{rank}(\mathfrak{g})$. Each $F^{(\kappa)}$ is homogeneous as an element of $S(\mathfrak{h})^W$ of some degree $d_\kappa + 1 \geq 2$.

(4f) The algebra $\mathcal{W}_{\text{fin}}(\mathfrak{g})$ is isomorphic to the algebra $\mathcal{W}_{\text{fin}}(\mathfrak{g}^\vee)$.

Feigin and Frenkel (cf. [**30**] and references therein) have generalized the above results to the case of affine Lie algebras. Namely, let $\mathfrak{g}((t))$ denote the Lie algebra of $\mathfrak{g}$-valued formal loops. It has a natural central extension

$$0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0$$

(this extension depends on a choice of an invariant form on $\mathfrak{g}$ which we choose so that the squared length of every short coroot is equal to 2). The group $\mathbb{C}^*$ acts naturally on $\hat{\mathfrak{g}}$ by "loop rotation" and the same is true for its Lie algebra $\mathbb{C}$. We let $\mathfrak{g}_{\text{aff}}$ be the semi-direct product of $\hat{\mathfrak{g}}$ and $\mathbb{C}$ (for the above action).

For every $k \in \mathbb{C}$ one can consider the algebra $U_k(\hat{\mathfrak{g}})$ — this is the quotient of $U(\hat{\mathfrak{g}})$ by the ideal generated by $\mathbf{1} - k$ where $\mathbf{1}$ denotes the generator of the central $\mathbb{C} \subset \mathfrak{g}_{\text{aff}}$. Let us also extend χ to $\mathfrak{n}_+((t))$ by taking the composition of the residue map $\mathfrak{n}_+((t)) \to \mathfrak{n}_+$ with $\chi : \mathfrak{n}_+ \to \mathbb{C}$. Abusing slightly the notation, we shall denote this map again by χ.

The $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g})$ is roughly speaking the Hamiltonian reduction of $U_k(\hat{\mathfrak{g}})$ with respect to $(\mathfrak{n}_+((t)), \chi)$. However, the reader must be warned that rigorously this reduction must be performed in the language of vertex operator algebras; in particular, $\mathcal{W}_k(\mathfrak{g})$ is a vertex operator algebra (cf. again [**30**] for the relevant definitions).

Unlike in the finite case, the algebra $\mathcal{W}_k(\mathfrak{g})$ is usually non-commutative (unless $k = -h^\vee$). The main results of Feigin and Frenkel about $\mathcal{W}_k(\mathfrak{g})$ can be summarized as follows (notice the similarities between (1f)-(4f) and (1w)-(4w)):

(1w) The algebra $\mathcal{W}_{-h^\vee}(\mathfrak{g})$ can be naturally identified with the center of the (vertex operator algebra version of) $U_{-h^\vee}(\hat{\mathfrak{g}})$.

(2w) Let $\mathfrak{Heis}(\mathfrak{h})$ denote the central extension of $\mathfrak{h}((t))$ corresponding to the bilinear form on $\mathfrak{h}$ chosen above. Abusing the notation we shall use the same symbol for the corresponding vertex operator algebra. Also for any $k \in \mathbb{C}$ we can consider the corresponding algebra $\mathfrak{Heis}_k(\mathfrak{h})$ ("Heisenberg algebra of level k").(1) Then for generic k there exists a canonical embedding $\mathcal{W}_k(\mathfrak{g}) \hookrightarrow \mathfrak{Heis}_{k+h^\vee}(\mathfrak{h})$.

---

(1) Note that for all $k \neq 0$ these algebras are isomorphic.

(3w) The algebra $\mathscr{W}_k(\mathfrak{g})$ is generated (in the sense of [**30**, 15.1.9]) by certain elements $W^{(\kappa)}$, $\kappa = 1, \dots, \ell$ of conformal dimension $d_\kappa + 1$. This (among other things) means that for every module M over $\mathscr{W}_k(\mathfrak{g})$ and every $\kappa = 1, \dots, \ell$ there is a well defined field $Y(W^{(\kappa)}, z) = \sum\limits_{n \in \mathbb{Z}} W_n^{(\kappa)} z^{-n-d_\kappa-1}$ where $W_n^{(\kappa)}$ can be regarded as a linear endomorphism of M.

(4w) Suppose k is generic. There is a natural isomorphism $\mathscr{W}_k(\mathfrak{g}) \simeq \mathscr{W}_{k^\vee}(\mathfrak{g}^\vee)$ where $(k + h_{\mathfrak{g}}^\vee)(k^\vee + h_{\mathfrak{g}^\vee}^\vee) = r^\vee$ where $r^\vee$ is the lacing number of $\mathfrak{g}$ (i.e., the maximal number of edges between two vertices of the Dynkin diagram of $\mathfrak{g}$). We shall call this isomorphism *the Feigin-Frenkel duality.*

The representation theory of $\mathscr{W}_k(\mathfrak{g})$ has been extensively studied (cf. for example [**2**]). In particular, to any $\lambda \in \mathfrak{h}^*$ one can attach a Verma module $M(\lambda)$ over $\mathscr{W}_k(\mathfrak{g})$ and $M(\lambda_1)$ is isomorphic to $M(\lambda_2)$ if $\lambda_1 + \rho$ and $\lambda_2 + \rho$ are on the same orbit of the Weyl group. This module carries a natural (Kac-Shapovalov) bilinear form, with respect to which the operator $W_n^{(\kappa)}$ is conjugate to $W_{-n}^{(\kappa)}$ (up to sign). This module can be obtained as the Hamiltonian reduction of the corresponding Verma module for $\mathfrak{g}$.

1.4. The main result: localized form

Let us set

$$M_{\mathbf{F}_G}^d(\boldsymbol{a}) = \mathrm{IH}^*_{\mathbb{G}}(\mathscr{U}_G^d) \underset{\mathbf{A}_G}{\otimes} \mathbf{F}_G; \quad M_{\mathbf{F}_G}(\boldsymbol{a}) = \bigoplus_{d=0}^{\infty} M_{\mathbf{F}_G}^d(\boldsymbol{a}).$$

It is easy to see that $M_{\mathbf{F}_G}^d(\boldsymbol{a})$ is also naturally isomorphic to $\mathrm{IH}^*_{\mathbb{G},c}(\mathscr{U}_G^d) \underset{\mathbf{A}_G}{\otimes} \mathbf{F}_G$ where the subscript $_c$ stands for cohomology with compact support.

Let us also set

$$k = -h^\vee - \frac{\varepsilon_2}{\varepsilon_1}.$$

Then (a somewhat weakened) form of our main result is the following:

Theorem 1.4.1. — *Assume that G is simply laced and let us identify $\mathfrak{h}$ with $\mathfrak{h}^*$ by means of the invariant form such that $(\alpha, \alpha) = 2$ for every root of $\mathfrak{g}$. Then there exists an action of the algebra $\mathscr{W}_k(\mathfrak{g})$ on $M_{\mathbf{F}_G}(\boldsymbol{a})$ such that*

1. *The resulting module is isomorphic to the Verma module $M(\lambda)$ over $\mathscr{W}_k(\mathfrak{g})$ where*
$$\lambda = \frac{\boldsymbol{a}}{\varepsilon_1} - \rho$$
(here we take $\mathbf{F}_T = \mathrm{Frac}(H_T^*(\mathrm{pt}))$ as our field of scalars).
2. *Under the above identification a twisted Poincaré pairing on $M_{\mathbf{F}_G}(\boldsymbol{a})$ goes over to the Kac-Shapovalov form on $M(\lambda)$. (The twisting will be explained in §6.8.)*
3. *Under the above identification the grading by d corresponds to the grading by eigenvalues of L_0.*

4. *Let* $d \geq 1$, $n > 0$. *We have*

$$W_n^{(\kappa)}|1^d\rangle = \begin{cases} \pm\varepsilon_1^{-1}\varepsilon_2^{-h^\vee+1}|1^{d-1}\rangle & \textit{if } \kappa = \ell \textit{ and } n = 1, \\ 0 & \textit{otherwise.} \end{cases} \tag{1.4.2}$$

Remarks 1.4.3. — 1) We believe that the sign in (1.4.2) is actually always "+," however, currently we don't know how to eliminate the sign issue. Note, however, that (1.4.2) still defines the scalar product $\langle 1^d|1^d\rangle$ unambiguously. Also (assuming that the above sign issue can be settled) it follows from (1.4.2) that if we formally set $w = \sum_d |1^d\rangle$ then we have

$$W_n^{(\kappa)}(w) = \begin{cases} \varepsilon_1^{-1}\varepsilon_2^{-h^\vee+1} w & \text{if } \kappa = \ell \text{ and } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Sometimes we shall write $w_{\boldsymbol{a},\varepsilon_1,\varepsilon_2}$ to emphasize the dependence on the corresponding parameters.

2) The assumption that G is simply laced is essential for Theorem 1.4.1 to hold as stated. However, we believe that a certain modified version of Theorem 1.4.1 holds in the non-simply laced case as well, although at the moment we don't have a proof of this modified statement (cf. Subsection 1.10 for a brief discussion of the non-simply laced case).

3) Since $\mathcal{U}_G^d$ is acted on by the full $GL(2)$ and not just by $\mathbb{C}^* \times \mathbb{C}^*$, it follows that the vector space $M_{\mathbf{F}_G}(\boldsymbol{a})$ has a natural automorphism which induces the involution $\varepsilon_1 \leftrightarrow \varepsilon_2$ on $\mathbf{F}$ (and leaves $\boldsymbol{a}$ untouched). Note that changing ε_1 to ε_2 amounts to changing $k = -h^\vee - \frac{\varepsilon_2}{\varepsilon_1}$ to $k^\vee = -h^\vee - \frac{\varepsilon_1}{\varepsilon_2}$ and we have $(k + h^\vee)(k^\vee + h^\vee) = 1$. Note also that we are assuming that $\mathfrak{g}$ is simply laced, so $\mathfrak{g}$ is isomorphic to $\mathfrak{g}^\vee$ and the above geometrically defined automorphism is in fact a corollary of the Feigin-Frenkel duality (cf. (1w)–(4w)).

1.5. Relation to previous works

We discuss previous works related to the above result here and later in §1.8. This subsection is devoted for those works related to statements themselves, and §1.8 is for those which give us a strategy and techniques of the proof.

First we discuss the statements (1),(2),(3). There are many previous works in almost the same pattern: We consider moduli spaces of instantons or variants on complex surfaces, and their homology groups or similar theory. Then some algebras similar to affine Lie algebras act on direct sums of homology groups, where we sum over various Chern classes.

The first example of such a result was given by the third-named author [**51, 53**]. The 4-manifold is $\mathbb{C}^2/\Gamma$ for a nontrivial finite subgroup $\Gamma \subset SU(2)$, and the gauge group is $U(r)$. The direct sum of homology groups of symplectic resolutions of Uhlenbeck spaces, called *quiver varieties* in more general context, is an integrable representation of the affine Lie algebra $\mathfrak{g}_{\Gamma,\mathrm{aff}}$ of level r. Here $\mathfrak{g}_\Gamma$ is a simple Lie algebra of type ADE corresponding to Γ via the McKay correspondence, and $\mathfrak{g}_{\Gamma,\mathrm{aff}}$ is its affine Lie algebra.

This result nicely fitted with the S-duality conjecture on the modular invariance of the partition function of $4d$ $N = 4$ supersymmetric gauge theory by Vafa-Witten [**70**], as characters of integrable representations are modular forms. It was understood that the correspondence [**51, 53**] should be understood in the framework of a duality in string theories [**68**]. There are lots of subsequent developments in physics literature since then.

In mathematics, the case $\Gamma = \{e\}$ was subsequently treated by [**52**] and Grojnowski [**35**] for $r = 1$, and by Baranovsky [**6**] for general r. The corresponding $\mathfrak{g}_{\Gamma,\mathrm{aff}}$ is the Heisenberg algebra, i.e., the affine Lie algebra associated with the trivial Lie algebra $\mathfrak{gl}_1$, in this case.

For $\Gamma = \{e\}$, the symplectic resolution $\widetilde{\mathcal{U}}_r^d \to \mathcal{U}_G^d$ of the Uhlenbeck space $\mathcal{U}_G^d$ is given by the moduli space of torsion-free sheaves on $\mathbb{P}^2$ together with a trivialization at ℓ_∞ of generic rank r and of second Chern class d. We call it *the Gieseker space* in this paper. For general Γ, we have its variant. All have description in terms of representations of quivers by variants of the ADHM description, and hence are examples of quiver varieties. (See Remark 1.5.2 for historical comments.)

This result was extended to an action of the quantum toroidal algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g}_{\Gamma,\mathrm{aff}})$ on the equivariant K-theory of the moduli spaces when $\Gamma \neq \{e\}$ [**55, 57**]. A variant for equivariant homology groups was given by Varagnolo [**71**].

In all these works, the action was given by introducing correspondences in products of moduli spaces, which give generators of the algebra. In particular, the constructions depend on *good* presentations of algebras. The case $\Gamma = \{e\}$ was studied much later, as we explain below, as the corresponding algebra, which would be $\mathbf{U}_q(\mathbf{L}(\mathfrak{gl}_1)_{\mathrm{aff}})$, was considerably more difficult.

Let us also mention that the second-named author with Kuznetsov [**28**] constructed an action of the affine Lie algebra $\widehat{\mathfrak{gl}}_r$ on the homology group of moduli spaces of parabolic sheaves on a surface, called *flag Gieseker spaces* or *affine Laumon spaces* when the surface is $\mathbb{P}^2$, the parabolic structure is put on a line and the framing is added. (Strictly speaking, the action was constructed on the homology group of the fibers of morphisms from flag Gieseker spaces to flag Uhlenbeck spaces. The action for the whole variety is constructed much later by Negut [**63**] in the equivariant K-theory framework.)

Let us turn to works on the inner product $\langle 1^d | 1^d \rangle$, which motivate the statement (4). It is given by the equivariant integration of 1 over $\mathcal{U}_G^d$, and their generating function

$$Z(Q, \boldsymbol{a}, \varepsilon_1, \varepsilon_2) = \sum_{d=0}^{\infty} Q^d \langle 1^d | 1^d \rangle \tag{1.5.1}$$

is called "the instanton part of the Nekrasov partition function for pure $N = 2$ supersymmetric gauge theory" [**64**]. This partition function has been studied intensively in both mathematical and physical literature. In particular, a result, which is very similar to Theorem 1.4.1(1)∼(4) (but technically much simpler) was proved by the first-named author [**14**]. Namely, in the situation of [**14**] on the representation theory side one deals with the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ instead of the corresponding $\mathscr{W}$-algebra, and on the geometric side one needs to replace the Uhlenbeck spaces $\mathcal{U}_G^d$ by *flag Uhlenbeck spaces* $\mathcal{Z}_G^\alpha$. In fact, it is important to note that when the original group G is not simply laced, the main result of [**14**] relates the equivariant intersection cohomology of the flag Uhlenbeck spaces for the group G with the representation theory of the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}^\vee$, whose root system is dual to that of $\mathfrak{g}_{\mathrm{aff}}$. A somewhat simpler construction exists also for the finite-dimensional Lie algebra $\mathfrak{g}^\vee$ – in that case on the geometric side one has to work with the so called space of *based quasi-maps into the flag variety of* $\mathfrak{g}$, also known as Zastava spaces (cf. [**15**] for a survey on these spaces).

The Nekrasov partition functions are equal for $\mathcal{U}_G^d$ and for flag Uhlenbeck spaces at $\varepsilon_2 = 0$, and it is enough for some purposes, say to determine Seiberg-Witten curves, but they are different in general. Therefore it was clear that we must replace $\mathfrak{g}_{\mathrm{aff}}^\vee$ by something else, but we did not know what it is.

A breakthrough was given in a physics context by Alday-Gaiotto-Tachikawa [**1**] (AGT for short). They conjectured that the partition functions for $G = SL(2)$ with four fundamental matters and adjoint matters are conformal blocks of the Virasoro algebra. They provided enough mathematically rigorous evidence, say numerical checks for small instanton numbers. They also give physical intuition that this correspondence is coming from an observation that $4d$ $N = 2$ supersymmetric gauge theories are obtained by compactifying the $6d$ theory on a Riemann surface: the Virasoro algebra naturally lives on the Riemann surface, which cannot be directly seen from the $4d$ side. They also guessed that the Virasoro algebra is replaced by the $\mathscr{W}$-algebra for a group G of type ADE.

There is a large literature in physics after AGT, especially for type A. We do not give the list, though those works are implicitly related to ours. We mention only one which was most relevant for us, it is [**40**] by Keller et al, where the statement (4) was written down for the first time for general G. (There is an earlier work by Gaiotto for $G = SL(2)$ [**32**], and various others for classical groups.)

Around the same time when [**1**] appeared in a physics context, there was an independent advance on the understanding of the algebra $\mathbf{U}_q(\mathbf{L}(\mathfrak{gl}_1)_{\mathrm{aff}})$ acting on the K-theory of resolutions of Uhlenbeck spaces of type A by Feigin-Tsymbaliuk [**27**] and Schiffmann-Vasserot [**67**]. They noticed that $\mathbf{U}_q(\mathbf{L}(\mathfrak{gl}_1)_{\mathrm{aff}})$ is isomorphic to various algebras, which had been studied in different contexts: a Ding-Iohara algebra, a shuffle algebra with the wheel conditions, the Hall algebra for elliptic curves, and an algebra studied by Miki [**47**]. Combined with the AGT picture, we understand that $\mathbf{U}_q(\mathbf{L}(\mathfrak{gl}_1)_{\mathrm{aff}})$ is the limit of the deformed $\mathscr{W}(\mathfrak{sl}_r)$, or $\mathscr{W}(\mathfrak{gl}_r)$ by the reason explained below, when $r \to \infty$.

In [**17**] a similar result is conjectured (and proved in type A) for *finite W-algebras* associated with a nilpotent element $e \in \mathfrak{g}^\vee$, which is principal in some Levi subalgebra (in that case on the geometric side one works with the so called parabolic Zastava spaces - cf. [**15**] for the relevant definitions).

Finally Maulik-Okounkov [**46**] and Schiffmann-Vasserot [**66**] proved Theorem 1.4.1 in the case when $G = SL(r)$. More precisely, they work with the equivariant cohomology of $\widetilde{\mathcal{U}}_r^d$ rather than with equivariant intersection cohomology of $\mathcal{U}_G^d$, which is slightly bigger. As a result on the representation theory side they get a Verma module over $\mathscr{W}(\mathfrak{gl}_r)$ (this algebra is isomorphic to the tensor product of $\mathscr{W}(\mathfrak{sl}_r)$ with a (rank 1) Heisenberg algebra). We should also mention that we use the construction of [**46**] for $r = 2$ in a crucial way for the proof of Theorem 1.4.1.

Remark 1.5.2. — Gieseker constructed a moduli space of semistable sheaves on a projective surface [**33**]. A morphism from Gieseker's moduli space to Uhlenbeck compactification was constructed by Li and Morgan [**44, 50**]. See [**38**, Ch. 8] as a modern reference.

There is an alternative approach for the case of bundles with trivialization over $\mathbb{P}^2$: The ADHM description [**3**] of instantons on S^4 describes the moduli space as a space of certain linear maps modulo the action of the unitary group. The Uhlenbeck space naturally arises by dropping an open condition, and considering a larger space (see [**25**, Ch. 3]). Furthermore this description is an affine algebro-geometric quotient [**24**], and one can introduce a GIT quotient by perturbing the stability condition [**54**, Ch. 3]. It gives the moduli space of torsion free sheaves with trivialization. The morphism from Gieseker space to Uhlenbeck space is also naturally defined.

1.6. Hyperbolic restriction

One of the main technical tools used in the proof of Theorem 1.4.1 is the notion of *hyperbolic restriction*. Let us recall the general definition of this notion.

Let X be an algebraic variety endowed with an action of $\mathbb{C}^*$. Then $X^{\mathbb{C}^*}$ is a closed subvariety of X. Let $\mathcal{A}_X$ denote the corresponding attracting set. Let $i\colon X^{\mathbb{C}^*} \to \mathcal{A}_X$ and $j\colon \mathcal{A}_X \to X$ be the natural embeddings. Then we have the functor $\Phi = i^* j^!$ from the derived category of constructible sheaves on X to the derived category of constructible sheaves on $X^{\mathbb{C}^*}$. This functor has been extensively studied by Braden in [**13**]. In particular, the main result of [**13**] says that Φ preserves the semi-simplicities of complexes.

Assume that we have a symplectic resolution $\pi\colon Y \to X$ in the sense of [**46**] and assume in addition that the above $\mathbb{C}^*$-action lifts to Y preserving the symplectic structure. Let $\mathcal{F} = \pi_* \mathbb{C}_Y[\dim X]$ (where $\mathbb{C}_Y$ denotes the constant sheaf on Y). Then we have

Theorem 1.6.1. — 1. [**72**] *$\Phi(\mathcal{F})$ is isomorphic to* $\pi_* \mathbb{C}_{Y^{\mathbb{C}^*}}[\dim X^{\mathbb{C}^*}]$.

2. *Maulik-Okounkov's stable envelope* [**46**] *gives us a choice of an isomorphism in* (1).

See **[60]** for the proof. Though both $\Phi(\mathscr{F})$ and $\pi_*\mathbb{C}_{Y^{\mathbb{C}^*}}[\dim X^{\mathbb{C}^*}]$ are isomorphic semi-simple perverse sheaves, the proof of **[72]** only gives us a canonical filtration on the former whose associated graded is canonically isomorphic to the latter. Then the stable envelope **[46]** gives us a choice of a splitting.

Now we specialize the above discussion to the following situation. Let $P \subset G$ be a parabolic subgroup of G with Levi subgroup L. Let us choose a subgroup $\mathbb{C}^* \subset Z(L)$ (here $Z(L)$ stands for the center of L) such that the fixed point set of its adjoint action on P is L and the attracting set is equal to all of P. Let now $X = \mathscr{U}_G^d$. We denote by $\mathscr{U}_L^d$ the fixed point set of the above $\mathbb{C}^*$ on $\mathscr{U}_G^d$ and by $\mathscr{U}_P^d$ the corresponding attracting set. It is easy to see that if L is not a torus, then $\mathscr{U}_L^d$ is just homeomorphic to $\mathscr{U}_{[L,L]}^d$ (and if L is a torus, then $\mathscr{U}_L^d$ is just $S^d(\mathbb{C}^2)$). See §4.2. Often we are going to drop the instanton number d from the notation, when there is no fear of confusion. We let i and p denote the corresponding maps from $\mathscr{U}_L$ to $\mathscr{U}_P$ and from $\mathscr{U}_P$ to $\mathscr{U}_L$. Also we denote by j the embedding of $\mathscr{U}_P$ to $\mathscr{U}_G$. We have the diagram

$$\mathscr{U}_L \overset{p}{\underset{i}{\leftrightarrows}} \mathscr{U}_P \xrightarrow{j} \mathscr{U}_G, \tag{1.6.2}$$

Thus we can consider the corresponding hyperbolic restriction functor $\Phi_{L,G} = i^* j^!$ (note that the functor actually depends on P and not just on L, but it depend on the choice of $\mathbb{C}^* \subset Z(L)$ made above, as we will explain in §4.4).

The following is one of the main technical results used in the proof of Theorem 1.4.1:

Theorem 1.6.3. —

1. *Let $P_1 \subset P_2$ be two parabolic subgroups and let $L_1 \subset L_2$ be the corresponding Levi subgroups. Then we have a natural isomorphism of functors $\Phi_{L_1,G} \simeq \Phi_{L_1,L_2} \circ \Phi_{L_2,G}$.*
2. *For P and L as above the complex $\Phi_{L,G}(\mathrm{IC}(\mathscr{U}_G^d))$ is perverse and semi-simple. Moreover, the same is true for any semi-simple perverse sheaf on $\mathscr{U}_G^d$ which is constructible with respect to the natural stratification.*

Note that when $G = SL(r)$, it is easy to deduce Theorem 1.6.3 from Theorem 1.6.1(1), since in this case the scheme $\mathscr{U}_G^d$ has a symplectic resolution $\widetilde{\mathscr{U}}_r^d$.

1.7. Sketch of the proof

The proof of Theorem 1.4.1 will follow the following plan:

1) Replace $\mathbb{G} = G \times \mathbb{C}^* \times \mathbb{C}^*$-equivariant cohomology with $\mathbb{T} = T \times \mathbb{C}^* \times \mathbb{C}^*$-equivariant cohomology. Note that the former is just equal to the space of W-invariants in the latter, so if we define an action of $\mathscr{W}_k(\mathfrak{g})$ on $\oplus \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}_G^d) \underset{\mathbf{A}_T}{\otimes} \mathbf{F}_T$ (where $\mathbf{A}_T = H^*_{T\times\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{pt})$ and $\mathbf{F}_T$ is its field of fractions) and check that if commutes with the action of W, we get an action of $\mathscr{W}_k(\mathfrak{g})$ on $\bigoplus_d \mathrm{IH}^*_{\mathbb{G}}(\mathscr{U}_G^d) \underset{\mathbf{A}_G}{\otimes} \mathbf{F}_G$.

2) We are going to construct an action of $\mathfrak{Heis}_{k+h^\vee}(\mathfrak{h})$ on $\bigoplus_d \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}_G^d) \underset{\mathbf{A}_T}{\otimes} \mathbf{F}_T$ and then get the action of $\mathscr{W}_k(\mathfrak{g})$ by using the embedding $\mathscr{W}_k(\mathfrak{g}) \hookrightarrow \mathfrak{Heis}_{k+h^\vee}(\mathfrak{h})$. It

should be noted that the above $\mathfrak{Heis}_{k+h^\vee}(\mathfrak{h})$-action will have several "disadvantages" that will disappear when we restrict ourselves to $\mathscr{W}_k(\mathfrak{g})$. For example, this action will depend on a certain auxiliary choice (a choice of a Weyl chamber).

3) The action of the Heisenberg algebra on $\bigoplus_d \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G) \underset{\mathbf{A}_T}{\otimes} \mathbf{F}_T$ will be constructed in the following way. Let us choose a Borel subgroup B containing the chosen maximal torus T. We can identify $\bigoplus_d \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G) \underset{\mathbf{A}_T}{\otimes} \mathbf{F}_T$ with $\bigoplus_d H^*_{\mathbb{T}}\Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G)) \underset{\mathbf{A}_T}{\otimes} \mathbf{F}_T$, so it is enough to define an action of the Heisenberg algebra on the latter. For this it is enough to define the action of $\mathfrak{Heis}(\mathbb{C}\alpha_i^\vee)$ for every simple coroot $\alpha_i^\vee$ of G (and then check the corresponding relations). Let P_i denote the corresponding subminimal parabolic subgroup containing B. Let also L_i be its Levi subgroup (it is canonical after the choice of T). Note that $[L_i, L_i] \simeq SL(2)$. Using the isomorphism $\Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G)) \simeq \Phi_{T,L_i} \circ \Phi_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G))$ and Theorem 1.6.1, we define the action of $\mathfrak{Heis}(\mathbb{C}\alpha_i^\vee)$ on $\bigoplus_d H^*_{\mathbb{T}}\Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G)) \underset{\mathbf{A}_T}{\otimes} \mathbf{F}_T$ using the results of [**46**] for $G = SL(2)$.

Here it is important for us to write down $\Phi_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G))$ in terms of $\mathrm{IC}(\mathscr{U}^{d'}_{L_i})$ ($d' \leq d$) and local systems on symmetric products in a 'canonical' way. In particular, we need to construct a base in the multiplicity space of $\mathrm{IC}(\mathscr{U}^{d'}_{L_i})$ in $\Phi_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G))$. For $G = SL(r)$, this follows from the stable envelope, thanks to Theorem 1.6.1(2). For general G, this argument does not work, and we use the factorization property of Uhlenbeck spaces together with the special case $G = SL(2)$. A further detail is too complicated to be explained in Introduction, so we ask an interested reader to proceed to the main text.

4) We now need to check the relations between various $\mathfrak{Heis}(\mathbb{C}\alpha_i^\vee)$. For this we have two proofs. One reduces it again to the results of [**46**] for $G = SL(3)$ (note that since we assume that G is simply laced, any connected rank 2 subdiagram of the Dynkin diagram of G is of type A_2). The other goes through the theory of certain "geometric" R-matrices (cf. Section 7). The proof of assertions (2) and (3) of Theorem 1.4.1 is more or less straightforward. The proof of assertion (4) is more technical and we are not going to discuss it in the Introduction. Let us just mention that for that proof we need a stronger form of the first 3 statements of Theorem 1.4.1 which is briefly discussed below.

1.8. Relation to previous works – technical parts

Let us mention previous works which give us a strategy and techniques of the proof.

First of all, we should mention that the overall framework of the proof is the same as those in [**46, 66**]. We realize the Feigin-Frenkel embedding of $\mathscr{W}_k(\mathfrak{g})$ into $\mathfrak{Heis}_{k+h^\vee}(\mathfrak{h})$ in a geometric way via the fixed point $(\mathscr{U}^d_G)^{\mathbb{C}^*} = \mathscr{U}^d_L$, as is explained the geometric realization in 3),4) in §1.7. This was first used in [**46, 66**] for type A.

What we do here is to replace the equivariant homology of Gieseker spaces $\widetilde{\mathcal{U}}_r^d$ by intersection cohomology of $\mathcal{U}_G^d$ as the former exists only in type A. Various foundational issues were discussed in the joint work of the first and second-named authors with Gaitsgory [**21**]. In particular, the fact that the character of $M_{\mathbf{F}_G}(\boldsymbol{a})$ is equal to the character of a Verma module over $\mathcal{W}_k(\mathfrak{g})$ follows from the main result of [**21**]. (For type A, it was done earlier in the joint work of the third-named author with Yoshioka. See [**54**, Exercise 5.15] and its solution in [**61**].)

A search of a replacement of Maulik-Okounkov's stable envelope [**46**] was initiated by the third-named author [**60**]. In particular, the relevance of the hyperbolic restriction functor Φ and the statement Theorem 1.6.1(2) were found. Therefore our technical aim is to find a 'canonical' isomorphism between $\Phi_{L,G}(\mathrm{IC}(\mathcal{U}_G^d))$ and a certain perverse sheaf on $\mathcal{U}_L^d$.

Let us also mention that Theorem 1.6.1(1) was proved much earlier by Varagnolo-Vasserot [**72**] in their study of quiver varieties. The functor Φ realized tensor products of representations of $\mathfrak{g}_{\Gamma,\mathrm{aff}}$. (Strictly speaking, only quiver varieties of finite types were considered in [**72**]. A slight complication occurs for quiver varieties of affine types which give $\mathfrak{g}_{\Gamma,\mathrm{aff}}$. See [**60**, Remark 1] for detail.)

When we do not have a symplectic resolution like $\widetilde{\mathcal{U}}_d^r$, we need another tool to analyze Φ. Fortunately the hyperbolic restriction functor was studied by Mirković-Vilonen [**48, 49**] in the context of the geometric Satake isomorphism, which asserts the category of $G(\mathbb{C}[[t]])$-equivariant perverse sheaves on the affine Grassmannian $\mathrm{Gr}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ is equivalent to the category of finite dimensional representations of the Langlands dual $G^\vee$ of G as tensor categories. The hyperbolic restriction functor realizes the restriction from $G^\vee$ to its Levi subgroup.

In particular, it was proved that Φ sends perverse sheaves to perverse sheaves. This was proved by estimating dimension of certain subvarieties of Gr_G, now called Mirković-Vilonen cycles. The proof of Theorem 1.6.3 is given in the same manner, replacing Mirković-Vilonen cycles by attracting sets of the $\mathbb{C}^*$-action.

It is clear that we should mimic the geometric Satake isomorphism from the conjecture of the first and second-named authors [**18**] which roughly says the following: it is difficult to make sense of perverse sheaves on the double affine Grassmannian, i.e., the affine Grassmannian $\mathrm{Gr}_{G_{\mathrm{aff}}}$ for the affine Kac-Moody group G_{aff}. But perverse sheaves on $\mathcal{U}_G^d$ (and more generally instanton moduli spaces on $\mathbb{C}^2/\Gamma$ with $\Gamma = \mathbb{Z}/k\mathbb{Z}$) serve as their substitute. Then they control the representation theory of $G_{\mathrm{aff}}^\vee$ at level k.

This conjecture nicely fits with the third-named author's works [**51, 53**] on quiver varieties via I. Frenkel's level-rank duality for the affine Lie algebra of type A [**31**]. Namely in the correspondence between moduli spaces and representation theory, the gauge group determines the rank, and Γ the level respectively in the double affine Grassmannian. And the role is reversed in quiver varieties.

In [**20**], the first and second-named authors proposed a functor, acting on perverse sheaves, which conjecturally gives tensor products of $G_{\mathrm{aff}}^\vee$. This proposal was checked in [**58**] for type A, by observing that the same functor gives the branching from $\mathfrak{g}_{\Gamma,\mathrm{aff}}$

to the affine Lie algebra of a Levi subalgebra. The interchange of tensor products and branching is again compatible with the level-rank duality.

Here in this paper, tensor products and branching appear in the opposite side: The hyperbolic restriction functor Φ realizes the tensor product in the quiver variety side, as we mentioned above. Therefore it should correspond to branching in the dual affine Grassmannian side. This is a philosophical explanation why the study of analog of Mirković-Vilonen cycles is relevant here.

1.9. The main result: integral form

The formulation of Theorem 1.4.1 has an obvious drawback: it is only formulated in terms of *localized* equivariant cohomology. First of all, it is clear that as stated Theorem 1.4.1 only has a chance to work over the localized field $\mathbf{F} = \mathbb{C}(\varepsilon_1, \varepsilon_2)$ rather than over $\mathbf{A} = \mathbb{C}[\varepsilon_1, \varepsilon_2]$. The reason is that our formula for the level $k = -h^\vee - \frac{\varepsilon_2}{\varepsilon_1}$ and the highest weight $\lambda = \frac{\boldsymbol{a}}{\varepsilon_1} - \rho$ are not elements of $\mathbf{A}$. For many purposes, it is convenient to have an $\mathbf{A}$-version of Theorem 1.4.1. In fact, technically in order to prove the last assertion of Theorem 1.4.1 we need such a refinement of the first 3 assertions (the reason is that we need to use the cohomological grading which is lost after localization). In earlier works [**46, 66**] for type A, the $\mathbf{A}$-version appears only implicitly, as operators $W_n^{(\kappa)}$ are given by cup products on Gieseker spaces. But in our case, Uhlenbeck spaces are singular, and we need to work with intersection cohomology groups. Hence $W_n^{(\kappa)}$ do not have such descriptions.

So, in order to formulate a non-localized version of Theorem 1.4.1 one needs to define an $\mathbf{A}$-version $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ of the $\mathscr{W}$-algebra (such that after tensoring with $\mathbf{F}$ we get the algebra $\mathscr{W}_k(\mathfrak{g})$ with $k = -h^\vee - \frac{\varepsilon_2}{\varepsilon_1}$). We also want this algebra to be graded (such that the degrees of ε_1 and ε_2 are equal to 2); in addition we need analogs of statements (2w) and (3w). This is performed in the Appendix B. Let us note, that although this $\mathbf{A}$-form is motivated by geometry, it can be defined purely in an algebraic way, following the work of Feigin and Frenkel. As far as we know, this $\mathbf{A}$-form does not appear in the literature before. As a purely algebraic application, we can remove the genericity assumption in (4w). The third named author learns from Arakawa that this was known to him before, but the proof is not written. After this we prove an $\mathbf{A}$-version of Theorem 1.4.1 in Section 8.

The non-localized equivariant cohomology groups also give us a refined structure in our construction. We construct $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$-module structures on four modules

$$\bigoplus_d \mathrm{IH}^*_{\mathbb{G},c}(\mathscr{U}^d_G), \quad \bigoplus_d H^*_{\mathbb{T},c}(\Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))) \qquad \bigoplus_d H^*_{\mathbb{T}}(\Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))), \quad \bigoplus_d \mathrm{IH}^*_{\mathbb{G}}(\mathscr{U}^d_G),$$

where the subscript c stands for cohomology with compact support. They become isomorphic if we take tensor products with $\mathbf{F}_T$, i.e., in the localized equivariant cohomology. But they are different over $\mathbf{A}_G$ and $\mathbf{A}_T$. We show that they are *universal* Verma, Wakimoto modules $M_{\mathbf{A}}(\boldsymbol{a})$, $N_{\mathbf{A}}(\boldsymbol{a})$, and their duals respectively. Here by a Wakimoto module, we mean the pull-back of a Fock space via the embedding of $\mathscr{W}(\mathfrak{g})$ in $\mathfrak{Heis}(\mathfrak{h})$. They are universal in the sense that we can specialize to Verma/Wakimoto and their duals at any evaluation $\mathbf{A}_G \to \mathbb{C}$, $\mathbf{A}_T \to \mathbb{C}$. This will be important for us to derive character formulas for simple modules, which will be discussed in a separate publication.

The importance of the integral form and the application to character formulas were first noticed in the context of the equivariant K-theory of the Steinberg variety and the affine Hecke algebra (see [**23**]), and then in quiver varieties [**55**] and parabolic Laumon spaces (= handsaw quiver varieties) [**59**].

1.10. Remarks about non-simply laced case

We have already mentioned above that verbatim Theorem 1.4.1 doesn't hold for non-simply laced G. However, we expect that the following modification of Theorem 1.4.1 should hold.

First, let $\mathscr{G}$ be any affine Lie algebra in the sense of [**39**] with connected Dynkin diagram. For example, $\mathscr{G}$ can be untwisted, and in this case it is isomorphic to a Lie algebra of the form $\mathfrak{g}_{\text{aff}}$ for some simple finite-dimensional Lie algebra. But in addition there exist twisted affine Lie algebras. We refer the reader to [**39**] for the relevant definitions; let us just mention that every twisted $\mathscr{G}$ comes from a pair $(\mathscr{G}', \sigma)$ where $\mathscr{G}' = \mathfrak{g}_{\text{aff}}$ for some simply laced simple finite-dimensional Lie algebra $\mathfrak{g}$ and σ is a certain automorphism of $\mathfrak{g}$ of finite order.

The Dynkin diagram of $\mathscr{G}$ comes equipped with a special "affine" vertex. We let $G_{\mathscr{G}}$ denote the semi-simple and simply connected group whose Dynkin diagram is obtained from that of $\mathscr{G}$ by removing that vertex.

To such an algebra one can attach another affine Lie algebra $\mathscr{G}^\vee$ — "the Langlands dual Lie algebra". By definition, this is just the Lie algebra whose generalized Cartan matrix is transposed to that of $\mathscr{G}$. It is worthwhile to note that:

1) If $\mathfrak{g}$ is a simply laced finite-dimensional simple Lie algebra, then $\mathfrak{g}_{\text{aff}}^\vee$ is isomorphic to $\mathfrak{g}_{\text{aff}}$ (which is also the same as $(\mathfrak{g}^\vee)_{\text{aff}}$ in this case).

2) In general, if $\mathfrak{g}$ is not simply laced, then $\mathfrak{g}_{\text{aff}}^\vee$ is not isomorphic to $(\mathfrak{g}^\vee)_{\text{aff}}$. In fact, if $\mathfrak{g}$ is not simply laced, then $\mathfrak{g}_{\text{aff}}^\vee$ is always a twisted Lie algebra.

It turns our that one can define the Uhlenbeck spaces $\mathscr{U}^d_{\mathscr{G}}$ for any affine Lie algebra $\mathscr{G}$ in such a way that that $\mathscr{U}^d_{\mathscr{G}} = \mathscr{U}^d_G$ when $\mathscr{G} = \mathfrak{g}_{\text{aff}}$ and $\mathfrak{g} = \text{Lie}(G)$ (the definition uses the corresponding simply laced algebra $\mathfrak{g}$ and its automorphism σ mentioned above). We are not going to explain the definition here (we shall postpone it for a later publication). This scheme is endowed with an action of the group $G_{\mathscr{G}} \times \mathbb{C}^* \times \mathbb{C}^*$.

In addition to $\mathscr{G}$ as above one can also attach a W-algebra $\mathcal{W}(\mathscr{G})$. Then we expect the following to be true:

Conjecture 1.10.1. — There exists an action of $\mathcal{W}(\mathscr{G})$ on $\oplus\, \mathrm{IH}^*_{G(\mathscr{G}^\vee)\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{U}^d_{\mathscr{G}^\vee})$ satisfying properties similar to those of Theorem 1.4.1.

Let us discuss one curious corollary of the above conjecture. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Set $\mathscr{G}_1 = \mathfrak{g}^\vee_{\text{aff}}, \mathscr{G}_2 = (\mathfrak{g}^\vee)^\vee_{\text{aff}}$. Then Conjecture 1.10.1 together with Feigin-Frenkel duality imply that there should be an isomorphism between $\mathrm{IH}^*_{G(\mathscr{G}_1)\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{U}^d_{\mathscr{G}_1})$ and $\mathrm{IH}^*_{G(\mathscr{G}_2)\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{U}^d_{\mathscr{G}_2})$ which sends $\frac{\varepsilon_2}{\varepsilon_1}$ and to $r^\vee\frac{\varepsilon_1}{\varepsilon_2}$. It would be interesting to see whether this isomorphism can be constructed geometrically (let us note that the naive guess that there exists an isomorphism between $\mathcal{U}^d_{\mathscr{G}_1}$ and $\mathcal{U}^d_{\mathscr{G}_2}$ giving rise to the above isomorphism between $\mathrm{IH}^*_{G(\mathscr{G}_1)\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{U}^d_{\mathscr{G}_1})$ and $\mathrm{IH}^*_{G(\mathscr{G}_2)\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{U}^d_{\mathscr{G}_2})$ is probably wrong). This question might be related to the work **[69]** where the author explains how to derive the 4-dimensional Montonen-Olive duality for non-simply laced groups from 6-dimensional (2,0) theory.

1.11. Further questions and open problems

In this subsection we indicate some possible directions for future research on the subject (apart from generalizing everything to the non-simply laced case, which was discussed before).

1.11(roman@subsection). VOA structure and CFT. — Our results imply that the space $M_{\mathbf{F}_G}(\boldsymbol{a})$ has a natural vertex operator algebra structure. It would be extremely interesting to construct this structure geometrically.

The AGT conjecture predicts a duality between $N = 2$ $4d$ gauge theories and $2d$ conformal field theories (CFT). The equivariant intersection cohomology group $M_{\mathbf{F}_G}(\boldsymbol{a})$ is just the quantum Hilbert space associated with S^1, appeared as a boundary of a Riemann surface. We should further explore the $4d$ gauge theory from CFT perspective, as almost nothing is known so far.

1.11(roman@subsection). Gauge theories with matter. — Our results give a representation-theoretic interpretation of the Nekrasov partition function of the *pure* $N = 2$ supersymmetric gauge theory on $\mathbb{R}^4$. For physical reasons it is also interesting to study gauge theories with matter. Mathematically it usually means that in the definition of the partition function (1.5.1) one should replace the equivariant integral of 1 by the equivariant integral of some other (intersection) cohomology class. However, when G is not of type A even the definition of the partition function is not clear to us. Namely, for $G = SL(r)$ one usually works with the Gieseker space $\widetilde{\mathcal{U}}^d_r$ instead of $\mathcal{U}^d_G$. In this case the cohomology classes in question are usually defined as Chern classes of certain natural sheaves $\widetilde{\mathcal{U}}^d_r$ (such as, for example, the tangent sheaf). Since $\mathcal{U}^d_G$ is singular and we work with *intersection cohomology* such constructions don't literally make sense for $\mathscr{U}_G^d$.

1.11(roman@subsection). The case of $\mathbb{C}^2/\Gamma$. — It would be interesting to try and generalize our results to Uhlenbeck space of $\mathbb{C}^2/\Gamma$. Here we expect the case when Γ is a cyclic group to be more accessible than the general case; in fact, in this case one should be able to see connections with **[18]**,**[20]** and on the other hand with **[10, 9]**. On the other hand the theory of quiver varieties deals with general Γ, but the group G is of type A, as we mentioned in §1.5. The case when both Γ and G are not of type A seems more difficult. Note that we must impose $\varepsilon_1 = \varepsilon_2$, therefore the level $k = -h^\vee - \varepsilon_2/\varepsilon_1$ cannot be deformed. In particular, the would-be $\mathscr{W}$-algebra does not have a classical limit.

1.11(roman@subsection). Surface operators. — As we have already mentioned in §1.5, there are *flag Uhlenbeck spaces* parametrizing (generalized) G-bundles on $\mathbb{P}^2$ with parabolic structure on the line $\mathbb{P}^1$. A type of parabolic structure corresponds to a parabolic subgroup P of G. Generalizing results in two extreme cases, $P = B$ in **[14]** and $P = G$ in this paper, it is expected that the equivariant intersection cohomology group admits a representation of the $\mathscr{W}$-algebra associated with the principal nilpotent element in the Lie algebra $\mathfrak{l}$ of the Levi part of P. (We assume G is of type ADE, and the issue of Langlands duality does not occur, for brevity.) This is an affine version of the conjecture in **[17]** mentioned before. Moduli of G-bundles with parabolic structure of type P is called a surface operator of Levi type $\mathfrak{l}$ in the context of $N = 4$ supersymmetric gauge theory **[36]**.

However there is a surface operator corresponding to *arbitrary* nilpotent element e in $\operatorname{Lie} G$ proposed in **[22]**, which is supposed to have the symmetry of $\mathscr{W}(\mathfrak{g}, e)$, the $\mathscr{W}$-algebra associated with e. We do not understand what kind of parabolic structures nor equivariant intersection cohomology groups we should consider if e is not regular in Levi.

1.12. Organization of the paper

In Section 2 we discuss some generalities about Uhlenbeck spaces. Section 3 is devoted to the general discussion of hyperbolic restriction and Section 4 — to hyperbolic restriction on Uhlenbeck spaces. In Section 5 we relate the constructions and results of Section 4 to certain constructions of **[46]** in the case when G is of type A. Section 6 is devoted to the construction of the action of the algebra $\mathscr{W}_k(\mathfrak{g})$ on $M_{\mathbf{F}_G}(\boldsymbol{a})$ along the lines presented above. Section 7 is devoted to the discussion of "geometric R-matrices".

1.13. Some notational conventions

(i) A partition λ is a nonincreasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots$ of nonnegative integers with $\lambda_N = 0$ for sufficiently large N. We set $|\lambda| = \sum \lambda_i$, $l(\lambda) = \#\{i \mid \lambda_i \neq 0\}$. We also write $\lambda = (1^{n_1} 2^{n_2} \cdots)$ with $n_k = \#\{i \mid \lambda_i = k\}$.

(ii) The equivariant cohomology group $H^*_{\mathbb{G}}(\mathrm{pt})$ of a point is canonically identified with the ring of invariant polynomials on the Lie algebra $\operatorname{Lie}\mathbb{G}$ of $\mathbb{G}$. The coordinate functions for the two factors $\mathbb{C}^*$ are denoted by ε_1, ε_2 respectively. We identify the ring of invariant polynomials on $\mathfrak{g} = \operatorname{Lie} G$ with the ring of the Weyl group invariant polynomials on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. When we consider the simple root α_i as a polynomial on $\mathfrak{h}$, we denote it by a^i.

(iii) For a variety X, let $D^b(X)$ denote the bounded derived category of complexes of constructible $\mathbb{C}$-sheaves on X. Let $\mathrm{IC}(X_0, \mathscr{L})$ denote the intersection cohomology complex associated with a local system $\mathscr{L}$ over a Zariski open subvariety X_0 in the smooth locus of X. We denote it also by $\mathrm{IC}(X)$ if $\mathscr{L}$ is trivial. When X is smooth and irreducible, $\mathscr{C}_X$ denotes the constant sheaf on X shifted by $\dim X$. If X is a disjoint union of irreducible smooth varieties X_α, we understand $\mathscr{C}_X$ as the direct sum of $\mathscr{C}_{X_\alpha}$.

(iv) We make a preferred degree shift for the Borel-Moore homology group (with complex coefficients), and denote it by $H_{[*]}(X)$. The shift is coming from a related perverse sheaf, which is clear from the context. For example, if X is smooth, $\mathscr{C}_X$ is a perverse sheaf. Hence $H_{[*]}(X) = H_{*+\dim X}(X)$ is a natural degree shift, as it is isomorphic to $H^{-*}(X, \mathscr{C}_X)$. More generally, if L is a closed subvariety in a smooth variety X, we consider $H_{[*]}(L) = H_{*+\dim X}(L) = H^{-*}(L, j^! \mathscr{C}_X)$, where $j \colon L \to X$ is the inclusion.

(v) We use the ADHM description of framed torsion free sheaves on $\mathbb{P}^2$ at several places. We change the notation (B_1, B_2, i, j) in [**54**, Ch. 2] to (B_1, B_2, I, J) as i, j are used for different things.

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CHAPTER 2

PRELIMINARIES

A basic reference to results in this section is [**21**], where [**4, 54**] are quoted occasionally.

2.1. Instanton number

We define an instanton number of a G-bundle $\mathscr{F}$ over $\mathbb{P}^2$. It is explained in, for example, [**4**]. Since it is related to our assumption that G is simply-laced, we briefly recall the definition.

The instanton number is the characteristic class associated with an invariant bilinear form $(\ ,\ )$ on the Lie algebra $\mathfrak{g}$ of G. Since we assume G is simple, the bilinear form is unique up to scalar. We normalize it so that the square length of the highest root θ is 2.

When $G = SL(r)$, it is nothing but the second Chern class of the associated complex vector bundle.

For an embedding $SL(2) \to G$ corresponding to a root α, we can induce a G-bundle $\mathscr{F}$ from an $SL(2)$-bundle $\mathscr{F}_{SL(2)}$. Then the corresponding instanton numbers are related by

$$d(\mathscr{F}) = d(\mathscr{F}_{SL(2)}) \times \frac{2}{(\alpha,\alpha)}. \tag{2.1.1}$$

Since we assume G is simply-laced, we have $(\alpha,\alpha) = 2$ for any root α. Thus the instanton number is preserved under the induction.

2.2. Moduli of framed G-bundles

Let Bun_G^d be the moduli space of G-bundles with trivialization at ℓ_∞ of instanton number d as before. We often call them *framed G-bundles*.

The tangent space of Bun_G^d at $\mathscr{F}$ is equal to the cohomology group $H^1(\mathbb{P}^2, \mathfrak{g}_{\mathscr{F}}(-\ell_\infty))$, where $\mathfrak{g}_{\mathscr{F}}$ is the vector bundle associated with $\mathscr{F}$ by the adjoint representation $G \to GL(\mathfrak{g})$ ([**21**, 3.5]). Other degree cohomology groups vanish, and hence the dimension of H^1 is given by the Riemann-Roch formula. It is equal to $2dh^\vee$ ([**4**]).

Here $h^\vee$ is the dual Coxeter number of G, appearing as the ratio of the Killing form and our normalized inner product $(\ ,\ )$.

It is known that Bun_G^d is connected, and hence irreducible ([**21**, Prop. 2.25]).

It is also known that Bun_G^d is a holomorphic symplectic manifold. Here the symplectic form is given by the isomorphism

$$H^1(\mathbb{P}^2, \mathfrak{g}_{\mathscr{F}}(-2\ell_\infty)) \xrightarrow{\cong} H^1(\mathbb{P}^2, \mathfrak{g}^*_{\mathscr{F}}(-\ell_\infty)), \tag{2.2.1}$$

where $\mathfrak{g} \cong \mathfrak{g}^*$ is induced by the invariant bilinear form, and $\mathcal{O}_{\mathbb{P}^2}(-\ell_\infty) \to \mathcal{O}_{\mathbb{P}^2}(-2\ell_\infty)$ is given by the multiplication by the coordinate z_0 corresponding to ℓ_∞. The tangent space $T_{\mathscr{F}} \mathrm{Bun}_G^d \cong H^1(\mathbb{P}^2, \mathfrak{g}_{\mathscr{F}}(-\ell_\infty))$ is isomorphic also to $H^1(\mathbb{P}^2, \mathfrak{g}_{\mathscr{F}}(-2\ell_\infty))$ and the above isomorphism can be regarded as $T_{\mathscr{F}} \mathrm{Bun}_G^d \to T^*_{\mathscr{F}} \mathrm{Bun}_G^d$. It is nondegenerate and closed. (See [**54**, Ch. 2, 3] for $G = SL(r)$. General cases can be deduced from the $SL(r)$-case by a faithful embedding $G \to SL(r)$.)

2.3. Stratification

Let $\mathcal{U}_G^d$ be the Uhlenbeck space for G. It has a stratification

$$\mathcal{U}_G^d = \bigsqcup \mathrm{Bun}_{G,\lambda}^{d_1}, \quad \mathrm{Bun}_{G,\lambda}^{d_1} = \mathrm{Bun}_G^{d_1} \times S_\lambda \mathbb{A}^2, \tag{2.3.1}$$

where the sum runs over pairs of integers d_1 and partitions λ with $d_1 + |\lambda| = d$. Here $S_\lambda \mathbb{A}^2$ is a stratum of the symmetric product $S^{|\lambda|}\mathbb{A}^2$, consisting of configurations of points whose multiplicities are given by λ, that is

$$S_\lambda \mathbb{A}^2 = \left\{ \sum \lambda_i x_i \in S^{|\lambda|}\mathbb{A}^2 \,\middle|\, x_i \neq x_j \text{ for } i \neq j \right\} \tag{2.3.2}$$

for $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$. We have

$$\dim \mathrm{Bun}_{G,\lambda}^{d_1} = 2(d_1 h^\vee + l(\lambda)). \tag{2.3.3}$$

Let $\mathcal{U}_{G,\lambda}^{d_1}$ be the closure of $\mathrm{Bun}_{G,\lambda}^{d_1}$. We have a finite morphism

$$\mathcal{U}_G^{d_1} \times \overline{S_\lambda \mathbb{A}^2} \to \mathcal{U}_{G,\lambda}^{d_1}, \tag{2.3.4}$$

extending the identification $\mathrm{Bun}_G^{d_1} \times S_\lambda \mathbb{A}^2 = \mathrm{Bun}_{G,\lambda}^{d_1}$, where $\overline{S_\lambda \mathbb{A}^2}$ is the closure of $S_\lambda \mathbb{A}^2$ in $S^{|\lambda|}\mathbb{A}^2$.

2.4. Factorization

For any projection $a \colon \mathbb{A}^2 \to \mathbb{A}^1$ we have a natural map $\pi_{a,G}^d \colon \mathcal{U}_G^d \to S^d\mathbb{A}^1$. See [**21**, §6.4]. It is equivariant under $\mathbb{G} = G \times \mathbb{C}^* \times \mathbb{C}^*$: it is purely invariant under G. We also change the projection a according to the $\mathbb{C}^* \times \mathbb{C}^*$-action.

Let us explain a few properties. Let $\mathscr{F} \in \mathrm{Bun}_G^d$. It is a principal G-bundle over $\mathbb{P}^2$ trivialized at ℓ_∞, but can be also considered as a G-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ trivialized at the union of two lines $\{\infty\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{\infty\}$. We extend a to $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Then $\pi_{a,G}^d(\mathscr{F})$ measures how the restriction of $\mathscr{F}$ to a projective line $a^{-1}(x)$ differs from the trivial G-bundle for $x \in \mathbb{P}^1$. If x is disjoint from $\pi^d_{a,G}(\mathcal{F})$, then $\mathcal{F}|_{a^{-1}(x)}$ is a trivial G-bundle. If not, the coefficient of x in $\pi^d_{a,G}(\mathcal{F})$ counts non-triviality with an appropriate multiplicity. (See [**21**, §4].)

On the stratum $\mathrm{Bun}^{d_1}_G \times S_\lambda\mathbb{A}^2$, $\pi^d_{a,G}$ is given as the sum of $\pi^{d_1}_{a,G}$ and the natural morphism $S_\lambda\mathbb{A}^2 \to S^{|\lambda|}\mathbb{A}^1$ induced from a. This property comes from the definition of the Uhlenbeck as a space of quasi-maps. (See [**21**, §§1,2].)

For type A, it is given as follows in terms of the ADHM description (B_1, B_2, I, J) (see [**54**, Ch. 2]): let B_a be the linear combination of B_1, B_2 corresponding to the projection $a\colon \mathbb{A}^2 \to \mathbb{A}^1$. Then $\pi^d_{a,G}$ is the characteristic polynomial of B_a. (See [**21**, Lem. 5.9].)

Moreover, most importantly, this map enjoys the factorization property, which says the following. Let us write $d = d_1 + d_2$ with $d_1, d_2 > 0$. Let $(S^{d_1}\mathbb{A}^1 \times S^{d_2}\mathbb{A}^1)_0$ be the open subset of $S^{d_1}\mathbb{A}^1 \times S^{d_2}\mathbb{A}^1$ where the first divisor is disjoint from the second divisor. Then we have a natural isomorphism

$$\mathcal{U}^d_G \times_{S^d\mathbb{A}^1} (S^{d_1}\mathbb{A}^1 \times S^{d_2}\mathbb{A}^1)_0 \cong (\pi^{d_1}_{a,G} \times \pi^{d_2}_{a,G})^{-1}((S^{d_1}\mathbb{A}^1 \times S^{d_2}\mathbb{A}^1)_0). \tag{2.4.1}$$

See [**21**, Prop. 6.5]. We call $\pi^d_{a,G}$ the *factorization morphism*. Often we are going to make statements about $\mathcal{U}^d_G$ and we are going to prove them by induction on d; (2.4.1) will usually allow us to say that the inductive step is trivial away from the preimage under $\pi^d_{a,G}$ of the main diagonal in $S^d\mathbb{A}^1$. In this case we are going to say that (the generic part of) the induction step “follows by the factorization argument”.

CHAPTER 3

LOCALIZATION

3.1. General Statement

Let T be a torus acting on X and Y be a closed invariant subset containing X^T. Let $\varphi\colon Y \to X$ be the inclusion. Let $U \overset{\text{def.}}{=} X \setminus Y$ and $\psi\colon U \to X$ be the inclusion. Let $\mathscr{F} \in D^b_T(X)$. We consider distinguished triangles

$$\begin{gathered}\varphi_!\varphi^!\mathscr{F} \to \mathscr{F} \to \psi_*\psi^*\mathscr{F} \xrightarrow{+1}, \\ \psi_!\psi^!\mathscr{F} \to \mathscr{F} \to \varphi_*\varphi^*\mathscr{F} \xrightarrow{+1}. \end{gathered} \tag{3.1.1}$$

Denote the Lie algebra of T by $\mathfrak{t}$. Natural homomorphisms

$$H^*_T(X,\mathscr{F}) \to H^*_T(X,\varphi_*\varphi^*\mathscr{F}) \cong H^*_T(Y,\varphi^*\mathscr{F}), \tag{3.1.2}$$

$$H^*_T(Y,\varphi^!\mathscr{F}) \cong H^*_T(X,\varphi_*\varphi^!\mathscr{F}) \cong H^*_T(X,\varphi_!\varphi^!\mathscr{F}) \to H^*_T(X,\mathscr{F}) \tag{3.1.3}$$

become isomorphisms after inverting an element $f \in \mathbb{C}[\mathfrak{t}]$ such that

$$\{x \in \mathfrak{t} \mid f(x) = 0\} \supset \bigcup_{x \in X\setminus Y} \operatorname{Lie}(\operatorname{Stab}_x). \tag{3.1.4}$$

See [**34**, (6.2)]. These assertions follow by observing $H^*_T(X;\psi_!\psi^!\mathscr{F}) = H^*_T(X,Y;\mathscr{F})$ and $H^*_T(X;\psi_*\psi^*\mathscr{F}) = H^*_T(U;\mathscr{F})$ are torsion in $\mathbb{C}[\mathfrak{t}]$. The same is true also for cohomology groups with compact supports. We call these statements the *localization theorem.*

We now suppose that we have an action of $\mathbb{C}^* \times \mathbb{C}^*$ commuting with the T-action such that

(3.1.5)
- — $X^{\mathbb{C}^*\times\mathbb{C}^*}$ is a single point, denoted by 0.
- — If $n_1, n_2 > 0$, $(t^{n_1}, t^{n_2}) \cdot x$ goes to 0 when $t \to 0$.

In fact, it is enough to have a $\mathbb{C}^*$-action for the result below, but we consider a $\mathbb{C}^* \times \mathbb{C}^*$-action, as the Uhlenbeck space has natural $\mathbb{C}^* \times \mathbb{C}^*$-action.

Let $\mathbb{T} = T \times \mathbb{C}^* \times \mathbb{C}^*$.

Lemma 3.1.6. — *The natural homomorphisms $H^*_{\mathbb{T}}(X,\mathscr{F}) \to H^*_{\mathbb{T}}(Y,\varphi^*\mathscr{F})$, $H^*_{\mathbb{T},c}(Y,\varphi^!\mathscr{F}) \to H^*_{\mathbb{T},c}(X,\mathscr{F})$ are isomorphisms for $\mathscr{F} \in D^b_{\mathbb{T}}(X)$.*

Proof. — Let $b_0^X : \{0\} \to X$, $b_0^Y : \{0\} \to Y$ be inclusions, and $a_X : X \to \{0\}$, $a_Y : Y \to \{0\}$ be the obvious morphisms. Since 0 is the unique fixed point of an attracting action of $\mathbb{C}^* \times \mathbb{C}^*$ by our assumption, adjunction gives us isomorphisms $(a_X)_* \xrightarrow{\cong} (b_0^X)^*$, $(a_Y)_* \xrightarrow{\cong} (b_0^Y)^*$ on equivariant objects by [**13**, Lemma 6]. Therefore we have a diagram

$$\begin{tikzcd} H^*_{\mathbb{T}}(X, \mathcal{F}) \arrow[r] \arrow[d, "\cong"'] & H^*_{\mathbb{T}}(Y, \varphi^*\mathcal{F}) \arrow[d, "\cong"] \\ H^*_{\mathbb{T}}((b_0^X)^*\mathcal{F}) \arrow[r, equal] & H^*_{\mathbb{T}}((b_0^Y)^*\varphi^*\mathcal{F}), \end{tikzcd} \tag{3.1.7}$$

where the lower horizontal equality follows from $\varphi b_0^Y = b_0^X$. If $\mathcal{F}$ is a sheaf, other three homomorphisms are given by restrictions, therefore the diagram is commutative. Hence it is also so for $\mathcal{F} \in D^b_{\mathbb{T}}(X)$ by a standard argument. Taking the dual spaces, we obtain the second assertion. □

3.2. The case of Ext algebras

Let $\mathcal{F}, \mathcal{G} \in D^b_T(X)$. We claim that

$$\operatorname{Ext}_{D^b_T(X)}(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}_{D^b_T(Y)}(\varphi^!\mathcal{F}, \varphi^!\mathcal{G}), \tag{3.2.1}$$

$$\operatorname{Ext}_{D^b_T(X)}(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}_{D^b_T(Y)}(\varphi^*\mathcal{F}, \varphi^*\mathcal{G}) \tag{3.2.2}$$

are isomorphisms after inverting an appropriate element f. Taking adjoint and considering (3.1.1), we see that it is enough to show that

$$\operatorname{Ext}_{D^b_T(X)}(\psi_*\psi^*\mathcal{F}, \mathcal{G}), \quad \operatorname{Ext}_{D^b_T(X)}(\mathcal{F}, \psi_!\psi^!\mathcal{G}) \tag{3.2.3}$$

are torsion. Let us observe that

$$\operatorname{Ext}_{D^b_T(X)}(\psi_*\psi^*\mathcal{F}, \psi_*\psi^*\mathcal{F}) \cong \operatorname{Ext}_{D^b_T(U)}(\psi^*\psi_*\psi^*\mathcal{F}, \psi^*\mathcal{F}) \tag{3.2.4}$$

is torsion, as it is an equivariant cohomology group over U. Then multiplying the identity endomorphism of $\psi_*\psi^*\mathcal{F}$ to $\operatorname{Ext}_{D^b_T(X)}(\psi_*\psi^*\mathcal{F}, \mathcal{G})$, we conclude that $\operatorname{Ext}_{D^b_T(X)}(\psi_*\psi^*\mathcal{F}, \mathcal{G})$ is torsion. The same argument applies also to $\operatorname{Ext}_{D^b_T(X)}(\mathcal{F}, \psi_!\psi^!\mathcal{G})$.

3.3. Attractors and repellents

Let X be a T-invariant closed subvariety in an affine space with a linear T-action. Let $A \subset T$ be a subtorus and X^A denote the fixed point set.

Let $X_*(A)$ be the space of cocharacters of A. It is a free $\mathbb{Z}$-module. Let

$$\mathfrak{a}_{\mathbb{R}} = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{3.3.1}$$

Let Stab_x be the stabilizer subgroup of a point $x \in X$. A *chamber* $\mathfrak{C}$ is a connected component of

$$\mathfrak{a}_{\mathbb{R}} \setminus \bigcup_{x \in X \setminus X^A} X_*(\operatorname{Stab}_x) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{3.3.2}$$

We fix a chamber $\mathfrak{C}$. Choose a cocharacter λ in $\mathfrak{C}$. Let $x \in X^A$. We introduce *attracting* and *repelling* sets:

$$\mathcal{A}_x = \left\{ y \in X \,\middle|\, \begin{array}{l} \text{the map } t \mapsto \lambda(t)(y) \text{ extends to a map } \mathbb{A}^1 \to X \\ \text{sending 0 to } x \end{array} \right\}, \tag{3.3.3}$$

$$\mathcal{R}_x = \left\{ y \in X \,\middle|\, \begin{array}{l} \text{the map } t \mapsto \lambda(t^{-1})(y) \text{ extends to a map } \mathbb{A}^1 \to X \\ \text{sending 0 to } x \end{array} \right\}.$$

These are closed subvarieties of X, and independent of the choice of $\lambda \in \mathfrak{C}$. Similarly we can define $\mathcal{A}_X$, $\mathcal{R}_X$ if we do not fix the point x as above. Note that X^A is a closed subvariety of both $\mathcal{A}_X$ and $\mathcal{R}_X$; in addition we have the natural morphisms $\mathcal{A}_X \to X^A$ and $\mathcal{R}_X \to X^A$.

3.4. Hyperbolic restriction

We continue the setting in the previous subsection. We choose a chamber in $\mathfrak{a}_\mathbb{R}$, and consider the diagram

$$\begin{tikzcd} X^A \arrow[r, shift right, "i"'] & \mathcal{A}_X \arrow[l, shift right, "p"'] \arrow[r, "j"] & X, \end{tikzcd} \tag{3.4.1}$$

where i, j are embeddings, and p is defined by $p(y) = \lim_{t\to 0} \lambda(t)y$.

We consider Braden's *hyperbolic restriction functor* [**13**] defined by $\Phi = i^* j^!$. (See also a recent paper [**26**].) Braden's theorem says that we have a canonical isomorphism

$$i^* j^! \cong i^!_- j^*_- \tag{3.4.2}$$

on weakly A-equivariant objects, where i_-, j_- are defined as in (3.4.1) for $\mathcal{R}_X$ instead of $\mathcal{A}_X$.

Braden proved his theorem for a normal algebraic variety. It is not known that $\mathcal{U}^d_G$ is normal or not. Therefore we use a more general result [**26**, Theorem 3.1.6].

Note also that i^* and p_* are isomorphic on weakly equivariant objects, we have $\Phi = p_* j^!$. (See [**13**, (1)].)

Let $\mathcal{F} \in D^b_T(X)$. A homomorphism

$$H^*_T(X^A, i^* j^! \mathcal{F}) \cong H^*_T(X^A, p_* j^! \mathcal{F}) = H^*_T(\mathcal{A}_X, j^! \mathcal{F}) \to H^*_T(X, \mathcal{F}) \tag{3.4.3}$$

becomes an isomorphism after inverting a certain element by the localization theorem in the previous subsection, applied to the pair $\mathcal{A}_X \subset X$.

We also have two naive restrictions

$$H^*_T(X^A, (j\circ i)^! \mathcal{F}), \quad H^*_T(X^A, (j\circ i)^* \mathcal{F}). \tag{3.4.4}$$

For the first one, we have a homomorphism to the hyperbolic restriction

$$H^*_T(X^A, (j\circ i)^! \mathcal{F}) \to H^*_T(X^A, i^* j^! \mathcal{F}), \tag{3.4.5}$$

which factors through $H^*(\mathcal{A}_X, j^! \mathcal{F})$. Then it also becomes an isomorphism after inverting an element.

The second one in (3.4.4) fits into a commutative diagram

$$\begin{tikzcd} H_T^*(X^A, i^* j^! \mathscr{F}) \arrow[r] & H_T^*(X^A, (j \circ i)^* \mathscr{F}) \\ H_T^*(\mathscr{C}_X, j^! \mathscr{F}) \arrow[r] \arrow[u] & H_T^*(\mathscr{C}_X, j^* \mathscr{F}). \arrow[u] \end{tikzcd} \tag{3.4.6}$$

Two vertical arrows are isomorphisms after inverting an element f. The lower horizontal homomorphism factors through $H_T^*(X, \mathscr{F})$ and the resulting two homomorphisms are isomorphisms after inverting an element, which we may assume equal to f. Therefore the upper arrow is also an isomorphism after inverting an element.

3.5. Hyperbolic semi-smallness

Braden's isomorphism $p_* j^! \cong (p_-)_! j_-^*$ implies that $p_* j^!$ preserves the purity of weakly equivariant mixed sheaves. ([**13**, Theorem 8]). In particular, $p_* j^! \operatorname{IC}(X)$ is isomorphic to a direct sum of shifts of intersection cohomology complexes ([**13**, Theorem 2]).

Braden's result could be viewed as a *formal* analog of the decomposition theorem (see [**23**, Theorem 8.4.8] for example). We give a sufficient condition so that $p_* j^! \operatorname{IC}(X)$ remains perverse (and semi-simple by the above discussion) in this subsection. This result is a formal analog of the decomposition theorem for *semi-small* morphisms (see [**23**, Proposition 8.9.3]). Therefore we call the condition the *hyperbolic semi-smallness*. This condition, without its naming, appeared in [**48, 49**] mentioned in the introduction. We give the statement in a general setting, as it might be useful also in other situations.

Let X, X^A as before. Let $X = \bigsqcup X_\alpha$ be a stratification of X such that $i_\alpha^! \operatorname{IC}(X)$, $i_\alpha^* \operatorname{IC}(X)$ are locally constant sheaves up to shifts. Here i_α denotes the inclusion $X_\alpha \to X$. We suppose that X_0 is the smooth locus of X as a convention.

We also suppose that the fixed point set X^A has a stratification $X^A = \bigsqcup Y_\beta$ such that the restriction of p to $p^{-1}(Y_\beta) \cap X_\alpha$ is a topologically locally trivial fibration over Y_β for any α, β (if it is nonempty). We assume the same is true for p_-. We take a point $y_\beta \in Y_\beta$.

Definition 3.5.1. — We say Φ is *hyperbolic semi-small* if the following two estimates hold

$$\begin{aligned} \dim p^{-1}(y_\beta) \cap X_\alpha &\le \frac{1}{2}(\dim X_\alpha - \dim Y_\beta), \\ \dim p_-^{-1}(y_\beta) \cap X_\alpha &\le \frac{1}{2}(\dim X_\alpha - \dim Y_\beta). \end{aligned} \tag{3.5.2}$$

In order to state the result, we need a little more notation. We have two local systems over Y_β, whose fibers at a point y_β are $H_{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)$ and $H_c^{\dim X - \dim Y_\beta}(p_-^{-1}(y_\beta) \cap X_0)$ respectively. Note that $p^{-1}(y_\beta) \cap X_0$ and $p_-^{-1}(y_\beta) \cap X_0$ are at most $(\dim X - \dim Y_\beta)/2$-dimensional if Φ is hyperbolic semi-small. In this case, cohomology groups have bases given by $(\dim X - \dim Y_\beta)/2$-dimensional irreducible components of $p^{-1}(y_\beta) \cap X_0$ and $p_-^{-1}(y_\beta) \cap X_0$ respectively. Let $H_{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)_\chi$ and $H_c^{\dim X - \dim Y_\beta}(p_-^{-1}(y_\beta) \cap X_0)_\chi$ denote the components corresponding to a simple local system χ on Y_β.

Theorem 3.5.3. — *Suppose Φ is hyperbolic semi-small. Then $\Phi(\mathrm{IC}(X))$ is perverse and it is isomorphic to*

$$\bigoplus_{\beta,\chi} \mathrm{IC}(Y_\beta, \chi) \otimes H_{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)_\chi.$$

Moreover, we have an isomorphism

$$H_{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)_\chi \cong H_c^{\dim X - \dim Y_\beta}(p_-^{-1}(y_\beta) \cap X_0)_\chi.$$

The proof is similar to one in [**49**, Theorem 3.5], hence the detail is left as an exercise for the reader. In fact, we only use the case when X^T is a point, and we explain the argument in detail for that case in Theorem A.7.1.

The same assertion holds for $\mathrm{IC}(X_0, \mathscr{L})$ the intersection cohomology complex with coefficients in a simple local system $\mathscr{L}$ over X_0, if we put $\mathscr{L}$ also to cohomology groups of fibers.

Note that $\Phi(\mathrm{IC}(X_\beta, \mathscr{L}_\beta))$ is also perverse for a local system L_β on X_β, and isomorphic to

$$\bigoplus_{\beta,\chi} \mathrm{IC}(Y_\beta, \chi) \otimes H_{\dim X_\beta - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_\beta)_\chi.$$

Conversely, if $\Phi(\mathrm{IC}(X_\beta, \mathscr{L}_\beta))$ is perverse, we have the dimension estimates (3.5.2). It is because the top degree cohomology groups are nonvanishing, and contribute to nonzero perverse degrees. See the argument in Corollary A.9.2 for detail.

3.6. Recovering the integral form

We assume (3.1.5) and also that X is affine. We consider the hyperbolic restriction with respect to T.

Let $\mathbf{A}_T = \mathbb{C}[\mathrm{Lie}(\mathbb{T})] = \mathbb{C}[\varepsilon_1, \varepsilon_2, \boldsymbol{a}]$ and $\mathbf{F}_T$ be its quotient field.

We further assume that $H^*_{\mathbb{T},c}(X, \mathscr{F})$ is torsion free over $H^*_{\mathbb{T}}(\mathrm{pt}) = \mathbf{A}_T$, i.e, $H^*_{\mathbb{T},c}(X, \mathscr{F}) \to H^*_{\mathbb{T},c}(X, \mathscr{F}) \otimes_{\mathbf{A}_T} \mathbf{F}_T$ is injective. This property for the Uhlenbeck space will be proved in Lemma 6.1.1.

We consider a homomorphism

$$H^*_{\mathbb{T},c}(X, \mathscr{F}) \cong H^*_{\mathbb{T},c}(X^T, i^! j^! \mathscr{F}) \to H^*_{\mathbb{T},c}(X^T, i^* j^! \mathscr{F}) \tag{3.6.1}$$

for $\mathscr{F} \in D^b_{\mathbb{T}}(X)$. The first isomorphism is given in Lemma 3.1.6. By the localization theorem, the second homomorphism becomes an isomorphism after inverting an element $f \in \mathbb{C}[\mathrm{Lie}\,\mathbb{T}]$ which vanishes on the union of the Lie algebras of the stabilizers of the points $x \in \mathscr{U}_X \setminus X^T$.

Theorem 3.6.2. — *Consider the intersection* $H^*_{\mathbb{T},c}(X^T, i^* j^! \mathscr{F}) \cap H^*_{\mathbb{T},c}(X^T, i^*_- j^!_- \mathscr{F})$ *in* $H^*_{\mathbb{T},c}(X, \mathscr{F}) \otimes_{\mathbf{A}_T} \mathbf{F}_T$. *It coincides with* $H^*_{\mathbb{T},c}(X, \mathscr{F})$.

The proof occupies the rest of this subsection. We first give a key lemma studying stabilizers of points in $\mathscr{A}_X \setminus X^T$.

Lemma 3.6.3. — *Suppose that* $(\lambda^\vee, n_1, n_2)$ *is a cocharacter of* $\mathbb{T}$ *such that either of the followings holds*

1. $\lambda^\vee$ *is dominant and* n_1, $n_2 > 0$.
2. $\lambda^\vee$ *is regular dominant and* n_1, $n_2 \geq 0$.

Then there is no point in $\mathscr{A}_X \setminus X^T$ *whose stabilizer contains* $(\lambda^\vee, n_1, n_2)(\mathbb{C}^*)$.

Proof. — Assume λ is dominant and n_1, $n_2 \geq 0$.

Suppose that $x \in \mathscr{A}_X$ is fixed by $(\lambda^\vee, n_1, n_2)(\mathbb{C}^*)$. Then we have

$$\lambda^\vee(t^{-1}) \cdot x = (t^{n_1}, t^{n_2}) \cdot x. \tag{3.6.4}$$

Since $\lambda^\vee$ is dominant, its attracting set contains $\mathscr{A}_X$. Therefore the left hand side has a limit when $t \to \infty$. On the other hand, the right hand side has a limit when $t \to 0$. Therefore $\mathbb{C}^* \ni t \mapsto \lambda^\vee(t^{-1}) \cdot x \in X$ extends to a morphism $\mathbb{P}^1 \to X$. As X is affine, such a morphism must be constant, i.e., (3.6.4) must be equal to x.

If n_1, $n_2 > 0$, x must be the unique $\mathbb{C}^* \times \mathbb{C}^*$ fixed point. It is contained in X^T.

If $\lambda^\vee$ is regular, x is fixed by T, that is $x \in X^T$. ☐

Proof of Theorem 3.6.2. — Let α be an element in $H^*_{\mathbb{T},c}(X, \mathscr{F})$ which is not divisible by any non-constant element of $\mathbf{A}_T$. Let $J^\pm_\alpha$ be two fractional ideals of $\mathbf{A}_T$ consisting of those rational functions f such that $f\alpha \in H^*_{\mathbb{T},c}(X^T, i^* j^! \mathscr{F})$ and $f\alpha \in H^*_{\mathbb{T},c}(X^T, i^*_- j^!_- \mathscr{F})$ respectively. We need to show that $J^+_\alpha \cap J^-_\alpha = \mathbf{A}_T$. Note that a priori the right hand side is embedded in the left hand side.

Let $f \in J^+_\alpha$. Then $f = g/h$ where g, $h \in \mathbf{A}_T$ and h is a product of linear factors of the form (μ, m_1, m_2) such that

— $\langle \lambda^\vee, \mu \rangle > 0$ for a regular dominant coweight $\lambda^\vee$, and

— m_1, $m_2 \geq 0$ with at least one of them nonzero.

In fact, we have $\langle (\lambda^\vee, n_1, n_2), (\mu, m_1, m_2) \rangle \neq 0$ for any $(\lambda^\vee, n_1, n_2)$ as in Lemma 3.6.3. Taking a regular dominant coweight $\lambda^\vee$ and n_1, $n_2 = 0$, we get the first condition. Next we take $\lambda = 0$ and n_1, $n_2 > 0$ and get the second condition.

Similarly for $f = g/h \in J^-_\alpha$, h is a product of (μ, m_1, m_2) with $\langle \lambda^\vee, \mu \rangle < 0$ for a regular dominant coweight $\lambda^\vee$, and the same conditions for (m_1, m_2) as above. Then there are no linear factors satisfying both conditions, hence we have $J^+_\alpha \cap J^-_\alpha = \mathbf{A}_T$. ☐

CHAPTER 4

HYPERBOLIC RESTRICTION ON UHLENBECK SPACES

This section is of technical nature, but will play a quite important role later. Feigin-Frenkel realized the $\mathscr{W}$-algebra $\mathscr{W}_k(\mathfrak{g})$ in the Heisenberg algebra $\mathfrak{Heis}(\mathfrak{h})$ associated with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. (See [**30**, Ch. 15].)

We will realize this picture in a geometric way. In [**46**] Maulik-Okounkov achieved it by *stable envelopes* which relate the cohomology group of Gieseker space to that of the fixed point set with respect to a torus. The former is a module over $\mathscr{W}_k(\mathfrak{g})$ and the latter is a Heisenberg module. In [**66**] Schiffmann-Vasserot also related two cohomology groups by a different method.

We will take a similar approach, but we need to use a sheaf theoretic language, as Uhlenbeck space is singular. We use the *hyperbolic restriction functor* in §3.4, and combine it with the theory of stable envelopes. This study was initiated by the third author [**60**]. A new and main result here is Theorem 4.6.1, which says that perversity is preserved under the hyperbolic restriction in our situation.

We fix a pair $T \subset B$ of a maximal torus T and a Borel subgroup B, and consider only parabolic subgroups P containing B, except we occasionally use opposite parabolic subgroups P_- until §4.13. In §4.13, we consider other parabolic subgroups also.

4.1. A category of semisimple perverse sheaves

Let $\mathrm{IC}(\mathrm{Bun}^d_{G,\lambda}, \rho)$ denote the intersection cohomology (IC) complexes, where ρ is a simple local system on $\mathrm{Bun}^d_{G,\lambda} = \mathrm{Bun}^d_G \times S_\lambda \mathbb{A}^2$ corresponding to an irreducible representation of $S_{n_1} \times S_{n_2} \times \cdots$ via the covering

$$(\mathbb{A}^2)^{n_1} \times (\mathbb{A}^2)^{n_2} \times \cdots \setminus \text{diagonal} \to S_\lambda \mathbb{A}^2, \tag{4.1.1}$$

where $\lambda = (1^{n_1} 2^{n_2} \cdots)$. (Recall $S_\lambda \mathbb{A}^2$ is a stratum of $S^{|\lambda|}\mathbb{A}^2$, see (2.3.2).)

Definition 4.1.2. — Let $\mathrm{Perv}(\mathscr{U}^d_G)$ be the additive subcategory of the abelian category of semisimple perverse sheaves on $\mathscr{U}^d_G$, consisting of finite direct sums of $\mathrm{IC}(\mathrm{Bun}^d_{G,\lambda}, \rho)$.

By abuse of notation, we use the same notation $\mathrm{IC}(\mathrm{Bun}^d_{G,\lambda}, \rho)$ even if ρ is a reducible representation of $S_{n_1} \times S_{n_2} \times \cdots$. It is the direct sum of the corresponding simple IC sheaves.

If ρ is the trivial rank 1 local system, we omit ρ from the notation and denote the corresponding IC complex by $\mathrm{IC}(\mathrm{Bun}^d_{G,\lambda})$, or $\mathrm{IC}(\mathcal{U}^d_{G,\lambda})$.

Furthermore, we omit λ from the notation when it is the empty partition $\varnothing$. Therefore $\mathrm{IC}(\mathcal{U}^d_G)$ means $\mathrm{IC}(\mathrm{Bun}^d_{G,\varnothing})$.

Objects in $\mathrm{Perv}(\mathcal{U}^d_G)$ naturally have structures of equivariant perverse sheaves in the sense of [**11**] with respect to the group action $\mathbb{G} = G \times \mathbb{C}^* \times \mathbb{C}^*$ on $\mathcal{U}^d_G$. We often view $\mathrm{Perv}(\mathcal{U}^d_G)$ as the subcategory of equivariant perverse sheaves.

4.2. Fixed points

Let P be a parabolic subgroup of G with a Levi subgroup L. Let $A = Z(L)^0$ denote the connected center of L. Let Bun^d_L denote the moduli space of L-bundles on $\mathbb{P}^2$ with trivialization at ℓ_∞ of 'instanton number d'. The latter expression makes sense, since the notion of instanton number, defined as in §2.1, corresponds to a choice of a bilinear form on the coweight lattice, which is the same for G and for L.

Suppose that $\mathcal{F} \in \mathrm{Bun}^d_G$ is fixed by the A-action. It means that bundle automorphisms at ℓ_∞ parametrized by A extend to the whole space $\mathbb{P}^2$. The extensions are unique. Therefore the structure group G of $\mathcal{F}$ reduces to the centralizer of A, which is L. Hence $(\mathrm{Bun}^d_G)^A = \mathrm{Bun}^d_L$.

Let us consider the fixed point subvariety

$$\mathcal{U}^d_L = (\mathcal{U}^d_G)^A \tag{4.2.1}$$

in the Uhlenbeck space. Then we have an induced stratification

$$\mathcal{U}^d_L = \bigsqcup_{d_1+d_2=d,\lambda\vdash d_2} \mathrm{Bun}^{d_1}_{L,\lambda}, \quad \mathrm{Bun}^{d_1}_{L,\lambda} = \mathrm{Bun}^{d_1}_L \times S_\lambda \mathbb{A}^2. \tag{4.2.2}$$

Strictly speaking, our $\mathcal{U}^d_L$ depends on the choice of the embedding $L \to G$, therefore should be denoted, say by $\mathcal{U}^d_{L,G}$. We think that there is no fear of confusion.

Note that $[L, L]$ is again semi-simple and simply-connected. (See [**12**, Cor. 4.4].) Suppose that we have only one simple factor. Since we assume G is simply-laced, $[L, L]$ is also. The instanton number is the same for G and $[L, L]$. Otherwise we define the instanton number for $[L, L]$ by the invariant form on $\mathrm{Lie}([L, L])$ induced from one on $\mathfrak{g}$.

We only have trivial framed $L/[L, L]$-bundles as $H^2(\mathbb{P}^2)$ is 1-dimensional hence the first Chern class of a framed bundle vanishes. Thus we have

$$\mathrm{Bun}^{d_1}_L = \mathrm{Bun}^{d_1}_{[L,L]}. \tag{4.2.3}$$

Since $[L, L]$ is a subgroup of G, we have the induced closed embedding $\mathcal{U}^d_{[L,L]} \to \mathcal{U}^d_G$ (see [**21**, Lem. 6.2]), which clearly factors as

$$\mathcal{U}^d_{[L,L]} \to \mathcal{U}^d_L. \tag{4.2.4}$$

By (4.2.3), this map is bijective. Since both spaces are closed subschemes of $\mathcal{U}^d_G$, we have

Proposition 4.2.5. — *The morphism $\mathcal{U}^d_{[L,L]} \to \mathcal{U}^d_L = (\mathcal{U}^d_G)^A$ is a homeomorphism between the underlying topological spaces.*

We are interested in perverse sheaves on $\mathcal{U}^d_L$, hence we only need underlying topological spaces. Hence we may identify $\mathcal{U}^d_L$ and $\mathcal{U}^d_{[L,L]}$. We define the category $\operatorname{Perv}(\mathcal{U}^d_L)$ in the same way as $\operatorname{Perv}(\mathcal{U}^d_G)$.

Example 4.2.6. — The case when L is a maximal torus T is most important. We have

$$\mathcal{U}^d_T = S^d\mathbb{A}^2 = \bigsqcup_{\lambda \vdash d} S_\lambda \mathbb{A}^2, \tag{4.2.7}$$

as we do not have nontrivial framed T-bundles.

4.3. Polarization

Following [**46**, §3.3.2], we introduce the notion of a *polarization* of a normal bundle of the smooth part of a fixed point component.

Let us give a definition in a general situation. Suppose a torus A acts on a holomorphic symplectic manifold X, preserving the symplectic structure. Let Z be a connected component of X^A and N_Z be its normal bundle in X. Consider A-weights of a fiber of N_Z. Let $e(N_Z)|_{H^*_A(\mathrm{pt})}$ be the $H^*_A(\mathrm{pt})$-part of the Euler class of the normal bundle, namely the product of all A-weights of a fiber of N_Z. Since A preserves the symplectic form, Z is a symplectic submanifold, and weights of N_Z appear in the pairs $(\alpha_i, -\alpha_i)$. Hence

$$(-1)^{(\operatorname{codim} Z)/2} e(N_Z)|_{H^*_A(\mathrm{pt})} = \prod \alpha_i^2 \tag{4.3.1}$$

is a perfect square. A choice of a square root δ of (4.3.1) is called a *polarization* of Z in X.

In the next subsection we consider attractors and repellents. We have a polarization δ_{rep} given by product of weights in repellent directions. However this will not be a right choice to save signs. Our choice of the polarization δ, which follows [**46**, Ex. 3.3.3], will be explained in §5.3 for Gieseker spaces, and in §6.2 for Uhlenbeck spaces. Then we understand $\delta = \pm 1$, depending on whether it is the same as or the opposite to δ_{rep}, in other words we identify δ with $\delta/\delta_{\mathrm{rep}}$, as δ_{rep} is clear from the context.

Note that a polarization does not make sense unless the variety X is smooth. Therefore we restrict the normal bundle to $Z \cap \operatorname{Bun}^d_G = Z \cap \operatorname{Bun}^d_L$ and consider a polarization there for Uhlenbeck spaces.

However a fixed point component Z, in general, does not intersect with Bun_G^d. Say $Z \cap \mathrm{Bun}_G^d = \varnothing$ if $L = T$. We do not consider a polarization of Z in this case, and smooth cases are enough for our purpose.

4.4. Definition of hyperbolic restriction functor

We now return to the situation when $X = \mathcal{U}_G^d$. We choose a parabolic subgroup P with a Levi subgroup L as before.

We consider the setting in §§3.3,3.4 with $A = Z(L)^0$. Then (3.3.2) is the hyperplane arrangement induced by roots:

$$\mathfrak{a}_{\mathbb{R}} \setminus \bigcup_{\alpha} \{ \alpha|_{\mathfrak{a}_{\mathbb{R}}} = 0 \}, \tag{4.4.1}$$

where the union runs over all positive roots α which do not vanish on $\mathfrak{a}_{\mathbb{R}}$. The chambers are in one to one correspondence to the parabolic subgroups containing L as their Levi (associated parabolics). Therefore the fixed P determines a 'positive' chamber.

We denote the corresponding attracting and repelling sets $\mathcal{A}_X$, $\mathcal{R}_X$ by $\mathcal{U}_P^d$ and $\mathcal{U}_{P_-}^d$. Often we are going to drop the instanton number d from the notation, when there is no fear of confusion. We let i and p denote the corresponding maps from $\mathcal{U}_L$ to $\mathcal{U}_P$ and from $\mathcal{U}_P$ to $\mathcal{U}_L$. Also we denote by j the embedding of $\mathcal{U}_P$ to $\mathcal{U}_G$. We shall sometimes also use similar maps i_-, j_- and p_- where $\mathcal{U}_P$ is replaced with $\mathcal{U}_{P_-}$. We have diagrams

$$\mathcal{U}_L \underset{i}{\overset{p}{\leftrightarrows}} \mathcal{U}_P \xrightarrow{j} \mathcal{U}_G, \qquad \mathcal{U}_L \underset{i_-}{\overset{p_-}{\leftrightarrows}} \mathcal{U}_{P_-} \xrightarrow{j_-} \mathcal{U}_G. \tag{4.4.2}$$

Definition 4.4.3. — We define the functor $\Phi_{L,G}$ by $i^* j^! = p_* j^!$.

We apply it to weakly A-equivariant objects, in particular on $\mathrm{Perv}(\mathcal{U}_G^d)$.

Warning. Of course, the functor $\Phi_{L,G}$ depends on P and not just on L. When we want to emphasize P, we write $\Phi_{L,G}^P$. Otherwise P is always chosen so that $P \supset B$ for the fixed Borel subgroup B.

Let us justify our notation $\mathcal{U}_P$ for the attracting set. We have a one parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that

$$\begin{aligned} P &= \left\{ g \in G \,\middle|\, \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists} \right\}, \\ L &= G^{\lambda(\mathbb{G}_m)} = \{ g \in G \mid \lambda(t) g = g \lambda(t) \text{ for any } t \in \mathbb{G}_m \}. \end{aligned} \tag{4.4.4}$$

Then we have

$$\begin{aligned} \mathcal{U}_P &\overset{\text{def.}}{=} \left\{ x \in \mathcal{U}_G \,\middle|\, \lim_{t\to 0} \lambda(t) \cdot x \text{ exists} \right\}, \\ \mathcal{U}_L &\overset{\text{def.}}{=} (\mathcal{U}_G)^{\lambda(\mathbb{G}_m)} = \{ x \in \mathcal{U}_G \mid \lambda(t) \cdot x = x \text{ for any } t \in \mathbb{G}_m \}. \end{aligned} \tag{4.4.5}$$

We embed G into $SL(N)$ and consider the corresponding space for $G = SL(N)$. We use the ADHM description for $\mathcal{U}_{SL(N)}$ to identify it with the affine GIT quotient as in

[**54**, Ch. 3]. Then $SL(N) = SL(W)$, and $\mathscr{U}_P$ coincides with the variety $\pi(\mathfrak{Z})$ studied in [**56**, §3]. Here π is Gieseker-Uhlenbeck morphism, and $\mathfrak{Z}$ is the attracting set in the Gieseker space, which will be denoted by $\widetilde{\mathscr{U}}_P$ later.

In [**56**, Rem. 3.16] it was remarked that $\mathfrak{Z}$ parametrizes framed torsion free sheaves having a filtration $E = E^0 \supset E^1 \supset \cdots \supset E^k \supset E^{k+1} = 0$. If all $F^i = E^i/E^{i+1}$ are locally free, E is a P-bundle. Thus $\mathscr{U}_P$ contains a possibly empty open subset $p^{-1}(\mathrm{Bun}_L)$ consisting of P-bundles.

Let us, however, note that $\mathscr{U}_P \cap \mathrm{Bun}_G$ is not entirely consisting of P-bundles, hence larger than $p^{-1}(\mathrm{Bun}_L)$: Consider a short exact sequence

$$0 \to F^2 \to E \to F^1 = \mathscr{I}_x \to 0,$$

arising from the Koszul resolution of the skyscraper sheaf at a point $x \in \mathbb{A}^2$. Here $\mathscr{I}_x$ is the ideal sheaf for x. Then $E \in \mathscr{U}_P \cap \mathrm{Bun}_G$, but E is not a P-bundle as F^1 is not locally free. More detailed analysis will be given in the proof of Proposition 5.8.9.

4.5. Associativity

Proposition 4.5.1. — *Let Q be another parabolic subgroup of G, contained in P and let M denote its Levi subgroup. Let Q_L be the image of Q in L and we identify M with the corresponding Levi group. Then we have a natural isomorphism of functors*

(4.5.2) $$\Phi_{M,L} \circ \Phi_{L,G} \cong \Phi_{M,G}.$$

Proof. — It is enough to show that

(4.5.3) $$\mathscr{U}_P \times_{\mathscr{U}_L} \mathscr{U}_{Q_L} = \mathscr{U}_Q,$$

as

$$p'_* j'^! p_* j^! = p'_* p''_* j''^! j^! = (p' \circ p'')_* (j \circ j'')^!$$

in the diagram

(4.5.4) $$\begin{tikzcd} \mathscr{U}_Q \arrow[r, "j''"] \arrow[d, "p''"'] & \mathscr{U}_P \arrow[r, "j"] \arrow[d, "p"] & \mathscr{U}_G \\ \mathscr{U}_{Q_L} \arrow[r, "j'"] \arrow[d, "p'"'] & \mathscr{U}_L & \\ \mathscr{U}_M & & \end{tikzcd}$$

The left hand side of (4.5.3) is just equal to $p^{-1}(\mathscr{U}_{Q_L})$. By embedding G into $SL(N)$ we may assume that $G = SL(N)$. In this case, we use the ADHM description to describe $\mathscr{U}_P$, $\mathscr{U}_Q$, $\mathscr{U}_{Q_L}$. By [**56**, Proof of Lemma 3.6], they are consisting of data (B_1, B_2, I, J) such that $JF(B_1, B_2)I$ are in P, Q, Q_L respectively, i.e., upper triangular in appropriate sense, for any products $F(B_1, B_2)$ of B_1, B_2 of arbitrary order. Now the assertion is clear. □

4.6. Preservation of perversity

The following is our first main result:

Theorem 4.6.1. — $\Phi_{L,G}(\mathrm{IC}(\mathcal{U}_G^d))$ *is perverse (and semi-simple, according to* [**13**, Theorem 2]*). Moreover, the same is true for any perverse sheaf in* $\mathrm{Perv}(\mathcal{U}_G^d)$.

The proof will be given in §A.

Let us remark that the result is easy to prove for type A, see [**60**, §4.4, Lemma 3]. The argument goes back to an earlier work by Varagnolo-Vasserot [**72**].

4.7. Hyperbolic restriction on Bun_L^d

Let us consider the restriction of $\Phi_{L,G}(\mathrm{IC}(\mathcal{U}_G^d))$ to the open subset Bun_L^d in this subsection.

For simplicity, suppose that $[L,L]$ has one simple factor so that the instanton numbers of L-bundles are the same as those of $[L,L]$-bundles. In particular, Bun_L^d is irreducible. Then $\mathrm{IC}(\mathcal{U}_L^d)$ is a simple perverse sheaf, and we study

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{U}_L^d)}(\mathrm{IC}(\mathcal{U}_L^d), \Phi_{L,G}(\mathrm{IC}(\mathcal{U}_G^d))). \tag{4.7.1}$$

We restrict (4.4.2) to the open subsets consisting of genuine bundles:

$$\mathrm{Bun}_L^d \overset{p}{\underset{i}{\leftrightarrows}} p^{-1}(\mathrm{Bun}_L^d) \xrightarrow{j} \mathrm{Bun}_G^d . \tag{4.7.2}$$

Let us take $\mathcal{F} \in \mathrm{Bun}_L^d$. Then the tangent space of Bun_L^d at $\mathcal{F}$ is $H^1(\mathbb{P}^2, \mathfrak{l}_{\mathcal{F}}(-\ell_\infty))$, where $\mathfrak{l}$ is the Lie algebra of L. This is the subspace of $H^1(\mathbb{P}^2, \mathfrak{g}_{\mathcal{F}}(-\ell_\infty)) = T_{\mathcal{F}} \mathrm{Bun}_G^d$, consisting of $Z(L)^0$-fixed elements. The normal bundle of Bun_L^d in Bun_G^d splits into the sum of $H^1(\mathbb{P}^2, \mathfrak{n}_{\mathcal{F}}(-\ell_\infty))$ and $H^1(\mathbb{P}^2, \mathfrak{n}^-_{\mathcal{F}}(-\ell_\infty))$, where $\mathfrak{n}$ is the nil radical of $\mathfrak{p} = \mathrm{Lie}\, P$, and $\mathfrak{n}^-$ is its opposite. They correspond to attracting and repellent directions respectively. Then $p^{-1}(\mathrm{Bun}_L^d)$ is a vector bundle over Bun_L^d, whose fiber at $\mathcal{F}$ is $H^1(\mathbb{P}^2, \mathfrak{n}_{\mathcal{F}}(-\ell_\infty))$. It parametrizes framed P-bundles. The morphism p is the projection and i is the inclusion of the zero section. Therefore we have the Thom isomorphism between $i^* j^!(\mathcal{C}_{\mathrm{Bun}_G^d})$ and $\mathcal{C}_{\mathrm{Bun}_L^d}$ up to shift.

Note further that $\dim p^{-1}(\mathrm{Bun}_L^d)$ is the half of the sum of dimensions of Bun_L^d and Bun_G^d, as $H^1(\mathbb{P}^2, \mathfrak{n}_{\mathcal{F}}(-\ell_\infty))$ and $H^1(\mathbb{P}^2, \mathfrak{n}^-_{\mathcal{F}}(-\ell_\infty))$ are dual to each other with respect to the symplectic form. Hence a shift is unnecessary, and the Thom isomorphism gives the canonical identification $i^* j^!(\mathcal{C}_{\mathrm{Bun}_G^d}) \cong \mathcal{C}_{\mathrm{Bun}_L^d}$. Therefore we normalize the canonical homomorphism

$$1^d_{L,G} \in \mathrm{Hom}_{\mathrm{Perv}(\mathcal{U}_L^d)}(\mathrm{IC}(\mathcal{U}_L^d), \Phi_{L,G}(\mathrm{IC}(\mathcal{U}_G^d))) \tag{4.7.3}$$

so that it is equal to the Thom isomorphism on the open subset.

Note also that a homomorphism in (4.7.1) is determined by its restriction to Bun_L^d, hence (4.7.1) is 1-dimensional from the above observation. And $1^d_{L,G}$ is its base.

If $[L, L]$ has more than one simple factors $G_1, G_2, \dots$, Bun_L^d is not irreducible as it is isomorphic to $\bigsqcup_{d_1+d_2+\cdots=d} \mathrm{Bun}_{G_1}^{d_1} \times \mathrm{Bun}_{G_2}^{d_2} \times \cdots$. Then $\mathrm{IC}(\mathcal{U}_L^d)$ must be understood as the direct sum

$$\bigoplus_{d_1+d_2+\cdots=d} \mathrm{IC}(\mathrm{Bun}_{G_1}^{d_1} \times \mathrm{Bun}_{G_2}^{d_2} \times \cdots). \tag{4.7.4}$$

In particular, (4.7.1) is not 1-dimensional. But it does not cause us any trouble. We have the canonical isomorphism for each summand, and $1_{L,G}^d$ is understood as their sum.

4.8. Space U^d and its base

We shall introduce the space U^d of homomorphisms from $\mathscr{C}_{S_{(d)}\mathbb{A}^2}$ to $\Phi_{L,G}(\mathrm{IC}(\mathcal{U}_G^d))$ and study its properties in this subsection. A part of computation is a byproduct of the proof of Theorem 4.6.1 (see Lemma 4.8.15). The study of U^d will be continued in the remainder of this section, and also in the next section.

Definition 4.8.1. — For $d > 0$, we define a vector space

$$\begin{aligned} U_{L,G}^d \equiv U^d &\overset{\text{def.}}{=} \mathrm{Hom}_{\mathrm{Perv}(\mathcal{U}_L^d)}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \Phi_{L,G}(\mathrm{IC}(\mathcal{U}_G^d))) \\ &= H^{-2}(S_{(d)}\mathbb{A}^2, \xi^! \Phi_{L,G}(\mathrm{IC}(\mathcal{U}_G^d))), \end{aligned} \tag{4.8.2}$$

where (d) is the partition of d consisting of a single entry d, and $\xi\colon S_{(d)}\mathbb{A}^2 \to \mathcal{U}_L^d$ is the inclusion.

We use the notation U^d, when L, G are clear from the context.

Since the hyperbolic restriction $\Phi_{L,G}$ depends on P, the space $U_{L,G}^d$ depends also on P. When we want to emphasize P, we denote it by $U_{L,G}^{d,P}$ or simply by $U^{d,P}$.

We have a natural evaluation homomorphism

$$U^d \otimes \mathscr{C}_{S_{(d)}\mathbb{A}^2} \to \Phi_{L,G}(\mathrm{IC}(\mathcal{U}_G^d)), \tag{4.8.3}$$

which gives the isotypical component of $\Phi_{L,G}(\mathrm{IC}(\mathcal{U}_G^d))$ corresponding to the simple perverse sheaf $\mathscr{C}_{S_{(d)}\mathbb{A}^2}$.

By the factorization §2.4 together with the Thom isomorphism $i^* j^!(\mathscr{C}_{\mathrm{Bun}_G^{d_1}}) \cong \mathscr{C}_{\mathrm{Bun}_L^{d_1}}$, we get

Proposition 4.8.4. — *We have the canonical isomorphism in* $\mathrm{Perv}(\mathcal{U}_L^d)$*:*

$$\Phi_{L,G}(\mathrm{IC}(\mathcal{U}_G^d)) \cong \bigoplus \mathrm{IC}(\mathrm{Bun}_{L,\lambda}^{d_1}, \rho). \tag{4.8.5}$$

Here ρ is the (semisimple) local system on $\mathrm{Bun}_{L,\lambda}^{d_1} = \mathrm{Bun}_L^{d_1} \times S_\lambda\mathbb{A}^2$ with $\lambda = (1^{n_1}2^{n_2}\cdots)$ corresponding to the representation of $S_{n_1} \times S_{n_2} \times \cdots$ on $(U^1)^{\otimes n_1} \otimes (U^2)^{\otimes n_2} \otimes \cdots$ given by permutation of factors.

Moreover the isomorphism is also in the equivariant category with respect to $L \times \mathbb{C}^* \times \mathbb{C}^*$.

For example, the isotypical component for the intersection cohomology complex $\mathrm{IC}(\mathrm{Bun}^{d_1}_{L,\lambda})$ for the trivial simple local system is

$$\mathrm{Sym}^{n_1} U^1 \otimes \mathrm{Sym}^{n_2} U^2 \otimes \cdots, \tag{4.8.6}$$

where Sym denotes the symmetric power.

The second statement is the consequence of the first as the spaces of homomorphisms between objects in $\mathrm{Perv}(\mathscr{U}^d_L)$ are canonically isomorphic for equivariant category with respect to $L \times \mathbb{C}^* \times \mathbb{C}^*$ and non-equivariant one. (See [**45**, 1.16(a)].) Therefore (4.8.5) is an isomorphism in the equivariant derived category, though we use the factorization, which is *not* equivariant with respect to $\mathbb{C}^* \times \mathbb{C}^*$.

Lemma 4.8.7. — *Suppose $L = T$. We have*

$$H^*(S^d\mathbb{A}^2, \Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))) \cong \bigoplus_{|\lambda|=d} \mathrm{Sym}^{n_1} U^1 \otimes \mathrm{Sym}^{n_2} U^2 \otimes \cdots \tag{4.8.8}$$

where $\lambda = (1^{n_1} 2^{n_2} \dots)$.

Proof. — Since $L = T$, we have $\mathscr{U}^d_T = S^d\mathbb{A}^2$. See Example 4.2.6. Then the assertion means that only trivial representation of $S_{n_1} \times S_{n_2} \times \cdots$ contribute to the global cohomology group.

Let U be an open subset of $(\mathbb{A}^2)^{n_1} \times (\mathbb{A}^2)^{n_2} \times \cdots$ consisting of pairwise disjoint n_1 ordered points, n_2 ordered points, and so on in $\mathbb{A}^2$. Forgetting orderings, we get an $(S_{n_1} \times S_{n_2} \times \cdots)$-covering $p\colon U \to S_\lambda\mathbb{A}^2$. The pushforward of the trivial rank 1 system with respect to p is the regular representation ρ_{reg} of $(S_{n_1} \times S_{n_2} \times \cdots)$.

Since p extends to a finite morphism $(\mathbb{A}^2)^{n_1} \times (\mathbb{A}^2)^{n_2} \times \cdots \to \overline{S_\lambda\mathbb{A}^2}$, we have $\mathrm{IC}(S_\lambda\mathbb{A}^2, \rho_{\mathrm{reg}}) = p_*(\mathscr{C}_{(\mathbb{A}^2)^{n_1} \times (\mathbb{A}^2)^{n_2} \times \cdots})$. By the Künneth theorem, the global cohomology group $H^*(\bullet)$ of the right hand side is $H^*((\mathbb{A}^2)^{n_1}) \otimes H^*((\mathbb{A}^2)^{n_2}) \otimes \cdots$. This is 1-dimensional, and corresponds to the trivial isotypical component of ρ_{reg}. Now the assertion follows. □

Let us continue the study of U^d. Let us note that all of our spaces $\mathscr{U}^d_G$, $\mathscr{U}^d_L$, $\mathscr{U}^d_P$ have trivial factors $\mathbb{A}^2$ given by the center of instantons, or the translation on the base space $\mathbb{A}^2$ except $d = 0$ where $\mathscr{U}^0_G = \mathscr{U}^0_L = \mathscr{U}^0_P = \mathrm{pt}$. We assume $d \neq 0$ hereafter. Let ${}^c\mathscr{U}^d_G$ denote the centered Uhlenbeck space at the origin, thus we have $\mathscr{U}^d_G = {}^c\mathscr{U}^d_G \times \mathbb{A}^2$. Let us compose factorization morphisms $\pi^d_{h,G}$, $\pi^d_{v,G}$ for the horizontal and vertical projections $h\colon \mathbb{A}^2 \to \mathbb{A}^1$, $v\colon \mathbb{A}^2 \to \mathbb{A}^1$ with the sum map $\sigma\colon S^d\mathbb{A}^1 \to \mathbb{A}^1$. Then ${}^c\mathscr{U}^d_G = (\sigma\pi^d_{h,G} \times \sigma\pi^d_{v,G})^{-1}(0,0)$. We use the notation ${}^c\mathscr{U}^d_L$, ${}^c\mathscr{U}^d_P$ for $\mathscr{U}^d_L$, $\mathscr{U}^d_P$ cases. The diagrams (4.4.2) factor and induce the diagrams for the centered spaces, and the factorization is compatible with the hyperbolic restriction. Let us use the same notation for i, j, p for the centered spaces. Then we have

$$U^d = H^0(\xi_0^! p_* j^! \, \mathrm{IC}({}^c\mathscr{U}^d_G)), \tag{4.8.9}$$

where ξ_0 is the inclusion of the single point $d\cdot 0$ in ${}^c\mathscr{U}^d_L$. Here $d\cdot 0$ is the point in $S_{(d)}\mathbb{A}^2$, the origin with multiplicity d.

By base change we get

$$U^d \cong H^0(p^{-1}(d\cdot 0), \tilde{j}^! \operatorname{IC}({}^c\mathscr{U}_G^d)), \tag{4.8.10}$$

where $\tilde{j}\colon p^{-1}(d\cdot 0) \to {}^c\mathscr{U}_G^d$ is the inclusion.

We have

Lemma 4.8.11. —

$$\dim U^d = \operatorname{rank} G - \operatorname{rank}[L, L]. \tag{4.8.12}$$

Proof. — According to a theorem of Laumon [**42**], given a constructible complex F on a complex algebraic variety X, and a morphism $f\colon X \to Y$, the classes $[Rf_*F]$ and $[Rf_!F]$ in the Grothendieck group of constructible complexes on Y coincide. In particular, $\chi(X, F) = \chi_c(X, F)$. It follows that the Euler characteristic of the stalk of $\operatorname{IC}(\mathscr{U}_G^d)$ at a point of $S_{(d)}\mathbb{A}^2$ is equal to the Euler characteristic of the stalk of $\Phi_{L,G}(\operatorname{IC}(\mathscr{U}_G^d))$ at the same point; the former was computed in Theorem 7.10 in [**21**].

Now let us give a proof in the case $L = T$. Then it is easy to see that Proposition 4.8.4 implies that the stalk of $\Phi_{T,G}(\operatorname{IC}(\mathscr{U}_G^d))$ at a point of $S_{(d)}\mathbb{A}^2$ is isomorphic to $\operatorname{Sym}^d(\bigoplus_i U_{T,G}^i)$, where we regard $\bigoplus_i U_{T,G}^i$ as a graded vector space (with the natural grading coming from i) and the super-script d means degree d with respect to that grading. On the other hand, [**21**, Theorem 7.10] implies that a similar description fits the stalk of $\operatorname{IC}(\mathscr{U}_G^d)$ at a point of $S_{(d)}\mathbb{A}^2$ if we disregard the cohomological grading (the "first" grading in the language of [**21**]) and take a $\operatorname{rank}(G)$-dimensional space V^i in place of $U_{T,G}^i$ above. We get $\dim U_{T,G}^d = \operatorname{rank}(G)$ for every d by induction in d.

Let us now consider the case of arbitrary L. Again, it is easy to deduce from Proposition 4.8.4 that the stalk of $\Phi_{T,L}(\Phi_{L,G}(\operatorname{IC}(\mathscr{U}_G^d))) \simeq \Phi_{T,G}(\operatorname{IC}(\mathscr{U}_G^d))$ at a point of $S_{(d)}\mathbb{A}^2$ is isomorphic to

$$\bigoplus_{d_1+d_2=d} \operatorname{Sym}^{d_1}\Big(\bigoplus_i U_{T,L}^i\Big) \otimes \operatorname{Sym}^{d_2}\Big(\bigoplus_j U_{L,G}^j\Big),$$

where the meaning of the super-scripts d_1 and d_2 is as above. In view of the preceding paragraph, we get $\dim U_{L,G}^d = \operatorname{rank}(G) - \operatorname{rank}([L, L])$. □

The dimension estimate Corollary A.9.2 and the argument in [**49**, Prop. 3.10] implies that

$$\begin{aligned} &H^0(p^{-1}(d\cdot 0), \tilde{j}^! \operatorname{IC}({}^c\mathscr{U}_G^d)) \\ &\cong H^0(p^{-1}(d\cdot 0) \cap \operatorname{Bun}_G^d, \tilde{j}^! \operatorname{IC}({}^c\mathscr{U}_G^d)) \\ &= H_{[0]}(p^{-1}(d\cdot 0) \cap \operatorname{Bun}_G^d, \mathbb{C}). \end{aligned} \tag{4.8.13}$$

Here we use the degree shift convention of the Borel-Moore homology group (see Convention (iv)), which is shift by $\dim {}^c\mathscr{U}_G^d = 2dh^\vee - 2$ in this case.

Let us set

$$\mathscr{U}_{P,0}^d \overset{\text{def.}}{=} p^{-1}(d\cdot 0). \tag{4.8.14}$$

The subscript 0 stands for $d \cdot 0$, and this convention will be also used later. More generally, we denote $p^{-1}(x)$ by $\mathscr{U}^d_{P,x}$ for $x \in \mathscr{U}^d_L$.

Then $H_{[0]}(\mathscr{U}^d_{P,0} \cap \mathrm{Bun}^d_G, \mathbb{C})$ has a base given by $(dh^\vee - 1)$-dimensional irreducible components of $\mathscr{U}^d_{P,0} \cap \mathrm{Bun}^d_G$. The dimension estimate Corollary A.9.2 implies that $\mathscr{U}^d_{P,0} \cap \mathrm{Bun}^{d'}_G$ $(d' < d)$ is lower-dimensional. Therefore

Lemma 4.8.15. — *We have*

$$U^d \cong H_{[0]}(\mathscr{U}^d_{P,0}). \tag{4.8.16}$$

This space has a base given by $(dh^\vee - 1)$*-dimensional irreducible components of* $\mathscr{U}^d_{P,0}$.

4.9. Irreducible components

Let us describe $(dh^\vee - 1)$-dimensional irreducible components of $\mathscr{U}^d_{P,0}$ for $P = B$ explicitly. We believe that there is no irreducible component of smaller dimension (see Remark A.7.3), but we do not have a proof.

First consider the case $G = SL(2)$. By Lemma 4.8.11 we have $\dim U^d = 1$, and hence $\mathscr{U}^d_{B,0}$ has only one $(2d - 1)$-dimensional irreducible component. As we have observed in the previous subsection, it is the closure of $\mathscr{U}^d_{B,0} \cap \mathrm{Bun}^d_G$. In §5.8, it will be shown that $\mathscr{U}^d_{B,0} \cap \mathrm{Bun}^d_G$ consists of rank 2 vector bundles E arising from a short exact sequence

$$0 \to \mathscr{O} \to E \to \mathscr{I} \to 0 \tag{4.9.1}$$

compatible with framing, where $\mathscr{I}$ is an ideal sheaf of colength d.

For a general G, consider the diagram (4.5.4) with $M = T$, $L = L_i$ the Levi subgroup corresponding to a simple root α_i. Note that $[L_i, L_i] \cong SL(2)$, and hence $\mathscr{U}^d_{L_i}$ is homeomorphic to $\mathscr{U}^d_{SL(2)}$. Therefore $\mathscr{U}^d_{B_{L_i},0} \cap \mathrm{Bun}^d_{L_i}$ is irreducible of dimension $2d - 1$ by the above consideration.

Proposition 4.9.2. — *The irreducible components of* $\mathscr{U}^d_{B,0}$ *of dimension* $dh^\vee - 1$ *are the closures of* $p^{-1}(\mathscr{U}^d_{B_{L_i},0} \cap \mathrm{Bun}^d_{L_i})$ *for* $i \in I$.

Definition 4.9.3. — Let us denote the closure of $p^{-1}(\mathscr{U}^d_{B_{L_i},0} \cap \mathrm{Bun}^d_{L_i})$ by Y_i.

Proof. — Consider the upper right part of (4.5.4), which is (4.4.2). Its restriction to the open subset $\mathrm{Bun}^d_{L_i}$ has been described in §4.7. As p is a vector bundle whose rank is equal to the half of the codimension of $\mathrm{Bun}^d_{L_i}$ in Bun^d_G, it follows that the inverse image $p^{-1}(\mathscr{U}^d_{B_{L_i},0} \cap \mathrm{Bun}^d_{L_i})$ is irreducible and has dimension $dh^\vee - 1$. Therefore the closure of $p^{-1}(\mathscr{U}^d_{B_{L_i},0} \cap \mathrm{Bun}^d_{L_i})$ is an irreducible component of $\mathscr{U}^d_{B,0}$.

Since $\dim U^d = \operatorname{rank} G$ by Lemma 4.8.11, it is enough to check that $p^{-1}(\mathscr{U}^d_{B_{L_i},0} \cap \mathrm{Bun}^d_{L_i}) \neq p^{-1}(\mathscr{U}^d_{B_{L_j},0} \cap \mathrm{Bun}^d_{L_j})$ if $i \neq j$. When $G = SL(r)$, $\mathscr{U}^d_{B,0} \cap \mathrm{Bun}^d_G$ consists of vector bundles E having a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$ compatible with framing. Moreover $p^{-1}(\mathscr{U}^d_{B_{L_i},0} \cap \operatorname{Bun}^d_{L_i})$ consists of those with $c_2(E_i/E_{i-1}) = d$ and $c_2(E_j/E_{j-1}) = 0$ for $j \neq i$. Therefore $p^{-1}(\mathscr{U}^d_{B_{L_i},0} \cap \operatorname{Bun}^d_{L_i}) \neq p^{-1}(\mathscr{U}^d_{B_{L_j},0} \cap \operatorname{Bun}^d_{L_j})$ for $i \neq j$. (See §5.8 for detail.) For a general G, we embed G into $SL(N)$. Then we need to replace B by a parabolic P, but $p^{-1}(\mathscr{U}^d_{B_{L_i},0} \cap \operatorname{Bun}^d_{L_i})$ is embedded into a corresponding space, and the same argument still works. □

4.10. A pairing on U^d

Let us introduce a pairing between $U^{d,P}$ and U^{d,P_-} in this subsection.

We combine Braden's isomorphism (3.4.2) with the natural homomorphism $\xi_0^! \to \xi_0^*$ to get

$$H^0(\xi_0^! i^* j^! \operatorname{IC}({}^c\mathscr{U}^d_G)) \to H^0(\xi_0^* i_-^! j_-^* \operatorname{IC}({}^c\mathscr{U}^d_G)). \tag{4.10.1}$$

The right hand side is dual to

$$U^{d,P_-} = H^0(\xi_0^! i_-^* j_-^! \operatorname{IC}({}^c\mathscr{U}^d_G)). \tag{4.10.2}$$

Thus we have a pairing between $U^{d,P}$ and U^{d,P_-}. Following the convention in [**46**, 3.1.3], we multiply the pairing by the sign $(-1)^{\dim {}^c\mathscr{U}^d_G/2} = (-1)^{dh^\vee - 1}$. Let us denote it by $\langle\ ,\ \rangle$. When we want to emphasize that it depends on the choice of the parabolic subgroup P, we denote it by $\langle\ ,\ \rangle_P$.

Since $\xi_0^! \mathbb{C}_{d\cdot 0} \to \xi_0^* \mathbb{C}_{d\cdot 0}$ is obviously an isomorphism, this pairing is nondegenerate.

The transpose of the homomorphism $U^{d,P} \to (U^{d,P_-})^\vee$ is a linear map $U^{d,P_-} \to (U^{d,P})^\vee$. It is

$$H^0(\xi_0^! i_-^* j_-^! \operatorname{IC}({}^c\mathscr{U}^d_G)) \to H^0(\xi_0^* i^! j^* \operatorname{IC}({}^c\mathscr{U}^d_G)), \tag{4.10.3}$$

given by the transpose of the composite of $\xi_0^! \to \xi_0^*$ and Braden's isomorphism $i^* j^! \to i_-^! j_-^*$. They are the same as original homomorphisms $\xi_0^! \to \xi_0^*$ and $i_-^* j_-^! \to i^! j^*$ respectively. It means that

$$\langle u, v\rangle_P = \langle v, u\rangle_{P_-} \quad \text{for } u \in U^{d,P},\ v \in U^{d,P_-}, \tag{4.10.4}$$

where $\langle\ ,\ \rangle_{P_-}$ is the pairing defined with respect to the opposite parabolic, i.e., one given after exchanging i, j and i_-, j_- respectively.

4.11. Another base of U^d

We next construct another base of $U^d = U^d_{T,G}$ for $L = T$, which is $(\operatorname{rank} G)$-dimensional by Lemma 4.8.11. This new base is better behaved under hyperbolic restrictions than the previous one given by irreducible components.

This subsection is preliminary, and the construction will be completed in §6.2.

We study $U^d_{T,G}$, using the associativity of the hyperbolic localization (Proposition 4.5.1) for $M = T$, $L = L_i$ the Levi subgroup corresponding to a simple root α_i. Since various Levi subgroups appear, we use the notation $U^d_{T,G}$ indicating groups we are considering.

Note that $[L_i, L_i] \cong SL(2)$, and hence $\mathscr{U}^d_{L_i}$ is homeomorphic to $\mathscr{U}^d_{SL(2)}$. We understand $\mathrm{IC}(\mathscr{U}^d_{L_i})$ as $\mathrm{IC}(\mathscr{U}^d_{SL(2)})$ and apply Lemma 4.8.11 to see that

$$U^d_{T,L_i} = \mathrm{Hom}_{\mathrm{Perv}(\mathscr{U}^d_T)}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \Phi_{T,L_i}(\mathrm{IC}(\mathscr{U}^d_{L_i}))) \tag{4.11.1}$$

is 1-dimensional. In the next section, we shall introduce an element $1^d_{L_i}$ in U^d_{T,L_i} using the theory of the stable envelope in [**46**].

Taking $L = L_i$ in the construction in §4.7, we apply the functor Φ_{T,L_i}. By Proposition 4.5.1 we get an element

$$\Phi_{T,L_i}(1^d_{L_i,G}) \in \mathrm{Hom}_{\mathrm{Perv}(\mathscr{U}^d_T)}(\Phi_{T,L_i}(\mathrm{IC}(\mathscr{U}^d_{L_i})), \Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))). \tag{4.11.2}$$

Composing with the element $1^d_{L_i}$ in U^d_{T,L_i} mentioned just above, we get

$$\Phi_{T,L_i}(1^d_{L_i,G}) \circ 1^d_{L_i} \in U^d_{T,G}. \tag{4.11.3}$$

We have (rank G)-choices of i. Then we will show that

$$\{\tilde{\alpha}^d_i \overset{\text{def.}}{=} \Phi_{T,L_i}(\delta 1^d_{L_i,G}) \circ 1^d_{L_i}\}_i \tag{4.11.4}$$

gives a basis of $U^d_{T,G}$ in the next subsection. Here we will introduce an appropriate polarization $\delta = \pm 1$, using a consideration of rank 2 case. See (6.2.1). Moreover, this will give us an identification $U^d_{T,G}$ with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\tilde{\alpha}^d_i$ is sent to the i^{th} simple coroot $\alpha^\vee_i$. See a remark after Proposition 6.3.8.

We normalize the inclusion $\mathrm{IC}(\mathscr{U}^d_{L_i}) \to \Phi_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G))$ by $\delta 1^d_{L_i,G}$ as above. Then the projection $\Phi_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G)) \to \mathrm{IC}(\mathscr{U}^d_{L_i})$ is also determined, as $\mathrm{IC}(\mathscr{U}^d_{L_i})$ has multiplicity 1 in $\Phi_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G))$ (see §4.7). Therefore we have the canonical isomorphism

$$\Phi_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G)) \cong \mathrm{IC}(\mathscr{U}^d_{L_i}) \oplus \mathrm{IC}(\mathscr{U}^d_{L_i})^\perp, \tag{4.11.5}$$

where $\mathrm{IC}(\mathscr{U}^d_{L_i})^\perp$ is the sum of isotypical components for simple factors not isomorphic to $\mathrm{IC}(\mathscr{U}^d_{L_i})$. Applying Φ_{T,L_i} and using $\Phi_{T,L_i}\Phi_{L_i,G} = \Phi_{T,G}$, we get an induced decomposition

$$U^d_{T,G} = U^d_{T,L_i} \oplus (U^d_{T,L_i})^\perp. \tag{4.11.6}$$

This decomposition is orthogonal with respect to the pairing in §4.10 in the following sense. We have the decomposition $U^{d,B_-}_{T,G} = U^{d,B_-\cap L_i}_{T,L_i} \oplus (U^{d,B_-\cap L_i}_{T,L_i})^\perp$ for the opposite Borel B_-, and

$$\langle U^d_{T,L_i}, (U^{d,B_-\cap L_i}_{T,L_i})^\perp \rangle = 0 = \langle (U^d_{T,L_i})^\perp, U^{d,B_-\cap L_i}_{T,L_i} \rangle. \tag{4.11.7}$$

Moreover the restriction of the pairing to $U^{d,B\cap L_i}_{T,L_i}$, $U^{d,B_-\cap L_i}_{T,L_i}$ coincides with one defined via $\mathscr{U}^d_{L_i}$.

4.12. Dual base

Let $\tilde{\alpha}_i^{d,-}$ denote the element defined as $\tilde{\alpha}_i^d$ for the opposite Borel. We shall prove

$$\langle [Y_j], \tilde{\alpha}_i^{d,-}\rangle = \pm\delta_{ij}(-1)^{d-1}d \tag{4.12.1}$$

modulo the computation for $G = SL(2)$, corresponding to the case $i = j$ in this subsection. The computation for $G = SL(2)$ will be given in Remark 5.13.9. This formula means that $\tilde{\alpha}_i^{d,-}$ is the dual base to the base given by irreducible components Y_j with respect to the pairing $\frac{(-1)^{d-1}}{d}\langle\ ,\ \rangle$ up to sign.

Consider the diagram (4.5.4) for the centered version, where we take $M = T$, $L = L_i$ as in §4.11. Let us consider the open embedding of ${}^c\operatorname{Bun}_{L_i}^d$ to ${}^c\mathscr{U}_{L_i}^d$. We have the corresponding restriction homomorphism

$$\begin{aligned}
U_{T,G}^d &= H^0(\xi_0^!(p'\circ p'')_*(j\circ j'')^!\operatorname{IC}({}^c\mathscr{U}_G^d))\\
&\cong H^0(\xi_0^! p'_* j'^!\Phi_{L_i,G}(\operatorname{IC}({}^c\mathscr{U}_G^d)))\\
&\cong H^0(p'^{-1}(d\cdot 0), \tilde{j}'^!\Phi_{L_i,G}(\operatorname{IC}({}^c\mathscr{U}_G^d)))\\
&\quad \to H^0(p'^{-1}(d\cdot 0)\cap {}^c\operatorname{Bun}_{L_i}^d, \tilde{j}'^!\Phi_{L_i,G}(\operatorname{IC}({}^c\mathscr{U}_G^d))),
\end{aligned} \tag{4.12.2}$$

where $\tilde{j}'$ is the restriction of j' to $p'^{-1}(d\cdot 0)$. When we restrict $\Phi_{L_i,G}(\operatorname{IC}({}^c\mathscr{U}_G^d))$ to the open set ${}^c\operatorname{Bun}_{L_i}^d$, the first summand $\operatorname{IC}(\mathscr{U}_{L_i}^d)$ in the decomposition (4.11.5) is replaced by the constant sheaf $\mathscr{C}_{{}^c\operatorname{Bun}_{L_i}^d}$, and the second summand is killed. Therefore we have an isomorphism

$$H^0(p'^{-1}(d\cdot 0)\cap {}^c\operatorname{Bun}_{L_i}^d, \tilde{j}'^!\Phi_{L_i,G}(\operatorname{IC}({}^c\mathscr{U}_G^d)))$$
$$\cong H_{[0]}(p'^{-1}(d\cdot 0)\cap {}^c\operatorname{Bun}_{L_i}^d, \mathbb{C}) \cong U_{T,L_i}^d,$$

where the second isomorphism is nothing but (4.8.13) for G replaced by L_i.

Thus the projection $U_{T,G}^d \to U_{T,L_i}^d$ to the first summand in (4.11.6) is nothing but the restriction homomorphism we have just constructed.

Let us further consider the restriction of the upper right corner of the diagram (4.5.4) to the open subset ${}^c\operatorname{Bun}_{L_i}^d$. Then

$$p^{-1}(p'^{-1}(d\cdot 0)\cap {}^c\operatorname{Bun}_{L_i}^d) = p^{-1}(\mathscr{U}_{B_{L_i},0}^d \cap {}^c\operatorname{Bun}_{L_i}^d)$$

has been studied in §4.9: Its closure is an irreducible component of $\mathscr{U}_{B,0}^d$. By the base change the restriction to ${}^c\operatorname{Bun}_{L_i}^d$ is replaced by one to $p^{-1}(\mathscr{U}_{B_{L_i},0}^d\cap {}^c\operatorname{Bun}_{L_i}^d)$, and we can replace relevant IC sheaves by constant sheaves. The Thom isomorphism gives us $p_* j^!\mathscr{C}_{\operatorname{Bun}_G^d} \cong \mathscr{C}_{\operatorname{Bun}_{L_i}^d}$ as in §4.7. Note that the intersection of an irreducible component Y_j of Proposition 4.9.2 with the open subset $p^{-1}(\mathscr{U}_{B_{L_i},0}^d \cap^c \operatorname{Bun}_{L_i}^d)$ is lower-dimensional if $i\neq j$, as $p^{-1}(\mathscr{U}_{B_{L_i},0}^d\cap^c\operatorname{Bun}_{L_i}^d)$ is irreducible. Therefore the fundamental class of Y_j goes to 0 under the restriction. Hence we have (4.12.1) for $i\neq j$ by (4.11.7). In fact, we will see that $Y_j\cap p^{-1}(\mathscr{U}_{B_{L_i},0}^d\cap {}^c\operatorname{Bun}_{L_i}^d) = \varnothing$ for type A in §5.8, and the same is true for any G thanks to an embedding $G\to SL(N)$.

The Thom isomorphism sends $[Y_i]$ to $[\mathcal{U}^d_{B_{L_i},0}]$ from the definition of Y_i. The sign in (4.12.1) appears as we multiply the Thom isomorphism by a polarization δ (see (6.2.1) below). Therefore the computation of (4.12.1) for $i = j$ is reduced to the case $G = SL(2)$. The relevant computation will be given in Remark 5.13.9 as we mentioned above.

4.13. $\operatorname{Aut}(G)$ invariance

Let $\operatorname{Aut}(G)$ be the group of automorphisms of G. Its natural action on Bun^d_G extends to $\mathcal{U}^d_G$ ([**21**, §6.1]).

Let us fix a cocharacter $\lambda\colon \mathbb{G}_m \to G$, and consider our construction with respect to $\sigma \circ \lambda$ for $\sigma \in \operatorname{Aut}(G)$. Here $L = G^{\lambda(\mathbb{G}_m)}$ is considered as a fixed Levi subgroup. Substituting $\sigma\circ\lambda$ into λ in the Formula (4.4.4), we define a pair (P^σ, L^σ) of a parabolic subgroup and its Levi part. The action $\varphi_\sigma\colon \mathcal{U}^d_G \to \mathcal{U}^d_G$ induces $\varphi_\sigma\colon \mathcal{U}^d_P \to \mathcal{U}^d_{P^\sigma}$, $\varphi_\sigma\colon \mathcal{U}^d_L \to \mathcal{U}^d_{L^\sigma}$, and we have a commutative diagram

(4.13.1)
$$\begin{tikzcd}
\mathcal{U}^d_L \arrow[r, "i"] \arrow[d, "\varphi_\sigma"'] & \mathcal{U}^d_P \arrow[r, "j"] \arrow[d, "\varphi_\sigma"'] & \mathcal{U}^d_G \arrow[d, "\varphi_\sigma"'] \\
\mathcal{U}^d_{L^\sigma} \arrow[r, "i_\sigma"'] & \mathcal{U}^d_{P^\sigma} \arrow[r, "j_\sigma"'] & \mathcal{U}^d_G,
\end{tikzcd}$$

where the subscript σ indicates morphisms between spaces for $\sigma \in \operatorname{Aut}(G)$.

Since $\operatorname{IC}(\mathcal{U}^d_G)$ is an $\operatorname{Aut}(G)$-equivariant perverse sheaf, we have an isomorphism $\varphi^*_\sigma \operatorname{IC}(\mathcal{U}^d_G) \cong \operatorname{IC}(\mathcal{U}^d_G)$. Therefore we have an isomorphism

(4.13.2)
$$i^* j^! \operatorname{IC}(\mathcal{U}^d_G) \cong \varphi^*_\sigma i^*_\sigma j^!_\sigma \operatorname{IC}(\mathcal{U}^d_G).$$

The isomorphism (4.13.2) is equivariant in the following sense: The right hand side is a $\mathbb{T}^\sigma = T^\sigma \times \mathbb{C}^* \times \mathbb{C}^*$-equivariant perverse sheaf, while the left hand side is $\mathbb{T}$-equivariant. The isomorphism (4.13.2) respects equivariant structures under the group isomorphism $\sigma\colon \mathbb{T} \xrightarrow{\cong} \mathbb{T}^\sigma$. In particular, we have an isomorphism

(4.13.3)
$$\varphi_\sigma\colon H^*_{\mathbb{T}}(\mathcal{U}^d_L, i^* j^! \operatorname{IC}(\mathcal{U}^d_G)) \xrightarrow{\cong} H^*_{\mathbb{T}^\sigma}(\mathcal{U}^d_{L^\sigma}, i^*_\sigma j^!_\sigma \operatorname{IC}(\mathcal{U}^d_G)),$$

which respects the $H^*_{\mathbb{T}}(\mathrm{pt})$ and $H^*_{\mathbb{T}^\sigma}(\mathrm{pt})$ structures via $\mathbb{T} \cong \mathbb{T}^\sigma$.

In the same way, we obtain a canonical isomorphism

(4.13.4)
$$U^{d,P}_{L,G} \xrightarrow{\cong} U^{d,P^\sigma}_{L^\sigma,G},$$

which is denoted also by φ_σ for brevity.

The pairing $\langle\ ,\ \rangle$ in §4.10 is compatible with φ_σ: Let us denote by $\langle\ ,\ \rangle_{P^\sigma}$ the pairing between $U^{d,P^\sigma}_{L^\sigma,G}$ and $U^{d,P^\sigma_-}_{L^\sigma,G}$. We have $\varphi_\sigma\colon U^{d,P_-}_{L,G} \xrightarrow{\cong} U^{d,P^\sigma_-}_{L^\sigma,G}$ as above, and the following holds

(4.13.5)
$$\langle \varphi_\sigma(u), \varphi_\sigma(v)\rangle_{P^\sigma} = \langle u, v\rangle_P, \qquad u \in U^{d,P}_{L,G}, v \in U^{d,P_-}_{L,G}.$$

The decomposition (4.11.5) is transferred under φ_σ to

$$i_\sigma^* j_\sigma^! \operatorname{IC}(\mathcal{U}_G^d) \cong \pm \operatorname{IC}(\mathcal{U}^d_{L_i^\sigma}) \oplus \operatorname{IC}(\mathcal{U}^d_{L_i^\sigma})^\perp. \tag{4.13.6}$$

Here the sign ± means that we multiply the projection to $\operatorname{IC}(\mathcal{U}^d_{L_i^\sigma})$ by ±, according to whether σ respects the polarization δ for $\mathcal{U}^d_{L_i}$ and $\mathcal{U}^d_{L_i^\sigma}$ or not. Our polarization will be invariant under inner automorphisms, so the sign depends on diagram automorphisms $\operatorname{Aut}(G)/\operatorname{Inn}(G)$. The decomposition (4.11.6) is mapped to

$$U_{T^\sigma,G}^{d,B^\sigma} = U_{T^\sigma,L_i^\sigma}^{d,B^\sigma\cap L_i^\sigma} \oplus (U_{T^\sigma,L_i^\sigma}^{d,B^\sigma\cap L_i^\sigma})^\perp. \tag{4.13.7}$$

Suppose $\sigma \in L$. We have $L^\sigma = L$, $P^\sigma = P$, $i_\sigma = i$, $j_\sigma = j$. Then $i^* j^! \operatorname{IC}(\mathcal{U}_G^d)$ is an L-equivariant perverse sheaf, and (4.13.2) is the isomorphism induced by the equivariant structure.

Let us further assume $L = T$. Then T acts trivially on $\mathcal{U}_T^d = S^d\mathbb{A}^2$, and $\varphi_\sigma|_{\mathcal{U}_T^d} = \mathrm{id}$. The equivariant structure of the T-equivariant perverse sheaf $i^* j^! \operatorname{IC}(\mathcal{U}_G^d)$ is *trivial*. In particular, the isomorphism (4.13.2) is the identity. Therefore (4.13.2) is well-defined for $\sigma \in \operatorname{Aut}(G)/(T/Z(G))$, where $Z(G)$ is the center of G.

Note that chambers of hyperbolic restrictions for $L = T$ are Weyl chambers. They appear as a subfamily for $W = N_G(T)/T$ in $\operatorname{Aut}(G)/(T/Z(G))$.

Let us take $\sigma = w_0$, the longest element of the Weyl group. Then $B^{w_0} = B_-$. We come back to B via (4.10.4), and hence we get

$$\langle u, v\rangle_B = \langle \varphi_{w_0}(u), \varphi_{w_0}(v)\rangle_{B_-} = \langle \varphi_{w_0}(v), \varphi_{w_0}(u)\rangle_B \tag{4.13.8}$$

for $u \in U_{T,G}^{d,B}$,$v \in U_{T,G}^{d,B_-}$.

We can take $\sigma \in \operatorname{Aut}(G)$, which preserves T and the set of positive roots, and induces a Dynkin diagram automorphism. Then $B^\sigma = B$. Hence $U_{T,G}^{d,B}$ is a representation of the group of Dynkin diagram automorphisms. The inner product is preserved.

We have $L_i^\sigma = L_{\sigma(i)}$, where $\sigma(i)$ is the vertex of the Dynkin diagram, the image of i under the corresponding Dynkin diagram automorphism. From (4.13.6) $\varphi_\sigma(\tilde\alpha_i^d)$ is equal to $\tilde\alpha^d_{\sigma(i)}$ up to scalar. We will prove the following in §5.14.

Lemma 4.13.9. — *We have*

$$\varphi_\sigma(\tilde\alpha_i^d) = \pm\tilde\alpha^d_{\sigma(i)}, \tag{4.13.10}$$

where ± is the ratio of the polarizations for $\operatorname{Bun}^d_{L_i}$ *and* $\operatorname{Bun}^d_{L_{\sigma(i)}}$, *compared under* φ_σ.

CHAPTER 5

HYPERBOLIC RESTRICTION IN TYPE A

We shall study the case $G = SL(r)$ in detail in this section.

We have the moduli space $\widetilde{\mathcal{U}}_r^d$ of framed torsion free sheaves (E, φ) of rank r, second Chern class d over $\mathbb{P}^2$. It is called the *Gieseker space*. We have a projective morphism π (the *Gieseker-Uhlenbeck morphism*) from $\widetilde{\mathcal{U}}_r^d$ to the corresponding Uhlenbeck space $\mathcal{U}_G^d$. It is known that $\widetilde{\mathcal{U}}_r^d$ is smooth and π is a semi-small resolution of singularities. Therefore we can study $\mathrm{IC}(\mathcal{U}_G^d)$ via the constant sheaf $\mathcal{C}_{\widetilde{\mathcal{U}}_r^d}$ over $\widetilde{\mathcal{U}}_r^d$.

If $r = 1$, we understand $\widetilde{\mathcal{U}}_1^d$ as the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{A}^2)$ of d points on $\mathbb{A}^2$, while $\mathcal{U}_{SL(1)}^d$ is the symmetric power $S^d\mathbb{A}^2$.

5.1. Gieseker-Uhlenbeck

Let us first explain the relation between $\mathrm{IC}(\mathcal{U}_G^d)$ and $\mathcal{C}_{\widetilde{\mathcal{U}}_r^d}$ in more detail.

Theorem 5.1.1 **([6, §3]).** — *The Gieseker-Uhlenbeck morphism* $\pi\colon \widetilde{\mathcal{U}}_r^d \to \mathcal{U}_G^d$ *is semi-small with respect to the standard stratification* (2.3.1). *All strata are relevant and fibers are irreducible. Therefore*

$$\pi_! \mathcal{C}_{\widetilde{\mathcal{U}}_r^d} \cong \bigoplus_{d_1+|\lambda|=d} H_{\mathrm{top}}(\pi^{-1}(x_\lambda^{d_1})) \otimes \mathrm{IC}(\mathrm{Bun}_{G,\lambda}^{d_1}), \tag{5.1.2}$$

where $x_\lambda^{d_1}$ *is a point in the stratum* $\mathrm{Bun}_{G,\lambda}^{d_1}$.

(See also [**54**, Ch. 3,5,6], where $\widetilde{\mathcal{U}}_r^d$, $\mathcal{U}_G^d$ are denoted by $\mathcal{M}(n,r)$, $\mathcal{M}_0(n,r)$ respectively. See also [**61**, Ch.3] for the detail on the irreducibility of fibers.)

Since $\mathrm{IC}(\mathcal{U}_{G,\lambda}^{d_1})$ is isomorphic to the pushforward of $\mathrm{IC}(\mathcal{U}_G^{d_1}) \boxtimes \mathcal{C}_{\overline{S_\lambda\mathbb{A}^2}}$ under the finite morphism (2.3.4), we have

$$H^{[*]}_{\mathbb{T}}(\widetilde{\mathcal{U}}_r^d) \cong \bigoplus \mathrm{IH}^{[*]}_{\mathbb{T}}(\mathcal{U}_G^{d_1}) \otimes H_{\mathrm{top}}(\pi^{-1}(x_\lambda^{d_1})) \otimes H^{[*]}_{\mathbb{T}}(\overline{S_\lambda\mathbb{A}^2}). \tag{5.1.3}$$

We also have the corresponding isomorphism for the cohomology with compact support.

5.2. Heisenberg operators

For $r = 1$, the third author and Grojnowski independently constructed operators acting on the direct sum of homology groups of $\widetilde{\mathcal{U}}_1^{d}$ satisfying the Heisenberg relation (see [**54**, Ch. 8]). It was extended by Baranovsky to higher rank case [**6**]. Let us review his construction in this subsection.

We consider here both $H_{\mathbb{T}}^{[*]}(\widetilde{\mathcal{U}}_r^{d})$ and $H_{\mathbb{T},c}^{[*]}(\widetilde{\mathcal{U}}_r^{d})$, the equivariant cohomology with arbitrary and compact support, which is Poincaré dual to Borel-Moore and the ordinary equivariant homology groups. To save the notation, we use the notation $H_{\mathbb{T}(,c)}^{[*]}(\widetilde{\mathcal{U}}_r^{d})$ meaning either of cohomology groups.

For $n > 0$ we consider subvariety

$$P_n \subset \bigsqcup_d \widetilde{\mathcal{U}}_r^{d} \times \widetilde{\mathcal{U}}_r^{d+n} \times \mathbb{A}^2, \tag{5.2.1}$$

consisting of triples (E_1, E_2, x) such that $E_1 \supset E_2$ and E_1/E_2 is supported at x. We have

Proposition 5.2.2. — *P_n is half-dimensional in $\widetilde{\mathcal{U}}_r^{d} \times \widetilde{\mathcal{U}}_r^{d+n} \times \mathbb{A}^2$ for each d.*

Let us denote the projection to the third factor by Π. For a cohomology class $\alpha \in H_{\mathbb{T}(,c)}^{[*]}(\mathbb{A}^2)$, we consider $P_{-n}^{\Delta}(\alpha) = [P_n] \cap \Pi^*(\alpha)$ as a correspondence in $\widetilde{\mathcal{U}}_r^{d} \times \widetilde{\mathcal{U}}_r^{d+k}$. Then we have the convolution product

$$P_{-n}^{\Delta}(\alpha)\colon H_{\mathbb{T}(,c)}^{[*]}(\widetilde{\mathcal{U}}_r^{d}) \to H_{\mathbb{T}(,c)}^{[*+\deg\alpha]}(\widetilde{\mathcal{U}}_r^{d+n}). \tag{5.2.3}$$

Thanks to the previous proposition, the shift of the degree is simple in our perverse degree convention. The reason why we put Δ in the notation will be clear later.

We define $P_n^{\Delta}(\alpha)$ as the adjoint operator

$$P_{n}^{\Delta}(\alpha)\colon H_{\mathbb{T}(,c)}^{[*]}(\widetilde{\mathcal{U}}_r^{d+n}) \to H_{\mathbb{T}(,c)}^{[*+\deg\alpha]}(\widetilde{\mathcal{U}}_r^{d}). \tag{5.2.4}$$

Here we have two remarks. First we follow the sign convention in [**46**, 3.1.3] for the intersection pairing

$$\langle \bullet, \bullet \rangle = (-1)^{\dim X/2} \int_X \bullet \cup \bullet. \tag{5.2.5}$$

Second, we take $\alpha \in H_{\mathbb{T},c}^{[*]}(\mathbb{A}^2)$ for $H_{\mathbb{T}}^{[*]}(\widetilde{\mathcal{U}}_r^{d})$ and $\alpha \in H_{\mathbb{T}}^{[*]}(\mathbb{A}^2)$ for $H_{\mathbb{T},c}^{[*]}(\widetilde{\mathcal{U}}_r^{d})$. Then the operators are well-defined, though various projections are not proper. (See [**54**, §8.3].)

We have the commutator relation

$$[P_m^{\Delta}(\alpha), P_n^{\Delta}(\beta)] = \langle \alpha, \beta \rangle m \delta_{m+n,0}\, r. \tag{5.2.6}$$

If $m + n = 0$, one of α or β is in $H_{\mathbb{T}}^{[*]}(\mathbb{A}^2)$ and another is in $H_{\mathbb{T},c}^{[*]}(\mathbb{A}^2)$. Hence $\langle \alpha, \beta \rangle$ is well-defined.

Since the construction is linear over $H_{\mathbb{T}}^{*}(\mathrm{pt})$, and $H_{\mathbb{T},c}^{[*]}(\mathbb{A}^2)$, $H_{\mathbb{T}}^{[*]}(\mathbb{A}^2)$ are free of rank 1, we can choose α to be their generators, i.e., the Poincaré dual of $[0]$ for $H_{\mathbb{T},c}^{[*]}(\mathbb{A}^2)$, and 1 (dual of $[\mathbb{A}^2]$) for $H^{[*]}_{\mathbb{T}}(\mathbb{A}^2)$. We assume these choices hereafter until §6. Note also that $\langle [0], 1\rangle = -1$ in our sign convention.

We take the direct sum over d in (5.1.3):

$$\bigoplus_d H^{[*]}_{\mathbb{T}}(\widetilde{\mathscr{U}}^d_r) \cong \bigoplus_d \mathrm{IH}^{[*]}_{\mathbb{T}}(\mathscr{U}^d_G) \otimes \bigoplus_\lambda H_{\mathrm{top}}(\pi^{-1}(x^d_\lambda)) \otimes H^{[*]}_{\mathbb{T}}(\overline{S_\lambda \mathbb{A}^2}). \tag{5.2.7}$$

Note that $H^{[*]}_{\mathbb{T}}(\overline{S_\lambda \mathbb{A}^2}) \cong H^*_{\mathbb{T}}(\mathrm{pt}) \cdot 1$, as $\overline{S_\lambda \mathbb{A}^2}$ is equivariantly contractible. Here $1 \in H^0_{\mathbb{T}}(\overline{S_\lambda \mathbb{A}^2}) = H^{[-2l(\lambda)]}_{\mathbb{T}}(\overline{S_\lambda \mathbb{A}^2})$.

From the definition of the Heisenberg operators, it acts only on the second factor of (5.2.7): $\lambda = \varnothing$ are killed by $P^\Delta_k([0])$ $(k > 0)$ and the summand for $\lambda = (1^{n_1} 2^{n_2} \cdots)$ is spanned by the monomial in $P_{-1}(1)^{n_1}/n_1! \cdot P_{-2}(1)^{n_2}/n_2! \cdots$. The second factor is isomorphic to the Fock space.

Let us give another representation of the Heisenberg algebra. Let 0 denote the point $d \cdot 0 \in S_{(d)}\mathbb{A}^2$, and consider the inverse image $\pi^{-1}(0) \subset \widetilde{\mathscr{U}}^d_r$, and denote it by $\widetilde{\mathscr{U}}^d_{r,0}$. It is the Quot scheme parametrizing quotients of $\mathscr{O}^{\oplus r}_{\mathbb{P}^2}$ of length d whose support is 0.

Let us restate Theorem 5.1.1 in a different form:

Proposition 5.2.8. — *$\widetilde{\mathscr{U}}^d_{r,0}$ is an irreducible $(dr-1)$-dimensional subvariety in $\widetilde{\mathscr{U}}^d_r$, unless $d = 0$.*

It is needless to say that we have $\widetilde{\mathscr{U}}^0_{r,0} = \widetilde{\mathscr{U}}^0_r = \mathrm{pt}$.

The convolution product by $P^\Delta_{\pm k}(\alpha)$ sends $H^{\mathbb{T}}_{[*]}(\widetilde{\mathscr{U}}^d_{r,0})$ to $H^{\mathbb{T}}_{[*-\deg\alpha]}(\widetilde{\mathscr{U}}^{d\pm k}_{r,0})$, where $\alpha \in H^*_{\mathbb{T},c}(\mathbb{A}^2)$ for $k < 0$, $\alpha \in H^*_{\mathbb{T}}(\mathbb{A}^2)$ for $k > 0$. Therefore

$$\bigoplus_d H^{\mathbb{T}}_{[*]}(\widetilde{\mathscr{U}}^d_{r,0}) \tag{5.2.9}$$

is a representation of the Heisenberg algebra. It is known that $\widetilde{\mathscr{U}}^d_{r,0}$ is homotopy equivalent to $\widetilde{\mathscr{U}}^d_r$, hence $H^{\mathbb{T}}_{[*]}(\widetilde{\mathscr{U}}^d_{r,0})$ is isomorphic to the ordinary homology group of $\widetilde{\mathscr{U}}^d_r$, and hence to $H^{[-*]}_{\mathbb{T},c}(\widetilde{\mathscr{U}}^d_r)$ by the Poincaré duality.

5.3. Fixed points and polarization

Let us take a decomposition $r = r_1 + r_2 + \cdots + r_N$. We have the corresponding $(N-1)$-dimensional torus, which is the connected center $A = Z(L)^0$ of the Levi subgroup $L = S(GL(r_1) \times \cdots \times GL(r_N)) \subset SL(r)$. We have the corresponding parabolic subgroup P consisting of block upper triangular elements.

Let us consider the fixed point set $\widetilde{\mathscr{U}}^d_L = (\widetilde{\mathscr{U}}^d_r)^A$. It consists of framed sheaves, which is a direct sum of sheaves of rank r_1, $r_2, \dots, r_N$. Thus we have

$$\widetilde{\mathscr{U}}^d_L = \bigsqcup_{d = d_1 + \cdots + d_N} \widetilde{\mathscr{U}}^{d_1}_{r_1} \times \cdots \times \widetilde{\mathscr{U}}^{d_N}_{r_N}. \tag{5.3.1}$$

We omit the superscript d, when there is no fear of confusion.

Following [**46**, Ex.3.3.3], we choose a polarization δ for each component of $\widetilde{\mathcal{U}}_L$, as a quiver variety associated with the Jordan quiver. Let us review the construction quickly. See the original paper for more detail: We represent $\widetilde{\mathcal{U}}_L$ as the space of quadruples (B_1, B_2, I, J) satisfying certain conditions. We decide to choose pairs, say (B_1, I), from quadruples. The choice gives us a decomposition of the tangent bundle of $\widetilde{\mathcal{U}}_r$ as

$$T\widetilde{\mathcal{U}}_r = T^{1/2} + (T^{1/2})^\vee \tag{5.3.2}$$

in the equivariant K-theory with respect to the A-action on $\widetilde{\mathcal{U}}^d_r$. We also have the decomposition of $T\widetilde{\mathcal{U}}_L$, and hence also of the normal bundle. Then we choose a polarization δ of $\widetilde{\mathcal{U}}_L$ in $\widetilde{\mathcal{U}}_r$ as product of weights in the normal bundle part of $(T^{1/2})^\vee$.

Let us also explain another description of the polarization δ given in [**46**, §12.1.5]. We consider the following Quot scheme

$$Q_r = \{(E, \varphi) \mid x_2\mathcal{O}^{\oplus r} \subset E \subset \mathcal{O}^{\oplus r}\} \subset \widetilde{\mathcal{U}}_r, \tag{5.3.3}$$

where x_2 is one of coordinates of $\mathbb{A}^2$. This is a fixed point component of a certain $\mathbb{C}^*$-action, and is a smooth lagrangian subvariety in $\widetilde{\mathcal{U}}_r$. In the ADHM description, it is given by the equation $B_2 = 0 = J$. Now $(T^{1/2})^\vee$ is the normal direction to Q_r at a point in Q_r. Since any component of $\widetilde{\mathcal{U}}_L$ intersects with Q_r, and the intersection is again a smooth lagrangian subvariety, Q_r gives us the polarization.

Note that the polarization is invariant under the action of $G = SL(r)$ on $\widetilde{\mathcal{U}}_G$, as we promised in §4.13.

We calculate the sign $\pm$ of the ratio of this polarization δ and the repellent one δ_{rep}, of $\widetilde{\mathcal{U}}^d_2 \times \widetilde{\mathcal{U}}^0_1$ and $\widetilde{\mathcal{U}}^0_1 \times \widetilde{\mathcal{U}}^d_2$ in $\widetilde{\mathcal{U}}^d_3$ for a later purpose. Here $L = S(GL(2) \times GL(1))$ in the first case and $L = S(GL(1) \times GL(2))$ for the latter case.

Lemma 5.3.4. — *We have $\delta_{\mathrm{rep}}/\delta = 1$ for $\widetilde{\mathcal{U}}^d_2 \times \widetilde{\mathcal{U}}^0_1$, $\delta_{\mathrm{rep}}/\delta = (-1)^d$ for $\widetilde{\mathcal{U}}^0_1 \times \widetilde{\mathcal{U}}^d_2$.*

Proof. — Both components $\widetilde{\mathcal{U}}^d_2 \times \widetilde{\mathcal{U}}^0_1$, $\widetilde{\mathcal{U}}^0_1 \times \widetilde{\mathcal{U}}^d_2$ intersect with the open set $\pi^{-1}_{a,G}(S_{(1^d)}\mathbb{A}^1)$, the inverse image of the open stratum under the factorization morphism. Since the normal bundle decomposes according to the factorization, the polarization is of the form $(\pm 1)^d$. Hence it is enough to determine the case $d = 1$.

We factor out $\mathbb{A}^2$ in $\widetilde{\mathcal{U}}^1_r$ and consider the centered Gieseker spaces. We have

$$^c\widetilde{\mathcal{U}}^1_3 \cong T^*\mathbb{P}^2, \tag{5.3.5}$$

$$^c\widetilde{\mathcal{U}}^1_2 \times {}^c\widetilde{\mathcal{U}}^0_1 \cong T^*(z_2 = 0), \quad {}^c\widetilde{\mathcal{U}}^0_1 \times {}^c\widetilde{\mathcal{U}}^2_1 \cong T^*(z_0 = 0), \tag{5.3.6}$$

where $[z_0 : z_1 : z_2]$ is the homogeneous coordinate system of $\mathbb{P}^2$. The polarization δ above is given by the base direction of the cotangent bundle.

On the other hand, the repellent directions are base in the first case and fibers in the second case. Therefore we have $\delta_{\mathrm{rep}}/\delta = 1$ in the first case and -1 in the second case. □

5.4. Stable envelope

Recall we considered the attracting set $\mathscr{U}_P$ in the Uhlenbeck space $\mathscr{U}_G$. Let us denote its inverse image $\pi^{-1}(\mathscr{U}_P)$ in $\widetilde{\mathscr{U}}_r$ by $\widetilde{\mathscr{U}}_P$. This is the *tensor product variety*, denoted by $\mathfrak{T}$ in [**60**], where $\mathscr{U}_P$ is denoted by $\mathfrak{T}_0$. (In [**56**] $\mathfrak{T}$ was denoted by $\mathfrak{Z}$.)

We have the following moduli theoretic description:

$$\widetilde{\mathscr{U}}_P = \left\{ (E,\varphi) \in \widetilde{\mathscr{U}}_r \;\middle|\; \begin{array}{l} E \text{ admits a filtration } 0 = E_0 \subset E_1 \subset \cdots \subset E_N = E \\ \text{with } \operatorname{rank} E_i/E_{i-1} = r_i, \text{ compatible with } \varphi. \end{array} \right\}. \tag{5.4.1}$$

See §4.4.

We consider the fiber product Z_P of $\widetilde{\mathscr{U}}_P$ and $\widetilde{\mathscr{U}}_L$ over $\mathscr{U}_L$:

$$Z_P = \widetilde{\mathscr{U}}_P \times_{\mathscr{U}_L} \widetilde{\mathscr{U}}_L, \tag{5.4.2}$$

where the map from $\widetilde{\mathscr{U}}_L$ to $\mathscr{U}_L$ is the restriction of π, and the map from $\widetilde{\mathscr{U}}_P$ to $\mathscr{U}_L$ is the composition of the restriction $\widetilde{\mathscr{U}}_P \to \mathscr{U}_P$ of π and the map p in §4.4. In the above description of $\widetilde{\mathscr{U}}_P$, it is just given as the direct sum $\bigoplus(E_i/E_{i-1})^{\vee\vee}$ plus the sum of singularities of E_i/E_{i-1}. One can show that Z_P is a lagrangian subvariety in $\widetilde{\mathscr{U}}_r \times \widetilde{\mathscr{U}}_L$. See [**60**, Prop. 1]. (There are no lower dimensional irreducible components, as all strata are relevant for the semismall morphism $\pi\colon \widetilde{\mathscr{U}}_r \to \mathscr{U}_G$.)

Maulik-Okounkov stable envelope is a 'canonical' lagrangian cycle class $\mathscr{L}$ in Z_P:

$$\mathscr{L} \in H_{[0]}(Z_P). \tag{5.4.3}$$

See [**46**, §3.5]. Note that $\mathscr{L}$ depends on the choice of the parabolic subgroup P as well as the polarization δ. Since they are canonically chosen, we suppress them in the notation $\mathscr{L}$.

The convolution by $\mathscr{L}$ defines a homomorphism

$$\mathscr{L} * - = p_{1*}(p_2^*(-) \cap \mathscr{L})\colon H_{[*]}(\widetilde{\mathscr{U}}_L) \to H_{[*]}(\widetilde{\mathscr{U}}_P). \tag{5.4.4}$$

It is known that $\mathscr{L} * -$ is an isomorphism (see [**60**, §4.2]), and it *does* also make sense for equivariant homology groups, as $H_{[0]}(Z_P) \cong H^{\mathbb{T}}_{[0]}(Z_P)$

We have $H_{[*]}(\widetilde{\mathscr{U}}_L) \cong H^{[*]}(\widetilde{\mathscr{U}}_L)$ by the Poincaré duality. Then we have

$$H^{[*]}_{\mathbb{T}}(\widetilde{\mathscr{U}}_L) \to H^{[*]}_{\mathbb{T}}(\widetilde{\mathscr{U}}_r) \tag{5.4.5}$$

as the composite of $\mathscr{L} * -$ and the pushforward with respect to the inclusion $\widetilde{\mathscr{U}}_P \subset \widetilde{\mathscr{U}}_r$. This is the original formulation of *stable envelope* in [**46**, Ch. 3], and properties of $\mathscr{L}$ are often stated in terms of this homomorphism there.

Let $x \in \mathcal{U}_L$. Let $\widetilde{\mathcal{U}}_{L,x}$ denote the inverse image of x under the Gieseker-Uhlenbeck morphism $\widetilde{\mathcal{U}}_L \to \mathcal{U}_L$. Similarly let $\widetilde{\mathcal{U}}_{P,x}$ denote the inverse image of x under the composition $\widetilde{\mathcal{U}}_P \to \mathcal{U}_P \to \mathcal{U}_L$. Then the convolution $\mathcal{L} * -$ also defines

$$\mathcal{L} * -\colon H^{\mathbb{T}_x}_{[*]}(\widetilde{\mathcal{U}}_{L,x}) \to H^{\mathbb{T}_x}_{[*]}(\widetilde{\mathcal{U}}_{P,x}), \tag{5.4.6}$$

where $\mathbb{T}_x$ is the stabilizer of x.

5.5. Tensor product module

Let $0 = d \cdot 0$ as before and consider the inverse image $\widetilde{\mathcal{U}}^d_{P,0}$ of 0 under $\widetilde{\mathcal{U}}^d_P \to \mathcal{U}^d_L$ as in the previous subsection.

We consider the direct sum

$$\bigoplus_d H^{\mathbb{T}}_{[*]}(\widetilde{\mathcal{U}}^d_{P,0}). \tag{5.5.1}$$

The Heisenberg algebra acts on the sum: This follows from a general theory of the convolution algebra: it is enough to check that $\widetilde{\mathcal{U}}^d_{P,0} \circ (P_n \cap \Pi^{-1}(0)) \subset \widetilde{\mathcal{U}}^{d+n}_{P,0}$ (for $k > 0$). If $(E_1, E_2, x) \in P_n \cap \Pi^{-1}(0)$, then $\pi(E_2) = \pi(E_1) + n \cdot 0$. Therefore the assertion follows.

The stable envelope $\mathcal{L} * -$ gives an isomorphism $\bigoplus_d H^{\mathbb{T}}_{[*]}(\widetilde{\mathcal{U}}^d_{L,0}) \cong \bigoplus_d H^{\mathbb{T}}_{[*]}(\widetilde{\mathcal{U}}^d_{P,0})$, where the left hand side is the tensor product

$$\bigoplus_{d_1,\dots,d_N} H^{\mathbb{T}}_{[*]}(\widetilde{\mathcal{U}}^{d_1}_{r_1,0}) \otimes \cdots \otimes H^{\mathbb{T}}_{[*]}(\widetilde{\mathcal{U}}^{d_N}_{r_N,0}) \tag{5.5.2}$$

by (5.3.1). This is a representation of N copies of Heisenberg algebras. Under the stable envelope, $P^{\Delta}_{-k}([0])$ on (5.5.1) is mapped to

$$\sum_{i=1}^{N} 1 \otimes \cdots \otimes \underbrace{P^{\Delta}_{-k}([0])}_{i^{\text{th}}\text{ factor}} \otimes \cdots \otimes 1. \tag{5.5.3}$$

This is [**46**, Th. 12.2.1]. Our Heisenberg generators are diagonal in this sense, and hence we put Δ in the notation. This result is compatible with the decomposition $\mathcal{W}(\mathfrak{gl}_r) = \mathcal{W}(\mathfrak{sl}_r) \otimes \mathfrak{Heis}$, where $\mathcal{W}(\mathfrak{sl}_r)$ is contained in the tensor product of the remaining $(N-1)$ copies of Heisenberg algebras, orthogonal to the diagonal one.

5.6. Sheaf theoretic analysis

By [**60**, §4, Lem. 4] we have a natural isomorphism

$$H_{[0]}(Z_P) \cong \operatorname{Hom}_{\operatorname{Perv}(\mathcal{U}_L)}\left(p_! j^* \pi_! \mathcal{C}_{\widetilde{\mathcal{U}}_r}, \pi_! \mathcal{C}_{\widetilde{\mathcal{U}}_L}\right) \tag{5.6.1}$$

where j, p are as in §4.4 and we use the same symbol π for Gieseker-Uhlenbeck morphisms for $\widetilde{\mathcal{U}}_r$ and $\widetilde{\mathcal{U}}_L$.

The Verdier duality gives us an isomorphism

$$\operatorname{Hom}(p_! j^* \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_r}, \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_L}) \cong \operatorname{Hom}(\pi_* \mathscr{C}_{\widetilde{\mathscr{U}}_L}, p_* j^! \pi_* \mathscr{C}_{\widetilde{\mathscr{U}}_r}). \tag{5.6.2}$$

Therefore the stable envelope gives us the canonical isomorphism

$$\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_L} \xrightarrow[\cong]{\mathscr{L}} \Phi_{L,G}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_r}) = p_* j^! \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_r}, \tag{5.6.3}$$

as $\pi_! = \pi_*$. This is nothing but Theorem 1.6.1(2) in Introduction.

Let $x \in \mathscr{U}_L$ and i_x denote the inclusion of x in $\mathscr{U}_L$. Then $\mathscr{L} \in \operatorname{Hom}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_L}, p_* j^! \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_r})$ defines an operator

(5.6.4)
$$\begin{tikzcd}
H^*(i_x^! \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_L}) \arrow[r] \arrow[d, equal] & H^*(i_x^! p_* j^! \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_r}) \arrow[d, equal] \\
H_{[-*]}(\widetilde{\mathscr{U}}_{L,x}) & H_{[-*]}(\widetilde{\mathscr{U}}_{P,x}).
\end{tikzcd}$$

This is equal to $\mathscr{L} * -$ in (5.4.6) under the isomorphism (5.6.1). See [**60**, §4.4].

5.7. The associativity of stable envelopes

Let us take parabolic subgroups $Q \subset P \subset G$ and the corresponding Levi subgroup $M \subset L$ as in §4.5. (G is still $SL(r)$.) Let Q_L be the image of Q in L.

Let us denote by $\mathscr{L}_{L,G}$ the isomorphism given by the stable envelope in (5.6.3):

$$\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_L} \xrightarrow[\cong]{\mathscr{L}_{L,G}} \Phi_{L,G}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_r}). \tag{5.7.1}$$

We similarly have isomorphisms

$$\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_M} \xrightarrow[\cong]{\mathscr{L}_{M,G}} \Phi_{M,G}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_r}), \quad \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_M} \xrightarrow[\cong]{\mathscr{L}_{M,L}} \Phi_{M,L}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_L}). \tag{5.7.2}$$

Then stable envelopes are compatible with the associativity (4.5.2) of the hyperbolic restriction:

Proposition 5.7.3. — *We have a commutative diagram*

(5.7.4)
$$\begin{tikzcd}
\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_M} \arrow[r, "\cong", "\mathscr{L}_{M,G}"'] \arrow[d, "\cong"', "\mathscr{L}_{M,L}"] & \Phi_{M,G}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_r}) \arrow[d, equal, "(4.5.2)"'] \\
\Phi_{M,L}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_L}) \arrow[r, "\Phi_{M,L}(\mathscr{L}_{L,G})", "\cong"'] & \Phi_{M,L}\Phi_{L,G}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_r}).
\end{tikzcd}$$

Let us check that this follows from the proof of [**46**, Lemma 3.6.1]. (To compare the following with the original paper, the reader should note that the tori $A \supset A'$ were used in [**46**], which correspond to $Z(M)^0 \supset Z(L)^0$ respectively in our situation.)

We consider

$$Z_P = \widetilde{\mathscr{U}}_P \times_{\mathscr{U}_L} \widetilde{\mathscr{U}}_L, \quad Z_Q = \widetilde{\mathscr{U}}_Q \times_{\mathscr{U}_M} \widetilde{\mathscr{U}}_M, \quad Z_{Q_L} = \widetilde{\mathscr{U}}_{Q_L} \times_{\mathscr{U}_M} \widetilde{\mathscr{U}}_M. \tag{5.7.5}$$

The stable envelopes $\mathcal{L}_{L,G}$, $\mathcal{L}_{M,G}$, $\mathcal{L}_{M,L}$ are classes in $H_{[0]}(Z_P)$, $H_{[0]}(Z_Q)$, $H_{[0]}(Z_{Q_L})$ respectively. We consider the convolution product

$$\mathcal{L}_{L,G} * \mathcal{L}_{M,L} \in H_{[0]}(Z_P \circ Z_{Q_L}). \tag{5.7.6}$$

Note that $Z_P \circ Z_{Q_L}$ consists of $(x_1, x_3) \in \widetilde{\mathcal{U}}_P \times \widetilde{\mathcal{U}}_M$ such that there exists $x_2 \in \widetilde{\mathcal{U}}_{Q_L} \subset \widetilde{\mathcal{U}}_L$ with $(x_1, x_2) \in Z_P$, $(x_2, x_3) \in Z_{Q_L}$ by definition. This is nothing but Z_Q. Therefore $\mathcal{L}^P * \mathcal{L}^{Q_L}$ is a class in $H_{[0]}(Z_Q)$. The proof in [**46**, Lemma 3.6.1] actually gives $\mathcal{L}_{L,G} * \mathcal{L}_{M,L} = \mathcal{L}_{M,G}$.

Therefore the commutativity of (5.7.4) follows, once we check that the convolution product corresponds to the composition of homomorphisms (Yoneda product) under the isomorphism (5.6.1). This is not covered by [**23**, Prop. 8.6.35], as the base spaces of fiber products are different: $\mathcal{U}_L$ and $\mathcal{U}_M$. But we can easily modify its proof to our situation.

5.8. Space V^d and its base given by irreducible components

Let us write d for the instanton number again. Similarly to (4.8.2) we define

$$\begin{aligned} V^d_{L,G} \equiv V^d &\overset{\text{def.}}{=} \operatorname{Hom}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \Phi_{L,G}(\pi_! \mathscr{C}_{\widetilde{\mathcal{U}}^d_r})) \\ &= H^{-2}(S_{(d)}\mathbb{A}^2, \xi^! \Phi_{L,G}(\pi_! \mathscr{C}_{\widetilde{\mathcal{U}}^d_r})), \end{aligned} \tag{5.8.1}$$

where $\xi \colon S_{(d)}\mathbb{A}^2 \to \mathcal{U}^d_L$ is as before. We denote by $V^{d,P}_{L,G}$ or $V^{d,P}$ when we want to emphasize P.

As in Lemma 4.8.15 we have

$$V^d \cong H_{[0]}(\widetilde{\mathcal{U}}^d_{P,0}), \tag{5.8.2}$$

and V^d has a base given by $(dh^\vee - 1)$-dimensional irreducible components of $\widetilde{\mathcal{U}}^d_{P,0}$.

On the other hand, $H_{[0]}(\widetilde{\mathcal{U}}^d_{P,0})$ is isomorphic to $H_{[0]}(\widetilde{\mathcal{U}}^d_{L,0})$ by the stable envelope. In the description (5.3.1), note that the fiber $\widetilde{\mathcal{U}}^{d_i}_{r_i,0}$ has $\dim = \dim \widetilde{\mathcal{U}}^{d_i}_{r_i}/2 - 1$ by Proposition 5.2.8 unless $d_i = 0$. Therefore we can achieve the degree $[0] = \dim \widetilde{\mathcal{U}}^d_L - 2 = \sum \dim \widetilde{\mathcal{U}}^{d_i}_{r_i} - 2$ only when all $d_i = 0$ except one. There are N choices $i = 1, \dots, N$. Therefore $\dim V^d = N$.

Let us study $V^d = H_{[0]}(\widetilde{\mathcal{U}}^d_{P,0})$ in more detail. This will give the detail left over from §4.9. By [**56**, §3] we have a decomposition

$$\widetilde{\mathcal{U}}^d_{P,0} = \bigsqcup_{d_1+\dots+d_N=n} \mathfrak{T}(d_1, \dots, d_N)_0, \tag{5.8.3}$$

where

$$\mathfrak{T}(d_1,\dots,d_N)_0 = \left\{ (E,\varphi) \,\middle|\, \begin{array}{l} E \text{ admits a filtration } 0 = E_0 \subset E_1 \subset \cdots \subset \\ E_N = E \text{ with } E_i/E_{i-1} \in \widetilde{\mathscr{U}}^{d_i}_{r_i,0} \text{ compatible with} \\ \varphi. \end{array} \right\}. \tag{5.8.4}$$

We have a projection

$$\mathfrak{T}(d_1,\dots,d_N)_0 \to \widetilde{\mathscr{U}}^{d_1}_{r_1,0} \times \cdots \times \widetilde{\mathscr{U}}^{d_N}_{r_N,0}, \tag{5.8.5}$$

which is a vector bundle of rank $dr - \sum d_i r_i$. Note that

$$\dim \widetilde{\mathscr{U}}^{d_i}_{r_i,0} = \begin{cases} 0 & \text{if } d_i = 0, \\ d_i r_i - 1 & \text{if } d_i \neq 0. \end{cases} \tag{5.8.6}$$

(See Proposition 5.2.8.) Therefore

$$\dim \mathfrak{T}(d_1,\dots,d_N)_0 = dr - \#\{i \mid d_i \neq 0\} \leq dr - 1. \tag{5.8.7}$$

The equality holds if and only if there is only one i with $d_i \neq 0$. Therefore $H_{[0]}(\widetilde{\mathscr{U}}^d_{P_-,0})$ is spanned by fundamental cycles

$$[\overline{\mathfrak{T}(d,0,\dots,0)_0}], \quad \cdots \quad, [\overline{\mathfrak{T}(0,\dots,0,d)_0}]. \tag{5.8.8}$$

Thus it is N-dimensional, as expected.

In the remainder of this subsection, we study the corresponding space $U^d = H_{[0]}(\mathscr{U}^d_{P,0})$ for the Uhlenbeck space. Note that we have projective morphism $\pi\colon \widetilde{\mathscr{U}}^d_{P,0} \to \mathscr{U}^d_{P,0}$, and the Quot scheme $\pi^{-1}(d\cdot 0) = \widetilde{\mathscr{U}}^d_{r,0}$ is contained in $\widetilde{\mathscr{U}}^d_{P,0}$. The class of fiber $\widetilde{\mathscr{U}}^d_{r,0}$ is given by $P_{-d}([0])[\mathscr{U}^0_G]$, and $H_{[0]}(\mathscr{U}^d_{P,0})$ is killed by Baranovsky's Heisenberg operators by the construction.

Proposition 5.8.9. — *Among N cycles in* (5.8.8), *the first one* $[\overline{\mathfrak{T}(d,0,\dots,0)_0}]$ *is* $[\widetilde{\mathscr{U}}^d_{r,0}]$. *The remaining cycles give a base of $U^d = H_{[0]}(\mathscr{U}^d_{P,0})$ under π_*.*

From the definition, this description of irreducible components of $\mathscr{U}^d_{P,0}$ is the same as one in Proposition 4.9.2 when $G = SL(r)$, $P = B$.

Proof. — Suppose that $E \in \mathfrak{T}(d,0,\dots,0)_0$. Then we have a short exact sequence

$$0 \to E_1 \to E \to \mathscr{O}^{\oplus r_2+\cdots+r_N} \to 0 \tag{5.8.10}$$

with $E_1 \in \widetilde{\mathscr{U}}^d_{r_1,0}$. Consider

$$0 \to \mathscr{O}^{\oplus r_2+\cdots+r_N} \to E^\vee \to E_1^\vee \to \mathscr{E}xt^1(\mathscr{O}^{\oplus r_2+\cdots+r_N}, \mathscr{O}) \tag{5.8.11}$$

Since $\mathscr{E}xt^1(\mathscr{O}^{\oplus r_2+\cdots+r_N}, \mathscr{O}) = 0$, the last homomorphism $E^\vee \to E_1^\vee$ is surjective. Therefore this is a short exact sequence. Dualizing again, we get

$$0 \to E_1^{\vee\vee} \to E^{\vee\vee} \to \mathscr{O}^{\oplus r_2+\cdots+r_N} \to 0. \tag{5.8.12}$$

The last homomorphism $E^{\vee\vee} \to \mathcal{O}^{\oplus r_2+\cdots+r_N}$ is surjective as $E \to \mathcal{O}^{\oplus r_2+\cdots+r_N}$ is so. (Or, we observe $E_1^{\vee\vee} \cong \mathcal{O}^{\oplus r_1}$ as $E_1 \in \widetilde{\mathcal{U}}^{d}_{r_1,0}$, and $\mathcal{E}xt^1(\mathcal{O}^{\oplus r_1}, \mathcal{O}) = 0$.) Therefore this is also exact. We have $E_1^{\vee\vee} = \mathcal{O}^{\oplus r_1}$ as $E_1 \in \widetilde{\mathcal{U}}^{d_1}_{r_1,0}$. Since the extension between the trivial sheaves is zero on $\mathbb{P}^2$, we have $E^{\vee\vee} = \mathcal{O}^{\oplus r}$. Therefore $E \in \widetilde{\mathcal{U}}^{d}_{r,0}$.

Thus we have $\overline{\mathfrak{T}(d, 0, \dots, 0)_0} \subset \widetilde{\mathcal{U}}^{d}_{r,0}$. Since both are $(dr-1)$-dimensional, and $\widetilde{\mathcal{U}}^{d}_{r,0}$ is irreducible, they must coincide. This shows the first claim.

The second claim follows as we have already shown $\dim U^d = N - 1$ in Lemma 4.8.11, hence other classes cannot be killed by π_*.

Let us directly check that any of $\mathfrak{T}(0, d, 0, \dots, 0)_0, \dots, \mathfrak{T}(0, \dots, 0, d)_0$ contains a locally free sheaf for definiteness. (It gives us another proof of Lemma 4.8.11, which does not dependent on [**21**, Theorem 7.10].) Then it is enough to consider the case $N = 2$ and check that $\mathfrak{T}(0, d)_0$ contains a locally free sheaf, as an extension of a locally free sheaf by a locally free sheaf is again locally free. Furthermore we may assume $r = 2$ and $r_1 = r_2 = 1$.

We use the ADHM description. Let

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & 1 & \dots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \\ \text{\huge 0} & & & & 0 \end{pmatrix}, \ B_2 = 0,$$

$$I = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \ J = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

We have $[B_1, B_2] + IJ = 0$. We see (B_1, B_2, I, J) is stable, i.e., a subspace $S \subset \mathbb{C}^d$ containing the image of I and invariant under B_1, B_2 must be $S = \mathbb{C}^d$. We also see that (B_1, B_2, I, J) is costable, i.e., a subspace $S \subset \mathbb{C}^d$ contained in the kernel of J and invariant under B_1, B_2 must be $S = 0$. Therefore (B_1, B_2, I, J) defines a framed locally free sheaf (E, φ), i.e., an element in $\mathrm{Bun}^d_{SL(2)}$. We consider a subspace $\{\begin{pmatrix} 0 \\ * \end{pmatrix}\} \subset \mathbb{C}^2$, which is the kernel of a. Taking 0 as a subspace in $\mathbb{C}^d$, we have a subrepresentation of a quiver. Therefore E contains the trivial rank 1 sheaf $\mathcal{O}_{\mathbb{P}^2}$ correspondingly. The quotient $E/\mathscr{O}_{\mathbb{P}^2}$ is given by the data

$$I = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and $J = 0$, B_1, B_2 the same as above. This is the ideal sheaf (x^d, y), and hence in $\widetilde{\mathscr{U}}^d_{1,0}$. Thus E is a point in $\mathfrak{T}(0,d)_0$. $\square$

5.9. A pairing on V^d

In the same way as §4.10, we can define a nondegenerate pairing between $V^{d,P}$ and V^{d,P_-}.

We have an isomorphism

$$H^0(\xi_0^! i^* j^! \pi_* \mathscr{C}_{{}^c\widetilde{\mathscr{U}}^d_r}) \xrightarrow{\cong} H^0(\xi_0^* i_-^! j_-^* \pi_! \mathscr{C}_{{}^c\widetilde{\mathscr{U}}^d_r}), \tag{5.9.1}$$

where we also used $\pi_* = \pi_!$ as π is proper. By the base change and the replacement i^*, $i_-^!$ to p_*, $(p_-)_!$, we can identify this with

$$H_{[0]}(\widetilde{\mathscr{U}}^d_{P,0}) \to H_c^{[0]}(\widetilde{\mathscr{U}}^d_{P_-,0}), \tag{5.9.2}$$

and we have a pairing

$$\langle\ ,\ \rangle \colon H_{[0]}(\widetilde{\mathscr{U}}^d_{P,0}) \otimes H_{[0]}(\widetilde{\mathscr{U}}^d_{P_-,0}) \to \mathbb{C}. \tag{5.9.3}$$

Note that we also have the intersection pairing in the centered Gieseker space ${}^c\widetilde{\mathscr{U}}^d_r$. As the intersection $\widetilde{\mathscr{U}}^d_{P,0} \cap \widetilde{\mathscr{U}}^d_{P_-,0}$ consists of a *compact* space $\widetilde{\mathscr{U}}^d_{L,0}$, the pairing is well-defined, and takes values in $\mathbb{C}$. We multiply the sign $(-1)^{\dim {}^c\widetilde{\mathscr{U}}^d_r/2} = (-1)^{dr-1}$ as before.

Lemma 5.9.4. — *The pairing is equal to the intersection pairing.*

Proof. — The pairing is the restriction of that on equivariant cohomology groups:

$$\langle\ ,\ \rangle \colon H^*_{\mathbb{T}}(\xi_0^! i^* j^! \pi_* \mathscr{C}_{{}^c\widetilde{\mathscr{U}}^d_r}) \otimes H^*_{\mathbb{T}}(\xi_0^! i_-^* j_-^! \pi_* \mathscr{C}_{{}^c\widetilde{\mathscr{U}}^d_r}) \to H^*_{\mathbb{T}}(\mathrm{pt}). \tag{5.9.5}$$

By the localization theorem, natural homomorphisms

$$H^*_{\mathbb{T}}(\xi_0^! i^! j^! \pi_* \mathscr{C}_{{}^c\widetilde{\mathscr{U}}^d_r}) \to H^*_{\mathbb{T}}(\xi_0^! i^* j^! \pi_* \mathscr{C}_{{}^c\widetilde{\mathscr{U}}^d_r}), \tag{5.9.6}$$

$$H^*_{\mathbb{T}}(\xi_0^! i_-^* j_-^! \pi_* \mathscr{C}_{{}^c\widetilde{\mathscr{U}}^d_r}) \to H^*_{\mathbb{T}}(\xi_0^* i_-^* j_-^* \pi_* \mathscr{C}_{{}^c\widetilde{\mathscr{U}}^d_r}) \tag{5.9.7}$$

become isomorphisms over the fractional field of $H^*_{\mathbb{T}}(\mathrm{pt})$. Then the pairing between $H^*_{\mathbb{T}}(\xi_0^! i^! j^! \pi_* \mathscr{C}_{c\widetilde{\mathscr{U}}_r^d})$ and $H^*_{\mathbb{T}}(\xi_0^* i_-^* j_-^* \pi_* \mathscr{C}_{c\widetilde{\mathscr{U}}_r^d})$ is equal to the intersection pairing by [**23**, §8.5]. Therefore we only need to show that the composition

$$i^! j^! \to i^* j^! \to i_-^! j_-^* \tag{5.9.8}$$

is equal to $i^! j^! = i_-^! j_-^! \to i_-^! j_-^*$. This is a consequence of the following general statement : Let T be a torus action on X and $Y = X^T$ (more generally, it can be a closed invariant subset containing X^T). Let $a\colon Y \to X$ be the embedding. Let F be a functor from $D_T(X)$ to $D_T(Y)$. Assume that we have two morphisms of functors α, $\beta\colon a^! \to F$. Then $\alpha = \beta$ if and only if it is so on the image of $a_!\colon D_T(Y) \to D_T(X)$. We apply this claim to $a = ji$, $F = i_-^! j_-^*$. In our case, $\alpha = \beta$ on $a_! D_T(Y)$ is evident, as all the involved morphisms are identities on the fixed point set Y.

Let us give the proof of the claim. We consider a natural map $a_! a^! \mathscr{F} \to \mathscr{F}$ for $\mathscr{F} \in D_T(X)$. It becomes an isomorphism if we apply $a^!$ by the base change. We set $\mathscr{G} = a_! a^! \mathscr{F}$. We have $\alpha_{\mathscr{G}} = \beta_{\mathscr{G}}$, as homomorphisms $a^! \mathscr{G} \to F(\mathscr{G})$, from the assumption. Then we have $\alpha_{\mathscr{F}} = \beta_{\mathscr{F}}$ as the composition of $\alpha_{\mathscr{G}} = \beta_{\mathscr{G}}$ and $F(\mathscr{F}) \to F(\mathscr{G})$. ☐

5.10. Another base of V^d

Recall we have the canonical isomorphism $\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_L^d} \xrightarrow[\cong]{\mathscr{L}} \Phi_{L,G}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_r^d})$ in (5.6.3). Thanks to the decomposition (5.3.1), the morphism $\pi\colon \widetilde{\mathscr{U}}_L^d \to \mathscr{U}_L^d$ is the composite of

$$\pi \times \cdots \times \pi\colon \widetilde{\mathscr{U}}_{r_1}^{d_1} \times \cdots \times \widetilde{\mathscr{U}}_{r_N}^{d_N} \to \mathscr{U}_{SL(r_1)}^{d_1} \times \cdots \times \mathscr{U}_{SL(r_N)}^{d_N} \tag{5.10.1}$$

with the sum map

$$\kappa\colon \mathscr{U}_{SL(r_1)}^{d_1} \times \cdots \times \mathscr{U}_{SL(r_N)}^{d_N} \to \mathscr{U}_L^d. \tag{5.10.2}$$

The latter is a finite birational morphism. Then $\mathscr{C}_{\widetilde{\mathscr{U}}_L^d}$ decomposes under (5.10.1) as in (5.1.2):

$$\begin{aligned}\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_L^d} \cong \bigoplus & H_{\mathrm{top}}(\pi^{-1}(x_{\lambda_1}^{d_1}) \times \cdots \times \pi^{-1}(x_{\lambda_N}^{d_N})) \\ & \otimes \kappa_! \operatorname{IC}(\mathrm{Bun}_{SL(r_1),\lambda_1}^{d_1} \times \cdots \times \mathrm{Bun}_{SL(r_N),\lambda_N}^{d_N}),\end{aligned} \tag{5.10.3}$$

where $\lambda_1,\ldots, \lambda_N$ are partitions with $d = d_1 + |\lambda_1| + \cdots + d_N + |\lambda_N|$. (These $d_1,\ldots, d_N$ are different from above.) The image of the closure of $\mathrm{Bun}_{SL(r_1),\lambda_1}^{d_1} \times \cdots \times \mathrm{Bun}_{SL(r_N),\lambda_N}^{d_N}$ under κ is the closure of

$$\mathrm{Bun}_{SL(r_1)}^{d_1} \times \cdots \times \mathrm{Bun}_{SL(r_N)}^{d_N} \times S_\mu \mathbb{A}^2, \tag{5.10.4}$$

where $\mu = \lambda_1 \sqcup \cdots \sqcup \lambda_N$. Let us denote this stratum by $\mathrm{Bun}_{L,\mu}^{d_1,\ldots,d_N}$. Then as κ is a finite morphism, we have

$$\kappa_! \operatorname{IC}(\mathrm{Bun}_{SL(r_1),\lambda_1}^{d_1} \times \cdots \times \mathrm{Bun}_{SL(r_N),\lambda_N}^{d_N}) \cong \operatorname{IC}(\mathrm{Bun}_{L,\mu}^{d_1,\ldots,d_N}, \rho), \tag{5.10.5}$$

where ρ is the local system corresponding to the covering

$$S_{\lambda_1}\mathbb{A}^2 \times \cdots \times S_{\lambda_N}\mathbb{A}^2 \setminus \text{diagonal} \to S_\mu \mathbb{A}^2 \tag{5.10.6}$$

and $\mu = \lambda_1 \sqcup \cdots \sqcup \lambda_N$. Taking sum over λ_1, λ_2, … which give the same μ, we get

$$\begin{multline}\bigoplus_{\lambda_1 \sqcup \cdots \sqcup \lambda_N = \mu} \kappa_! \operatorname{IC}(\operatorname{Bun}^{d_1}_{SL(r_1),\lambda_1} \times \cdots \times \operatorname{Bun}^{d_N}_{SL(r_N),\lambda_N}) \\ \cong \operatorname{IC}(\operatorname{Bun}^{d_1,\dots,d_N}_{L,\mu}, \rho),\end{multline} \tag{5.10.7}$$

where ρ is now given by the permutation representation

$$('V^1)^{\otimes n_1} \otimes ('V^2)^{\otimes n_2} \otimes \cdots \tag{5.10.8}$$

of $S_{n_1} \times S_{n_2} \times \cdots$ if $\mu = (1^{n_1} 2^{n_2} \cdots)$ with $\dim {}'V^d = N$. Here we define $'V^d$ as the cohomology of the union of the fibers of (5.10.6) for the special case when μ is the partition (d) with the single entry d, where the union runs over $\lambda_1, \dots, \lambda_N$:

$$\sigma\colon \bigsqcup_{\lambda_1 \sqcup \cdots \sqcup \lambda_N = (d)} S_{\lambda_1}\mathbb{A}^2 \times \cdots \times S_{\lambda_N}\mathbb{A}^2 \setminus \text{diagonal} \to S_{(d)}\mathbb{A}^2, \tag{5.10.9}$$

and

$$'V^d = H_0(\sigma^{-1}(d \cdot 0)). \tag{5.10.10}$$

Since $\mu = (d)$, one of $\lambda_1, \dots, \lambda_N$ is (d) and others are the empty partition $\varnothing$. Therefore the fiber $\sigma^{-1}(d \cdot 0)$ consists of N distinct points, hence we have $\dim {}'V^d = N$.

Moreover $\operatorname{Hom}_{\operatorname{Perv}(\mathscr{U}^d_L)}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_L})$ is given by the component $\operatorname{IC}(\operatorname{Bun}^{0,\dots,0}_{L,(d)}, \rho)$, where ρ is the trivial representation of S_1 on $'V^d$. Therefore we have a canonical isomorphism

$$\begin{aligned} 'V^d &\cong \operatorname{Hom}_{\operatorname{Perv}(\mathscr{U}^d_L)}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_L}) \\ &\cong \operatorname{Hom}_{\operatorname{Perv}(\mathscr{U}^d_L)}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \Phi_{L,G}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_r})), \end{aligned} \tag{5.10.11}$$

where the first isomorphism is via $H_{\text{top}}(\pi^{-1}(x^{d_1}_{\lambda_1}) \times \cdots \times \pi^{-1}(x^{d_N}_{\lambda_N})) \cong \mathbb{C}$ ($d_1 = \cdots = d_N = 0$, one of $\lambda_1, \dots, \lambda_N$ is (d) and others are the empty partition) given by the fundamental class, and the second isomorphism is given by the stable envelope $\mathscr{L}$. Thus our $'V^d$ is isomorphic to V^d in (5.8.1). We will identify $'V^d$ with V^d hereafter.

We have just shown

$$\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_L} \cong \bigoplus \operatorname{IC}(\operatorname{Bun}^{d_1,\dots,d_N}_{L,\lambda}, \rho). \tag{5.10.12}$$

This is similar to Proposition 4.8.4, where we used the factorization argument to construct an isomorphism. Our argument looks slightly different, as we have not used the projection $a\colon \mathbb{A}^2 \to \mathbb{A}^1$. But the isomorphism is the same as one given by the factorization argument from the above construction, together with the observation that a, i, j commute with the projection $\pi^d_{a,?}$ $(? = G, P, L)$.

Note that we have $e_i \in V^d = H_0(\sigma^{-1}(d\cdot 0))$ corresponding to the component of $\sigma^{-1}(d\cdot 0)$ in

$$S_{\varnothing}\mathbb{A}^2 \times \cdots \times \underbrace{S_{(d)}\mathbb{A}^2}_{i^{\text{th}}\text{ factor}} \times \cdots \times S_{\varnothing}\mathbb{A}^2. \tag{5.10.13}$$

Then $e_1, \dots, e_N$ gives a base of V^d.

If we view V^d as $\operatorname{Hom}_{\operatorname{Perv}(\mathscr{U}^d_L)}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_L})$, e_i is the composite of homomorphisms

$$\mathscr{C}_{S_{(d)}\mathbb{A}^2} \to \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_{r_i}} \to \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_L}, \tag{5.10.14}$$

where the left homomorphism is given by the fundamental class $[\pi^{-1}(d\cdot 0)]$, and the right one is given by the inclusion of the component $d_i = d$, $d_j = 0$ $(j \neq i)$ in the decomposition (5.3.1).

Example 5.10.15. — For $d = 1$, $\widetilde{\mathscr{U}}^1_r$ (resp. $\mathscr{U}^1_G$) is isomorphic to the product of $\mathbb{A}^2$ and the cotangent bundle of $\mathbb{P}^{r-1}$ (resp. the closure of the minimal nilpotent orbit of $\mathfrak{sl}_r$). Further suppose $N = r$ and $r_1 = \cdots = r_N = 1$. Then [**46**, Remark 3.5.3] gives us the relation:

$$[\overline{\mathfrak{T}(0, \dots, 0, \underbrace{1}_{k^{\text{th}}\text{ factor}}, 0, \dots, 0)}] = (-1)^{k-1}(e_k + e_{k+1} + \cdots + e_r). \tag{5.10.16}$$

Here the sign $(-1)^{k-1}$ comes from the polarization, mentioned in §5.3.

Example 5.10.17. — We know that $[\overline{\mathfrak{T}(d, 0, \dots, 0)_0}] = [\widetilde{\mathscr{U}}^d_{r,0}]$ (Proposition 5.8.9), and hence

$$[\overline{\mathfrak{T}(d, 0, \dots, 0)_0}] = e_1 + \cdots + e_N \tag{5.10.18}$$

by (5.5.3).

On the other hand, the opposite extreme $[\overline{\mathfrak{T}(0, 0, \dots, d)_0}]$ is equal to e_N up to sign by the support property of the stable envelope [**46**, Th. 3.3.4 (i)]. The polarization is opposite, therefore the sign is the half of the codimension of the corresponding fixed point component. We get

$$[\overline{\mathfrak{T}(0, 0, \dots, d)_0}] = (-1)^{d(r - r_N)} e_N. \tag{5.10.19}$$

If $N = 2$, two elements exhaust the base.

The transition matrix between two bases for $d > 1$, $N > 2$ can be calculated from (4.12.1) together with (5.11.5) below. Though (4.12.1) determines $[\overline{\mathfrak{T}(0, \dots, 0, d, 0, \dots, 0)}]$ (d is in the k^{th} entry) up to $\mathbb{C}[\overline{\mathfrak{T}(d, 0, \dots, 0)_0}]$, it is a linear span of e_k, $\dots$, e_N thanks to the support property of the stable envelope. Therefore we can fix the ambiguity.

5.11. Computation of the pairing

Let us relate the pairing in §5.9 to the pairing defined on $\widetilde{\mathscr{U}}^d_L$ using the stable envelope

$$\mathscr{L}\colon H_{[0]}(\widetilde{\mathscr{U}}^d_{L,0}) \xrightarrow{\cong} H_{[0]}(\widetilde{\mathscr{U}}^d_{P,0}). \tag{5.11.1}$$

Let us temporarily denote the stable envelope with respect to the opposite parabolic by $\mathscr{L}^-$. Then we want to compute

$$\langle \mathscr{L}(\alpha), \mathscr{L}^-(\beta)\rangle, \tag{5.11.2}$$

which is equal to the intersection pairing times $(-1)^{\dim {}^c\widetilde{\mathscr{U}}^d_r}$ by Lemma 5.9.4.

Suppose that α, β are classes on a component Z of $\widetilde{\mathscr{U}}^d_{L,0}$. Let us take equivariant lifts of α, β to $Z(L)^0$-equivariant cohomology. Since the supports of $\mathscr{L}(\alpha)$ and $\mathscr{L}^-(\beta)$ intersect along Z by one of characterizing properties of the stable envelope [**46**, Th. 3.3.4(i)], we need to compute the restriction of the (Poincaré dual of) $\mathscr{L}(\alpha)$, $\mathscr{L}^-(\beta)$ to the fixed point component Z. Again by a property of the stable envelop [**46**, Th. 3.3.4(ii)], we have $\mathscr{L}(\alpha)|_Z = (\delta_{\mathrm{rep}}/\delta)e(N^-)\cup\alpha$ and $\mathscr{L}^-(\beta)|_Z = (\delta_{\mathrm{att}}/\delta)e(N^+)\cup\beta$, where δ_{rep}, δ_{att} are the polarizations given by attracting and repellent directions. Then we have

$$\int_{{}^c\widetilde{\mathscr{U}}^d_r} \mathscr{L}(\alpha)\cup\mathscr{L}^-(\beta) = \frac{\delta_{\mathrm{rep}}\delta_{\mathrm{att}}}{\delta^2}\int_{{}^c\widetilde{\mathscr{U}}^d_r} e(N)\cup\alpha\cup\beta = (-1)^{\operatorname{codim} Z/2}\int_Z \alpha\cup\beta \tag{5.11.3}$$

by the fixed point formula. Therefore if we multiply $(-1)^{\dim {}^c\widetilde{\mathscr{U}}^d_r/2}$, we get $(-1)^{\dim Z/2}\int_X \alpha\cup\beta = \langle\alpha,\beta\rangle$.

If α, β are supported on different components Z, Z' of $\widetilde{\mathscr{U}}^d_{L,0}$ respectively, we use a property [**46**, Th. 3.7.5], which says the restrictions of $\mathscr{L}(\alpha)$, $\mathscr{L}(\beta)$ to components other than Z, Z' are zero. Then it is clear that $\langle\mathscr{L}(\alpha),\mathscr{L}^-(\beta)\rangle = 0$.

As an application of this formula, we compute $\langle e_i, e_j^-\rangle$, where $e_i\in V^d$ as in the previous subsection, and $e_j^-\in V^{d,P_-}$ is defined in the same way using the opposite hyperbolic restriction $\mathscr{L}^-$. This is reduced to the computation of the self-intersection number of the punctual Quot scheme $\widetilde{\mathscr{U}}^d_{r_i,0}$ in the centered Gieseker space ${}^c\widetilde{\mathscr{U}}^d_{r_i}$. This is given by $(-1)^{r_id-1}dr_i = (-1)^{\dim {}^c\widetilde{\mathscr{U}}^d_{r_i}/2}dr_i$ ([**6**, §4]). Therefore we get

Proposition 5.11.4. — *We have*

$$\langle e_i, e_j^-\rangle = dr_i\delta_{ij}. \tag{5.11.5}$$

5.12. Relation between V^d and U^d

Let us apply the decomposition (5.1.2) to (5.10.11). We have

$$\begin{aligned} V^d &= \operatorname{Hom}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \Phi_{L,G}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_r})) \\ &= \bigoplus_{d_1+|\lambda|=d} H_{\mathrm{top}}(\pi^{-1}(x^{d_1}_\lambda)) \otimes \operatorname{Hom}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \Phi_{L,G}(\mathrm{IC}(\mathrm{Bun}^{d_1}_{G,\lambda}))). \end{aligned} \tag{5.12.1}$$

Then $\operatorname{Hom}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \Phi_{L,G}(\mathrm{IC}(\mathrm{Bun}^{d_1}_{G,\lambda})))$ is nonzero only in either of the following cases:

1. $d_1 = d$ and $\lambda = \varnothing$,
2. $d_1 = 0$ and $\lambda = (d)$.

In the first case, it is U^d by definition. And in the second case, it is

$$\operatorname{Hom}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \Phi_{L,G}(\mathscr{C}_{S_{(d)}\mathbb{A}^2})) = \operatorname{Hom}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \mathscr{C}_{S_{(d)}\mathbb{A}^2}) \cong \mathbb{C}\,\mathrm{id}\,. \tag{5.12.2}$$

Thus

$$V^d \cong \left(H_{\mathrm{top}}(\pi^{-1}(x^d_\varnothing)) \otimes U^d\right) \oplus H_{\mathrm{top}}(\pi^{-1}(x^0_{(d)})). \tag{5.12.3}$$

Note that $\pi^{-1}(x^d_\varnothing)$ is a single point. Therefore we have the canonical isomorphism $H_{\mathrm{top}}(\pi^{-1}(x^d_\varnothing)) \cong \mathbb{C}$. Now the homomorphism $\pi_*\colon V^d \cong H_{[0]}(\widetilde{\mathscr{U}}^d_{P,0}) \to U^d \cong H_{[0]}(\mathscr{U}^d_{P,0})$ is identified with the projection to the first component in (5.12.3). In particular, bases of U^d and V^d given by irreducible components (see Lemma 4.8.15 and Proposition 5.8.9) are related by the projection.

The subspace $H_{\mathrm{top}}(\pi^{-1}(x^0_{(d)}))$ is 1-dimensional space spanned by the fundamental class $[\pi^{-1}(x^0_{(d)})]$, or equivalently $P_{-d}([0]) \cdot [\mathscr{U}^0_G]$ where $P_{-d}([0])$ is the Heisenberg operator, and $[\mathscr{U}^0_G] = 1 \in H^0_{\mathbb{T}}(\mathscr{U}^0_G)$. Recall that the Baranovsky's Heisenberg operator is mapped to the diagonal operator under the stable envelope, see §5.5. It means that $[\pi^{-1}(x^0_{(d)})]$ is equal to

$$e_1 + \cdots + e_N, \tag{5.12.4}$$

where $\{e_i\}$ is the base of V^d in the previous subsection.

And U^d is the subspace killed by the Heisenberg operator $P_d(1)$. Therefore

$$U^d \cong \{\lambda_1 e_1 + \cdots + \lambda_N e_N \mid \lambda_1 + \cdots + \lambda_N = 0\}\,. \tag{5.12.5}$$

We have a base $\{e_i - e_{i+1}\}_{i=1,\dots,N-1}$ of U^d.

It is also clear that the decomposition (5.12.3) is orthogonal with respect to the pairing in §5.9. And the restriction of the pairing to U^d is equal to one in §4.10. Therefore we can calculate the pairing between $U^{d,P}$ and U^{d,P_-}. Let us consider the case $P = B$ for brevity. We have

$$\langle e_i - e_{i+1}, e^-_j - e^-_{j+1}\rangle = \begin{cases} 2d & \text{if } i = j, \\ -d & \text{if } |i-j| = 1, \\ 0 & \text{otherwise} \end{cases} \tag{5.12.6}$$

by Proposition 5.11.4. Thus the pairing between $U^{d,B}$ and U^{d,B_-} is identified with the natural pairing on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_r$ multiplied by d, under the identification $e_i - e_{i+1}$ and $e_i^- - e_{i+1}^-$ with the simple coroot $\alpha_i^\vee$.

5.13. Compatibility

Let us take $L = T$. We shall show that the base $\{e_i - e_{i+1}\}_i$ of U^d is compatible with the construction in §4.11 in this subsection.

We fix the Borel subgroup B consisting of upper triangular matrices, and let P_i be the parabolic subgroup corresponding to a simple root α_i and L_i be the Levi subgroup $(i = 1, \dots r-1)$. Recall that we have taken

$$1^d_{L_i,G} \in \operatorname{Hom}_{\operatorname{Perv}(\mathcal{U}^d_{L_i})}(\operatorname{IC}(\mathcal{U}^d_{L_i}), \Phi_{L_i,G}(\operatorname{IC}(\mathcal{U}^d_G))). \tag{5.13.1}$$

(See (4.7.3).)

Let us consider the corresponding fixed point set $\widetilde{\mathcal{U}}^d_{L_i} = (\widetilde{\mathcal{U}}^d_r)^{Z(L_i)}$ in the Gieseker space. The decomposition (5.3.1) in our case is

$$\bigsqcup_{d_1+\cdots+\widehat{d_{i+1}}+\cdots+d_r=d} \widetilde{\mathcal{U}}^{d_1}_1 \times \cdots \times \widetilde{\mathcal{U}}^{d_{i-1}}_1 \times \widetilde{\mathcal{U}}^{d_i}_2 \times \widetilde{\mathcal{U}}^{d_{i+2}}_1 \times \cdots \times \widetilde{\mathcal{U}}^{d_r}_1 . \tag{5.13.2}$$

There is a distinguished connected component, isomorphic to $\widetilde{\mathcal{U}}^d_2$ with $d_i = d$, $d_j = 0$ for $j \neq i$. Let us denote it by Z.

Recall that $\mathcal{U}^d_{L_i}$ is equal to $\mathcal{U}^d_{SL(2)}$ as a topological space and the open subvariety $\operatorname{Bun}^d_{L_i}$ is equal to $\operatorname{Bun}^d_{SL(2)}$. The connected component Z is characterized among all components of $\widetilde{\mathcal{U}}^d_{L_i}$, as it contains $\operatorname{Bun}^d_{L_i}$.

We denote by δ the polarization of Z in $\widetilde{\mathcal{U}}^d_r$ in §5.3. We understand it is ± 1, according to whether it is equal to the polarization given by attracting directions or not, as in §4.3. We correct $1^d_{L_i,G}$ by $\delta 1^d_{L_i,G}$ so that it will be compatible with the stable envelope.

Let us consider the diagram

$$\begin{tikzcd}
\operatorname{IC}(\mathcal{U}^d_{L_i}) \arrow[r, "\delta 1^d_{L_i,G}"] & \Phi_{L_i,G}(\operatorname{IC}(\mathcal{U}^d_G)) \\
\pi_! \mathcal{C}_{\widetilde{\mathcal{U}}^d_{L_i}} \arrow[u] \arrow[r, "\cong", "\mathcal{L}_{L_i,G}"'] & \Phi_{L_i,G}(\pi_! \mathcal{C}_{\widetilde{\mathcal{U}}^d_r}). \arrow[u]
\end{tikzcd} \tag{5.13.3}$$

The upper arrow is given just above, and the bottom arrow is the stable envelope. The right vertical arrow comes from the natural projection to the direct summand $\pi_!(\mathcal{C}_{\widetilde{\mathcal{U}}^d_r}) \to \operatorname{IC}(\mathcal{U}^d_G)$ in (5.1.2), which is the identity homomorphism on the open subset Bun^d_G of $\mathcal{U}^d_G$. The left vertical arrow is defined as follows. We have the distinguished component Z of $\widetilde{\mathcal{U}}^d_{L_i}$ isomorphic to $\widetilde{\mathcal{U}}^d_2$. We have $\operatorname{IC}(\mathcal{U}^d_{L_i}) = \operatorname{IC}(\mathcal{U}^d_{SL(2)})$, and hence have a natural projection $\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}_2^d} \to \mathrm{IC}(\mathscr{U}^d_{L_i})$, as for the right vertical arrow. Composing with the restriction to the distinguished component $\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_{L_i}} \to \pi_! \mathscr{C}_Z$, we define the left vertical arrow.

Proposition 5.13.4. — *The diagram* (5.13.3) *is commutative.*

Proof. — From the construction of the diagram, it is clear that we need to check the commutativity on the open subset $\mathrm{Bun}^d_{L_i}$. Then the commutativity is clear, as two constructions $\delta 1^d_{L_i,G}$ and $\mathscr{L}_{L_i,G}$ are the same: Both are given by the Thom isomorphism corrected by polarization. See [**46**, Th. 3.3.4(ii)] for the stable envelope. ☐

Recall also that we have proposed that there exists a canonical element

$$\begin{aligned} 1^d_{L_i} &\in \mathrm{Hom}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \Phi_{T,L_i}(\mathrm{IC}(\mathscr{U}^d_{L_i}))) \\ &\cong \mathrm{Hom}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \Phi_{\mathbb{C}^*,SL_2}(\mathrm{IC}(\mathscr{U}^d_{SL_2}))) \end{aligned} \tag{5.13.5}$$

in §4.11. We define it so that the following diagram is commutative:

$$\begin{tikzcd} \mathscr{C}_{S_{(d)}\mathbb{A}^2} \arrow[r, "1^d_{L_i}"] \arrow[d] & \Phi_{\mathbb{C}^*,SL(2)}(\mathrm{IC}(\mathscr{U}^d_{SL(2)})) \\ \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_{\mathbb{C}^*}} \arrow[r, "\cong", "\mathscr{L}_{\mathbb{C}^*,SL(2)}"'] & \Phi_{\mathbb{C}^*,SL(2)}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_2}), \arrow[u] \end{tikzcd} \tag{5.13.6}$$

where we choose the parabolic subgroup in $SL(2) \cong [L_i, L_i]$ corresponding to the chosen Borel subgroup B to define the hyperbolic restriction $\mathscr{L}_{\mathbb{C}^*,SL(2)}$. The right vertical arrow is the projection to the direct summand as before. The left vertical arrow is $e_i - e_{i+1}$, where $\{e_i, e_{i+1}\}$ is the base of $V^d_{\mathbb{C}^*,SL(2)} \cong \mathrm{Hom}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_{\mathbb{C}^*}})$, i.e., e_i corresponds to $\widetilde{\mathscr{U}}_1^d \times \widetilde{\mathscr{U}}_1^0 \subset \widetilde{\mathscr{U}}^d_{\mathbb{C}^*} = \bigsqcup \widetilde{\mathscr{U}}_1^{d_1} \times \widetilde{\mathscr{U}}_1^{d_2}$, and e_{i+1} corresponds to $\widetilde{\mathscr{U}}_1^0 \times \widetilde{\mathscr{U}}_1^d$.

We enlarge the bottom row as

$$\begin{tikzcd} \mathscr{C}_{S_{(d)}\mathbb{A}^2} \arrow[r, "1^d_{L_i}"] \arrow[d, "e_i - e_{i+1}"] & \Phi_{T,L_i}(\mathrm{IC}(\mathscr{U}^d_{L_i})) \\ \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_T} \arrow[r, "\cong", "\mathscr{L}_{T,L_i}"'] & \Phi_{T,L_i}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_{L_i}}). \arrow[u] \end{tikzcd} \tag{5.13.7}$$

Here we identify $\widetilde{\mathscr{U}}_2^d$ with the distinguished component Z. We similarly consider $\widetilde{\mathscr{U}}^d_{\mathbb{C}^*}$ as a union of components of $\widetilde{\mathscr{U}}^d_T$, putting it in i and $(i+1)^{\text{th}}$ components. The left vertical arrow is $e_i - e_{i+1}$, where $\{e_1, \ldots, e_r\}$ is the base of $V^d_{T,G}$. Two bases are obviously compatible, so it is safe to use the same notation.

We apply Φ_{T,L_i} to the commutative diagram (5.13.3) and combine it with (5.13.7):

(5.13.8)
$$\begin{tikzcd}
\mathscr{C}_{S_{(d)}\mathbb{A}^2} \arrow[r, "1^d_{L_i}"] \arrow[d, "e_i - e_{i+1}"] & \Phi_{T,L_i}(\mathrm{IC}(\mathscr{U}^d_{L_i})) \arrow[r, "\Phi_{T,L_i}(\delta 1_{L_i,G})"] & \Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G)) \\
\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^n_T} \arrow[r, "\cong", "\mathscr{L}_{T,L_i}"'] & \Phi_{T,L_i}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_{L_i}}) \arrow[u] \arrow[r, "\cong", "\Phi_{T,L_i}(\mathscr{L}_{L_i,G})"'] & \Phi_{T,G}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_r}). \arrow[u]
\end{tikzcd}$$

The composite of lower horizontal arrows is $\mathscr{L}_{T,G}$ by the commutativity (5.7.4). Recall we made an identification of V^d by $\mathscr{L}_{T,G}$ (see (5.10.11)). Therefore $e_i - e_{i+1} \in V^d$ considered as a homomorphism in $\mathrm{Hom}(\mathscr{C}_{S_{(d)}\mathbb{A}^2}, \Phi_{T,G}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_r}))$ is the composition of arrows from the upper left corner to the lower right corner.

It is also clear that the homomorphism $V^d \to U^d$ given by the composition of the rightmost upper arrow coincides with the projection in (5.12.3).

We thus see that $\{\tilde{\alpha}^d_i = \Phi_{T,L_i}(\delta 1^d_{L_i,G}) \circ 1^d_{L_i}\}_i$ coincides with the base $\{e_i - e_{i+1}\}$ of U^d. This gives the construction promised in §4.11 when G is of type A.

Remark 5.13.9. — Suppose $G = SL(2)$. Thanks to Example 5.10.17, we have $[\overline{\mathfrak{T}(0,d)_0}] = (-1)^d e^d_2$. (Here $r_1 = r_2 = 1$.) Therefore we have

$$\langle [\overline{\mathfrak{T}(0,d)_0}], \tilde{\alpha}^{d,-}_1 \rangle = (-1)^d \langle e^d_2, e^{d,-}_1 - e^{d,-}_2 \rangle = (-1)^{d+1} d$$

by Proposition 5.11.4. This completes the proof of (4.12.1).

5.14. Aut(G) invariance

Recall that we have studied $\mathrm{Aut}(G)$ invariance of various constructions for $\mathscr{U}^d_G$ in §4.13. The same applies also to the Gieseker space $\widetilde{\mathscr{U}}^d_r$, if we restrict to the inner automorphism $\mathrm{Inn}(G)$. This is because $\mathrm{Inn}(G)$ acts on $\widetilde{\mathscr{U}}^d_r$, and hence the same applies.

Let us consider $\mathrm{Aut}(G)/\mathrm{Inn}(G)$. It is $\{\pm 1\}$ for type A, and is the Dynkin diagram automorphism given by the reflection at the center. It is represented modulo inner automorphisms by a group automorphism $g \mapsto {}^t g^{-1}$. In terms of Bun^d_G, it corresponds to taking the *dual* vector bundle. In particular, it does not extend to an action on the Gieseker space $\widetilde{\mathscr{U}}^d_r$, as the second Chern class may drop when we take the dual of a sheaf.

In the ADHM description, the diagram automorphism is given by

(5.14.1)
$$[(B_1, B_2, I, J)] \mapsto [(B_1^t, B_2^t, -J^t, I^t)].$$

This does not preserve the stability condition. Therefore we must be careful when we study what happens under this automorphism.

Nevertheless we give

Proof of Lemma 4.13.9. — Recall $\sigma \in \mathrm{Aut}(G)$ preserves T, B, and corresponds to a Dynkin diagram automorphism. Recall also $\tilde{\alpha}^d_i = \Phi_{T,L_i}(\delta 1^d_{L_i,G}) \circ 1^d_{L_i}$.

It is clear that $\Phi_{T,L_i}(1^d_{L_i,G})$ is sent to $\Phi_{T,L_{\sigma(i)}}(1^d_{L_{\sigma(i)},G})$ under φ_σ from its definition.

Next consider $1^d_{L_i} \in U^d_{T,L_i}$. In view of Lemma 4.8.15, U^d_{T,L_i} is $H_{[0]}(\mathcal{U}^d_{B\cap L_i,0})$, which is 1-dimensional space spanned by the irreducible component $[\mathcal{U}^d_{B\cap L_i,0}]$. The class $[\mathcal{U}^d_{B\cap L_i,0}]$ is sent to $[\mathcal{U}^d_{B\cap L_{\sigma(i)},0}]$ under φ_σ, as it is induced from the isomorphism $\mathcal{U}^d_{B\cap L_i,0} \to \mathcal{U}^d_{B\cap L_{\sigma(i)},0}$.

On the other hand, $[\mathcal{U}^d_{B\cap L_i,0}]$ is the image of $[\overline{\mathfrak{T}(0,d)}_0]$ under $\pi_*\colon H_{[0]}(\widetilde{\mathcal{U}}^d_{B\cap L_i,0}) \to H_{[0]}(\mathcal{U}^d_{B\cap L_i,0})$. We have $[\overline{\mathfrak{T}(0,d)}_0] = (-1)^d e_2$ by Example 5.10.17. Hence $[\mathcal{U}^d_{B\cap L_i,0}] = (-1)^{d+1} 1^d_{L_i}/2$. Combining with the above observation, we deduce the assertion. □

CHAPTER 6

$\mathscr{W}$-ALGEBRA REPRESENTATION ON LOCALIZED EQUIVARIANT COHOMOLOGY

The goal of this section is to define a representation of the $\mathscr{W}$-algebra $\mathscr{W}_k(\mathfrak{g})$ on the direct sum of equivariant intersection cohomology groups $\mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G)$ over d, isomorphic to the Verma module with the level and highest weight, given by the equivariant variables by

$$k + h^\vee = -\frac{\varepsilon_2}{\varepsilon_1}, \quad \lambda = \frac{\boldsymbol{a}}{\varepsilon_1} - \rho, \quad \text{where } \boldsymbol{a} = (a^1, \dots, a^\ell) \tag{6.0.1}$$

respectively. Here $\boldsymbol{a}$ is a collection of variables, but will be regarded also as a variable in the Cartan subalgebra $\mathfrak{h}$ so that $a^i = \alpha_i(\boldsymbol{a})$ for a simple root α_i.

Since the level is a rational function in ε_1, ε_2, we must be careful over which ring the representation is defined. In geometric terms, it corresponds to that we need to consider *localized* equivariant cohomology groups. The equivariant cohomology group $H^*_{\mathbb{T}}(\ )$ is a module over $H^*_{\mathbb{T}}(\mathrm{pt}) = \mathbb{C}[\mathrm{Lie}\,\mathbb{T}] = \mathbb{C}[\varepsilon_1, \varepsilon_2, \boldsymbol{a}]$. Let us denote this polynomial ring by $\mathbf{A}_T$ and its quotient field by $\mathbf{F}_T$. In algebraic terms, it means that our $\mathscr{W}$-algebra is defined over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$. Then the level k is a *generic point* in $\mathbb{A}^1$. Moreover we consider a Verma module whose highest weight is in $\mathfrak{h}^* \otimes \mathbf{F}_T$. This means that the highest weight is also generic. More precisely, we regard $\boldsymbol{a}$ as a canonical element in $\mathfrak{h}^* \otimes \mathbf{F}_T = \mathfrak{h}^* \otimes \mathrm{Frac}(S(\mathfrak{h}^*)[\varepsilon_1, \varepsilon_2])$ given by the inner product on $\mathfrak{h}$. Here we have used the Langlands duality implicitly : we first consider $\boldsymbol{a}$ as the identity element in $\mathfrak{h} \otimes \mathfrak{h}^* \subset \mathfrak{h} \otimes \mathbf{F}_T$. Then we regard the first $\mathfrak{h}$ as the dual of the Cartan subalgebra of the Langlands dual of $\mathfrak{g}$. But the Langlands dual is $\mathfrak{g}$ itself as we are considering ADE cases.

We will construct a representation on

$$\bigoplus_d \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G) \otimes_{\mathbf{A}_T} \mathbf{F}_T = \bigoplus_d H^*_{\mathbb{T}}(\mathscr{U}^d_G, \mathrm{IC}(\mathscr{U}^d_G)) \otimes_{\mathbf{A}_T} \mathbf{F}_T. \tag{6.0.2}$$

By the localization theorem and Lemma 3.1.6, natural homomorphisms

$$\begin{aligned}
&\mathrm{IH}^*_{\mathbb{T},c}(\mathscr{U}^d_G) \cong H^*_{\mathbb{T},c}(\mathscr{U}^d_T, i^! j^! (\mathrm{IC}(\mathscr{U}^d_G)))\\
&\to H^*_{\mathbb{T},c}(\mathscr{U}^d_T, \Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G)))\\
&\to H^*_{\mathbb{T}}(\mathscr{U}^d_T, \Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G)))\\
&\to H^*_{\mathbb{T}}(\mathscr{U}^d_T, i^* j^* (\mathrm{IC}(\mathscr{U}^d_G))) \cong \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G)
\end{aligned} \tag{6.0.3}$$

all become isomorphisms over $\mathbf{F}_T$. Thus over $\mathbf{F}_T$, we could use any of these four spaces. Let us denote its direct sum by $M_{\mathbf{F}}(\boldsymbol{a})$:

$$M_{\mathbf{F}}(\boldsymbol{a}) = \bigoplus_d \mathrm{IH}^*_{\mathbb{T},c}(\mathscr{U}^d_G) \otimes_{\mathbf{A}_T} \mathbf{F}_T. \tag{6.0.4}$$

In fact, we will construct representations of *integral forms* (i.e., $\mathbf{A}_T$-forms) of Heisenberg and Virasoro algebras on *non-localized* equivariant cohomology groups $\bigoplus_d H^*_{\mathbb{T},c}(\mathscr{U}^d_T, \Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G)))$ of hyperbolic restrictions in this section. This construction will be the first step towards a construction of the $\mathscr{W}$-algebra representation on *non-localized* equivariant cohomology groups. To follow the remaining argument, the reader needs to read our definition of an integral form of the $\mathscr{W}$-algebra given in §B. Therefore the whole construction will be postponed to §8.1.

Let us denote the fundamental class $1 \in \mathrm{IH}^0_{\mathbb{T}}(\mathscr{U}^0_G) = \mathrm{IH}^0_{\mathbb{T},c}(\mathscr{U}^0_G) = H^0_{\mathbb{T}}(\mathrm{pt})$ by $|\boldsymbol{a}\rangle$. It will be identified with the highest weight vector (or the vacuum vector) of the Verma module. See Proposition 6.7.9 below.

We also use the following notation:

$$\mathbf{A} = \mathbb{C}[\varepsilon_1, \varepsilon_2], \quad \mathbf{F} = \mathbb{C}(\varepsilon_1, \varepsilon_2).$$

6.1. Freeness

Lemma 6.1.1. — *Four modules appearing in* (6.0.3) *are free over* $\mathbf{A}_T$.

Proof. — By Lemma 3.1.6 all four modules are pure, as $(\mathscr{U}^d_G)^{\mathbb{T}}$ is a single point, and they are stalks at the point. Now freeness follows as in [**34**, Th. 14.1(8)].

Or we have odd cohomology vanishing by [**21**, Th. 7.10]. So it also follows from [**34**, Th. 14.1(1)]. ☐

In particular, homomorphisms in (6.0.3) are all injective.

6.2. Another base of U^d, continued

Let $U^d = U^d_{T,G}$ be as in §4.8. Let L_i be the Levi subgroup corresponding to a simple root α_i and consider U^d_{T,L_i} as in §4.11. We identify $\mathrm{IC}(\mathscr{U}^d_{L_i})$ with $\mathrm{IC}(\mathscr{U}^d_{SL(2)})$ by the bijective morphism $\mathscr{U}^d_{SL(2)} \to \mathscr{U}^d_{L_i}$ (see Proposition 4.2.5). We have a maximal torus and a Borel subgroup induced from those of G. Then U^d_{T,L_i} has the base in (5.12.5), where it consists of a single element as $N = 2$. Let us denote the element by $1^d_{L_i}$, as we promised in §4.11.

Next consider $1^d_{L_i,G}$ given by the Thom isomorphism as in §4.7. We have the repellent polarization δ_{rep} of $\mathrm{Bun}^d_{L_i}$ in Bun^d_G. We modify it to δ according to Lemma 5.3.4. We choose and fix a bipartite coloring of the vertices of the Dynkin diagram, i.e., $o\colon I \to \{\pm 1\}$ such that $o(i) = -o(j)$ if i and j are connected in the diagram. Then we set

$$\delta = o(i)^d \delta_{\mathrm{rep}}. \tag{6.2.1}$$

This is our polarization, which was promised in (4.11.4). Let us write

$$\tilde{\alpha}^d_i \overset{\text{def.}}{=} \Phi_{T,L_i}(\delta 1^d_{L_i,G}) \circ 1^d_{L_i}. \tag{6.2.2}$$

This gives us a collection $\{\tilde{\alpha}^d_i\}_i$ of elements in U^d labeled by I. Thanks to (4.12.1), it is a base of U^d. This will follow also from Proposition 6.3.8.

6.3. Heisenberg algebra associated with the Cartan subalgebra

We construct a representation of the Heisenberg algebra associated with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ on the direct sum of (6.0.2) in this subsection. It will be the first step towards the $\mathscr{W}$-algebra representation.

Let us first review the construction of the Heisenberg algebra representation in §5.5 for the case $r = 2$ and $L = S(GL(1) \times GL(1)) = \mathbb{C}^*$. We consider Heisenberg operators $P^\Delta_n \equiv P^\Delta_n(1)$ associated with the cohomology class $1 \in H^{[*]}_{\mathbb{T}}(\mathbb{A}^2)$. We omit (1) hereafter. They are not well-defined on $\bigoplus_d H^{\mathbb{T}}_{[*]}(\widetilde{\mathscr{U}}^d_P)$ if $d > 0$, but are well-defined on the localized equivariant homology group $\bigoplus_d H^{\mathbb{T}}_{[*]}(\widetilde{\mathscr{U}}^d_P) \otimes_{\mathbf{A}_T} \mathbf{F}_T$, and satisfy the commutation relations

$$[P^\Delta_m, P^\Delta_n] = -2m\delta_{m,-n}\frac{1}{\varepsilon_1\varepsilon_2}. \tag{6.3.1}$$

Via the stable envelope, we have the isomorphism

$$\bigoplus_d H^{\mathbb{T}}_{[*]}(\widetilde{\mathscr{U}}^d_P) \cong \bigoplus_d H^{\mathbb{T}}_{[*]}(\widetilde{\mathscr{U}}^d_L) = \bigoplus_{d_1,d_2} H^{\mathbb{T}}_{[*]}(\widetilde{\mathscr{U}}^{d_1}_1) \otimes H^{\mathbb{T}}_{[*]}(\widetilde{\mathscr{U}}^{d_1}_1), \tag{6.3.2}$$

and we have the representation of the tensor product of two copies of Heisenberg algebras, given by $P^{(1)}_n = P_n \otimes 1$ and $P^{(2)}_n = 1 \otimes P_n$ on the localized equivariant homology group, where P_n is the Heisenberg generator for $r = 1$. The above Heisenberg generator P^Δ_n is the diagonal $P^{(1)}_n + P^{(2)}_n$. See §5.5.

We have

$$H^{\mathbb{T}}_{[-*]}(\widetilde{\mathscr{U}}^d_P) \cong H^*_{\mathbb{T}}(\mathscr{U}^d_{\mathbb{C}^*}, p_* j^! \pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_2}) = H^*_{\mathbb{T}}(\mathscr{U}^d_{\mathbb{C}^*}, \Phi_{\mathbb{C}^*,SL(2)}(\pi_! \mathscr{C}_{\widetilde{\mathscr{U}}^d_2})) \tag{6.3.3}$$

by §5.6 and $\pi_! = \pi_*$. This homology group contains

$$H^*_{\mathbb{T}}(\mathscr{U}^d_{\mathbb{C}^*}, \Phi_{\mathbb{C}^*,SL(2)}(\mathrm{IC}(\mathscr{U}^d_{SL(2)}))) \tag{6.3.4}$$

as a direct summand, and the *anti-diagonal* Heisenberg algebra generated by $P^{(1)}_n - P^{(2)}_n$ acts on its direct sum over d. (See §5.12.)

Let us return back to general G. Let L_i be the Levi subgroup as in the previous subsection. We identify $\mathrm{IC}(\mathscr{U}^d_{SL(2)})$ with $\mathrm{IC}(\mathscr{U}^d_{L_i})$ as before, and we have a(n anti-diagonal) Heisenberg algebra representation on

$$\bigoplus_d H^*_{\mathbb{T}}(\mathscr{U}^d_T, \Phi_{T,L_i}(\mathrm{IC}(\mathscr{U}^d_{L_i}))) \otimes_{\mathbf{A}_T} \mathbf{F}_T. \tag{6.3.5}$$

Using the decomposition (4.11.5) and $\Phi_{T,L_i}\Phi_{L_i,G} = \Phi_{T,G}$, we have an induced Heisenberg algebra representation on $M_{\mathbf{F}}(\boldsymbol{a})$ in (6.0.4). Let us denote the Heisenberg generator by P^i_n.

By Lemma 4.8.7, the space $M_{\mathbf{F}}(\boldsymbol{a})$ is isomorphic to

$$\mathrm{Sym}((U^1 \oplus U^2 \oplus \cdots) \otimes_{\mathbb{C}} \mathbf{F}_T), \tag{6.3.6}$$

where Sym denotes the symmetric power. ($U^d = U^d_{T,G}$ as before.)

Let us describe P^i_n in this space. Recall that we have the orthogonal decomposition $U^d = U^d_{T,L_i} \oplus (U^d_{T,L_i})^\perp$ in (4.11.6). Then we have the factorization

$$\begin{aligned} \mathrm{Sym}((U^1 \oplus U^2 \oplus \cdots) \otimes_{\mathbb{C}} \mathbf{F}_T) &\cong \mathrm{Sym}((U^1_{T,L_i} \oplus U^2_{T,L_i} \oplus \cdots) \otimes_{\mathbb{C}} \mathbf{F}_T) \otimes_{\mathbf{F}_T} \\ &\qquad \mathrm{Sym}(((U^1_{T,L_i})^\perp \oplus (U^2_{T,L_i})^\perp \oplus \cdots) \otimes_{\mathbb{C}} \mathbf{F}_T) \end{aligned} \tag{6.3.7}$$

The first factor of the right hand side is the usual Fock space associated with the Cartan subalgebra $\mathfrak{h}_{\mathfrak{sl}_2}$ of $\mathfrak{sl}_2$. In fact, using $U^d_{T,L_i} \cong \mathbb{C}1^d_{L_i}$, we identify U^d_{T,L_i} with $\mathfrak{h}_{\mathfrak{sl}_2}$. The pairing is multiplied by $-1/\varepsilon_1\varepsilon_2$ from the natural one. Then the factor is $\mathrm{Sym}(z^{-1}\mathfrak{h}_{\mathfrak{sl}_2}[z^{-1}])$ and the Heisenberg algebra acts in the standard way. From its definition, our Heisenberg operator P^i_n is given by the tensor product of the Heisenberg operator for $\mathrm{Sym}(z^{-1}\mathfrak{h}_{\mathfrak{sl}_2}[z^{-1}])$, and the identity.

The following means that the operators P^i_n define the Heisenberg algebra $\mathfrak{Heis}(\mathfrak{h})$ associated with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Proposition 6.3.8. — *Heisenberg generators satisfy commutation relations*

$$[P^i_m, P^j_n] = -m\delta_{m,-n}(\alpha_i, \alpha_j)\frac{1}{\varepsilon_1\varepsilon_2}. \tag{6.3.9}$$

If we normalize the generator by $\widehat{h}^i_n = \varepsilon_2 P^i_n$, the relations match with a standard convention with level $-\varepsilon_2/\varepsilon_1 = k + h^\vee$. See (B.1.5).

From the construction, P^i_{-d} applied to the vacuum vector $|\boldsymbol{a}\rangle \in H^0_{\mathbb{T}}(\mathscr{U}^0_T, \Phi_{T,G}(\mathrm{IC}(\mathscr{U}^0_G)))$ is equal to $\Phi_{T,L_i}(\delta 1^d_{L_i,G}) \circ 1^d_{L_i} \in U^d$ divided by $\varepsilon_1\varepsilon_2$, considered as an element in (6.3.6).

From the construction, (6.3.6) is the Fock space of the Heisenberg algebra associated with the Cartan subalgebra $\mathfrak{h}$. It is $\mathrm{Sym}(z^{-1}\mathfrak{h}[z^{-1}])$, where the base field is $\mathbf{F}_T$. The element $\tilde{\alpha}^d_i$ is a linear function, living on $z^{-d}\mathfrak{h}$.

Proof of Proposition 6.3.8. — The case $(\alpha_i, \alpha_j) = 2$, i.e., $i = j$ is obvious from the construction.

Next consider the case $(\alpha_i, \alpha_j) = -1$. Then i and j are connected by an edge in the Dynkin diagram. Let us take the parabolic subgroup P corresponding to the subset consisting of two vertices i and j, and the corresponding Levi subgroup L. We have $[L, L] \cong SL(3)$. Then from our construction and the compatibility of the stable envelope with the hyperbolic restriction functor in §5.13, the assertion follows from the $SL(3)$-case, which is clear as Heisenberg algebra generators are given by

$$P^i_n = P_n \otimes 1 \otimes 1 - 1 \otimes P_n \otimes 1, \quad P^j_n = 1 \otimes P_n \otimes 1 - 1 \otimes 1 \otimes P_n. \tag{6.3.10}$$

Note also that our polarization δ in (6.2.1) was chosen so that it is the same as the polarization for $\widetilde{\mathcal{U}}^{d}_{SL(3)}$ via Lemma 5.3.4 up to overall sign independent of d.

Finally consider the case $(\alpha_i, \alpha_j) = 0$. We argue as above by taking the corresponding Levi subgroup L with $[L, L] \cong SL(2) \times SL(2)$. Then it is clear that Heisenberg generators commute.

If a reader would wonder that $SL(2) \times SL(2)$ is not considered in §5, we instead take a type A_k subdiagram containing i, j and take the corresponding Levi subgroup L with $[L, L] \cong SL(k + 1)$. Then it is clear that the Heisenberg generators P^i_m, P^j_n commute for $SL(k + 1)$. Therefore they commute also for G. □

Let us consider Heisenberg operators $P^i_n([0]) = \varepsilon_1\varepsilon_2 P^i_n$, coupled with the Poincaré dual of $[0] \in H^{\mathbb{T}}_0(\mathbb{A}^2)$, and denote them by $\widetilde{P}^i_n$. Then they are well-defined on non-localized equivariant cohomology groups

$$\bigoplus_d H^*_{\mathbb{T}}(\mathcal{U}^d_T, \Phi_{T,G}(\mathrm{IC}(\mathcal{U}^d_G))), \tag{6.3.11}$$

and satisfy the commutation relations

$$[\widetilde{P}^i_m, \widetilde{P}^j_n] = -m\delta_{m,-n}(\alpha_i, \alpha_j)\varepsilon_1\varepsilon_2. \tag{6.3.12}$$

The same is true for the non-localized equivariant cohomology with compact supports.

We define the $\mathbf{A}$-form $\mathfrak{Heis}_{\mathbf{A}}(\mathfrak{h})$ of the Heisenberg vertex algebra as the vertex $\mathbf{A}$-subalgebra of $\mathfrak{Heis}(\mathfrak{h})$ generated by $\widetilde{P}^i_m$.

6.4. Virasoro algebra

Let us introduce 0-mode operators P^i_0. In §5.2 we did not introduce them. Since they commute with all other operators, we can set them any scalars. We follow the convention in [**46**, §13.1.5, §14.3.1], that is

$$P^i_0 = \frac{a^i}{\varepsilon_1\varepsilon_2}. \tag{6.4.1}$$

Here a^i is the i^{th} simple root, and should be identified with $a_i - a_{i+1}$ in [**46**] in the Fock space $\mathbf{F}(a_1) \otimes \cdots \otimes \mathbf{F}(a_r)$ corresponding to the equivariant cohomology of Gieseker spaces for rank r sheaves. We also set $\widetilde{P}_0^i = \varepsilon_1\varepsilon_2 P_0^i = a^i$.

We then introduce Virasoro generators by

$$L_n^i = -\frac{1}{4}\varepsilon_1\varepsilon_2 \sum_m :P_m^i P_{n-m}^i: - \frac{n}{2}(\varepsilon_1+\varepsilon_2)P_n^i + \frac{(\varepsilon_1+\varepsilon_2)^2}{4\varepsilon_1\varepsilon_2}\delta_{n,0}. \tag{6.4.2}$$

See [**46**, (13.10),(14.10)]. Let us briefly explain how to derive the above expression from [**46**]: The Virasoro field $T(\gamma,\kappa) = \sum L_n(\gamma,\kappa)z^{-n}$ in [**46**, (13.10)] is given by

$$T(\gamma,\kappa) = \frac{1}{2}:\boldsymbol{\alpha}^2:(\gamma) + \partial\boldsymbol{\alpha}(\gamma\kappa) - \frac{1}{2}\tau(\gamma\kappa^2), \tag{6.4.3}$$

where $\boldsymbol{\alpha}(\gamma) = \sum \alpha_n(\gamma)z^{-n}$ is the free field. Note that T and $\boldsymbol{\alpha}$ are different from the usual convention, as the exponents are not $-n-1$, $-n-2$ respectively. Also $\partial = z\partial_z$.

We take $\gamma = 1$, the fundamental class of $H^0_{\mathbb{T}}(\mathbb{A}^2)$. Next note that $\boldsymbol{\alpha} = \boldsymbol{\alpha}^-/\sqrt{2}$ [**46**, (14.8)], and our P^i is identified with $\boldsymbol{\alpha}^-$. This is the reason we have $1/4$ instead of $1/2$. The remaining factor $-\varepsilon_1\varepsilon_2$ comes from $1^\Delta = -1 \otimes \mathrm{pt}$ in [**46**, §13.3.2].

For the second term, note $\kappa = \hbar/\sqrt{2}$ (see [**46**, (14.8)]), $\hbar = -t_1 - t_2$ (see [**46**, §17.1.1,(18.10)] for example). We denote their t_1, t_2 by ε_1, ε_2 instead.

For the last constant term, we have $-\gamma\kappa^2 = -(\varepsilon_1+\varepsilon_2)^2/2$ and $\tau(1) = -\int_{\mathbb{A}^2} 1 = -1/\varepsilon_1\varepsilon_2$.

The Virasoro algebra commutation relations are

$$[L_m^i, L_n^i] = (m-n)L_{m+n}^i + \left(1 + \frac{6(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}\right)\delta_{m,-n}\frac{m^3-m}{12}. \tag{6.4.4}$$

See [**46**, §13.3.2]. And the highest weight is given by

$$L_0^i|\boldsymbol{a}\rangle = -\frac{1}{4}\left(\frac{(a^i)^2}{\varepsilon_1\varepsilon_2} - \frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}\right)|\boldsymbol{a}\rangle. \tag{6.4.5}$$

See [**46**, §13.3.5].

In order to apply the result of Feigin-Frenkel to our situation later, we shift P_n^i in (6.4.2) as $P_n^i - (\varepsilon_1+\varepsilon_2)/\varepsilon_1\varepsilon_2\delta_{n,0}$ (see [**46**, §19.2.5]) so that

$$L_n^i = -\frac{1}{4}\varepsilon_1\varepsilon_2 \sum_m :P_m^i P_{n-m}^i: - \frac{n+1}{2}(\varepsilon_1+\varepsilon_2)P_n^i. \tag{6.4.6}$$

This is a standard embedding of the Virasoro algebra in the Heisenberg algebra, given as the kernel of the screening operator (see [**30**, §15.4.14]). We have

$$P_0^i = \frac{1}{\varepsilon_1\varepsilon_2}\left(a^i - (\varepsilon_1+\varepsilon_2)\right) \tag{6.4.7}$$

in this convention.

We modify (6.4.2) as

$$\begin{aligned} L^i_n = &\ -\frac{1}{4}\varepsilon_1\varepsilon_2 \sum_m {:}(P^i_m - \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}\delta_{m,0})(P^i_{n-m} - \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}\delta_{m,n}){:} \\ &\qquad - \frac{1}{2}(\varepsilon_1+\varepsilon_2)P^i_n + \frac{(\varepsilon_1+\varepsilon_2)^2}{4\varepsilon_1\varepsilon_2}\delta_{n,0} \\ &\qquad - \frac{n}{2}(\varepsilon_1+\varepsilon_2)P^i_n + \frac{(\varepsilon_1+\varepsilon_2)^2}{4\varepsilon_1\varepsilon_2}\delta_{n,0} \\ = &\ -\frac{1}{4}\varepsilon_1\varepsilon_2 \sum_m {:}(P^i_m - \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}\delta_{m,0})(P^i_{n-m} - \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}\delta_{m,n}){:} \\ &\qquad - \frac{n+1}{2}(\varepsilon_1+\varepsilon_2)(P^i_n - \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}\delta_{n,0}). \end{aligned} \tag{6.4.8}$$

Therefore if we replace P^i_n by $P^i_n - (\varepsilon_1+\varepsilon_2)/\varepsilon_1\varepsilon_2\delta_{n,0}$, we get the above expression.

We denote by $\mathfrak{Vir}_i$ the Virasoro vertex subalgebra of $\mathfrak{Heis}(\mathfrak{h})$ generated by L^i_n.

Let us introduce a modified Virasoro generator $\widetilde{L}^i_n = \varepsilon_1\varepsilon_2 L^i_n$. We have

$$\widetilde{L}^i_n = -\frac{1}{4}\sum_m {:}\widetilde{P}^i_m\widetilde{P}^i_{n-m}{:} - \frac{n+1}{2}(\varepsilon_1+\varepsilon_2)\widetilde{P}^i_n. \tag{6.4.9}$$

Hence $\widetilde{L}^i_n$ is an element in $\mathfrak{Heis}_{\mathbf{A}}(\mathfrak{h})$. We denote the corresponding vertex $\mathbf{A}$-subalgebra by $\mathfrak{Vir}_{i,\mathbf{A}}$.

Note that the central charge $1 + 6(\varepsilon_1+\varepsilon_2)^2/\varepsilon_1\varepsilon_2$ is equal to that of Virasoro algebras, appearing in the construction of the $\mathscr{W}$-algebra $\mathscr{W}_k(\mathfrak{g})$ as the BRST reduction of the affine vertex algebra at level k, if we have the relation

$$-\frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2} = k + h^\vee - 2 + \frac{1}{k+h^\vee}, \tag{6.4.10}$$

see [**30**, §15.4.14] and Corollary B.6.11 below. In other words, $k + h^\vee = -\varepsilon_2/\varepsilon_1$ or $-\varepsilon_1/\varepsilon_2$. It is known that the $\mathscr{W}$-algebra for type ADE has a symmetry under $k + h^\vee \leftrightarrow (k+h^\vee)^{-1}$ [**30**, Prop. 15.4.16]. Therefore either choice gives the same result. We here take $k + h^\vee = -\varepsilon_2/\varepsilon_1$, see (6.0.1). It is remarkable that the symmetry $k + h^\vee \leftrightarrow (k+h^\vee)^{-1}$ corresponds to a trivial symmetry $\varepsilon_1 \leftrightarrow \varepsilon_2$ in geometry.

6.5. The first Chern class of the tautological bundle

Let us explain a geometric meaning of the Virasoro generators in the previous subsection. It was obtained in [**46**, Th. 14.2.3], based on an earlier work by Lehn [**43**] for the rank 1 case. Let us first consider the rank 2 case.

Consider the Gieseker space $\widetilde{\mathscr{U}}^d_2$ of rank 2 framed sheaves on $\mathbb{P}^2$ with $c_2 = d$. For $(E,\varphi) \in \widetilde{\mathscr{U}}^d_2$, consider $H^1(\mathbb{P}^2, E(-\ell_\infty))$. Other cohomology groups vanish, and hence it has dimension equal to d by the Riemann-Roch formula. In the ADHM description, it is identified with the vector space V. When we vary E, it forms a vector bundle over $\widetilde{\mathscr{U}}^d_2$, which we denote by $\mathscr{V}$. Its first Chern class $c_1(\mathscr{V})$ can be considered as an operator on $H^*_{\mathbb{T}}(\widetilde{\mathscr{U}}_2^d)$ acting by the cup product. Then its commutator with the diagonal Heisenberg generator, restricted to $\mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_{SL_2})$, is the Virasoro generator up to constant:

$$[c_1(\mathscr{V}), P^\Delta_n]\big|_{\mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_{SL_2})} = nL_n, \tag{6.5.1}$$

where we denote L^i_n in the previous subsection by L_n since $G = SL_2$. (See [**46**, Th. 14.2.3].)

Let us remark that $c_1(\mathscr{V})$ is defined on *non-localized* equivariant cohomology groups $\mathrm{IH}^*_{\mathbb{T},c}(\widetilde{\mathscr{U}}_2^d)$. Therefore $\widetilde{L}_n = \varepsilon_1\varepsilon_2 L_n$ is also well-defined on non-localized equivariant cohomology groups for $n \neq 0$. The operator $\widetilde{L}_0 = \varepsilon_1\varepsilon_2 L_0$ is also well-defined as it is the grading operator ([**46**, Lem. 13.1.1]).

Returning back to general G, we see that $\widetilde{L}^i_n$ is well-defined on

$$\bigoplus_d H^*_{\mathbb{T},c}(\mathscr{U}^d_{L_i}, \Phi_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G))) \tag{6.5.2}$$

thanks to the decomposition (4.11.5). Namely this space is a module over $\mathfrak{Vir}_{i,\mathbf{A}}$. It lies in between the first two spaces in (6.0.3):

$$\mathrm{IH}^*_{\mathbb{T},c}(\mathscr{U}^d_G) \to H^*_{\mathbb{T},c}(\mathscr{U}^d_{L_i}, \Phi_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G))) \to H^*_{\mathbb{T},c}(\mathscr{U}^d_T, \Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))). \tag{6.5.3}$$

The Formula (6.4.9) relates operators $\widetilde{L}^i_n$ and $\widetilde{P}^i_n$ acting on the middle and right spaces respectively via the second homomorphism.

6.6. $\mathscr{W}$-algebra representation

Let us consider the vertex algebra associated with the Heisenberg algebra, and denote it by the same notation $\mathfrak{Heis}(\mathfrak{h})$ for brevity. It is regarded as a vertex algebra over $\mathbf{F}$.

We have the Virasoro vertex subalgebra $\mathfrak{Vir}_i$ corresponding to each simple root α_i as in §6.4. Consider the orthogonal complement $\alpha_i^\perp$ of $\mathbb{C}\alpha_i$ in $\mathfrak{h}$, and the corresponding Heisenberg vertex algebra $\mathfrak{Heis}(\alpha_i^\perp)$. It commutes with $\mathfrak{Vir}_i$, and the tensor product $\mathfrak{Vir}_i \otimes \mathfrak{Heis}(\alpha_i^\perp)$ is a vertex subalgebra of $\mathfrak{Heis}(\mathfrak{h})$.

By a result of Feigin-Frenkel (see [**30**, Th. 15.4.12]), the $\mathscr{W}$-algebra $\mathscr{W}_k(\mathfrak{g})$ is identified with the intersection

$$\bigcap_i \mathfrak{Vir}_i \otimes \mathfrak{Heis}(\alpha_i^\perp) \tag{6.6.1}$$

in $\mathfrak{Heis}(\mathfrak{h})$ when the level k is generic. More precisely, $\mathfrak{Vir}_i \otimes \mathfrak{Heis}(\alpha_i^\perp)$ is given by the kernel of a screening operator on $\mathfrak{Heis}(\mathfrak{h})$, and $\mathscr{W}_k(\mathfrak{g})$ is the intersection of the kernel of screening operators.

Now $\mathscr{W}_k(\mathfrak{g})$ has a representation on the direct sum of localized equivariant cohomology groups $M_{\mathbf{F}}(\boldsymbol{a})$ (see (6.0.4)), as a vertex subalgebra of $\mathfrak{Heis}(\mathfrak{h})$.

6.7. Highest weight

In this subsection we explain that we can identify $\boldsymbol{a}$ with the highest weight of the $\mathscr{W}_k(\mathfrak{g})$-module $M_{\mathbf{F}}(\boldsymbol{a})$, where the highest weight vector is $|\boldsymbol{a}\rangle$.

Let us first briefly review the definition of Verma modules of the $\mathscr{W}$-algebra to set up the notation. See [**2**, §5] for detail.

Let $\mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))$ be the current algebra of the $\mathscr{W}$-algebra as in [**2**, §4]. (The finite dimensional Lie algebra is denoted by $\bar{\mathfrak{g}}$, while $\mathfrak{g}$ is the corresponding untwisted affine Lie algebra in [**2**].) We denote the current algebra of the Heisenberg algebra by $\mathfrak{U}(\mathfrak{Heis}(\mathfrak{h}))$. It is a completion of the universal enveloping algebra of the Heisenberg Lie algebra. The embedding $\mathscr{W}_k(\mathfrak{g}) \subset \mathfrak{Heis}(\mathfrak{h})$ induces an embedding $\mathfrak{U}(\mathscr{W}_k(\mathfrak{g})) \to \mathfrak{U}(\mathfrak{Heis}(\mathfrak{h}))$.

We have decompositions $\mathfrak{U}(\mathscr{W}_k(\mathfrak{g})) = \bigoplus_d \mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_d$, $\mathfrak{U}(\mathfrak{Heis}(\mathfrak{h})) = \bigoplus_d \mathfrak{U}(\mathfrak{Heis}(\mathfrak{h}))_d$ by degree. Two decompositions are compatible under the embedding. Let

$$\mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_{\geq 0} \overset{\text{def.}}{=} \bigoplus_{d\geq 0} \mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_d, \quad \mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_{>0} \overset{\text{def.}}{=} \bigoplus_{d>0} \mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_d. \tag{6.7.1}$$

The *Zhu algebra* of $\mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))$ is given by

$$\mathfrak{Zh}(\mathscr{W}_k(\mathfrak{g})) \overset{\text{def.}}{=} \mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_0 / \overline{\sum_{r>0} \mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_{-r} \mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_r}. \tag{6.7.2}$$

Then it is isomorphic to the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ ([**2**, Th. 4.16.3]). We further identify it with the Weyl group invariant part of the symmetric algebra of $\mathfrak{h}$ ([**2**, (55)]):

$$\mathfrak{Zh}(\mathscr{W}_k(\mathfrak{g})) \cong Z(\mathfrak{g}) \cong S(\mathfrak{h})^W. \tag{6.7.3}$$

We have an induced embedding $\mathfrak{Zh}(\mathscr{W}_k(\mathfrak{g})) \to \mathfrak{Zh}(\mathfrak{U}(\mathfrak{Heis}(\mathfrak{h})))$, where the latter is the subalgebra generated by zero modes. We have

$$\mathfrak{Zh}(\mathfrak{U}(\mathfrak{Heis}(\mathfrak{h}))) \cong S(\mathfrak{h}). \tag{6.7.4}$$

Lemma 6.7.5. — *Under the identifications* (6.7.3), 6.7.4, *the embedding* $\mathfrak{Zh}(\mathscr{W}_k(\mathfrak{g})) \to \mathfrak{Zh}(\mathfrak{U}(\mathfrak{Heis}(\mathfrak{h})))$ *is induced by*

$$h^i \mapsto h^i + (k + h^\vee), \tag{6.7.6}$$

where h^i *is a simple coroot of* $\mathfrak{h}$.

Proof. — The assertion follows from [**2**, Th. 4.16.4], together with an isomorphism $\widehat{t}_{-\bar{\rho}^\vee}$ which sends the old Zhu algebra, denoted by $H^0(\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}))$ there, to a new one $H^0(\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})''_{\mathrm{new}}))$. The zero mode is written as $\widehat{J}_i(0)$ there. We can calculate $\widehat{t}_{-\bar{\rho}^\vee}(\widehat{J}_i(0)) = \widehat{J}_i(0) + k + h^\vee$ by formulas in [**2**, bottom of p.276]. ☐

We regard $\lambda \in \mathfrak{h}^*$ as a homomorphism $S(\mathfrak{h})^W \to \mathbb{C}$ by the evaluation at $\lambda + \rho$, where ρ is the half sum of positive roots of $\mathfrak{g}$. (It is denoted by $\gamma_{\bar{\lambda}}$ in [**2**, §5].) We further regard $\mathbb{C}$ as a $\mathfrak{Zh}(\mathscr{W}_k(\mathfrak{g}))$-module by the above isomorphism, and denote it by $\mathbb{C}_\lambda$. We extend it to a $\mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_{\geq 0}$-module on which $\mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_{>0}$ acts trivially. Then we define

$$M(\lambda) \overset{\text{def.}}{=} \mathfrak{U}(\mathscr{W}_k(\mathfrak{g})) \otimes_{\mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_{\geq 0}} \mathbb{C}_\lambda. \tag{6.7.7}$$

This is called the *Verma module with highest weight* λ.

Now we turn to our $\mathscr{W}_k(\mathfrak{g})$-module $M_{\mathbf{F}}(\boldsymbol{a})$. We identify $\mathfrak{h} = \operatorname{Lie} T$ with $\mathfrak{h}^*$ by the invariant bilinear form $(\ ,\ )$. Then we have an identification

$$S(\mathfrak{h})^W \cong \mathbb{C}[\operatorname{Lie} T]^W = H^*_T(\mathrm{pt})^W. \tag{6.7.8}$$

We regard the collection $\boldsymbol{a} = (a^1, \dots, a^\ell)$ as a variable in $\operatorname{Lie} T$ by considering a^i its coordinate. Hence $\boldsymbol{a}$ has value in $\mathfrak{h}^*$ by the above identification.

Recall that $|\boldsymbol{a}\rangle$ is the fundamental class $1 \in \mathrm{IH}^0_{\mathbb{T}}(\mathscr{U}^0_G)$. Since the degree d corresponds to an instanton number, $\mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_{\geq 0}$ acts via a homomorphism $\mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))_{\geq 0} \to \mathbf{F}_T$ induced from $\mathfrak{Zh}(\mathscr{W}_k(\mathfrak{g})) \to \mathbf{F}_T$ on $\mathbf{F}_T|\boldsymbol{a}\rangle = \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^0_G) \otimes_{\mathbf{A}_T} \mathbf{F}_T$. Hence we have a $\mathscr{W}_k(\mathfrak{g})$-homomorphism $M(\lambda) \to M_{\mathbf{F}}(\boldsymbol{a})$, sending $1 \in \mathbb{C}_\lambda \subset M(\lambda)$ to $|\boldsymbol{a}\rangle \in M_{\mathbf{F}}(\boldsymbol{a})$. Here we generalize the above definition to $\lambda\colon \mathfrak{Zh}(\mathscr{W}_k(\mathfrak{g})) \to \mathbf{F}_T$.

Proposition 6.7.9. — (1) *The highest weight* λ *is given by*

$$\lambda = \frac{\boldsymbol{a}}{\varepsilon_1} - \rho. \tag{6.7.10}$$

(2) $M_{\mathbf{F}}(\boldsymbol{a})$ *is irreducible as a* $\mathscr{W}_k(\mathfrak{g})$*-module, and isomorphic to* $M(\lambda)$.

Note that the Weyl group action on $\boldsymbol{a}$ corresponds to the dot action on λ, $w \circ \lambda = w(\lambda + \rho) - \rho$.

Proof. — (1) Recall that our Heisenberg generators and standard generators are related by $\widehat{h}^i_n = \varepsilon_2 P^i_n$. Then the zero mode acts by

$$\frac{a^i}{\varepsilon_1} - 1 - \frac{\varepsilon_2}{\varepsilon_1} = (\alpha_i, \frac{\boldsymbol{a}}{\varepsilon_1} - \rho) + k + h^\vee \tag{6.7.11}$$

thanks to (6.4.7).

We compare this formula with a realization of $M(\lambda)$ in [**2**, §5.2]. Our $\widehat{h}^i_n$ is $\widehat{t}_{-\bar\rho^\vee}(\widehat{J}_i(n)) \in \mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}})$, and $\widehat{t}_{-\bar\rho^\vee}(\widehat{J}_i(0)) = \widehat{J}_i(0) + k + h^\vee$ as in Lemma 6.7.5. Since $\widehat{J}_i(0)$ acts by $\lambda(J_i)$ on $M(\lambda)$, we obtain $\lambda = \boldsymbol{a}/\varepsilon_1 - \rho$.

(2) It is well-known that $M(\lambda)$ is irreducible when λ is generic. It follows, for example, from the fact that the determinant of the Kac-Shapovalov form is a nonzero rational function, hence the form is nondegenerate if λ is neither a zero nor a pole. (See below for the Kac-Shapovalov form.) It also means that the form is nondegenerate when one views λ as a rational function like us. Therefore $M(\lambda) \to M_{\mathbf{F}}(\boldsymbol{a})$ is injective.

Now we compare the graded characters. The character of $M(\lambda)$ is the same as the character of $S(t\mathfrak{h}[t])$ where $\deg(t) = 1$. We have $M_{\mathbf{F}}(\boldsymbol{a}) = \bigoplus_{d \in \mathbb{N}} \mathrm{IH}^*(\mathscr{U}^d_G) \otimes \mathbf{F}_T$. According to [**21**, Theorem 7.10], the character of $M_{\mathbf{F}}(\boldsymbol{a})$ (with grading by the instanton number) is the same as the character of $S(t\mathfrak{g}^f[t])$ where f is a principal nilpotent. Since $\dim \mathfrak{g}^f = \dim \mathfrak{h}$, the graded characters of $M(\lambda)$ and $M_{\mathbf{F}}(\boldsymbol{a})$ coincide. $\square$

6.8. Kac-Shapovalov form

We shall identify the Kac-Shapovalov form on $M(\lambda)$ with a natural pairing on $M_{\mathbf{F}}(\boldsymbol{a})$ given by the Verdier duality in this subsection.

Let σ be the Dynkin diagram automorphism given by $-\alpha_{\sigma(i)} = w_0(\alpha_i)$. We denote the corresponding element in $\operatorname{Aut}(G)$ also by σ. We have an induced isomorphism

$$\varphi_\sigma\varphi_{w_0}\colon \operatorname{IH}^*_{\mathbb{T},c}(\mathscr{U}^d_G) \to \operatorname{IH}^*_{\mathbb{T},c}(\mathscr{U}^d_G),$$

which is $\mathbf{A}_T = H^*_{\mathbb{T}}(\mathrm{pt})$-linear if we twist the $\mathbf{A}_T$-structure on the second $\operatorname{IH}^*_{\mathbb{T},c}(\mathscr{U}^d_G)$ by composing the automorphism $\boldsymbol{a} \mapsto -\boldsymbol{a}$ of $\mathbf{A}_T$. This is explained in the paragraph after (4.13.2).

Let us denote the natural perfect pairing by

$$\langle\ ,\ \rangle\colon \operatorname{IH}^*_{\mathbb{T},c}(\mathscr{U}^d_G) \otimes_{\mathbf{A}_T} \operatorname{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G) \to \mathbf{A}_T, \tag{6.8.1}$$

where we compose the above $\varphi_\sigma\varphi_{w_0}$ for the first factor. We also multiply it by $(-1)^{dh^\vee}$ as in (5.2.5). The notation conflicts with the pairing between $U^{d,P}$ and U^{d,P_-} in §4.10. But the two pairings are closely related, so the same notation does not give us any confusion. (See §8.1 for a more precise relation.)

By the localization theorem and Lemma 3.1.6 we extend it to a perfect pairing

$$\langle\ ,\ \rangle\colon M_{\mathbf{F}}(-\boldsymbol{a}) \otimes M_{\mathbf{F}}(\boldsymbol{a}) \to \mathbf{F}_T. \tag{6.8.2}$$

(cf. [**14**, §2.6].) Here the highest weight of the first factor is $-\boldsymbol{a}$ since we compose the automorphism $\boldsymbol{a} \mapsto -\boldsymbol{a}$.

When we localize the equivariant cohomology groups, there is no distinction between compact support and arbitrary support. We then see that (6.8.2) is symmetric in the sense as in (4.10.4).

We also have the pairing

$$\langle\ ,\ \rangle\colon H^*_{\mathbb{T},c}(\mathscr{U}^d_T, \Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))) \otimes_{\mathbf{A}_T} H^*_{\mathbb{T}}(\mathscr{U}^d_T, \Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))) \to \mathbf{A}_T, \tag{6.8.3}$$

where we compose $\varphi_\sigma\varphi_{w_0}$ on the first factor as above. Since σw_0 sends B to the opposite Borel B_-, the above is coming from the pairing between $H^*_{\mathbb{T},c}(\mathscr{U}^d_T, \Phi^{B_-}_{T,G}(\mathrm{IC}(\mathscr{U}^d_G)))$ and $H^*_{\mathbb{T}}(\mathscr{U}^d_T, \Phi_{T,G}(\mathrm{IC}(\mathscr{U}^d_G)))$. Therefore it is a perfect pairing thanks to Braden's isomorphism (3.4.2). This pairing also extends to a pairing (6.8.2), which is the same as defined above thanks to the compatibility between Braden's isomorphism and $i^! j^! \to i^* j^!$ as in the proof of Lemma 5.9.4.

The Heisenberg generator P^i_n satisfies

$$\langle u, P^i_n v\rangle = \langle \theta(P^i_n) u, v\rangle, \tag{6.8.4}$$

where θ is an anti-involution on the Heisenberg algebra given by

$$\theta(P^i_n) = -P^i_{-n} - \frac{2(\varepsilon_1+\varepsilon_2)}{\varepsilon_1\varepsilon_2}\delta_{n0}. \tag{6.8.5}$$

Let us explain the reason for this formula of θ. Thanks to a standard property of convolution algebras, the diagonal Heisenberg generator P^Δ_n in §5.2 was defined so that P^Δ_n is adjoint to P^Δ_{-n}. Since the intersection pairing (5.2.5) is compatible with the above one, we change n to $-n$. Moreover, since P^i_n is defined via the stable envelope and we must use the opposite Borel as in §4.10, we need to swap $P^{(1)}_n$ and $P^{(2)}_n$ in §6.3. Therefore we need to change the sign of P^i_{-n}. The zero mode P^i_0 was defined by hand as (6.4.7). We must also change the sign of a^i, as the $\mathbf{A}_T$-module structure is twisted by $\boldsymbol{a} \mapsto -\boldsymbol{a}$ on the first factor. Then we must correct $-P^i_0$ by $-2(\varepsilon_1+\varepsilon_2)/\varepsilon_1\varepsilon_2$.

The Virasoro generator L^i_n is mapped to L^i_{-n} by θ. This is clear from (6.5.1): $c_1(\mathscr{V})$ is self adjoint and $\theta(P^\Delta_n) = P^\Delta_{-n}$ as we have just explained. It can be also checked by the Formula (6.4.6).

Therefore θ preserves $\mathscr{W}_k(\mathfrak{g})$, more precisely the associated Lie algebra $\mathfrak{L}(\mathscr{W}_k(\mathfrak{g}))$ and the current algebra $\mathfrak{U}(\mathscr{W}_k(\mathfrak{g}))$, thanks to (6.6.1). We have

$$\langle u, xv\rangle = \langle \theta(x)u, v\rangle \tag{6.8.6}$$

for $x \in \mathfrak{L}(\mathscr{W}_k(\mathfrak{g}))$, $u, v \in M_{\mathbf{F}}(\boldsymbol{a})$. On the other hand, $\mathfrak{L}(\mathscr{W}_k(\mathfrak{g}))$ has an anti-involution as in [**2**, §5.5], denoted also by θ.

Proposition 6.8.7. — *Our θ coincides with one in* [**2**, §5.5].

Proof. — We use the formula [**2**, Prop. 3.9.1] for the Heisenberg vertex algebra. We follow various notation in [**2**].

Since $\widehat{J}_i(n)$ is a Fourier mode of the vertex operator $Y(v,z) = \sum \widehat{J}_i(n) z^{-n-1}$ with $v = \widehat{J}_i(-1)|0\rangle$, we have

$$\theta(\widehat{J}_i(n)) = -(e^{T^*}v)_{-n}. \tag{6.8.8}$$

Here T^* must be substituted by T^*_{new} in [**2**, (173)]. Using

$$v = \widehat{J}_i(-1)|0\rangle = J_i(-1)|0\rangle - \sum_{\alpha\in\Delta} \alpha(h^i)\psi_{-\alpha}(0)\psi_\alpha(-1)|0\rangle \tag{6.8.9}$$

(see [**2**, the beginning of §4.8]), we can check

$$e^{T^*}v = \widehat{J}_i(-1)|0\rangle + 2(1-(k+h^\vee))|0\rangle. \tag{6.8.10}$$

Therefore we get the same formula as (6.8.5) under the identification $\widehat{J}_i(-1) = \varepsilon_2 P^i_n$. (This $\widehat{J}_i(-1)$ is in $\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{new}})$ and we do not need to apply $\widehat{t}_{-\bar\rho^\vee}$ in the proof of Proposition 6.7.9, as it is in T^*_{new}.) $\square$

Remark 6.8.11. — We can identify the graded dual $D(M_{\mathbf{F}}(\boldsymbol{a}))$ of $M_{\mathbf{F}}(\boldsymbol{a})$ with $M_{\mathbf{F}}(-\boldsymbol{a})$ via $\langle\ ,\ \rangle$. The graded dual has a $\mathscr{W}_k(\mathfrak{g})$-module structure via θ and the Formula (6.8.6). This is the duality functor D in [**2**, §5.5]. The isomorphism $D(M_{\mathbf{F}}(\boldsymbol{a})) \cong M_{\mathbf{F}}(-\boldsymbol{a})$ respects $\mathscr{W}_k(\mathfrak{g})$-module structures.

When λ is generic and $M(\lambda)$ is irreducible, the dual module $D(M(\lambda))$ is isomorphic to $M(-w_0(\lambda))$, where w_0 is the longest element in the Weyl group by [**2**, Th. 5.5.4]. Under the correspondence in Proposition 6.7.9(1), we have

$$-w_0(\lambda) = -w_0(\frac{\boldsymbol{a}}{\varepsilon_1}) - \rho, \tag{6.8.12}$$

as $w_0(\rho) = -\rho$. This means that the equivariant variable $\boldsymbol{a}$ is replaced by $-w_0(\boldsymbol{a})$. Since the highest weight module is invariant under the Weyl group action, we can omit w_0. So the equivariant variable is $-\boldsymbol{a}$ for $D(M_{\mathbf{F}}(\boldsymbol{a}))$. Therefore we have $D(M_{\mathbf{F}}(\boldsymbol{a})) \cong M_{\mathbf{F}}(-\boldsymbol{a})$. This is what we already observed in a geometric way above.

The pairing $\langle \cdot, \cdot \rangle$ is uniquely determined from (6.8.6) and the normalization $\langle -\boldsymbol{a} | \boldsymbol{a} \rangle = 1$ for generic $\boldsymbol{a}$. It is called the *Kac-Shapovalov form*. We thus see that the Poincaré pairing twisted by $\varphi_\sigma \varphi_{w_0}$ on $M_{\mathbf{F}}(\boldsymbol{a})$ coincides with the Kac-Shapovalov form.

CHAPTER 7

R-MATRIX

Recall that our hyperbolic restriction $\Phi_{L,G}$ depends on the choice of a parabolic subgroup P. Following [**46**, Ch. 4] (see also [**19**, §1.3]), we introduce R-matrices giving isomorphisms between various hyperbolic restrictions, and study their properties. They are defined as rational functions in equivariant variables, and their existence is an immediate corollary to localization theorem in the previous section.

As for the usual R-matrices for Yangians, they satisfy the Yang-Baxter equation and are ultimately related to the $\mathscr{W}$-algebra.

As an application, we give a different proof of the Heisenberg commutation relation (Proposition 6.3.8) up to sign, which does not depend on Gieseker spaces for $SL(3)$. We hope that this proof could be generalized to other rank 2 cases B_2, G_2.

Since the dependence on a parabolic subgroup is important, we denote the hyperbolic restriction by $\Phi^P_{L,G}$ in this section.

7.1. Definition

Let us consider the diagram (4.4.2) with respect to a parabolic subgroup P. Let us consider the homomorphism in (3.4.3)

$$\mathscr{I}_P\colon H^*_{\mathbb{T}}(\mathscr{U}^d_L, \Phi^P_{L,G}(\mathscr{F})) \to H^*_{\mathbb{T}}(\mathscr{U}^d_L, i^*j^*\mathscr{F}) \cong H^*_{\mathbb{T}}(\mathscr{U}^d_G, \mathscr{F}) \tag{7.1.1}$$

for $\mathscr{F} \in D^b_{\mathbb{T}}(\mathscr{U}^d_G)$. This is an isomorphism over the quotient field $\mathbf{F}_T$ of $\mathbf{A}_T = \mathbb{C}[\mathrm{Lie}(\mathbb{T})]$. When we want to emphasize $\mathscr{F}$, we write $\mathscr{I}^{\mathscr{F}}_P$.

Definition 7.1.2. — Let P_1, P_2 be two parabolic subgroups compatible with (G, L). Let us introduce the *R-matrix*

$$\begin{aligned} R_{P_1,P_2} = (\mathscr{I}_{P_1})^{-1}\mathscr{I}_{P_2}\colon H^*_{\mathbb{T}}(\mathscr{U}^d_L, \Phi^{P_2}_{L,G}(\mathscr{F})) \otimes_{\mathbf{A}_T} \mathbf{F}_T \\ \to H^*_{\mathbb{T}}(\mathscr{U}^d_L, \Phi^{P_1}_{L,G}(\mathscr{F})) \otimes_{\mathbf{A}_T} \mathbf{F}_T \end{aligned} \tag{7.1.3}$$

When we want to view R_{P_1,P_2} as a rational function in equivariant variables, we denote it by $R_{P_1P_2}(\boldsymbol{a})$. Dependence on ε_1, ε_2 are not important, so they are omitted. When we want to emphasize $\mathscr{F}$, we write $R^{\mathscr{F}}_{P_1,P_2}$.

From the definition, we have

$$R_{P_1,P_2}R_{P_2,P_3} = R_{P_1,P_3}. \tag{7.1.4}$$

7.2. Factorization

Suppose that $Q_1 \subset P$ be a pair of parabolic subgroups as in §4.5. Let $M \subset L$ be the corresponding Levi subgroups. We have $\Phi^{Q_{1,L}}_{M,L} \circ \Phi^P_{L,G} = \Phi^{Q_1}_{M,G}$ by Proposition 4.5.1.

We further suppose that there is another parabolic subgroup Q_2 contained in P such that the corresponding Levi subgroup is also M:

$$M \subset Q_1, Q_2 \subset P. \tag{7.2.1}$$

Then we also have the factorization $\Phi^{Q_{2,L}}_{M,L} \circ \Phi^P_{L,G} = \Phi^{Q_2}_{M,G}$. It is clear from the definition that we have

$$R^{\mathcal{F}}_{Q_1,Q_2} = R^{\Phi^P_{L,G}(\mathcal{F})}_{Q_{1,L},Q_{2,L}}. \tag{7.2.2}$$

Consider the case $L = T$. Note that Borel subgroups containing a fixed torus T are parametrized by the Weyl group W. Let us denote by B^w the Borel subgroup corresponding to $w \in W$, where $B^e = B$ is one which we have fixed at the beginning. From (7.1.4) $R^{\mathcal{F}}_{B^w,B^y}$ factors to a composition of R-matrices for two Borel subgroups related by a simple reflection, i.e., $y = ws_i$. Then we choose $P = P^w_i \supset B^w, B^{ws_i}$ for the parabolic subgroup to use (7.2.2). We have

$$R^{\mathcal{F}}_{B^w,B^{ws_i}} = R^{\Phi^P_{L,G}(\mathcal{F})}_{B_{1,L},B_{2,L}}, \tag{7.2.3}$$

where L is the Levi subgroup of P and $B_{1,L}$, $B_{2,L}$ are images of B^w, B^{ws_i} in L respectively. As $[L,L] \cong SL(2)$, we are reduced to study the $SL(2)$ case. The R-matrix for $SL(2)$ was computed in [**46**, Th. 14.3.1] and will be explained in §7.5.

7.3. Intertwiner property

Let $\mathcal{F} \in D^b_{\mathbb{T}}(\mathcal{U}^d_G)$. We have representations of the Ext algebra $\operatorname{Ext}_{D^b_{\mathbb{T}}(\mathcal{U}^d_G)}(\mathcal{F},\mathcal{F})$ on two cohomology groups in (7.1.1). This is thanks to (3.2.1), 3.2.2. Since $\mathcal{J}^{\mathcal{F}}_P$ is defined by a natural transformation of functors, it is a homomorphism of the Ext algebra. Therefore

Proposition 7.3.1. — *The R-matrix $R^{\mathcal{F}}_{P_1,P_2}$ is a homomorphism of modules over the Ext algebra* $\operatorname{Ext}_{D^b_{\mathbb{T}}(\mathcal{U}^d_G)}(\mathcal{F},\mathcal{F})$.

7.4. Yang-Baxter equation

Take $L = T$ and $\mathscr{F} = \mathrm{IC}(\mathscr{U}^d_G)$ in this subsection.

By (4.13.3) we can map all cohomology groups in (7.1.3) to the fixed one $H^*_{\mathbb{T}}(\mathscr{U}^d_T, \Phi^B_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))) \otimes_{\mathbf{A}_T} \mathbf{F}_T \cong M_{\mathbf{F}}(\boldsymbol{a})$ by φ_w. We conjugate the R-matrix as

$$\varphi_{w_1}^{-1} R_{B^{w_1},B^{w_2}} \varphi_{w_2} \in \mathrm{End}(H^*_{\mathbb{T}}(\mathscr{U}^d_T, \Phi^B_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))) \otimes_{\mathbf{A}_T} \mathbf{F}_T). \tag{7.4.1}$$

Remark that $H^*_{\mathbb{T}}(\mathrm{pt})$-structures are twisted by isomorphisms $w_1, w_2 \colon \mathbb{T} \to \mathbb{T}$, as mentioned after (4.13.2). In practice, we change the equivariant variable $\boldsymbol{a}$ according to w_1 ,w_2.

Since $\mathscr{I}_P$ is φ_w-equivariant, (7.4.1) depends only on $w_1 w_2^{-1}$. Moreover by (7.1.4) it is enough to consider the case $w_1 w_2^{-1}$ is a simple reflection s_i. Therefore we define

$$\check{R}_i \overset{\text{def.}}{=} \varphi_{s_i}^{-1} R_{B^{s_i},B} \varphi_e. \tag{7.4.2}$$

By the factorization (§7.2), this is the R-matrix for $SL(2)$. Since we only have two chambers, (7.1.4) implies

$$\check{R}_i(s_i \boldsymbol{a}) \check{R}_i(\boldsymbol{a}) = 1. \tag{7.4.3}$$

We change the equivariant variable to $s_i \boldsymbol{a}$, as it is the R-matrix from the opposite Borel to the original Borel. In the conventional notation for the R-matrix, we write $u = \langle \alpha_i, \boldsymbol{a} \rangle$ for the variable. Then $\langle \alpha_i, s_i \boldsymbol{a} \rangle = -u$, so this equation means the unitarity of the R-matrix.

Consider R-matrices $\check{R}_i$, $\check{R}_j$. By the factorization (§7.2), we consider them as the R-matrices for the rank 2 Levi subgroup L containing $SL(2)$ for i and j. We compute the R-matrix from a Borel subgroup of L to the opposite Borel by (7.1.4) in two ways to get

Theorem 7.4.4. —

$$\check{R}_i(s_j \boldsymbol{a}) \check{R}_j(\boldsymbol{a}) = \check{R}_j(s_i \boldsymbol{a}) \check{R}_i(\boldsymbol{a}) \quad \textit{if } (\alpha_i, \alpha_j) = 0, \tag{7.4.5}$$

$$\check{R}_j(s_i s_j \boldsymbol{a}) \check{R}_i(s_j \boldsymbol{a}) \check{R}_j(\boldsymbol{a}) = \check{R}_i(s_j s_i \boldsymbol{a}) \check{R}_j(s_i \boldsymbol{a}) \check{R}_i(\boldsymbol{a}) \quad \textit{if } (\alpha_i, \alpha_j) = -1. \tag{7.4.6}$$

7.5. $SL(2)$-case

As we mentioned earlier, it is enough to compute the R-matrix for $SL(2)$, which was given in [**46**, Th. 14.3.1]. We briefly recall the result, and point out a slight difference for the formulation.

By Proposition 7.3.1 and the observation that the left hand side of the Formula (6.5.1) is contained in the Ext algebra, we deduce that the R-matrix is an intertwiner of the Virasoro algebra. This is a fundamental observation due to Maulik-Okounkov [**46**].

The highest weight is generic, since we work over $\mathbf{F}_T$. Therefore the intertwiner is unique up to scalar, and we normalize it so that it preserves the highest weight vector $|\boldsymbol{a}\rangle$.

In [**46**] the R-matrix is given as an endomorphism of the localized equivariant cohomology group of the fixed point set via the stable envelop. On the other hand, our R is an endomorphism of $H^*_{\mathbb{T}}(\mathscr{U}^d_T, \Phi^B_{T,G}(\mathrm{IC}(\mathscr{U}^d_G)))$. Concretely

$$\check{R} = P_{12}R^{\mathrm{MO}}\big|_{\text{anti-diagonal part}}, \tag{7.5.1}$$

where P_{12} is the exchange of factors of the Fock space $F \otimes F$, as $s_i = P_{12}$.

By [**46**, Prop. 4.1.3] we have

$$\check{R} = -1 + O(\boldsymbol{a}^{-1}), \qquad \boldsymbol{a} \to \infty. \tag{7.5.2}$$

7.6. $\mathbb{G}$-equivariant cohomology

Recall that a larger group $\mathbb{G} = G \times \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathscr{U}^d_G$ so that $\mathrm{IC}(\mathscr{U}^d_G)$ is a $\mathbb{G}$-equivariant perverse sheaf. Therefore we can consider $\mathrm{IH}^*_{\mathbb{G}}(\mathscr{U}^d_G) = H^*_{\mathbb{G}}(\mathscr{U}^d_G, \mathrm{IC}(\mathscr{U}^d_G))$. It is related to the $\mathbb{T}$-equivariant cohomology $\mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G)$ as follows.

Let $N(\mathbb{T})$ (resp. $N(T)$) be the normalizer of $\mathbb{T}$ (resp. T) in $\mathbb{G}$ (resp. G). Then we have forgetful homomorphisms $\mathrm{IH}^*_{\mathbb{G}}(\mathscr{U}^d_G) \to \mathrm{IH}^*_{N(\mathbb{T})}(\mathscr{U}^d_G) \to \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G)$. It is well-known that the first homomorphism is an isomorphism, as the cohomology of $\mathbb{G}/N(\mathbb{T}) = G/N(T)$ is 1-dimensional (see e.g., [**37**]). The Weyl group $W = N(T)/T$ acts naturally on $\mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G)$, induced from the $N(T)$-action on $\mathscr{U}^d_G$. Moreover we have

$$\mathrm{IH}^*_{\mathbb{G}}(\mathscr{U}^d_G) \xrightarrow{\cong} \mathrm{IH}^*_{N(\mathbb{T})}(\mathscr{U}^d_G) \xrightarrow{\cong} \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G)^W. \tag{7.6.1}$$

Let us consider the following diagram

$$\begin{tikzcd}
H^*_{\mathbb{T}}(\mathscr{U}^d_T, \Phi^B_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))) \otimes_{\mathbf{A}_T} \mathbf{F}_T \arrow[r, "\mathscr{I}_B", "\cong"'] \arrow[d, "\check{R}_i"'] & \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G) \otimes_{\mathbf{A}_T} \mathbf{F}_T \arrow[d, "s_i"] \\
H^*_{\mathbb{T}}(\mathscr{U}^d_T, \Phi^B_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))) \otimes_{\mathbf{A}_T} \mathbf{F}_T \arrow[r, "\mathscr{I}_B", "\cong"'] & \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G) \otimes_{\mathbf{A}_T} \mathbf{F}_T,
\end{tikzcd} \tag{7.6.2}$$

where s_i is a simple reflection of the above W-action.

Lemma 7.6.3. — *The diagram* (7.6.2) *is commutative.*

Proof. — We have $\check{R}_i = \varphi_{s_i}^{-1}\mathscr{I}_{B_i}^{-1}\mathscr{I}_B\varphi_e$. As an endomorphism of $\mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G) \otimes_{\mathbf{A}_T} \mathbf{F}_T$, it is replaced by $\mathscr{I}_B\varphi_{s_i}^{-1}\mathscr{I}_{B_i}^{-1}$, as $\varphi_e = \mathrm{id}$.

From the definition of $\mathscr{I}_{B_i}$ and the commutativity of the diagram (4.13.1), we have $\mathscr{I}_{B_i}\varphi_{s_i} = \varphi_{s_i}\mathscr{I}_B$, where φ_{s_i} in the right hand side is the action on $\mathscr{U}^d_G$, the rightmost arrow in (4.13.1). Since the W-action is induced from φ_σ, the assertion follows. ☐

Proposition 7.6.4. — *The Weyl group action on $M_{\mathbf{F}}(\boldsymbol{a}) = \bigoplus_d \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G) \otimes_{\mathbf{A}_T} \mathbf{F}_T$ commutes with the $\mathscr{W}_k(\mathfrak{g})$ action. Hence $\mathscr{W}_k(\mathfrak{g})$ acts on the W-invariant part $M_{\mathbf{F}}(\boldsymbol{a})^W = \bigoplus_d \mathrm{IH}^*_{\mathbb{G}}(\mathscr{U}^d_G) \otimes_{\mathbf{A}_G} \mathbf{F}_G$.*

Proof. — Since $\mathscr{W}_k(\mathfrak{g})$ is the intersection of $\mathfrak{Vir}_i \otimes \mathfrak{Heis}(\alpha_i^\perp)$ (see (6.6.1)), it is enough to show that $\mathfrak{Vir}_i \otimes \mathfrak{Heis}(\alpha_i^\perp)$ commutes with s_i. By the previous lemma, s_i is given by the R-matrix.

Let us first factorize the hyperbolic restriction functors $\Phi^B_{T,G}$, $\Phi^{B^{s_i}}_{T,G}$ as

$$\Phi^B_{T,G} = \Phi^{B_{L_i}}_{T,L_i} \Phi^{P_i}_{L_i,G}, \qquad \Phi^{B^{s_i}}_{T,G} = \Phi^{B^{s_i}_{L_i}}_{T,L_i} \Phi^{P_i}_{L_i,G}$$

by Proposition 4.5.1. Then the same argument as in Proposition 7.3.1 shows that $\check{R}_i$ commutes with the action of the Ext algebra of $\Phi^{P_i}_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G))$. Since the Virasoro generators $\widetilde{L}^i_n$ are in this Ext algebra, the first assertion follows.

For the second assertion, we only need to check

$$(\mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G) \otimes_{\mathbf{A}_T} \mathbf{F}_T)^W \cong \mathrm{IH}^*_{\mathbb{G}}(\mathscr{U}^d_G) \otimes_{\mathbf{A}_G} \mathbf{F}_G.$$

By (7.6.1) we have a natural injective homomorphism form the right hand side to the left. On the other hand, if m/f ($f \in \mathbf{A}_T$, $m \in \mathrm{IH}^*_{\mathbb{T}}(\mathscr{U}^d_G)$) is fixed by W, we have

$$\frac{m}{f} = \frac{1}{|W|} \sum_{\sigma\in W} \frac{\sigma m}{\sigma f} = \frac{1}{|W|} \left(\prod_{\sigma\in W} \sigma f \right)^{-1} \sum_{\sigma \in W} \sigma m \prod_{\tau\neq\sigma} \tau f.$$

This is contained in the right hand side. Therefore the above follows. ☐

7.7. A different proof of the Heisenberg commutation relation

We give a different proof of Proposition 6.3.8.

Let $\tilde{\alpha}^{d,-}_i$ be the element defined as in $\tilde{\alpha}^d_i$ for the opposite Borel. Since the pairing can be computed from the $SL(2) \cong [L_i, L_i]$ case, we already know that

$$\langle \tilde{\alpha}^d_i, \tilde{\alpha}^{d,-}_i \rangle = 2d. \tag{7.7.1}$$

We generalize this to

Proposition 7.7.2. —

$$\langle \tilde{\alpha}^d_i, \tilde{\alpha}^{d,-}_j \rangle = \pm d(\alpha_i, \alpha_j). \tag{7.7.3}$$

The following proof does not determine ±, though we know that it is + by the reduction to the $SL(3)$ case and the Formula (5.12.6), which has been proved via Gieseker spaces.

Proof. — We consider the case $(\alpha_i, \alpha_j) = -1$. The proof for the case $(\alpha_i, \alpha_j) = 0$ is similar (and simpler).

Let us study the leading part of Yang-Baxter Equation (7.4.6). We consider R-matrices as endomorphisms of the space (6.3.6). By the factorization (7.2.2), we can use the expansion (7.5.2) for $SL(2)$. Then '-1' in (7.5.2) is replaced by the direct sum of (-1) on $U^d_{T,L_i} \cong \mathfrak{h}_{\mathfrak{sl}_2}$ and the identity on $(U^d_{T,L_i})^\perp$ in (6.3.7). Let us denote it by $\tilde{s}_i$.

Since $(U^d_{T,L_i})^\perp$ is the orthogonal complement of $\mathbb{C}\tilde{\alpha}_i^{d,-}$, we have

$$\tilde{s}_i(x) = x - \langle x, \tilde{\alpha}_i^{d,-}\rangle \frac{\tilde{\alpha}_i^d}{d}, \quad \text{for } x \in U^d. \tag{7.7.4}$$

From the Yang-Baxter equation, we have the braid relation

$$\tilde{s}_i\tilde{s}_j\tilde{s}_i = \tilde{s}_j\tilde{s}_i\tilde{s}_j. \tag{7.7.5}$$

Since we are considering the $SL(3)$-case, there is the diagram automorphism σ exchanging i and j. By Lemma 4.13.9, we have $\varphi_\sigma(\tilde{\alpha}_i^d) = (-1)^d\tilde{\alpha}_j^d$. Since φ_σ preserves the inner product, we get

$$\langle \tilde{\alpha}_i^d, \tilde{\alpha}_j^{d,-}\rangle = \langle \tilde{\alpha}_j^d, \tilde{\alpha}_i^{d,-}\rangle. \tag{7.7.6}$$

Now $\tilde{s}_i$ is the usual reflection with respect to the hyperplane $\tilde{\alpha}_i^{d,-} = 0$. Hence we conclude $\langle \tilde{\alpha}_i^d, \tilde{\alpha}_j^{d,-}\rangle = \pm d$.

Note that $\tilde{\alpha}_i^d = \pm\tilde{\alpha}_j^d$ are excluded thanks to (4.12.1), which has been proved without using Gieseker spaces for $SL(3)$. □

Once we compute the inner product, the Heisenberg relation is a consequence of the factorization (6.3.7). The generator P_n^i is the tensor product of the Heisenberg generator for the first factor and the identity in (6.3.7).

CHAPTER 8

WHITTAKER STATE

8.1. Universal Verma/Wakimoto modules

Let us denote the direct sum of four $\mathbf{A}_T$-modules over $d \in \mathbb{Z}_{\geq 0}$ in (6.0.3) by $M_{\mathbf{A}}(\boldsymbol{a})$, $N_{\mathbf{A}}(\boldsymbol{a})$, $D(N_{\mathbf{A}}(-\boldsymbol{a}))$, $D(M_{\mathbf{A}}(-\boldsymbol{a}))$ respectively. Thus we have

$$M_{\mathbf{A}}(\boldsymbol{a}) \subset N_{\mathbf{A}}(\boldsymbol{a}) \subset D(N_{\mathbf{A}}(-\boldsymbol{a})) \subset D(M_{\mathbf{A}}(-\boldsymbol{a})). \tag{8.1.1}$$

The reason for notation will be clear shortly.

The pairing (6.8.2) restricts to a perfect pairing

$$\langle\ ,\ \rangle\colon M_{\mathbf{A}}(-\boldsymbol{a}) \otimes D(M_{\mathbf{A}}(-\boldsymbol{a})) \to \mathbf{A}_T, \tag{8.1.2}$$

given by the Verdier duality, where the $\mathbf{A}_T$-structure is twisted by the automorphism $\boldsymbol{a} \mapsto -\boldsymbol{a}$ as in §6.8, and hence the notation is changed to $M_{\mathbf{A}}(-\boldsymbol{a})$. Then $D(M_{\mathbf{A}}(-\boldsymbol{a}))$ is identified with the graded dual of $M_{\mathbf{A}}(-\boldsymbol{a})$ by (8.1.2), hence our notation is compatible with the convention in Remark 6.8.11. Similarly if we twist $N_{\mathbf{A}}(\boldsymbol{a})$, we have an isomorphism

$$\varphi_\sigma \varphi_{w_0}\colon N_{\mathbf{A}}(-\boldsymbol{a}) \xrightarrow{\cong} \bigoplus_d H^*_{\mathbb{T},c}(\mathcal{U}^d_T, \Phi^{B_-}_{T,G}(\mathrm{IC}(\mathcal{U}^d_G))), \tag{8.1.3}$$

where $\Phi^{B_-}_{T,G}$ is the hyperbolic restriction with respect to the opposite Borel B_-. Then we have a perfect pairing

$$\langle\ ,\ \rangle\colon N_{\mathbf{A}}(-\boldsymbol{a}) \otimes D(N_{\mathbf{A}}(-\boldsymbol{a})) \to \mathbf{A}_T. \tag{8.1.4}$$

Recall that $N_{\mathbf{A}}(\boldsymbol{a})$, $D(N_{\mathbf{A}}(-\boldsymbol{a}))$ are modules over the integral form of the Heisenberg algebra $\mathfrak{Heis}_{\mathbf{A}}(\mathfrak{h})$, as we remarked at the end of §6.3.

Using Lemma 4.8.7, we make an identification

$$N_{\mathbf{A}}(\boldsymbol{a}) \cong \bigoplus_\lambda \operatorname{Sym}^{n_1} U^1 \otimes \operatorname{Sym}^{n_2} U^2 \otimes \cdots \otimes H^*_{\mathbb{T},c}(\overline{S_\lambda \mathbb{A}^2}), \tag{8.1.5}$$

where $U^d = U^{d,B}_{T,G}$ and $\lambda = (1^{n_1} 2^{n_2} \cdots)$. We also have an identification for the opposite Borel B_-:

$$D(N_{\mathbf{A}}(\boldsymbol{a})) \cong \bigoplus_\lambda \operatorname{Sym}^{n_1} U^{1,-} \otimes \operatorname{Sym}^{n_2} U^{2,-} \otimes \cdots \otimes H^*_{\mathbb{T}}(\overline{S_\lambda \mathbb{A}^2}), \tag{8.1.6}$$

where $U^{d,-} = U^{d,B_-}_{T,G}$. Then the pairing (8.1.4) is the product of the pairing between U^d and $U^{d,-}$ in §4.10 and one between $H^*_{c,\mathbb{T}}(\overline{S_\lambda \mathbb{A}^2})$ and $H^*_{\mathbb{T}}(\overline{S_\lambda \mathbb{A}^2})$.

Moreover, two pairings (8.1.2) and (8.1.4) are compatible with the embeddings (8.1.1).

Let $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ be the **A**-form of the W-algebra in §B.

Proposition 8.1.7. — *$M_{\mathbf{A}}(\boldsymbol{a})$, $D(M_{\mathbf{A}}(\boldsymbol{a}))$ are $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$-modules.*

Proof. — Note that $D(M_{\mathbf{A}}(\boldsymbol{a}))$ is characterized as

$$\{m \in M_{\mathbf{F}}(\boldsymbol{a}) \mid \langle m, M_{\mathbf{A}}(\boldsymbol{a})\rangle \in \mathbf{A}_T\}. \tag{8.1.8}$$

Therefore it is enough to show the assertion for $M_{\mathbf{A}}(\boldsymbol{a})$.

We consider $M_{\mathbf{A}}(\boldsymbol{a})$ as a subspace of $N_{\mathbf{A}}(\boldsymbol{a})$. The latter is a module over $\mathfrak{Heis}_{\mathbf{A}}(\mathfrak{h})$, and hence over $\mathfrak{Vir}_{i,\mathbf{A}}$. By Theorem B.6.1, it is enough to check that $M_{\mathbf{A}}(\boldsymbol{a})$ is invariant under the intersection of $\mathfrak{Vir}_{i,\mathbf{A}}$ for all i. Recall that we know that (6.5.2) is a $\mathfrak{Vir}_{i,\mathbf{A}}$-module, as $\widetilde{L}^i_n$ is well-defined. Therefore it is enough to show that

$$\mathrm{IH}^*_{\mathbb{T},c}(\mathscr{U}^d_G) = \bigcap_i H^*_{\mathbb{T},c}(\mathscr{U}^d_{L_i}, \Phi_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G))). \tag{8.1.9}$$

By Theorem 3.6.2 we have

$$\begin{aligned} &H^*_{\mathbb{T},c}(\mathscr{U}^d_{L_i}, \Phi_{L_i,G}(\mathrm{IC}(\mathscr{U}^d_G))) \\ &\qquad = H^*_{\mathbb{T},c}(\mathscr{U}^d_T, \Phi^B_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))) \cap H^*_{\mathbb{T},c}(\mathscr{U}^d_T, \Phi^{B^{s_i}}_{T,G}(\mathrm{IC}(\mathscr{U}^d_G))), \end{aligned} \tag{8.1.10}$$

where B^{s_i} is the Borel subgroup corresponding to a simple reflection s_i. Therefore it is enough to show that the intersection of the right hand side of (8.1.10) for all i is $\mathrm{IH}^*_{\mathbb{T},c}(\mathscr{U}^d_G)$. This is proved in a similar manner as Theorem 3.6.2. The only thing we need to use is the fact for any non-zero dominant λ there exists $i \in I$ such that $s_i(\lambda)$ is not dominant. ☐

Proposition 8.1.11. — *The $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$-submodule of $M_{\mathbf{F}}(\boldsymbol{a})$ generated by $|\boldsymbol{a}\rangle$ is $M_{\mathbf{A}}(\boldsymbol{a})$, i.e.,*

$$M_{\mathbf{A}}(\boldsymbol{a}) = \mathscr{W}_{\mathbf{A}}(\mathfrak{g})|\boldsymbol{a}\rangle. \tag{8.1.12}$$

Proof. — Comparison of bigraded dimensions: $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})|\boldsymbol{a}\rangle$ is bigraded by the usual degree and ${}^{c}\!\deg$, so that the bidegree of $\widetilde{W}^{(\kappa)}_n$ is $(n, d_\kappa + 1)$, see §B.2. According to *loc. cit.*, $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})|\boldsymbol{a}\rangle$ is a free **A**-module (the bidegree of $\varepsilon_1, \varepsilon_2, \mathfrak{h}$ equals $(0,1)$) with the space of generators $S(t\mathfrak{w}[t])$ where $\mathfrak{w} = \bigoplus_{\kappa=1}^{\ell} \mathfrak{w}^{(\kappa)}$ with the bidegree of $\mathfrak{w}^{(\kappa)}$ equal to $(0, d_\kappa + 1)$, and the bidegree of t equal to $(1,0)$.

On the other hand, $M_{\mathbf{A}}(\boldsymbol{a})$ is bigraded by the instanton number and half the cohomological degree. It is a free **A**-module with the space of generators equal to $\bigoplus_{d\in\mathbb{N}} \mathrm{IH}^*_c(\mathscr{U}^d_G)$. According to [**21**, Theorem 7.10], $\bigoplus_{d\in\mathbb{N}} \mathrm{IH}^*_c(\mathscr{U}^d_G) \simeq S(t\mathfrak{g}^f[t])$ where $\mathfrak{g}^f = \bigoplus_{\kappa=1}^{\ell} \mathfrak{g}^f_{(\kappa)}$ with the bidegree of $\mathfrak{g}^f_{(\kappa)}$ equal to $(0, d_\kappa + 1)$, and the bidegree of t equal to $(1,0)$. ☐

On the other hand, it is clear from (8.1.5) that

$$N_{\mathbf{A}}(\boldsymbol{a}) = \mathfrak{Heis}_{\mathbf{A}}(\mathfrak{h})|\boldsymbol{a}\rangle \tag{8.1.13}$$

For a homomorphism $\chi\colon \mathbf{A}_T \to \mathbb{C} \equiv \mathbb{C}_\chi$, the specialization

$$M_{\mathbf{A}}(\boldsymbol{a}) \otimes_{\mathbf{A}} \mathbb{C}_\chi \tag{8.1.14}$$

is a module over $\mathscr{W}_k(\mathfrak{g})$ with level $k = \chi(-\varepsilon_2/\varepsilon_1) - h^\vee$. It is a Verma module with highest weight $\chi(\boldsymbol{a}/\varepsilon_1) - \rho$, see §6.7. Here χ is regarded as the assignment of variables $\boldsymbol{a}$, ε_1, ε_2, or more concretely $\chi(\boldsymbol{a}) = \sum \chi(a^i)\varpi_i$ for fundamental weights ϖ_i.

Definition 8.1.15. — We call $M_{\mathbf{A}}(\boldsymbol{a})$ the *universal Verma module.*

Similarly $N_{\mathbf{A}}(\boldsymbol{a})$ is specialized to the Fock representation of the Heisenberg algebra by χ. We call $N_{\mathbf{A}}(\boldsymbol{a})$ the *universal Wakimoto module.* Similarly $D(M_{\mathbf{A}}(\boldsymbol{a}))$ is the universal dual Verma module, and $D(N_{\mathbf{A}}(\boldsymbol{a}))$ the universal dual Wakimoto module.

8.2. $\mathbb{G}$-equivariant cohomology

Let us consider the $\mathbb{G}$-equivariant intersection cohomology groups as in §7.6. We have

$$\bigoplus_d \mathrm{IH}^*_{\mathbb{G},c}(\mathcal{U}^d_G) = M_{\mathbf{A}}(\boldsymbol{a})^W, \qquad \bigoplus_d \mathrm{IH}^*_{\mathbb{G}}(\mathcal{U}^d_G) = D(M_{\mathbf{A}}(-\boldsymbol{a}))^W \tag{8.2.1}$$

by (7.6.1). Since the W-action commutes with the $\mathscr{W}_k(\mathfrak{g})$-action by Proposition 7.6.4, we see that both of (8.2.1) are modules over $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$.

8.3. Whittaker condition

Let $\widetilde{W}^{(\kappa)}$ be as in §B.2, which generates $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ in the sense of the reconstruction theorem. Let $|1^d\rangle \overset{\text{def.}}{=} [\mathcal{U}^d_G] \in \mathrm{IH}^0_{\mathbb{T}}(\mathcal{U}^d_G)$ be the fundamental class. It conjecturally satisfies the following *Whittaker conditions*

Conjecture 8.3.1. — Let $d \geq 1$, $n > 0$. We have

$$\widetilde{W}^{(\kappa)}_n|1^d\rangle = \begin{cases} |1^{d-1}\rangle & \text{if } \kappa = \ell \text{ and } n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8.3.2}$$

Since $\widetilde{W}^{(\kappa)}_n$ is contained in $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$, it is a well-defined operator on $D(M_{\mathbf{A}}(-\boldsymbol{a})) = \bigoplus \mathrm{IH}^*_{\mathbb{T}}(\mathcal{U}^d_G)$. Since $\widetilde{W}^{(\kappa)}_n$ has ${}^c\mathrm{deg} = d_\kappa + 1$ (d_κ is an exponent as in §B), it sends $|1^d\rangle \in \mathrm{IH}^0_{\mathbb{T}}(\mathcal{U}^d_G)$ into $\mathrm{IH}^{2(d_\kappa+1-nh^\vee)}_{\mathbb{T}}(\mathcal{U}^{d-n}_G)$. Since $d_\kappa \leq d_\ell = h^\vee - 1$, we have $\widetilde{W}^{(\kappa)}_n|1^d\rangle = 0$ unless $n = 1$, $\kappa = \ell$. Also we see that $\widetilde{W}^{(\ell)}_1|1^d\rangle$ is a multiple of $|1^{d-1}\rangle$ with the multiple constant of degree 0, i.e., a complex number. Moreover, if the multiple constant would be 0, it is a highest weight vector and generates a nontrivial submodule. Since $M_{\mathbf{F}}$ is irreducible, it is a contradiction. Therefore the constant cannot be zero. In particular, if we divide $|1^d\rangle$ by the constant, it satisfies the Whittaker condition (8.3.2).

Let $|w^d\rangle$ be the vector determined by with the normalization $|w^0\rangle = |1^0\rangle = |\boldsymbol{a}\rangle \in \mathrm{IH}^0_{\mathbb{T}}(\mathcal{U}^0_G) = \mathrm{IH}^0_{\mathbb{T}}(\mathrm{pt})$. Its existence and uniqueness will follow from the discussion in §8.4 below. (However it is not *a priori* clear that $|w^d\rangle \in D(M_{\mathbf{A}}(-\boldsymbol{a}))$, as for $|1^d\rangle$.) Therefore we already know that $|1^d\rangle = c_d|w^d\rangle$ for some $c_d \in \mathbb{C}$ by the above observation. The goal of this section is to prove a slightly weaker version of (8.3.2).

Theorem 8.3.3. — *Conjecture 8.3.1 holds up to sign.*

Our strategy of the proof is as follows. To determine c_d up to sign, it is enough to compare pairings $\langle 1^d|1^d\rangle$ with $\langle w^d|w^d\rangle$. Moreover, as c_d is a complex number, we may do it after specifying equivariant variables ε_1, ε_2. We will show that

$$\begin{aligned}(\varepsilon_1\varepsilon_2)^d\langle 1^d|1^d\rangle\big|_{\varepsilon_1,\varepsilon_2=0} &= \frac{1}{d!}\left(\varepsilon_1\varepsilon_2\langle 1^1|1^1\rangle\big|_{\varepsilon_1,\varepsilon_2=0}\right)^d,\\ (\varepsilon_1\varepsilon_2)^d\langle w^d|w^d\rangle\big|_{\varepsilon_1,\varepsilon_2=0} &= \frac{1}{d!}\left(\varepsilon_1\varepsilon_2\langle w^1|w^1\rangle\big|_{\varepsilon_1,\varepsilon_2=0}\right)^d.\end{aligned} \tag{8.3.4}$$

It implies that

$$c_d^2 = c_1^{2d}.$$

Recall that the top degree field $\widetilde{W}^{(\ell)}$ in §B.2 is well-defined only up to nonzero multiple even ignoring lower degree terms, as we just take it as a highest weight vector of a certain $\mathfrak{sl}_2$ representation. Therefore if we divide $\widetilde{W}^{(\ell)}$ by c_1, (8.3.2) holds up to sign.

Since $|1^d\rangle$ is canonically determined from geometry, it means that the top degree generator $\widetilde{W}^{(\ell)}$ is fixed without constant multiple ambiguity (up to sign). In particular, when we applied $\widetilde{W}_0^{(\ell)}$ to the highest weight vector $|\boldsymbol{a}\rangle$, we get an invariant polynomial in $\boldsymbol{a}$ of degree $h^\vee$. (See 6.7.) We do not study what this natural choice of the highest degree generator of the invariant polynomial $S(\mathfrak{h})^W$ is in general. But we will check that it is indeed a natural one for $\mathfrak{g} = \mathfrak{sl}_{\ell+1}$ in §8.9.

8.4. Whittaker vector and Kac-Shapovalov form

In this subsection, we shall prove that the Whittaker vector exists and is unique in the localized equivariant cohomology $M_{\mathbf{F}}(\boldsymbol{a})$, which we think of Verma module with generic highest weight by Proposition 6.7.9. The argument is more or less standard (see e.g., [**40**]), but we give the detail, as we will use similar one later in §8.8.

We have a nondegenerate Kac-Shapovalov form $\langle\ ,\ \rangle$ on $M_{\mathbf{F}}(\boldsymbol{a})$. Let θ denote the anti-involution on $\mathfrak{U}(\mathscr{W}_{\mathbf{F}}(\mathfrak{g}))$ as in §6.8. We have

$$\theta(\widetilde{W}_n^{(\kappa)}) = (-1)^{d_\kappa+1}\widetilde{W}_{-n}^{(\kappa)}. \tag{8.4.1}$$

See [**2**, §5.5]. In particular, $\mathfrak{U}(\mathscr{W}_{\mathbf{A}}(\mathfrak{g}))$ is invariant under θ.

Let us denote the highest weight vector of $D(M_{\mathbf{F}}(\boldsymbol{a}))$ by $\langle -\boldsymbol{a}|$. See Remark 6.8.11 to see that its highest weight is $-\boldsymbol{a}$.

Let $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^\ell)$ be an ℓ-partition, i.e., it is an ℓ-tuple of partitions $\lambda^i = (\lambda^i_1, \lambda^i_2, \dots)$. We consider the corresponding operator

$$\widetilde{W}[\boldsymbol{\lambda}] \overset{\text{def.}}{=} \widetilde{W}^{(1)}_{-\lambda^1_1}\widetilde{W}^{(1)}_{-\lambda^1_2}\cdots\widetilde{W}^{(\ell)}_{-\lambda^\ell_1}\widetilde{W}^{(\ell)}_{-\lambda^\ell_2}\cdots \tag{8.4.2}$$

in the current algebra of the W-algebra. Then

$$\widetilde{W}[\boldsymbol{\lambda}]|\boldsymbol{a}\rangle \tag{8.4.3}$$

form a PBW base of $M_{\mathbf{F}}(\boldsymbol{a})$. We define the Kac-Shapovalov form

$$K \equiv K^d \overset{\text{def.}}{=} (\langle -\boldsymbol{a}|\theta(\widetilde{W}[\boldsymbol{\lambda}])\widetilde{W}[\boldsymbol{\mu}]|\boldsymbol{a}\rangle)_{\boldsymbol{\lambda}\boldsymbol{\mu}}, \tag{8.4.4}$$

where $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ runs over ℓ-partitions whose total sizes are d. We consider it as a matrix, and an entry is denoted by $K_{\boldsymbol{\lambda}\boldsymbol{\mu}}$.

Let $(1^d) = (1, \dots, 1)$ be the partition of n whose all entries are 1. Let $\boldsymbol{\lambda}_0 = (\varnothing, \dots, \varnothing, (1^d))$ be the ℓ-partition where the first $(\ell - 1)$ partitions are all $\varnothing$ and the last one is (1^d). The corresponding operator $\widetilde{W}[\boldsymbol{\lambda}_0]$ is $(\widetilde{W}^{(\ell)}_{-1})^d$.

We have

$$\langle -\boldsymbol{a}|\theta(\widetilde{W}[\boldsymbol{\lambda}])|w^d\rangle = \begin{cases} 1 & \text{if } \boldsymbol{\lambda} = \boldsymbol{\lambda}_0, \\ 0 & \text{otherwise} \end{cases} \tag{8.4.5}$$

from (8.3.2) by the induction on d. Note that $|w^0\rangle = |\boldsymbol{a}\rangle$, and hence $\langle -\boldsymbol{a}|\boldsymbol{a}\rangle = 1$.

Let us write the Whittaker vector $|w^d\rangle$ in the PBW base as

$$|w^d\rangle = \sum_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}} \widetilde{W}[\boldsymbol{\mu}]|\boldsymbol{a}\rangle. \tag{8.4.6}$$

By (8.4.5) we have

$$\sum_{\boldsymbol{\mu}} K_{\boldsymbol{\lambda}\boldsymbol{\mu}} a_{\boldsymbol{\mu}} = \delta_{\boldsymbol{\lambda}\boldsymbol{\lambda}_0}. \tag{8.4.7}$$

In other words,

$$a_{\boldsymbol{\mu}} = K^{\boldsymbol{\mu}\boldsymbol{\lambda}_0}, \tag{8.4.8}$$

where $K^{-1} = (K^{\boldsymbol{\mu}\boldsymbol{\lambda}})$ is the inverse of K. In particular, the existence and the uniqueness of $|w^d\rangle$ follow.

We also get

$$\langle w^d|w^d\rangle = K^{\boldsymbol{\lambda}_0\boldsymbol{\lambda}_0}. \tag{8.4.9}$$

8.5. Lattices

Let

$$\widehat{W}^{(\kappa)}_n = (\varepsilon_1\varepsilon_2)^{-1}\widetilde{W}^{(\kappa)}_n \tag{8.5.1}$$

for $\kappa = 1, \dots, \ell$, $n \in \mathbb{Z}$.

Lemma 8.5.2. — *$M_{\mathbf{A}}(\boldsymbol{a})$ is invariant under $\widehat{W}^{(\kappa)}_m$ with $m > 0$. Equivalently $D(M_{\mathbf{A}}(\boldsymbol{a}))$ is invariant under $\widehat{W}^{(\kappa)}_{-m}$ with $m > 0$.*

Proof. — Recall that $M_{\mathbf{A}}(\boldsymbol{a})$ is graded by the instanton number d: $M_{\mathbf{A}}(\boldsymbol{a}) = \bigoplus_d M_{d,\mathbf{A}}$. In algebraic terms, it is the grading by L_0. Let us take $\widehat{W}^{(\kappa)}_m$ with $m > 0$. We show

$$\widehat{W}^{(\kappa)}_m x \in M_{d-m,\mathbf{A}} \tag{8.5.3}$$

for any $x \in M_{d,\mathbf{A}}$ by an induction on d. If $d = 0$, we have $\widehat{W}^{(\kappa)}_m x = 0$. Therefore the assertion is true.

Suppose that the statement is true for $d' < d$. We may assume $x = \widetilde{W}^{(\kappa')}_{-n} x'$ with $n > 0$, $x' \in M_{d-n,\mathbf{A}}$ by Proposition 8.1.11. Since $\widehat{W}^{(\kappa)}_m x' \in M_{\mathbf{A}}(\boldsymbol{a})$ by the induction hypothesis, it is enough to show that $[\widehat{W}^{(\kappa)}_m, \widetilde{W}^{(\kappa')}_{-n}]x' \in M_{\mathbf{A}}(\boldsymbol{a})$. In the Heisenberg algebra, we have $[a,b] \in \varepsilon_1\varepsilon_2 \widetilde{H}^0_{\mathbf{A}}(\mathfrak{g})$ for a, $b \in \widetilde{H}^0_{\mathbf{A}}(\mathfrak{g})$ from the relation (6.3.12). Since $\mathscr{W}_{\mathbf{A}}(\mathfrak{g}) \to \widetilde{H}^0_{\mathbf{A}}(\mathfrak{g})$ is an embedding, we have the same assertion for $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$. Therefore the assertion follows. ☐

Let $\mathbf{R} \subset \mathbf{F} = \mathbb{Q}(\varepsilon_1, \varepsilon_2)$ be the local ring of regular functions at $\varepsilon_1 = \varepsilon_2 = 0$. Let $\mathbf{R}_T = \mathbf{R}(\boldsymbol{a})$. We set

$$\begin{aligned} M_{\mathbf{R}}(\boldsymbol{a}) &= M_{\mathbf{A}}(\boldsymbol{a}) \otimes_{\mathbf{A}_T} \mathbf{R}_T, & D(M_{\mathbf{R}}(-\boldsymbol{a})) &= D(M_{\mathbf{A}}(-\boldsymbol{a})) \otimes_{\mathbf{A}_T} \mathbf{R}_T, \\ N_{\mathbf{R}}(\boldsymbol{a}) &= N_{\mathbf{A}}(\boldsymbol{a}) \otimes_{\mathbf{A}_T} \mathbf{R}_T, & D(N_{\mathbf{R}}(-\boldsymbol{a})) &= D(N_{\mathbf{A}}(-\boldsymbol{a})) \otimes_{\mathbf{A}_T} \mathbf{R}_T. \end{aligned} \tag{8.5.4}$$

These modules are the localization with respect to the ideal

$$\operatorname{Ker}(\mathbf{A}_T = \mathbb{C}[\varepsilon_1, \varepsilon_2, \boldsymbol{a}] = \mathbb{C}[\operatorname{Lie}\mathbb{T}] \to \mathbb{C}[\boldsymbol{a}] = \mathbb{C}[\operatorname{Lie} T]) \tag{8.5.5}$$

consisting of polynomials vanishing on $\operatorname{Lie} T$.

From the definition, operators $\widetilde{W}^{(\kappa)}_n$ are well-defined on four modules in (8.5.4). Moreover operators $\widehat{W}^{(\kappa)}_n$ and $\widehat{W}^{(\kappa)}_{-n}$ are well-defined on $M_{\mathbf{R}}(\boldsymbol{a})$ and $D(M_{\mathbf{R}}(\boldsymbol{a}))$ respectively if $n > 0$ by Lemma 8.5.2.

By the localization theorem, the first and the third homomorphisms in (6.0.3) become isomorphisms over $\mathbf{R}_T$. Therefore

$$M_{\mathbf{R}}(\boldsymbol{a}) \xrightarrow{\cong} N_{\mathbf{R}}(\boldsymbol{a}), \qquad D(N_{\mathbf{R}}(-\boldsymbol{a})) \xrightarrow{\cong} D(M_{\mathbf{R}}(-\boldsymbol{a})). \tag{8.5.6}$$

Recall that we have Heisenberg operators $P^i_n = (\varepsilon_1\varepsilon_2)^{-1}\widetilde{P}^i_n$, coupled with the fundamental class $1 \in H^0_{\mathbb{T}}(\mathbb{A}^2)$. Let

$$P[\boldsymbol{\lambda}] = P^1_{-\lambda^1_1} P^1_{-\lambda^1_2} \cdots P^\ell_{-\lambda^\ell_1} P^\ell_{-\lambda^\ell_2} \cdots \tag{8.5.7}$$

for $i = 1, \ldots, \ell$, $n \in \mathbb{Z}$, and an ℓ-partition $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^\ell)$. It is a well-defined operator on $D(M_{\mathbf{R}}(-\boldsymbol{a}))$ by the proof of Lemma 8.5.2.

Replacing P^i_m by $\widetilde{P}^i_m$, we introduce similar operators $\widetilde{P}[\boldsymbol{\lambda}]$.

Proposition 8.5.8. — *We have*

$$D(M_{\mathbf{R}}(-\boldsymbol{a})) = \operatorname{Span}_{\mathbf{R}_T}\{P[\boldsymbol{\lambda}]|\boldsymbol{a}\rangle\}, \tag{8.5.9}$$

where $\boldsymbol{\lambda}$ runs all ℓ-partitions.

Proof. — Thanks to (8.5.6), it is enough to show the assertion for $D(N_{\mathbf{R}}(-\boldsymbol{a}))$. We shall prove that $D(N_{\mathbf{A}}(-\boldsymbol{a}))$ is spanned by $P[\boldsymbol{\lambda}]$ over $\mathbf{A}$.

Recall that $N_{\mathbf{A}}(\boldsymbol{a}) = \operatorname{Span}_{\mathbf{A}_T}\{\widetilde{P}[\boldsymbol{\lambda}]|\boldsymbol{a}\rangle\}$, see (8.1.13). From the commutation relation

$$[P^i_m, \widetilde{P}^j_n] = -m\delta_{m,-n}(\alpha_i, \alpha_j), \tag{8.5.10}$$

we clearly have a perfect pairing between $N_{\mathbf{A}}(-\boldsymbol{a})$ and $\operatorname{Span}_{\mathbf{A}_T}\{P[\boldsymbol{\lambda}]|\boldsymbol{a}\rangle\}$. The assertion follows. ☐

8.6. Pairing at $\varepsilon_1, \varepsilon_2 = 0$

We consider the pairing $\langle\ ,\ \rangle$ on $M_{\mathbf{F}}(-\boldsymbol{a}) \otimes_{\mathbf{F}_T} M_{\mathbf{F}}(\boldsymbol{a})$ in §6.8, and restrict it to $D(M_{\mathbf{R}}(\boldsymbol{a})) \otimes_{\mathbf{R}_T} D(M_{\mathbf{R}}(-\boldsymbol{a}))$.

Lemma 8.6.1. — *We decompose $D(M_{\mathbf{R}}(\pm\boldsymbol{a}))$ as $\bigoplus D(M^{\pm}_{d,\mathbf{R}})$ by the instanton number d as before.*

(1) $(\varepsilon_1\varepsilon_2)^d\langle\ ,\ \rangle$ takes values in $\mathbf{R}_T$ on $D(M^-_{d,\mathbf{R}}) \otimes D(M^+_{d,\mathbf{R}})$.

(2) Let $\langle\ ,\ \rangle_0$ be its specialization at $\varepsilon_1 = \varepsilon_2 = 0$. For $m > 0$, we have

$$\langle x, \widehat{W}^{(\kappa)}_{-m} y\rangle_0 = \begin{cases} (-1)^{d_\kappa+1}\langle \widetilde{W}^{(\kappa)}_m x, y\rangle_0 & \text{if } m = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8.6.2}$$

Since $\langle\ ,\ \rangle$ is symmetric, (2) remains true when we exchange the first and second entries.

Proof. — (1) Thanks to (8.5.6), it is enough to show the assertion for $D(N_{\mathbf{R}}(\boldsymbol{a})) \otimes D(N_{\mathbf{R}}(-\boldsymbol{a}))$. By (8.1.3) and $\mathscr{U}^d_T = S^d\mathbb{A}^2$, it is enough to show that the intersection pairing $\langle\ ,\ \rangle$ on $H^*_{\mathbb{T}}(S^d\mathbb{A}^2)$ satisfies the same property. Note that $S^d\mathbb{A}^2$ is a smooth orbifold. Since we only have a single fixed point $d\cdot 0$ in $S^d\mathbb{A}^2$ and the weight of the tangent space there is $\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \dots$ (d times), the fixed point formula implies the assertion.

(2) Suppose $x \in D(M^+_{d,\mathbf{R}})$, $y \in D(M^-_{d-m,\mathbf{R}})$ with $m > 0$. Then

$$(\varepsilon_1\varepsilon_2)^d\langle x, \widehat{W}^{(\kappa)}_{-m} y\rangle = (-1)^{d_i+1}(\varepsilon_1\varepsilon_2)^{m-1}(\varepsilon_1\varepsilon_2)^{d-m}\langle \widetilde{W}^{(\kappa)}_m x, y\rangle \tag{8.6.3}$$

by (8.4.1). Now we specialize $\varepsilon_1, \varepsilon_2 = 0$ to get the assertion. ☐

Let us consider $M_0(\pm\boldsymbol{a}) \overset{\text{def.}}{=} D(M_{\mathbf{R}}(\pm\boldsymbol{a})) \otimes_{\mathbf{R}_T} \mathbb{C}/\operatorname{Rad}\langle\ ,\ \rangle_0$, where $\mathbf{R}_T \to \mathbb{C}$ is the evaluation at $\varepsilon_1 = \varepsilon_2 = 0$, and $\operatorname{Rad}\langle\ ,\ \rangle_0$ is the radical of $\langle\ ,\ \rangle_0$. Then (8.6.2) implies that $\widehat{W}^{(\kappa)}_{-m} = 0$ if $m > 1$, and $\widehat{W}^{(\kappa)}_{-1}$, $\widetilde{W}^{(\kappa)}_1$ are well-defined on $M_0(\pm\boldsymbol{a})$.

Proposition 8.6.4. — *(1) $P^{(i)}_{-m} = 0$ if $m > 1$, and $P^{(i)}_{-1}$, $\widetilde{P}^{(i)}_1$ are well-defined on $M_0(\pm\boldsymbol{a})$. And we have*

$$\langle x, P^{(i)}_{-1} y\rangle_0 = -\langle \widetilde{P}^{(i)}_1 x, y\rangle_0. \tag{8.6.5}$$

(2) *We have commutation relations*

$$[P^{(i)}_{-1}, P^{(j)}_{-1}] = 0, \quad [\widetilde{P}^{(i)}_{1}, \widetilde{P}^{(j)}_{1}] = 0, \quad [\widetilde{P}^{(i)}_{1}, P^{(j)}_{-1}] = -(\alpha_i, \alpha_j). \tag{8.6.6}$$

(3) $M_0(\pm\boldsymbol{a})$ *is isomorphic to the polynomial ring in* $P^{(i)}_{-1}$ $(i = 1, \dots, \ell)$. *The pairing* $\langle\ ,\ \rangle_0$ *is the induced pairing on the symmetric power from the pairing*

$$\langle -\boldsymbol{a} | P^{(i)}_{1} P^{(j)}_{-1} | \boldsymbol{a} \rangle_0 = (\alpha_i, \alpha_j). \tag{8.6.7}$$

Proof. — The same argument as above shows (1).

By Proposition 8.5.8 and (1), $M_0(\boldsymbol{a})$ is spanned by monomials in $P^{(i)}_{-1}$ applied to $|\boldsymbol{a}\rangle$.

(2) follows from Proposition 6.3.8.

(3) Let us replace $P^{(i)}_{-1}$, $\widetilde{P}^{(i)}_{1}$ by $Q^{(i)}_{-1}$, $\widetilde{Q}^{(i)}_{1}$ corresponding to an orthonormal basis of $\mathfrak{h}$ so that the commutation relation is $[\widetilde{Q}^{(i)}_{1}, Q^{(j)}_{-1}] = -\delta_{ij}$. Then (8.6.5) implies that monomials in $Q^{(i)}_{-1}$ are orthogonal. More precisely, the pairing is the standard one on $\mathbb{C}[Q^{(i)}_{-1}]$

$$\langle -\boldsymbol{a} | (Q^{(i)}_{1})^n (Q^{(i)}_{-1})^m | \boldsymbol{a} \rangle_0 = n! \delta_{mn}, \tag{8.6.8}$$

and the pairing factors on $M_0(\boldsymbol{a}) = \mathbb{C}[Q^{(1)}_{-1}] \otimes \cdots \otimes \mathbb{C}[Q^{(\ell)}_{-1}]$. This proves the assertion. □

8.7. Proof, a geometric part

Lemma 8.7.1. — *The first equality of* (8.3.4) *is true.*

Proof. — We have a natural homomorphism $\mathrm{IH}^*_{\mathbb{T}}(\mathcal{U}^d_G) \to H^{\mathbb{T}}_*(\mathcal{U}^d_G)$ and the image of 1^d is the fundamental class $[\mathcal{U}^d_G]$. Then $\langle 1^d | 1^d \rangle$ is equal to $\iota_*^{-1}[\mathcal{U}^d_G]$, where $\iota\colon \{d \cdot 0\} \to \mathcal{U}^d_G$ is the embedding of the $\mathbb{T}$-fixed point $d \cdot 0$, and we use the localization theorem to invert $\iota_*\colon H^{\mathbb{T}}_*(\{d \cdot 0\}) \to H^{\mathbb{T}}_*(\mathcal{U}^d_G)$ over $\mathbf{F}_T$.

Let us consider the embedding $\xi\colon (\mathcal{U}^d_G)^T = S^d\mathbb{A}^2 \to \mathcal{U}^d_G$ of the T-fixed point set. Then

$$\xi_*\colon H^{\mathbb{T}}_*(S^d\mathbb{A}^2) \to H^{\mathbb{T}}_*(\mathcal{U}^d_G) \tag{8.7.2}$$

is an isomorphism over $\mathbf{R}_T$. Since $H^{\mathbb{T}}_*(S^d\mathbb{A}^2) \cong \mathbf{A}_T[S^d\mathbb{A}^2]$, we have

$$\xi_*^{-1}[\mathcal{U}^d_G] = f_d(\boldsymbol{a}, \varepsilon_1, \varepsilon_2)[S^d\mathbb{A}^2] \tag{8.7.3}$$

for $f_d(\boldsymbol{a}, \varepsilon_1, \varepsilon_2) \in \mathbf{R}_T$.

We have $\iota_* = \xi_*\zeta_*$ for $\zeta\colon \{d \cdot 0\} \to S^d\mathbb{A}^2$, and $\zeta_*^{-1}[S^d\mathbb{A}^2] = (\varepsilon_1\varepsilon_2)^{-d}/d!$. Therefore

$$d!\ (\varepsilon_1\varepsilon_2)^d \langle 1^d | 1^d \rangle\big|_{\varepsilon_1, \varepsilon_2 = 0} = f_d(\boldsymbol{a}, 0, 0). \tag{8.7.4}$$

We replace the group $\mathbb{T}$ by T in (8.7.2) and denote the homomorphism by ξ^T_*, i.e., $\xi^T_*\colon H^T_*(S^d\mathbb{A}^2) \to H^T_*(\mathcal{U}^d_G)$. It is an isomorphism over $\mathbb{C}(\boldsymbol{a})$. Then we have

$$(\xi^T_*)^{-1}[\mathcal{U}^d_G] = f_d(\boldsymbol{a}, 0, 0)[S^d\mathbb{A}^2], \tag{8.7.5}$$

where $[\mathcal{U}_G^d]$, $[S^d\mathbb{A}^2]$ are considered in T-equivariant homology groups.

Let us take the projection $a\colon \mathbb{A}^2 \to \mathbb{A}^1$ and the factorization morphism $\pi_{a,G}^d\colon \mathcal{U}_G^d \to S^d\mathbb{A}^1$. Let $S^d a\colon S^d\mathbb{A}^2 \to S^d\mathbb{A}^1$ denote the induced projection. Let $(S^d\mathbb{A}^1)_0$ be the open subset of $S^d\mathbb{A}^1$ consisting of distinct d points. Then ξ induces a morphism between inverse images $(S^d a)^{-1}(S^d\mathbb{A}^1)_0$ and $(\pi_{a,G}^d)^{-1}(S^d\mathbb{A}^1)_0$. We get

$$(\xi_*^T)^{-1}[(\pi_{a,G}^d)^{-1}(S^d\mathbb{A}^1)_0] = f_d(\boldsymbol{a},0,0)[(S^d a)^{-1}(S^d\mathbb{A}^1)_0] \tag{8.7.6}$$

by restricting (8.7.5) to open subsets. Now by the factorization we deduce $f_d(\boldsymbol{a},0,0) = f_1(\boldsymbol{a},0,0)^d$. □

Remark 8.7.7. — This result is also a simple consequence of a property of Nekrasov's partition function

$$Z^{\text{inst}}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\Lambda) \overset{\text{def.}}{=} \sum_{d=0}^{\infty} \langle 1^d | 1^d \rangle \Lambda^{2h^\vee d} \tag{8.7.8}$$

stating that

$$\varepsilon_1\varepsilon_2 \log Z^{\text{inst}}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\Lambda) = F_0^{\text{inst}}(\boldsymbol{a},\Lambda) + o(\varepsilon_1,\varepsilon_2) \tag{8.7.9}$$

at $\varepsilon_1 = \varepsilon_2 = 0$. This property was proved by **[62, 65]** for type A and by **[16]** for general G.

8.8. Proof, a representation theoretic part

We shall complete the proof of the second equation in (8.3.4) in this subsection.

Let $F^{(\kappa)} \in S(\mathfrak{h})^W$ be one of generators as in §B.5. It has degree $d_\kappa + 1$.

Lemma 8.8.1. — *Following relations hold as operators on* $D(M_{\mathbf{R}}(-\boldsymbol{a})) \otimes_{\mathbf{R}_T} \mathbb{C}$:

$$\widetilde{W}_1^{(\kappa)} = \sum_i F^{(\kappa)}(a^1, \dots, \underbrace{\widetilde{P}_1^{(i)}}_{i^{\text{th}}\text{ factor}}, \dots, a^\ell), \tag{8.8.2}$$

$$\widetilde{W}_{-1}^{(\kappa)} = \sum_i F^{(\kappa)}(a^1, \dots, \underbrace{P_{-1}^{(i)}}_{i^{\text{th}}\text{ factor}}, \dots, a^\ell). \tag{8.8.3}$$

Proof. — At first sight, the Formula (B.5.24) seems to imply $\widetilde{W}_{-1}^{(\kappa)} = 0$, and hence also $\widetilde{W}_1^{(\kappa)} = 0$ thanks to the anti-involution θ. But (B.5.24) is the formula in the W-algebra at $\varepsilon_1 = \varepsilon_2 = 0$, and we want to consider $\widetilde{W}_1^{(\kappa)}$ on $D(M_{\mathbf{R}}(-\boldsymbol{a}))$. Since the highest weight $\lambda = \boldsymbol{a}/\varepsilon_1 - \rho$ cannot be specialized at $\varepsilon_1 = 0$, it could be nontrivial.

Let $\widetilde{W}^{(\kappa)}$ be the state corresponding to the field $Y(\widetilde{W}^{(\kappa)}, z) = \sum \widetilde{W}_n^{(\kappa)} z^{-n-d_\kappa-1}$ as in (B.2.1). By (B.5.24) we have

$$\widetilde{W}^{(\kappa)} = \widetilde{W}_{-d_\kappa-1}^{(\kappa)}|0\rangle = F^{(\kappa)}(\widetilde{P}_{-1}^{(i)})|0\rangle \tag{8.8.4}$$

at $\varepsilon_1 = \varepsilon_2 = 0$. It implies that

$$Y(\widetilde{W}^{(\kappa)}, z) = {:}F^{(\kappa)}(\widetilde{P}^{(i)}(z)){:} + o(\varepsilon_1,\varepsilon_2), \tag{8.8.5}$$

where $o(\varepsilon_1, \varepsilon_2)$ is a field in $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ which vanishes at $\varepsilon_1 = \varepsilon_2 = 0$.

Let the field act on $M_{\mathbf{R}}(\boldsymbol{a})$ and specialize at $\varepsilon_1 = \varepsilon_2 = 0$. The point is that $\widetilde{P}_0^{(i)}$ acts on $M_{\mathbf{R}}(\boldsymbol{a})$ by a^i at $\varepsilon_1 = \varepsilon_2 = 0$. Therefore the field $\widetilde{P}^{(i)}(z) = \sum_n \widetilde{P}_n^{(i)} z^{-n-1}$ is specialized to

$$a^i z^{-1} + \sum_{n<0} \widetilde{P}_n^{(i)} z^{-n-1} \tag{8.8.6}$$

on $M_{\mathbf{R}}(\boldsymbol{a})$.

Let us specialize (8.8.5) at $\varepsilon_1 = \varepsilon_2 = 0$. Then $\widetilde{P}^{(i)}(z)$ is replaced by (8.8.6), and the normal ordering by the usual multiplication. Therefore we obtain

$$Y(\widetilde{W}^{(\kappa)}, z) = F^{(\kappa)}(a^i z^{-1} + \sum_{n<0} \widetilde{P}_n^{(i)} z^{-n-1}). \tag{8.8.7}$$

Taking coefficients of z^{-d_κ} and then applying θ, we obtain (8.8.2).

Next we study the action of $Y(\widetilde{W}^{(\kappa)}, z)$ on $D(M_{\mathbf{R}}(-\boldsymbol{a}))$. Let us consider $\widetilde{W}_{-1}^{(\kappa)}$ in (8.8.5). So we take coefficients of z^{-d_κ}. The term $o(\varepsilon_1, \varepsilon_2)$ can be represented as a linear combination of monomials in $\widetilde{P}_m^{(i)}$ with coefficients in the maximal ideal of $\mathbf{R}$. We have at least one $\widetilde{P}_m^{(i)}$ with $m < 0$ in each monomial. It can be divided, as an operator on $D(M_{\mathbf{R}}(-\boldsymbol{a}))$, by $\varepsilon_1\varepsilon_2$ thanks to Lemma 8.5.2. Therefore $o(\varepsilon_1, \varepsilon_2)/\varepsilon_1\varepsilon_2$ still specialized to 0 at $\varepsilon_1 = \varepsilon_2 = 0$. Therefore (8.8.5) implies (8.8.3). $\square$

Lemma 8.8.8. — *The determinant of the matrix*

$$\left(\frac{\partial F^{(\kappa)}(a^i)}{\partial a^i}\right)_{i,\kappa=1,\dots,\ell}$$

is a nonzero constant multiple of the discriminant $\Delta(\boldsymbol{a})$.

Proof. — Consider $F = (F^{(1)}, \dots, F^{(\ell)})$ as the morphism from $\mathfrak{h}$ to $\mathfrak{h}/W$, written in a coordinate system on $\mathfrak{h}/W$. Then the matrix in question is the differential of F. Since $\mathfrak{h} \to \mathfrak{h}/W$ is a covering branched along root hyperplanes, we deduce that a) its determinant is nonzero, and b) it is divisible by $\Delta(\boldsymbol{a})$. The degree of the determinant is the sum $\sum d_\kappa$, which is equal to the number of positive roots. Therefore we get the assertion. $\square$

Since $\Delta(\boldsymbol{a})$ is invertible in $\mathbb{C}(\boldsymbol{a})$, we deduce

Lemma 8.8.9. — *$M_0(\boldsymbol{a})$ is isomorphic to the polynomial ring in $\widehat{W}_{-1}^{(\kappa)}$ $(\kappa = 1, \dots, \ell)$.*

Now the specialization of the Whittaker vector $|w^d\rangle$ in $M_0(\boldsymbol{a})$ is characterized by the conditions

$$\widetilde{W}_1^{(\kappa)}|w^d\rangle = \begin{cases} |w^{d-1}\rangle & \text{if } \kappa = \ell, \\ 0 & \text{if } \kappa \neq l. \end{cases} \tag{8.8.10}$$

The existence and the uniqueness in $M_0(\boldsymbol{a})$ are proved exactly as in §8.4. Moreover the pairing $\langle w^d | w^d \rangle_0$ is an entry of the inverse of the matrix

$$K_0^d \overset{\text{def.}}{=} (\langle -\boldsymbol{a} | \widetilde{W}[\boldsymbol{m}] \widehat{W}[\boldsymbol{n}] | \boldsymbol{a} \rangle_0)_{\boldsymbol{m},\boldsymbol{n}}, \tag{8.8.11}$$

where $\boldsymbol{m} = (m_1, \dots, m_\ell)$, $\boldsymbol{n} = (n_1, \dots, n_\ell) \in \mathbb{Z}_{\geq 0}^\ell$ and

$$\begin{aligned} \widetilde{W}[\boldsymbol{m}] &:= (\widehat{W}_{-1}^{(1)})^{m_1} \cdots (\widehat{W}_{-1}^{(\ell)})^{m_\ell}, \\ \widehat{W}[\boldsymbol{n}] &:= (\widetilde{W}_1^{(1)})^{n_1} \cdots (\widetilde{W}_1^{(\ell)})^{n_\ell}. \end{aligned} \tag{8.8.12}$$

Here multi-indices $\boldsymbol{m}$, $\boldsymbol{n}$ runs over $\sum m_\kappa = \sum n_\kappa = d$ for each d.

Now the matrix K_0^d is the d^{th} symmetric power of $K_0^{d=1}$, and hence we complete the proof of (8.3.4).

8.9. Type A

Let us consider the special case $\mathfrak{g} = \mathfrak{sl}_r$ in this section. Let us switch to the notation for $\mathfrak{gl}_r$. We have standard generators of the invariant polynomial ring:

$$F^{(\kappa)} = \sum_{i_1 < i_2 < \cdots < i_\kappa} h^{i_1} h^{i_2} \cdots h^{i_p}, \tag{8.9.1}$$

where $(h^1, \dots, h^r)$ is the standard coordinate system of the Cartan subalgebra of $\mathfrak{gl}_r$ such that $(h^i, h^j) = \delta_{ij}$.

Let us denote by $\widetilde{Q}_n^{(i)}$, $Q_n^{(i)}$ the Heisenberg algebra generators corresponding to $\widetilde{P}_n^{(i)}$, $P_n^{(i)}$. Then

$$\begin{aligned} \widehat{W}_{-1}^{(\kappa)} | \boldsymbol{a} \rangle &= \sum_{i_1 < i_2 < \cdots < i_\kappa} \sum_{l=1}^{p} \widetilde{Q}_0^{(i_1)} \widetilde{Q}_0^{(i_2)} \cdots Q_{-1}^{(i_l)} \cdots \widetilde{Q}_0^{i_\kappa} | \boldsymbol{a} \rangle \\ &= \sum_{i_1 < i_2 < \cdots < i_\kappa} \sum_{l=1}^{p} a_{i_1} a_{i_2} \cdots \widehat{a_{i_l}} \cdots a_{i_\kappa} Q_{-1}^{i_l} | \boldsymbol{a} \rangle. \end{aligned} \tag{8.9.2}$$

We use the Heisenberg algebra commutation relation

$$[\widetilde{Q}_1^i, Q_{-1}^j] = \delta_{ij} \tag{8.9.3}$$

to get

$$\begin{aligned} \widetilde{Q}_1^i W_{-1}^{(p)} | \boldsymbol{a} \rangle &= \sum_{\substack{i_1 < i_2 < \cdots < i_p \\ i_l = i}} a_{i_1} a_{i_2} \cdots \widehat{a_{i_l}} \cdots a_{i_p} | \boldsymbol{a} \rangle \\ &= \frac{\partial}{\partial a_i} e_p(\boldsymbol{a}) | \boldsymbol{a} \rangle, \end{aligned} \tag{8.9.4}$$

where $e_p(\boldsymbol{a})$ is the p^{th} elementary symmetric polynomial in $\boldsymbol{a}$.

The determinant of the $r \times r$-matrix $(\partial e_p(\boldsymbol{a}) / \partial a_i)_{i,p=1,\dots,r}$ is equal to $\prod_{i<j} (a_i - a_j)$. Therefore the matrix is invertible. This, in particular, implies that $\{W_{-1}^{(p)} | \boldsymbol{a} \rangle\}_{p=1,\dots,r}$ form a basis of $(M(\boldsymbol{a})_0)_1$.

Proposition 8.9.5. — *The Whittaker vector* $|w_1\rangle$ *at the instanton number* 1 *is given by*

$$\sum_i \frac{Q^i_{-1}|\boldsymbol{a}\rangle}{\prod_{j:j\neq i} a_j - a_i}. \tag{8.9.6}$$

Proof. — We have

$$W_1^{(p)} Q^i_{-1}|\boldsymbol{a}\rangle = \frac{\partial}{\partial a_i} e_p(\boldsymbol{a})|\boldsymbol{a}\rangle \tag{8.9.7}$$

as above. Now it is elementary to check that

$$\sum_i \frac{\frac{\partial}{\partial a_i} e_p(\boldsymbol{a})}{\prod_{j:j\neq i} a_j - a_i} = 0 \tag{8.9.8}$$

if $p < r$. If $p = r$, we have

$$\sum_i \frac{\frac{\partial}{\partial a_i} e_p(\boldsymbol{a})}{\prod_{j:j\neq i} a_j - a_i} = \sum_i \prod_{j:j\neq i} \frac{a_j}{a_j - a_i} = 1. \tag{8.9.9}$$

□

Now we have

$$(w_1|w_1)_0 = \sum_i \prod_{j:j\neq i} \frac{1}{(a_j - a_i)^2}. \tag{8.9.10}$$

This coincides with what is known from geometry.

APPENDIX A

APPENDIX: EXACTNESS OF HYPERBOLIC RESTRICTION

A.1. Zastava spaces

Let us denote by $\mathrm{Bun}_{G,B}$ the moduli space of G-bundles endowed with the following structures:

a) A trivialization at the infinite line $\mathbb{P}^1_\infty = \ell_\infty$.

b) A B-structure on the horizontal line $\mathbb{P}^1_h = \{y = 0\}$.

These two structures are required to be compatible at the intersection of $\mathbb{P}^1_\infty$ and $\mathbb{P}^1_h$ in the obvious way.

The connected components of $\mathrm{Bun}_{G,B}$ are numbered by positive elements of the coroot lattice of G_{aff} (cf. [**21**, §9]); for such element α we denote by $\mathrm{Bun}^\alpha_{G,B}$ the corresponding connected component.

We will also denote by $\mathscr{Z}^\alpha_G$ the corresponding "Zastava" space (a.k.a. "flag Uhlenbeck space") defined in [**21**]. We are going to need the following properties of $\mathscr{Z}^\alpha_G$. (Some of them are proved for the space $\mathrm{QMap}(\mathbb{P}^1_h, \mathscr{G}_{\mathfrak{g},\mathfrak{p}})$ of based quasi-maps to a flag scheme $\mathscr{G}_{\mathfrak{g},\mathfrak{p}}$ of a Kac-Moody Lie algebra $\mathfrak{g}$ associated with its parabolic $\mathfrak{p}$. Since $\mathscr{Z}^\alpha_G$ is the fiber product $\mathrm{QMap}(\mathbb{P}^1_h, \mathscr{G}_{\mathfrak{g},\mathfrak{b}}) \times_{\mathrm{QMap}(\mathbb{P}^1_h, \mathscr{G}_{\mathfrak{g},\mathfrak{p}})} \mathscr{U}^d_G$ for a Borel subalgebra $\mathfrak{b}$ of an affine Lie algebra $\mathfrak{g}$ and a maximal parabolic $\mathfrak{p}$, we can deduce assertions for $\mathscr{Z}^\alpha_G$ from those for $\mathrm{QMap}(\mathbb{P}^1_h, \mathscr{G}_{\mathfrak{g},\mathfrak{p}})$.)

(Z1) $\mathscr{Z}^\alpha_G$ is an irreducible affine scheme of dimension $2|\alpha|$ endowed with an action of $T \times \mathbb{C}^* \times \mathbb{C}^*$ which contains $\mathrm{Bun}^\alpha_{G,B}$ as an open subset (here we set $|\alpha| = \sum a_i$ if $\alpha = \sum a_i \alpha_i$ where α_i are the simple coroots of G_{aff}).

(Z2) There is a (factorization) map $\pi^\alpha_{\mathscr{Z}} \colon \mathscr{Z}^\alpha_G \to S^\alpha(\mathbb{A}^1_h)$. This map is $T \times \mathbb{C}^* \times \mathbb{C}^*$-equivariant if we let $T \times \mathbb{C}^* \times \mathbb{C}^*$ act on $S^\alpha(\mathbb{A}^1_h)$ just through the horizontal $\mathbb{C}^*$ (denoted by $\mathbb{C}^*_h$) and it admits a $T \times \mathbb{C}^* \times \mathbb{C}^*$-equivariant section ι^α. In particular, the fibers of $\pi^\alpha_{\mathscr{Z}}$ are stable under $T \times \mathbb{C}^*_v$ where the $\mathbb{C}^*_v = \mathbb{C}^*$-action comes from the vertical action on $\mathbb{A}^2$. All of these fibers have dimension $|\alpha|$. (See Conjecture 2.27, which is reduced to Conjecture 15.3 and proved for affine Lie algebras in §15.6 in [**21**].)

(Z3) Let set $\mathscr{F}^\alpha = (\pi^\alpha_{\mathscr{Z}})^{-1}(\alpha \cdot 0)$. Let $\rho \colon \mathbb{C}^* \to \widetilde{T} = T \times \mathbb{C}^*_v$ be any one-parameter subgroup which is a regular dominant coweight of G_{aff} (i.e., such that $\langle \rho, \beta \rangle > 0$ for any affine positive root β). Then the corresponding $\mathbb{C}^*$-action contracts $\mathcal{Z}^\alpha_G$ to $\iota^\alpha(S^\alpha(\mathbb{A}^1_h))$, and hence $\mathcal{F}^\alpha$ to $\iota^\alpha(\alpha\cdot 0)$. (cf. Proposition 2.6 and Corollary 10.4 in [**21**]).

(Z4) Let α_0 denote the affine simple coroot and let d be the coefficient of α_0 in α (in other words, $d=\langle\alpha,\omega_0\rangle$ where ω_0 denotes the corresponding fundamental weight of G_{aff}). Then there is a ("forgetting the B-structure") $T\times\mathbb{C}^*\times\mathbb{C}^*$-equivariant map $f_\alpha\colon\mathcal{Z}^\alpha_G\to\mathcal{U}^d_G$ which fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{Z}^\alpha_G \arrow[r, "f_\alpha"] \arrow[d, "\pi^\alpha_{\mathcal{Z}}"'] & \mathcal{U}^d_G \arrow[d, "\pi^d_G"] \\
S^\alpha\mathbb{A}^1_h \arrow[r] & S^d\mathbb{A}^1_h
\end{tikzcd}$$

where the bottom horizontal map sends a divisor $\sum\beta_i x_i$ to $\sum\langle\beta_i,\omega_0\rangle x_i$.

A.2. Plan of the proof

Let us discuss our strategy for proving Theorem 4.6.1. As we have explained in §3.5, it follows from dimension estimates of attracting and repelling sets by using arguments similar to those of [**49**]. However, at the moment we do not know how to prove estimates directly. So, our actual strategy will be slightly different. First, recall that we have

$$(\mathcal{U}^d_G)^T=S^d(\mathbb{A}^2),$$

and that we denote by $\mathcal{U}^d_B$, $\mathcal{U}^d_{B_-}$ the corresponding attracting and repelling sets. Also we denote by $p\colon\mathcal{U}^d_B\to S^d(\mathbb{A}^2)$ the corresponding map (sometimes we shall denote it by p^d when dependence on d is important). Then we are going to proceed in the following way:

1) Prove that the preimage of $S^d(\mathbb{A}^1)\subset S^d(\mathbb{A}^2)$ under the map $p\colon\mathcal{U}^d_B\to S^d(\mathbb{A}^2)=\mathcal{U}^d_{T,G}$ has dimension $\frac{\dim\mathcal{U}^d_G}{2}$ (here $\mathbb{A}^1\subset\mathbb{A}^2$ is any line). The proof will involve some facts about the Zastava spaces from [**21**].

2) Deduce Theorem 4.6.1 for $L=T$ from 1).

3) Using Proposition 4.5.1 deduce Theorem 4.6.1 for arbitrary L from the case $L=T$.

A.3. Attractors and repellents on the Uhlenbeck space: maximal torus case

Let us first look more closely at the case when $P=B$: a Borel subgroup of G. In this case $L=T$: a maximal torus of G.

Let us also define the set $\mathcal{S}^d\subset\mathcal{U}^d_G$ to be the attracting set in $\mathcal{U}^d_G$ with respect to the torus T to $S^d(\mathbb{A}^1_v\backslash 0)$ where $\mathbb{A}^1_v$ is the vertical line. In other words, $\mathcal{S}^d=p^{-1}(S^d(\mathbb{A}^1_v\backslash 0))$.

Proposition A.3.1. — *We have*

$$\dim\mathcal{S}^d\le dh^\vee=\frac{\dim\mathcal{U}^d_G}{2}.$$

Corollary A.3.2. — *Let* $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2$ *be any linear embedding. Then*

$$\dim p^{-1}(S^d(\mathbb{A}^1)) \leq \frac{\dim \mathscr{U}_G^d}{2}.$$

Corollary A.3.2 clearly follows from A.3.1. Indeed, first of all, it is clear that it is enough to prove Corollary A.3.2 when $\mathbb{A}^1 = \mathbb{A}^1_v$. In this case, $\mathscr{S}^d$ is open in $p^{-1}(S^d(\mathbb{A}^1_v))$, hence we have $\dim \mathscr{S}^d \leq \dim p^{-1}(S^d(\mathbb{A}^1_v))$. On the other hand, (the vertical) $\mathbb{A}^1$ acts naturally on $p^{-1}(S^d(\mathbb{A}^1_v))$ by shifts and any point of $p^{-1}(S^d(\mathbb{A}^1_v))$ lies in an open subset of the form $x(\mathscr{S}^d)$ for some $x \in \mathbb{A}^1$, hence the opposite inequality follows.

Let us now pass to the proof of Proposition A.3.1.

A.4. The map f_d

We have the natural (forgetting the flag) birational map $f_{d\delta} \colon \mathscr{Z}_G^{d\delta} \to \mathscr{U}_G^d$, which we shall simply denote by f_d. This map gives an isomorphism between the open subset of $\mathscr{U}_G^d$ consisting of (generalized) bundles which are trivial on the horizontal $\mathbb{P}^1_h$ and the open subset of $\mathscr{Z}_G^{d\delta}$ consisting of (generalized) bundles which are trivial on the horizontal $\mathbb{P}^1_h$ (and then the B-structure on the horizontal $\mathbb{P}^1_h$ is automatically trivial).

A.5. The central fiber

Recall that $\mathscr{F}^{d\delta}$ denotes the preimage of $d\delta \cdot 0$ under the map $\pi_{\mathscr{Z}}^{d\delta} \colon \mathscr{Z}_G^{d\delta} \to S^{d\delta}(\mathbb{A}^1_h)$. Again, to simplify the notation, we shall just write $\mathscr{F}^d$ instead of $\mathscr{F}^{d\delta}$. According to (Z2), $\dim \mathscr{F}^d = dh^\vee$.

We claim that

1) $\mathscr{S}^d$ lies in the open subset of $\mathscr{U}_G^d$ over which f_d is an isomorphism.

2) $f_d^{-1}(\mathscr{S}^d) \subset \mathscr{F}^d$.

The first statement is clear, since the image of $\mathscr{S}^d$ in $S^d(\mathbb{A}^1_v)$ under the factorization morphism π_v^d (to the symmetric product of the vertical line) must lie in $S^d(\mathbb{A}^1_v \backslash 0)$. To prove the second statement, let us note that $f_d^{-1}(\mathscr{S}^d)$ must lie in the attracting set in $\mathscr{Z}_G^{d\delta}$ with respect to the torus T to $f_d^{-1}(S^d(\mathbb{A}^1_v \backslash 0))$. It is clear that $f_d^{-1}(S^d(\mathbb{A}^1_v \backslash 0)) \subset \mathscr{F}^d$ and thus the statement follows, since every fiber of the map $\pi_{\mathscr{Z}}^{d\delta} \colon \mathscr{Z}_G^{d\delta} \to S^{d\delta}(\mathbb{A}^1_h)$ is stable under the action of T.

Hence we get $\dim \mathscr{S}^d \leq dh^\vee = \dim \mathscr{F}^d$.

A.6. Good coweights

Let X be an affine variety endowed with an action of $T \times \mathbb{C}^*$ (here T can be any torus). Let x be any $T \times \mathbb{C}^*$-fixed point (in practice this point will always be unique, but this is not needed formally for what follows) and let $Y \subset X^T$ be the $\mathbb{C}^*$-attractor to x inside X^T. Let now $\lambda : \mathbb{C}^* \to T$ be any coweight. Let us denote by $\mathscr{A}_\lambda$ the attractor to Y with respect to the $\mathbb{C}^*$-action given by λ. Let us also denote by $\widetilde{\mathscr{A}}_\lambda$ the attractor to x with respect to the $\mathbb{C}^*$-action given by the cocharacter $(\lambda, 1)$ of $T \times \mathbb{C}^*$.

We say that λ is *good* if $\mathscr{A}_\lambda = \widetilde{\mathscr{A}}_\lambda$.

Lemma A.6.1. — *For any λ as above, the coweight $n\lambda$ is good for $n \in \mathbb{N}$ large enough.*

Proof. — Obviously, there exists a closed T-equivariant embedding of X into a vector space V such that the action of $T \times \mathbb{C}^*$ on V is linear and such that x corresponds to $0 \in V$. Then it is clear that if λ is good for V, then it is also good for X. Hence we may assume that $X = V$.

In this case, we see that $n\lambda$ is good if and only if for every weight of $T \times \mathbb{C}^*$ on V of the form (θ, k) the following condition is satisfied:

$$n\langle\lambda, \theta\rangle + k > 0 \text{ if and only if either } \langle\lambda, \theta\rangle > 0, \text{ or } \langle\lambda, \theta\rangle = 0 \text{ and } k > 0.$$

Now, every $n \in \mathbb{N}$ such that $n|\langle\lambda, \theta\rangle| > |k|$ for any (θ, k) as above such that $\langle\lambda, \theta\rangle \neq 0$ will satisfy the conditions of the lemma. □

Let λ be as before and assume in addition that

(i) x is the only fixed point of $\mathbb{C}^*$ acting by means of the coweight $(\lambda, 1)$;

(ii) $X^{\lambda(\mathbb{C}^*)} = X^T$

(in this case we automatically have $(X^T)^{\mathbb{C}^*} = \{x\}$). Let us denote by $\widetilde{\Phi}$ the hyperbolic restriction for $(\lambda, 1)$ (acting from sheaves on X to sheaves on $\{x\}$), by Φ the hyperbolic restriction for $\lambda\colon \mathbb{C}^* \to T$ (acting from sheaves on X to sheaves on X^T) and by Φ_0 the hyperbolic restriction for the action of $\mathbb{C}^*$ on X^T (from sheaves on X^T to sheaves on $\{x\}$). Then the definition of "goodness" implies

Lemma A.6.2. — *Assume that λ is good and satisfies the conditions (i) and (ii). Then we have $\widetilde{\Phi} = \Phi_0 \circ \Phi$.*

A.7. Exactness of twisted hyperbolic restriction

Let $\widetilde{T} = T \times \mathbb{C}^*$ and let us make it act on $\mathscr{U}_G^d$ so that the action of $\mathbb{C}^*$ comes from the hyperbolic action of $\mathbb{C}^*$ on $\mathbb{A}^2$ of the form $z(x, y) = (z^{-1}x, zy)$. Note that $(\mathscr{U}_G^d)^{\widetilde{T}}$ consists of one point.

Let us fix d and let us choose a dominant regular coweight $\lambda : \mathbb{C}^* \to T$ which is good in the sense of Subsection A.6 (such λ exists because of Lemma A.6.1). Then the fact that λ is regular implies that it satisfies the conditions (i) and (ii). Consider the corresponding functors $\widetilde{\Phi}, \Phi$ and Φ_0. Obviously we have $\Phi = \Phi_{T,G}^d$, so we shall write $\widetilde{\Phi}_{T,G}^d$ instead of $\widetilde{\Phi}$. Also, to emphasize the dependence on d we set Φ_0^d instead of Φ_0. According to Lemma A.6.2 we have $\widetilde{\Phi}_{T,G}^d = \Phi_0^d \circ \Phi_{T,G}^d$.

Theorem A.7.1. — *The complex of vector spaces $\widetilde{\Phi}_{T,G}^d(\mathrm{IC}(\mathscr{U}_G^d))$ is concentrated in degree 0.*

Proof. — We will use the same notations as before for $L = T$ replaced with $\widetilde{T}$, such as $i_{\widetilde{T},G}$, $j_{\widetilde{T},G}$, $p_{\widetilde{T},G}$, $i^-_{\widetilde{T},G}$, $j^-_{\widetilde{T},G}$, $p^-_{\widetilde{T},G}$. The attracting set is denoted by $\mathscr{C}^d_{\lambda,\widetilde{T},G}$. According to [**13**, Theorem 1], the natural morphism $(p^-_{\widetilde{T},G})_*(j^-_{\widetilde{T},G})^! \operatorname{IC}(\mathscr{U}^d_G) \to (p_{\widetilde{T},G})_!(j_{\widetilde{T},G})^* \operatorname{IC}(\mathscr{U}^d_G) = \widetilde{\Phi}^d_{T,G}(\operatorname{IC}(\mathscr{U}^d_G))$ is an isomorphism. We will prove that $(p_{\widetilde{T},G})_!(j_{\widetilde{T},G})^* \operatorname{IC}(\mathscr{U}^d_G)$ is concentrated in nonpositive degrees. A similar (dual) argument proves that $(p^-_{\widetilde{T},G})_*(j^-_{\widetilde{T},G})^! \operatorname{IC}(\mathscr{U}^d_G)$ is concentrated in nonnegative degrees. In other words, we must prove that $H^\bullet_c(\mathscr{C}^d_{\lambda,\widetilde{T},G}, \operatorname{IC}(\mathscr{U}^d_G))$ lives in nonpositive cohomological degrees.

Now $\operatorname{IC}(\mathscr{U}^d_G)$ is smooth along the stratification

$$\mathscr{U}^d_G = \bigsqcup_{m+|\lambda|=d} \operatorname{Bun}^m_G \times S_\lambda(\mathbb{A}^2),$$

the dimension of a stratum being equal to $2l(\lambda) + 2mh^\vee$. Here for a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ we set $l(\lambda) = l$. The perverse sheaf $\operatorname{IC}(\mathscr{U}^d_G)$ lives in cohomological degrees $\leq -2l(\lambda) - 2mh^\vee$ on the stratum $\operatorname{Bun}^m_G \times S_\lambda(\mathbb{A}^2)$. We have $\mathscr{C}^d_{\lambda,\widetilde{T},G} \cap \big(\operatorname{Bun}^m_G \times S_\lambda(\mathbb{A}^2)\big) = (\mathscr{C}^m_{\lambda,\widetilde{T},G} \cap \operatorname{Bun}^m_G) \times S_\lambda(\mathbb{A}^1_v)$. Now it follows from Corollary A.3.2 and the goodness assumption on λ that $\dim(\mathscr{C}^m_{\lambda,\widetilde{T},G}) \leq mh^\vee$. Evidently, $\dim S_\lambda(\mathbb{A}^1_v) = l(\lambda)$. So the restriction of $\operatorname{IC}(\mathscr{U}^d_G)$ to $\mathscr{C}^d_{\lambda,\widetilde{T},G} \cap \big(\operatorname{Bun}^m_G \times S_\lambda(\mathbb{A}^2)\big)$ lives in degrees $\leq -2\dim\Big(\mathscr{C}^d_{\lambda,\widetilde{T},G} \cap \big(\operatorname{Bun}^m_G \times S_\lambda(\mathbb{A}^2)\big)\Big)$. Now an application of the Cousin spectral sequence for the stratification of $\mathscr{C}^d_{\lambda,\widetilde{T},G}$ finishes the proof. □

The following corollary is not needed for the rest, but we include it for the sake of completeness.

Corollary A.7.2. — $\dim \mathscr{S}^d = \dim p^{-1}(S^d(\mathbb{A}^1)) = dh^\vee$.

Proof. — We need to show that $\dim \mathscr{C}^d_{\lambda,\widetilde{T},G}$ is at least $dh^\vee$. By induction on d we may assume that this is true for all $d' < d$. Assume that $\dim \mathscr{C}^d_{\lambda,\widetilde{T},G} < dh^\vee$. Then repeating the argument from the above proof we see that $\widetilde{\Phi}^d_{T,G}(\operatorname{IC}(\mathscr{U}^d_G))$ is concentrated in strictly negative cohomological degrees, which contradicts Theorem A.7.1. □

Remark A.7.3. — The above argument only shows that the dimension of the whole of $\mathscr{S}^d$ is equal to $dh^\vee$, but doesn't show that this is true for each of its irreducible components (however, we believe that this is true).

A.8. Exactness of $\Phi_{T,G}$

We can now show that $\Phi^d_{T,G}(\operatorname{IC}(\mathscr{U}^d_G))$ is perverse. Indeed, using the factorization argument and induction on d, we may assume that $\Phi^d_{T,G}(\operatorname{IC}(\mathscr{U}^d_G))$ is perverse away from the main diagonal $\mathbb{A}^2 \subset S^d(\mathbb{A}^2)$. Since according to [**13**] the complex $\Phi^d_{T,G}(\operatorname{IC}(\mathscr{U}^d_G))$ is semi-simple and since it is also equivariant with respect to the action of $\mathbb{A}^2$ on $S^d(\mathbb{A}^2)$

by shifts, it follows that we just need to prove that $\Phi^d_{T,G}(\mathrm{IC}(\mathcal{U}^d_G))$ doesn't have any direct summands which are isomorphic to constant sheaves on $\mathbb{A}^2$ sitting in cohomological degrees $\neq -2$. But if such a direct summand existed, it would imply that $\Phi^d_0(\Phi^d_{T,G}(\mathrm{IC}(\mathcal{U}^d_G))) = \widetilde{\Phi}^d_{T,G}(\mathrm{IC}(\mathcal{U}^d_G))$ has non-zero cohomology in degree $\neq 0$, which contradicts Theorem A.7.1.

A.9. Exactness of $\Phi_{L,G}$

Let us now show that $\Phi^d_{L,G}(\mathrm{IC}(\mathcal{U}^d_G))$ is perverse. Indeed, first of all, according to Braden's theorem [**13**], $\Phi^d_{L,G}(\mathrm{IC}(\mathcal{U}^d_G))$ is a semi-simple complex, which is constructible with respect to the stratification (2.3.1). In other words, it is a direct sum of (possibly shifted) simple perverse sheaves, where each such sheaf is isomorphic to the Goresky-MacPherson extension of a local system $\mathcal{E}$ on $\mathrm{Bun}^{d_1}_L \times S_\lambda(\mathbb{A}^2)$ for some d_1 and λ as in 2.3.1.

Lemma A.9.1. — *Any such $\mathcal{E}$ is necessarily of the form $\mathbb{C}_{\mathrm{Bun}^{d_1}_L} \boxtimes \mathcal{E}'$ where $\mathcal{E}'$ is some local system on $S_\lambda(\mathbb{A}^2)$.*

Proof. — To prove this it is enough to show that the restriction of $\Phi_{L,G}(\mathrm{IC}(\mathcal{U}^d_G))$ to $\mathrm{Bun}^{d_1}_L \times S^{d_2}(\mathbb{A}^2)$ (here $d = d_1 + d_2$) is isomorphic to the exterior tensor product of the constant sheaf of $\mathrm{Bun}^{d_1}_L$ and some complex on $S^{d_2}(\mathbb{A}^2)$. Moreover, it is enough to construct such an isomorphism on some Zariski open subset U of $\mathrm{Bun}^{d_1}_L \times S^{d_2}(\mathbb{A}^2)$ (this follows from the fact that a local system which is constant on a Zariski dense subset is constant everywhere). Let us choose a projection $a : \mathbb{A}^2 \to \mathbb{A}^1$ and let $\pi^{d_1}_{a,L} : \mathrm{Bun}^{d_1}_L \to S^{d_1}(\mathbb{A}^1)$ be the corresponding map. Let U be the open subset of $\mathrm{Bun}^{d_1}_L \times S^{d_2}(\mathbb{A}^2)$ consisting of pairs $(\mathcal{F}, x)$ such that $\pi^{d_1}_{a,L}$ is disjoint from the projection of x to $S^{d_2}(\mathbb{A}^1)$. Then locally in étale topology near every point of U the scheme $\mathcal{U}^d_G$ looks like the product $\mathrm{Bun}^{d_1}_G \times \mathcal{U}^{d_2}_G$ and the statement follows. □

Now, we can finish the proof. Indeed, recall that the closure of $\mathrm{Bun}^{d_1}_L \times S_\lambda(\mathbb{A}^2)$ admits a finite birational map from $\mathcal{U}^{d_1}_L \times \overline{S}^\lambda(\mathbb{A}^2)$, where $\overline{S}^\lambda(\mathbb{A}^2)$ stands for the closure of S_λ in $S^{d_2}(\mathbb{A}^2)$. Thus for any $\mathcal{E}$ as above we see that $\mathrm{IC}(\mathcal{E})$ is the direct image of $\mathrm{IC}(\mathcal{U}^{d_1}_L) \boxtimes \mathrm{IC}(\mathcal{E}')$ under this map. Moreover, the complex $\Phi_{T,L}(\mathrm{IC}(\mathcal{E}))$ is equal to the direct image of $\Phi_{T,L}(\mathrm{IC}(\mathcal{U}^{d_1}_L)) \boxtimes \mathrm{IC}(\mathcal{E}')$. Hence, we see that it is perverse and non-zero. Thus, if for some $i \neq 0$ the complex $\mathrm{IC}(\mathcal{E})[i]$ is a direct summand of $\Phi_{L,G}(\mathrm{IC}(\mathcal{U}^d_G))$, then $\Phi_{T,L}(\Phi_{L,G}(\mathcal{U}^d_G))$ is not perverse. Since $\Phi_{T,L} \circ \Phi_{L,G} \simeq \Phi_{T,G}$, this contradicts Subsection A.8. □

Recall $\mathcal{U}^d_{P,0} \overset{\mathrm{def.}}{=} p^{-1}(d \cdot 0)$, see (4.8.14).

Corollary A.9.2. — $\dim \mathcal{U}^d_{P,0} \leq dh^\vee - 1$.

Proof. — We will argue by induction in d. We assume the claim for all $d' < d$. We know that the dual space $(U^d)^* \simeq H_c^\bullet(p^{-1}(d\cdot 0), \tilde{j}^* \operatorname{IC}({}^c\mathcal{U}_G^d))$ lives in degree 0. We consider the Cousin spectral sequence for the stratification $\mathcal{U}_{P,0}^d = \bigsqcup_{d'\leq d}(\mathcal{U}_{P,0}^{d'} \cap \operatorname{Bun}_G^{d'})$. By the induction assumption, all the strata for $d' < d$ contribute to nonpositive degrees of $H_c^\bullet(p^{-1}(d\cdot 0), \tilde{j}^* \operatorname{IC}({}^c\mathcal{U}_G^d))$ only. If we had $\dim \mathcal{U}_{P,0}^d > dh^\vee - 1$, the fundamental classes of the top dimensional components of $\mathcal{U}_{P,0}^d$ would contribute to the strictly positive degrees in $H_c^\bullet(p^{-1}(d\cdot 0), \tilde{j}^* \operatorname{IC}({}^c\mathcal{U}_G^d))$, and nothing would cancel their contribution. This would contradict to $H_c^{>0}(p^{-1}(d\cdot 0), \tilde{j}^* \operatorname{IC}({}^c\mathcal{U}_G^d)) = 0$. □

Here is a more direct proof suggested by the referee. We choose a faithful representation $\varrho\colon G \hookrightarrow \operatorname{SL}(r)$. It gives rise to a closed embedding $\varrho_{\mathcal{U}}\colon {}^c\mathcal{U}_G^d \hookrightarrow {}^c\mathcal{U}_r^d$. We choose a dominant coweight $\check{\chi}$ of T such that L is the centralizer of $\check{\chi}(\mathbb{C}^\times)$. Let $L_{\varrho_*\check{\chi}} \subset P_{\varrho_*\check{\chi}} \subset \operatorname{SL}(r)$ be the corresponding Levi and parabolic subgroups. Then $\varrho_{\mathcal{U}}({}^c\mathcal{U}_{P,0}^d) \subset {}^c\mathcal{U}_{P_{\varrho_*\check{\chi}},0}^{d\phi(\varrho)}$, where $\phi(\varrho)$ is the Dynkin index of ϱ. Now ${}^c\mathcal{U}_r^{d\phi(\varrho)}$ is equipped with a Poisson structure compatible with the symplectic structure of ${}^c\widetilde{\mathcal{U}}_r^{d\phi(\varrho)}$. This Poisson structure has finitely many symplectic leaves (the strata of the diagonal stratification of ${}^c\mathcal{U}_r^{d\phi(\varrho)}$), and the intersection of ${}^c\mathcal{U}_{P_{\varrho_*\check{\chi}},0}^{d\phi(\varrho)}$ with any symplectic leaf is isotropic since the preimage of ${}^c\mathcal{U}_{P_{\varrho_*\check{\chi}},0}^{d\phi(\varrho)}$ in ${}^c\widetilde{\mathcal{U}}_r^{d\phi(\varrho)}$ is isotropic. Finally, $\varrho_{\mathcal{U}}\colon {}^c\mathcal{U}_G^d \hookrightarrow {}^c\mathcal{U}_r^d$ induces a Poisson structure on ${}^c\mathcal{U}_G^d$ whose symplectic leaves are the strata of the diagonal stratification of ${}^c\mathcal{U}_G^d$. It follows that the intersection of ${}^c\mathcal{U}_{P,0}^d$ with any symplectic leaf is isotropic, and hence $\dim \mathcal{U}_{P,0}^d \leq dh^\vee - 1$. □

This is the estimate of the attracting set for the most singular point $d\cdot 0$. The exactness also implies estimates for attracting sets of other points, more precisely their intersection with the open locus Bun_G^d. Since any stratum of $\mathcal{U}_G^d$ is of the form $\operatorname{Bun}_G^{d_1} \times S_\lambda(\mathbb{A}^2)$, we have the corresponding dimension estimate for other strata from the perversity of $\Phi_{L,G}(\operatorname{IC}(\mathcal{U}_G^{d_1}))$ for any d_1. Therefore we see that $\Phi_{L,G}$ is hyperbolic semi-small in the sense of Definition 3.5.1.

APPENDIX B

INTEGRAL FORM OF THE $\mathscr{W}$-ALGEBRA

The purpose of this section is to introduce an **A**-form of the $\mathscr{W}$-algebra, generalizing the **A**-form $\mathfrak{Vir}_{i,\mathbf{A}}$ of the Virasoro algebra in §6.4, where the commutation relations of integral generators of the Heisenberg algebra and the Virasoro algebra are (see (6.3.12), (6.4.9))

$$[\widetilde{P}^i_m, \widetilde{P}^j_n] = -m\delta_{m,-n}(\alpha_i,\alpha_j)\varepsilon_1\varepsilon_2, \tag{B.0.1}$$

$$[\widetilde{L}^i_m, \widetilde{L}^i_n] = \varepsilon_1\varepsilon_2\left\{(m-n)\widetilde{L}^i_{m+n} + \left(\varepsilon_1\varepsilon_2 + 6(\varepsilon_1+\varepsilon_2)^2\right)\delta_{m,-n}\frac{m^3-m}{12}\right\},$$

and they are related by

$$\widetilde{L}^i_n = -\frac{1}{4}\sum_m {:}\widetilde{P}^i_m\widetilde{P}^i_{n-m}{:} - \frac{n+1}{2}(\varepsilon_1+\varepsilon_2)\widetilde{P}^i_n.$$

Let $\mathfrak{g}$ be a complex simple Lie algebra. We do not assume $\mathfrak{g}$ is of type ADE in this section. Let $(\ ,\ )$ be the normalized bilinear form so that the square length of a long root is 2. Let ℓ be its rank and $d_1 \le \cdots \le d_\ell$ be the exponents of $\mathfrak{g}$, counted with multiplicities. For example, $\mathfrak{g} = \mathfrak{sl}_{\ell+1}$, we have $d_1 = 1, d_2 = 2, \dots, d_\ell = \ell$. We have $d_\ell = h^\vee - 1$. The multiplicity of the exponent is equal to 1, except $d_{\ell/2} = d_{\ell/2+1} = \ell - 1$ for D_ℓ with ℓ even.

B.1. Integral form of the BRST complex

In order to define an **A**-form of the $\mathscr{W}$-algebra, we need to recall briefly the BRST complex used in the definition of the $\mathscr{W}$-algebra in [**30**, Ch. 15]. We assume that the reader is familiar with [**30**, Ch. 15], as we skip details.

Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the Cartan decomposition of $\mathfrak{g}$. Let $\Delta_\pm$ denote the set of positive/negative roots. Let I be the set of simple roots.

We consider the vertex superalgebra $C^\bullet_k(\mathfrak{g})$, which is the tensor product of the affine vertex algebra $V_k(\mathfrak{g})$ of level k and the fermionic vertex superalgebra $\bigwedge^\bullet_{\mathfrak{n}_+}$. We have two anti-commuting differentials d_{st} and χ on $C^\bullet_k$ so that $\mathscr{W}_k(\mathfrak{g})$ is defined as the 0^{th} cohomology with respect to $d = d_{\mathrm{st}} + \chi$.

We do not need the definition of d_{st}, χ. We start with the subcomplex $C_k^\bullet(\mathfrak{g})_0$ as the cohomology of $C_k^\bullet(\mathfrak{g})$ is a tensor product of $C_k^\bullet(\mathfrak{g})_0$ and another complex, whose cohomology is trivial (see [**30**, Lem. 15.2.7]).

We take a basis $\{J^a\}$ of $\mathfrak{g}$ consisting of root vectors and vectors h^i, dual to simple roots α_i with respect to $(\ ,\ )$. Let c_d^{ab} be the structure constants of $\mathfrak{g}$ with respect to the basis $\{J^a\}$. Latin indices are used to denote arbitrary basis elements, Latin indices with bar are used to denote elements in $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$. Therefore $\{J^{\bar{a}}\}_{\bar{a}\in\Delta_-\cup I}$ is a basis of $\mathfrak{b}_-$. Greek indices are used to denote basis elements of $\mathfrak{n}_+$. We also have a basis $\{\psi^*_\alpha\}_{\alpha\in\Delta_+}$ of $\mathfrak{n}_+^*$. We denote the corresponding fields by $\widehat{J}^{\bar{a}}(z)$ and $\psi^*_\alpha(z)$, where the former has a correction term (see [**30**, (15.2.1)]). The field $\widehat{J}^{\bar{a}}(z)$ satisfies the commutation relation for the affine Lie algebra at the level $k + h^\vee$ instead of k because of the correction terms (cf. [**2**, (4.8.1)]):

$$[\widehat{J}^{\bar{a}}(z), \widehat{J}^{\bar{b}}(w)] = \sum_{\bar{c}} c_{\bar{c}}^{\bar{a}\bar{b}} \widehat{J}^{\bar{c}}(w)\delta(z-w) + (k+h^\vee)\partial_w\delta(z-w). \tag{B.1.1}$$

Now the complex $C_k^\bullet(\mathfrak{g})_0$ is spanned by monomials of the form

$$\widehat{J}^{\bar{a}(1)}_{n_1} \cdots \widehat{J}^{\bar{a}(r)}_{n_r} \psi^*_{\alpha(1),m_1} \cdots \psi^*_{\alpha(s),m_s} |0\rangle, \tag{B.1.2}$$

and the action of the differentials is given by the following formulas

(B.1.3)

$$\begin{aligned}
&[\chi, \widehat{J}^{\bar{a}}(z)] = \sum_{i\in I}\sum_{\beta\in\Delta_+} c_{\alpha_i}^{\bar{a}\beta}\psi^*_\beta(z),\\
&[\chi, \psi^*_\alpha(z)]_+ = 0,\\
&[d_{\mathrm{st}}, \widehat{J}^{\bar{a}}(z)] = \sum_{\bar{b},\alpha} c_{\bar{b}}^{\alpha\bar{a}} {:}\widehat{J}^{\bar{b}}(z)\psi^*_\alpha(z){:} + k\sum_\alpha (J^{\bar{a}}, J^\alpha)\partial_z\psi^*_\alpha(z) - \sum_{\alpha,\beta,b} c_\beta^{\alpha b} c_b^{\beta\bar{a}}\partial_z\psi^*_\alpha(z),\\
&[d_{\mathrm{st}}, \psi^*_\alpha(z)]_+ = -\frac{1}{2}\sum_{\beta,\gamma} c_\alpha^{\beta\gamma}\psi^*_\beta(z)\psi^*_\gamma(z),
\end{aligned}$$

together with $\chi|0\rangle = d_{\mathrm{st}}|0\rangle = 0$. Here the formulas are copied from [**30**, 15.2.4] except that the first one is simplified as we only consider a field for $J^{\bar{a}}$ in $\mathfrak{b}_-$.

The bidegree is defined by

$$\begin{aligned}
&\operatorname{bideg}\widehat{J}^{\bar{a}}(z) = (-n, n),\\
&\operatorname{bideg}\psi^*_\alpha(z) = (l, -l+1),
\end{aligned} \tag{B.1.4}$$

where n is the principal gradation of $J^{\bar{a}}$ and l is the height of the root α. (See [**30**, 15.1.7] for definitions of the principal gradation and the height.) Therefore χ has bidegree $(1,0)$, and d_{st} has bidegree $(0,1)$. We get the double complex $C_k^\bullet(\mathfrak{g})_0 = \bigoplus_{p,q} C_k^{p,q}(\mathfrak{g})_0$. From the definition of the bidegree, we see that $C_k^{p,q}(\mathfrak{g})_0 = 0$ unless $p \geq 0$, $-p \leq q \leq 0$.

Now we rewrite the complex suitable for our purpose. By (6.0.1) we replace k by $-(h^\vee + \varepsilon_2/\varepsilon_1)$.

Next let us introduce a modification $\widetilde{J}^{\bar{a}}(z)$ of $\widehat{J}^{\bar{a}}(z)$, like $\widetilde{P}^i_m$ of P^i_m in §6.3. There is a simple recipe for this. Reading formulas in [**30**, §15.4.10], we note that $\widehat{J}^{\bar{a}}(z)$ for $\bar{a} \in I$ is denoted by $\widehat{h}^i(z)$ and satisfies the commutation relation

(B.1.5) $$[\widehat{h}^i_m, \widehat{h}^j_n] = m\delta_{m,-n}(\alpha_i, \alpha_j)(k + h^\vee).$$

See also (B.1.1). This Heisenberg operator gives the embedding $\mathscr{W}_k(\mathfrak{g}) \to \mathfrak{Heis}(\mathfrak{h})$. Comparing (B.0.1) with (B.1.5), we find that it is natural to set

(B.1.6) $$\widetilde{J}^{\bar{a}}(z) = \varepsilon_1 \widehat{J}^{\bar{a}}(z).$$

We also rescale χ by a function φ in ε_1, ε_2 as $\widetilde{\chi} = \varphi\chi$. Unless φ vanishes, the cohomology group is independent of φ. However we will specialize ε_1, ε_2 to 0, the result will be different. Therefore the choice of φ is important. Remember that our goal is to realize a generator $\widetilde{W}^{(\kappa)}_n$ in geometry. We want to assign it with the perverse cohomological degree $2(d_\kappa + 1)$, as $\widetilde{L}^i_n$ in §6.4 is of degree 4. This generator is a sum of a main term X_0 of bidegree $(d_\kappa, -d_\kappa)$ plus correction terms X_1, X_2, …of bidegree $(p, -p)$ with $0 \le p < d_\kappa$ determined by the condition $\widetilde{\chi}X_\kappa = -d_{\mathrm{st}}X_{\kappa-1}$. (See [**30**, 15.2.11].) Therefore we want all X_0, X_1, …to have the same (perverse) cohomological degree. This is achieved if φ is of degree -2. We still have ambiguity, but look at the Formulas (B.1.3) and (B.1.6), the simplest solution is to absorb $1/\varepsilon_1$ in $\widetilde{J}^{\bar{a}}(z)$ to $\widetilde{\chi}$, i.e., $\widetilde{\chi} = \chi/\varepsilon_1$.

We thus arrive at the following:

(B.1.7) $$\begin{aligned}
[\widetilde{\chi}, \widetilde{J}^{\bar{a}}(z)] &= \sum_{i\in I}\sum_{\beta\in\Delta_+} c^{\bar{a}\beta}_{\alpha_i}\psi^*_\beta(z), \\
[\widetilde{\chi}, \psi^*_\alpha(z)]_+ &= 0, \\
[d_{\mathrm{st}}, \widetilde{J}^{\bar{a}}(z)] &= \sum_{\bar{b},\alpha} c^{\alpha\bar{a}}_{\bar{b}} {:}\widetilde{J}^{\bar{b}}(z)\psi^*_\alpha(z){:} - (h^\vee\varepsilon_1 + \varepsilon_2)\sum_\alpha (J^{\bar{a}}, J^\alpha)\partial_z\psi^*_\alpha(z) \\
&\qquad - \varepsilon_1 \sum_{\alpha,\beta,b} c^{\alpha b}_\beta c^{\beta\bar{a}}_b \partial_z\psi^*_\alpha(z), \\
[d_{\mathrm{st}}, \psi^*_\alpha(z)]_+ &= -\frac{1}{2}\sum_{\beta,\gamma} c^{\beta\gamma}_\alpha \psi^*_\beta(z)\psi^*_\gamma(z).
\end{aligned}$$

Definition B.1.8. — We consider an **A**-span of monomials of the form (B.1.2) replacing $\widehat{J}$ by $\widetilde{J}$. We define the differentials d_{st}, $\widetilde{\chi}$ by (B.1.7). We get a double complex $C^\bullet_{\mathbf{A}}(\mathfrak{g})_0$ defined over **A**. Its total cohomology group $H^\bullet_{\mathbf{A}}(\mathfrak{g})$ is a vertex superalgebra defined over **A**.

The argument in the proof of [**30**, Th. 15.1.9] goes over **A**, and we get

(B.1.9) $$H^i_{\mathbf{A}}(\mathfrak{g}) = 0 \quad \text{for } i \neq 0.$$

We have

(B.1.10) $$H^0_{\mathbf{A}}(\mathfrak{g}) \otimes_{\mathbf{A}} \mathbf{F} \cong H^0_{\mathbf{F}}(\mathfrak{g}),$$

as the localization is an exact functor. Here $H^0_{\mathbf{F}}(\mathfrak{g})$ is the cohomology group of the complex $C^\bullet_{\mathbf{A}}(\mathfrak{g})_0 \otimes_{\mathbf{A}} \mathbf{F}$. It is isomorphic to $\mathscr{W}_k(\mathfrak{g}) \otimes_{\mathbb{C}(k)} \mathbf{F}$ as $\varepsilon_1 \neq 0$ in $\mathbf{F}$, where $k = -h^\vee - \varepsilon_2/\varepsilon_1$ as before.

Proposition B.1.11. — *$H^0_{\mathbf{A}}(\mathfrak{g})$ is free over* $\mathbf{A}$.

Proof. — Note that the complex $C^\bullet_{\mathbf{A}}(\mathfrak{g})_0$ is a direct sum of its homogeneous components with respect to the $\mathbb{Z}$-gradation. Each component forms a subcomplex and is free of finite rank over $\mathbf{A}$. Hence results in the homological algebra can be applied. Since only the 0^{th} cohomology survives, a component M of $H^0_{\mathbf{A}}(\mathfrak{g})$ is quasi-isomorphic to a complex of projective modules $P^\bullet$ with $P^i = 0$ for $i < 0$. Then we compute $\operatorname{Ext}^\bullet_{\mathbf{A}}(M, N)$ via $P^\bullet$ to deduce $\operatorname{Ext}^{>0}_{\mathbf{A}}(M, N) = 0$ for any N. Therefore M is projective. Since $\mathbf{A}$ is a polynomial ring, $H^0_{\mathbf{A}}(\mathfrak{g})$ is free. □

Thus $H^0_{\mathbf{A}}(\mathfrak{g})$ is an $\mathbf{A}$-form of the $\mathscr{W}$-algebra.

Definition B.1.12. — We denote $H^0_{\mathbf{A}}(\mathfrak{g})$ by $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$. It is called an $\mathbf{A}$*-form of the $\mathscr{W}$-algebra.*

Let us introduce a new degree, which corresponds to the half of the (perverse) cohomological degree in the geometric side. Let us denote it by '${}^{\mathrm{c}}\!\deg$'. We set ${}^{\mathrm{c}}\!\deg |0\rangle = 0$, ${}^{\mathrm{c}}\!\deg \varepsilon_1 = {}^{\mathrm{c}}\!\deg \varepsilon_2 = 1$. The degree of operators $\widehat{J^{\bar{a}}}(z)$ and $\psi^*_\alpha(z)$ is the first component of the bidegree. Then we put ${}^{\mathrm{c}}\!\deg \widetilde{J^{\bar{a}}}(z) = {}^{\mathrm{c}}\!\deg \widehat{J^{\bar{a}}}(z) + 1$ by (B.1.6). For example, $\widetilde{P}^i_m$ in §6.3 is a Fourier mode of $\widetilde{J^{\bar{a}}}(z)$ for $J^{\bar{a}} = h^i$. Therefore ${}^{\mathrm{c}}\!\deg \widetilde{P}^i_m = 1$.

From the Definition (B.1.7) we see that both $\widetilde{\chi}$ and d_{st} have degree 0. Therefore this degree descends to the cohomology group $H^0_{\mathbf{A}}(\mathfrak{g}) = \mathscr{W}_{\mathbf{A}}(\mathfrak{g})$. Hence $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ is a graded $\mathbf{A}$-module, where $\mathbf{A} = \mathbb{C}[\varepsilon_1, \varepsilon_2]$ is graded in the same way.

Be warned that ${}^{\mathrm{c}}\!\deg$ is not a $\mathbb{Z}$-grading of the vertex algebra in the sense of [**30**, §1.3.1]. All Fourier modes of vertex operators $Y(A, z)$, say $\widetilde{J^{\bar{a}}}(z)$, have the same degree, which is equal to the degree of the corresponding states $A = Y(A, z)|0\rangle|_{z=0}$. The translation operator T is of degree 0.

B.2. Generators $\widetilde{W}^{(\kappa)}_n$

The $\mathscr{W}$-algebra $\mathscr{W}_k(\mathfrak{g})$ is generated by certain elements W_κ ($\kappa = 1, \dots, \ell$) in the sense of the reconstruction theorem. (See [**30**, 15.1.9].) Moreover the subspace spanned by W_κ generates a PBW basis of $\mathscr{W}_k(\mathfrak{g})$. (See [**2**, §3.6 and Prop. 4.12.1] for the meaning of this statement.)

We briefly recall the definition of W_κ and see that their simple modifications live in our integral form and generate a PBW base of $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$. Let us change notation from W_κ to $W^{(\kappa)}$ in order to avoid a possible conflict with Fourier modes.

We have a regular nilpotent element p_- in $\mathfrak{n}_-$ so that χ is given by $(p_-, \bullet) = \chi(\bullet)$. (See [**30**, 15.2.9].) Let $\mathfrak{a}_-$ be the kernel of $\operatorname{ad} p_-$. It is a maximal abelian Lie subalgebra of $\mathfrak{g}$.

The cohomology H^i of the complex $C_k^\bullet(\mathfrak{g})_0$ with respect to χ vanishes for $i \neq 0$ and H^0 is equal to $V(\mathfrak{a}_-)$, the vertex algebra associated with $\mathfrak{a}_-$. It is a commutative vertex algebra, and isomorphic to the symmetric algebra $\operatorname{Sym}(\mathfrak{a}_- \otimes t^{-1}\mathbb{C}[t^{-1}])$ of $\mathfrak{a}_- \otimes t^{-1}\mathbb{C}[t^{-1}]$. Therefore a basis of $\mathfrak{a}_-$ gives a PBW base of $V(\mathfrak{a}_-)$.

There is a standard choice of a base of $\mathfrak{a}_-$. We take an $\mathfrak{sl}_2$-triple $\{p_+, p_0, p_-\}$ for p_-, and decompose $\mathfrak{g}$ into a direct sum of $(2d_\kappa + 1)$-dimensional representations R_κ ($\kappa = 1, \dots, \ell$). We choose a decomposition for $\mathfrak{g} = D_\ell$ with ℓ even, $\kappa = \ell/2$, $\ell/2+1$. We then choose a lowest weight vector $p_-^{(\kappa)}$ in R_κ. Then $\{p_-^{(\kappa)}\}_{\kappa=1,\dots,\ell}$ is a base of $\mathfrak{a}_-$. The vectors $p_-^{(\kappa)}$ are unique up to constant multiple, and we fix them hereafter. In fact, our geometric consideration of the $\mathscr{W}$-algebra will give us a canonical choice of $p_-^{(\kappa)}$ for $\kappa = \ell$, at least up to sign. See several paragraphs after Theorem 8.3.3.

The same is true over $\mathbf{A}$. The cohomology of $C_{\mathbf{A}}^\bullet(\mathfrak{g})_0$ with respect to χ vanishes except the degree 0, and H^0 is equal to $V(\mathfrak{a}_-) \otimes_{\mathbb{C}} \mathbf{A}$. The PBW base is its $\mathbf{A}$-basis.

Let ${}^0\widetilde{W}^{(\kappa)}(z)$ be the linear combination of $\widetilde{J}^{\bar{a}}(z)$ corresponding to $p_-^{(\kappa)}$, and let ${}^0\widetilde{W}^{(\kappa)}_{(-1)}$ be its constant part. Then ${}^0\widetilde{W}^{(\kappa)}_{(-1)}|0\rangle$ is contained in the kernel of $\widetilde{\chi}$. We construct a cocycle $\widetilde{W}^{(\kappa)}$ with respect to $d = d_{\mathrm{st}} + \widetilde{\chi}$ which is the main term ${}^0\widetilde{W}^{(\kappa)}_{(-1)}|0\rangle$ of bidegree $(d_\kappa, -d_\kappa)$ plus a sum of terms of bidegree $(p, -p)$ with $0 \le p < d_\kappa$, as we mentioned above. It is unique up to an element in $\operatorname{Ker} \widetilde{\chi}$ of a lower degree. We fix $\widetilde{W}^{(\kappa)}$ hereafter. We write

(B.2.1)
$$Y(\widetilde{W}^{(\kappa)}, z) = \sum_{n\in\mathbb{Z}} \widetilde{W}^{(\kappa)}_n z^{-n-d_\kappa-1}.$$

Let us check that ${}^{\mathrm{c}}\!\deg \widetilde{W}^{(\kappa)} = d_\kappa + 1$. Since d_{st} and $\widetilde{\chi}$ preserve ${}^{\mathrm{c}}\!\deg$, we have ${}^{\mathrm{c}}\!\deg \widetilde{W}^{(\kappa)} = {}^{\mathrm{c}}\!\deg {}^0\widetilde{W}^{(\kappa)}|0\rangle$. (Remember that we modify χ to $\widetilde{\chi}$ so that this is achieved.) Now the latter does not contain $\psi_\alpha^*(z)$, its degree is equal to the first component of the bidegree plus 1, i.e., $d_\kappa + 1$. Thus ${}^{\mathrm{c}}\!\deg \widetilde{W}^{(\kappa)} = d_\kappa + 1$. This is what we want from a geometry side.

B.3. Grading vs filtration

Let us make the relation between $\mathscr{W}_k(\mathfrak{g})$ and $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ more precise so that we could easily transfer computation in the literature to our setting.

Recall that the complexes (B.1.3) and (B.1.7) become the same if we put $\varepsilon_1 = (k + h^\vee)^{-1}$, $\varepsilon_2 = -1$ and identify $\widetilde{\chi}$ (resp. $\widetilde{J}^{\bar{a}}(z)$) with χ/ε_1 (resp. $\varepsilon_1 \widehat{J}^{\bar{a}}(z)$). As $H^{>0}_{\mathbf{A}}(\mathfrak{g}) = 0$ and $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ is free, the Künneth spectral sequence degenerate at E_2, and hence the specialization commutes with the cohomology. In particular, the homomorphism $\widehat{J}^{\bar{a}}(z) \mapsto \widetilde{J}^{\bar{a}}(z)/\varepsilon_1$ induces an isomorphism

(B.3.1)
$$\mathscr{W}_k(\mathfrak{g}) \xrightarrow{\cong} \mathscr{W}_{\mathbf{A}}(\mathfrak{g}) \otimes \mathbf{A}/(\varepsilon_1 - (k + h^\vee)^{-1}, \varepsilon_2 + 1).$$

Under this isomorphism standard generators $W^{(\kappa)}_n$ and our $\widetilde{W}^{(\kappa)}_n$ are related by

(B.3.2)
$$\text{our } \widetilde{W}^{(\kappa)}_n = \varepsilon_1^{d_\kappa+1} \text{ standard } W^{(\kappa)}_n,$$

as they are defined in the same way.

From this consideration, we can recover $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})\otimes_{\mathbf{A}}\mathbf{B}_1$ with $\mathbf{B}_1 = \mathbb{C}[\varepsilon_1] = \mathbf{A}/(\varepsilon_2+1)$ from $\mathscr{W}_k(\mathfrak{g})$ as follows. Let us consider k as a variable and understand that $\mathscr{W}_k(\mathfrak{g})$ is a vertex algebra defined over $\mathbb{C}(k)$. We identify $\mathbb{C}(k) = \mathbb{C}(\varepsilon_1)$ via $\varepsilon_1 = (k+h^\vee)^{-1}$. Then $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})\otimes_{\mathbf{A}}\mathbf{B}_1\otimes_{\mathbf{B}_1}\mathbb{C}(k)$ is isomorphic to $\mathscr{W}_k(\mathfrak{g})$, the cohomology of the complex over $\mathbb{C}(k)$ by the Künneth spectral sequence as above. Then we have an embedding $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})\otimes_{\mathbf{A}}\mathbf{B}_1 \to \mathscr{W}_k(\mathfrak{g})$, and the image is the $\mathbf{B}_1$-submodule generated by $\varepsilon_1^{d_\kappa+1}W_n^{(\kappa)}$. We denote $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})\otimes_{\mathbf{A}}\mathbf{B}_1$ by $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})$ hereafter.

Note further that the entire $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ can be recovered from $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})$ as follows. Since $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ is graded by ${}^{\mathrm{c}}\mathrm{deg}$, we have an induced filtration $0 = F_{-1}\subset F_0\subset F_1\subset\cdots$ on $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})$ such that $\varepsilon_1 F_p \subset F_{p+1}$. Then we can recover $\mathscr{W}_{\mathbf{A}}(g)$ as the associated Rees algebra:

$$\mathscr{W}_{\mathbf{A}}(\mathfrak{g}) = \bigoplus_p \varepsilon_2^p F_p. \tag{B.3.3}$$

In fact, we have a natural surjective homomorphism from the left hand side to the right, and it is also injective as $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ is torsion free over $\mathbf{B}_2 = \mathbb{C}[\varepsilon_2]$. Note also the specialization at $\varepsilon_2 = 0$ can be also recovered as the associated graded of the filtration.

The filtration $F_\bullet$ on $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})$ can be defined directly. From its definition, we assign ${}^{\mathrm{c}}\mathrm{deg}(\varepsilon_1^{d_\kappa+1}W_n^{(\kappa)}) = d_\kappa+1$ and ${}^{\mathrm{c}}\mathrm{deg}\,\varepsilon_1 = 1$. This gives us the filtration on $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})$.

Let us explain how the formula for $W_n^{(1)}$ given in [**30**, (15.3.1)] can be understood in our framework, for example. The field $T(z)$ written there is already divided by $k+h^\vee$ so that its Fourier modes gives Virasoro generators L_n. Therefore $W_n^{(1)} = (k+h^\vee)L_n$ and hence $\widetilde{W}_n^{(1)} = \varepsilon_1^2(k+h^\vee)L_n = -\varepsilon_1\varepsilon_2 L_n$. This is compatible (up to sign) with modified Virasoro generators in §6.4, as $\widetilde{L}_n^{(i)} = \varepsilon_1\varepsilon_2 L_n^i$.

B.4. Specialization at $\varepsilon_1 = 0$

In this subsection, we study the specialization at $\varepsilon_1 = 0$. This is the classical limit of the $\mathscr{W}$-algebra, but it also contains ε_2 as a parameter. The relevant computation can be found in [**30**, §15.4.1∼6].

Let us set $\varepsilon_1 = 0$ in (B.1.7). Since $\widetilde{J}^{\bar a}(z)$ and $\widetilde{J}^{\bar b}(z)$ commute at $\varepsilon_1 = 0$ (see (B.1.1)), the complex is identified with polynomials in the commuting variables $\widetilde{J}_n^{\bar a}$ $(n<0)$ and anti-commuting variables $\psi^*_{\alpha,m}$ $(m\le 0)$. Therefore

$$C^\bullet_{\mathbf{A}}(\mathfrak{g})_0\otimes_{\mathbf{A}}\mathbf{B}_2 \cong \operatorname{Sym}\mathfrak{b}_-((t))/b_-[[t]]\otimes_{\mathbb{C}}\textstyle\bigwedge^\bullet\mathfrak{n}_+[[t]]^*\otimes\mathbf{B}_2, \tag{B.4.1}$$

where $\mathbf{B}_2 = \mathbb{C}[\varepsilon_2] = \mathbf{A}/\varepsilon_1\mathbf{A}$. The differential is specialized as

$$\begin{aligned}
&[\widetilde{\chi}, \widetilde{J}^{\bar{a}}(z)] = \sum_{i\in I}\sum_{\beta\in\Delta_+} c^{\bar{a}\beta}_{\alpha_i}\psi^*_\beta(z),\\
&[\widetilde{\chi}, \psi^*_\alpha(z)]_+ = 0,\\
&[d_{\mathrm{st}}, \widetilde{J}^{\bar{a}}(z)] = \sum_{\bar{b},\alpha} c^{\alpha\bar{a}}_{\bar{b}}\widetilde{J}^{\bar{b}}(z)\psi^*_\alpha(z) - \varepsilon_2\sum_\alpha (J^{\bar{a}}, J^\alpha)\partial_z\psi^*_\alpha(z),\\
&[d_{\mathrm{st}}, \psi^*_\alpha(z)]_+ = -\frac{1}{2}\sum_{\beta,\gamma} c^{\beta\gamma}_\alpha \psi^*_\beta(z)\psi^*_\gamma(z),
\end{aligned} \tag{B.4.2}$$

where power series in z contain only terms with non-negative degrees in z. This is exactly the same complex as in [**30**, §15.4.2], if we set $\varepsilon_2 = -1$. It is the complex at the classical limit $k \to \infty$.

By [**30**, Cor. 15.4.6], the cohomology group $H^i_{\varepsilon_1=0}(\mathfrak{g})$ of this complex (at $\varepsilon_2 = -1$) vanishes for $i \neq 0$, and $H^0_{\varepsilon_1=0}(\mathfrak{g})$ is isomorphic to the ring of functions on $\mathfrak{a}_+[[t]]$, where $\mathfrak{a}_+$ is the kernel of $\operatorname{ad} p_+$. Here p_+ is as in the previous subsection.

In fact, $\mathfrak{a}_+[[t]]$ is obtained as the quotient of the space of connections of the form

$$\nabla = \partial_t + p_- + A(t), \qquad A(t) \in \mathfrak{b}_+[[t]], \tag{B.4.3}$$

modulo the action of the gauge transformations $N_+[[t]]$. This is the space $\mathrm{Op}_G(D)$ of G-opers on the formal disk $D = \operatorname{Spec}\mathbb{C}[[t]]$. There exists a unique gauge transformation in $N_+[[t]]$ so that ∇ is transformed into the same form with $A(t) \in \mathfrak{a}_+[[t]]$.

It is easy to put ε_2 in this picture. The term with ε_2 corresponds to the differential of the gauge transformation. Therefore the cohomology of our complex is the ring of functions on the quotient space of $(-\varepsilon_2)$-connections

$$\nabla = -\varepsilon_2\partial_t + p_- + A(t) \tag{B.4.4}$$

modulo $N_+[[t]]$. It is the space of $(-\varepsilon_2)$-opers on D. This notion appears for example in [**7**, §5.2]. See also §B.5 below.

We have a structure of a vertex Poisson algebra on $H^0_{\varepsilon_1=0}(\mathfrak{g})$ by [**30**, 16.2.4]. It is defined by renormalizing the polar part of vertex operators

$$Y_-(A, z) = \left.\frac{1}{\varepsilon_1}Y_-(\widetilde{A}, z)\right|_{\varepsilon_1=0}. \tag{B.4.5}$$

We can further make $\varepsilon_2 = 0$. Then we get $(p_- + \mathfrak{b}_+[[t]])/N_+[[t]]$. This space is also equal to $\mathfrak{a}_+[[t]]$. The proof in [**30**, 15.4.5] works also at $\varepsilon_2 = 0$. In fact, the result is a consequence of a classical result of Kostant: $(p_- + \mathfrak{b}_+)/N_+ \cong \mathfrak{a}_+$. See [**7**, §5.4] for further detail. Therefore the cohomology group $H^i_{\varepsilon_1,\varepsilon_2=0}(\mathfrak{g})$ of the complex at $\varepsilon_1 = \varepsilon_2 = 0$ vanishes for $i \neq 0$, and $H^0_{\varepsilon_1,\varepsilon_2=0}(\mathfrak{g}) \cong V(\mathfrak{a}_-)$.

The argument for (B.3.1) works also here, i.e., the specialization commutes with cohomology group. We have

$$\begin{aligned}
&\mathscr{W}_{\mathbf{A}}(\mathfrak{g}) \otimes_{\mathbf{A}} \mathbf{B}_2 \cong H^0_{\varepsilon_1=0}(\mathfrak{g}),\\
&\mathscr{W}_{\mathbf{A}}(\mathfrak{g}) \otimes_{\mathbf{A}} \mathbb{C} \cong H^0_{\varepsilon_1,\varepsilon_2=0}(\mathfrak{g}) \cong V(\mathfrak{a}_-),
\end{aligned} \tag{B.4.6}$$

where $\mathbf{B}_2 = \mathbf{A}/\varepsilon_1\mathbf{A}$, $\mathbb{C} = \mathbf{A}/(\varepsilon_1, \varepsilon_2)$.

B.5. The opposite spectral sequence

The embedding of the $\mathscr{W}$-algebra into the Heisenberg algebra is given by considering the 'opposite' spectral sequence associated with the double complex $C_k^\bullet(\mathfrak{g})_0$, where the E_1-term is the cohomology with respect to d_{st}. The detail is explained in [**30**, §15.4.10], and we give a brief review in order to see that the embedding is compatible with integral forms.

Let $\widetilde{H}_k^i(\mathfrak{g})$ be the i^{th} cohomology of the complex $C_k^\bullet(\mathfrak{g})_0$ with respect to d_{st}. This notation is taken from [**30**] and has nothing to do with our notation for elements in the integral form. Let $\widehat{h}^i(z)$ denote $\widehat{J}^{\bar{a}}(z)$ for $\bar{a} = i \in I$. Then we have

$$[d_{\mathrm{st}}, \widehat{h}^i(z)] = 0, \qquad [d_{\mathrm{st}}, \psi^*_{\alpha_i}(z)]_+ = 0 \tag{B.5.1}$$

by (B.1.3). Therefore we have linear maps $\mathbb{C}[\widehat{h}^i_n]_{i\in I, n<0}|0\rangle \to \widetilde{H}^0_k(\mathfrak{g})$, $\bigoplus_i \mathbb{C}[\widehat{h}^j_n]_{j\in I, n<0}\psi^*_{\alpha_i,0}|0\rangle \to \widetilde{H}^1_k(\mathfrak{g})$ respectively. In fact, they live in the uppermost row as $\operatorname{bideg}\widehat{h}^i(z) = (0,0)$, $\operatorname{bideg}\psi^*_{\alpha_i}(z) = (1,0)$. Then by considering the limit $k \to \infty$, one can see that both cohomology groups are exactly the same as the above spaces respectively if k is generic. Moreover one can identify $\widetilde{H}^0_k(\mathfrak{g})$ with the Heisenberg vertex algebra associated with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. This is because $\widehat{h}^i_n$ satisfies the commutation relation (B.1.5). Modified generators $\overline{h}^i_n = \widehat{h}^i_n/\sqrt{k+h^\vee}$ satisfy the usual commutation rule

$$[\overline{h}^i_m, \overline{h}^j_n] = m\delta_{m,-n}(\alpha_i, \alpha_j). \tag{B.5.2}$$

And $\widetilde{H}^1_k(\mathfrak{g})$ is its module. It is a direct sum of $(\#I)$ Fock modules. The highest weights are given by the formula

$$\overline{h}^i_0 \psi^*_{\alpha_j,0}|0\rangle = -\frac{(\alpha_i, \alpha_j)}{\sqrt{k+h^\vee}}\psi^*_{\alpha_j,0}|0\rangle. \tag{B.5.3}$$

Another differential χ induces a homomorphism $\widetilde{H}^0_k(\mathfrak{g}) \to \widetilde{H}^1_k(\mathfrak{g})$. Since $\widetilde{H}^1_k(\mathfrak{g})$ lives only at bidegree $(1,0)$, we have $\mathscr{W}_k(\mathfrak{g}) = H^0_k(\mathfrak{g}) \cong \operatorname{Ker}\chi$ for generic k.

Moreover χ is the sum of the residue of the field $\psi^*_{\alpha_i}(z)$, which is given by the vertex operator in terms of the Heisenberg algebra:

$$\psi^*_{\alpha_i}(z) = V_{-\alpha_i/\sqrt{k+h^\vee}}(z) \tag{B.5.4}$$

where

$$V_\lambda(z) = S_\lambda z^{\lambda b_0} \exp\left(-\lambda \sum_{n<0} \frac{b_n}{n} z^{-n}\right) \exp\left(-\lambda \sum_{n>0} \frac{b_n}{n} z^{-n}\right). \tag{B.5.5}$$

This formula is given in [**30**, (5.2.8)]. The operator S_λ sends the highest weight vector $|0\rangle$ to the highest weight vector $|\lambda\rangle$ and commutes with all b_n, $n \neq 0$. And λb_n is replaced by

$$\lambda b_n = -\frac{\overline{h}^i_n}{\sqrt{k+h^\vee}} = -\frac{\widehat{h}^i_n}{k+h^\vee}, \tag{B.5.6}$$

and S_λ sends $|0\rangle$ to $\psi^*_{\alpha_i,0}|0\rangle$ here.

Now we consider the cohomology group $\widetilde{H}^i_{\mathbf{A}}(\mathfrak{g})$ over $\mathbf{A}$. The 0th cohomology $\widetilde{H}^0_{\mathbf{A}}(\mathfrak{g}) = \operatorname{Ker} d_{\mathrm{st}}$ is a direct sum of $\mathbf{A}[\widetilde{P}^i_n]_{i\in I,n<0}$ with bidegree $(0,0)$ and the other parts with bidegree $(p,-p)$ with $p>0$. Here we put $\widetilde{P}^i_n = \varepsilon_1\widehat{h}^i_n$ so that they satisfy the commutation relation (6.3.12). Since d_{st} on $(p,-p)$ part is injective for generic $(\varepsilon_1,\varepsilon_2)$ by the above computation, it is injective as an $\mathbf{A}$-homomorphism. Therefore we have

Lemma B.5.7. —

$$\widetilde{H}^0_{\mathbf{A}}(\mathfrak{g}) = \mathbf{A}[\widetilde{P}^i_n]_{i\in I,n<0}|0\rangle. \tag{B.5.8}$$

This is an $\mathbf{A}$-form of the Heisenberg vertex algebra, denoted by $\mathfrak{Heis}_{\mathbf{A}}(\mathfrak{h})$ in §6.3.

We have an induced homomorphism $\mathscr{W}_{\mathbf{A}}(\mathfrak{g}) = H^0_{\mathbf{A}}(\mathfrak{g}) \to \widetilde{H}^0_{\mathbf{A}}(\mathfrak{g})$, taking the bidegree $(0,0)$ component. It is injective as $\operatorname{Ker} d_{\mathrm{st}} = 0$ on $(p,-p)$ with $p>0$. Therefore we can consider $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ as an $\mathbf{A}$-submodule of $\widetilde{H}^0_{\mathbf{A}}(\mathfrak{g})$. We have an induced homomorphism $\widetilde{\chi}\colon \widetilde{H}^0_{\mathbf{A}}(\mathfrak{g}) \to \widetilde{H}^1_{\mathbf{A}}(\mathfrak{g})$ and the double complex tells us that $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ is contained in $\operatorname{Ker}\widetilde{\chi}$.

When we compare the embedding with the usual one $\mathscr{W}_k(\mathfrak{g}) \to \widetilde{H}^0_k(\mathfrak{g})$ in the literature via the identification of $\mathscr{W}_k(\mathfrak{g})$ and $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ in §B.3, we use the relations $\widetilde{P}^i_n = \varepsilon_1\widehat{h}^i_n$ as before.

For example, consider $\widetilde{W}^{(1)}_n$ for $\mathfrak{g} = \mathfrak{sl}_2$. It is given by (6.4.9) up to sign, and is contained in $\widetilde{H}^0_{\mathbf{A}}(\mathfrak{g})$. The formula follows from the computation in the literature, say [**30**, §15.4.14], with the rule for the change of generators above.

Let us look at $\widetilde{H}^1_{\mathbf{A}}(\mathfrak{g})$ more closely. From the definition, we have

$$\begin{aligned} C^{1,-1}_{\mathbf{A}}(\mathfrak{g})_0 &= \bigoplus_{i,m<0} \mathbf{A}[\widetilde{P}^j_n]_{j\in I,n<0}\widetilde{f}_{i,m}|0\rangle, \\ C^{1,0}_{\mathbf{A}}(\mathfrak{g})_0 &= \bigoplus_{i,m\le 0} \mathbf{A}[\widetilde{P}^j_n]_{j\in I,n<0}\psi^*_{\alpha_i,m}|0\rangle, \end{aligned} \tag{B.5.9}$$

where $\widetilde{f}_{i,m}$ is the Fourier mode of $\widetilde{J}^{\bar{a}}(z)$ corresponding to the basis element $f_i = f^{\alpha_i}$. The differential $d_{\mathrm{st}}\colon C^{1,-1}_{\mathbf{A}}(\mathfrak{g})_0 \to C^{1,0}_{\mathbf{A}}(\mathfrak{g})_0$ can be calculated from (B.1.7), in particular we have

$$[d_{\mathrm{st}}, \widetilde{P}^i(z)] = 0, \tag{B.5.10}$$

$$[d_{\mathrm{st}}, \widetilde{f}_i(z)] = \frac{2}{(\alpha_i,\alpha_i)}\left(:\widetilde{P}^i(z)\psi^*_{\alpha_i}(z): - \varepsilon_2\partial_z\psi^*_{\alpha_i}(z)\right). \tag{B.5.11}$$

See the formula in the middle of [**30**, p.261]. From the second formula we have

$$-\varepsilon_2\partial_z\psi^*_{\alpha_i}(z) = :\widetilde{P}^i(z)\psi^*_{\alpha_i}(z): \tag{B.5.12}$$

modulo d_{st}-exact term. If ε_2 would be invertible, we could replace $\psi^*_{\alpha_i,m}|0\rangle$ with $m \neq 0$ in (B.5.9) by an element in $\mathbf{A}[\widetilde{P}^i_n]\psi^*_{\alpha_i,0}|0\rangle$ so that $\widetilde{H}^1_{\mathbf{A}}(\mathfrak{g})$ is isomorphic to $\bigoplus_i \mathbf{A}[\widetilde{P}^j_n]\psi^*_{\alpha_i,0}|0\rangle$. As ε_2 is not invertible in $\mathbf{A}$, this cannot be true.

From this consideration, we set $\varepsilon_2 = -1$ in the double complex (B.1.7), and consider it over $\mathbf{B}_1 = \mathbb{Q}[\varepsilon_1]$, as in §B.3. We denote it by $C^\bullet_{\mathbf{B}_1}(\mathfrak{g})_0$. This is not any loss of the information for our purpose, as $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ can be recovered from $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})$ together with its natural filtration, as explained in §B.3.

However, the higher cohomology groups $\widetilde{H}^{>0}_{\mathbf{A}}(\mathfrak{g})$ may not vanish nor be free. Hence the cohomology group $\widetilde{H}^\bullet_{\mathbf{B}_1}(\mathfrak{g})$ of $C^\bullet_{\mathbf{B}_1}(\mathfrak{g})_0$ with respect to d_{st} may be different from $\widetilde{H}^\bullet_{\mathbf{A}}(\mathfrak{g}) \otimes_{\mathbf{A}} \mathbf{B}_1$. We will see that $\widetilde{H}^\bullet_{\mathbf{B}_1}(\mathfrak{g})$ behaves better than $\widetilde{H}^\bullet_{\mathbf{A}}(\mathfrak{g})$ at $\varepsilon_2 = 0$ below.

Let us study first two terms of $\widetilde{H}^\bullet_{\mathbf{B}_1}(\mathfrak{g})$. We have $\widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g}) \cong \mathbf{B}_1[\widetilde{P}^i_n]_{i\in I, n<0}|0\rangle$ by the same argument as in (B.5.8). Let $\widetilde{H}^{1,0}_{\mathbf{B}_1}(\mathfrak{g})$ be the $(1,0)$ part of the cohomology. We do not know $\widetilde{H}^1_{\mathbf{B}_1}(\mathfrak{g}) \cong \widetilde{H}^{1,0}_{\mathbf{B}_1}(\mathfrak{g})$, but $\widetilde{\chi}$ maps $\widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g})$ to $\widetilde{H}^{1,0}_{\mathbf{B}_1}(\mathfrak{g})$ anyway. From the above argument we have a surjective homomorphism $\bigoplus \mathbf{B}_1[\widetilde{P}^j_n]\psi^*_{\alpha_i,0}|0\rangle \to \widetilde{H}^{1,0}_{\mathbf{B}_1}(\mathfrak{g})$. It is an isomorphism for generic ε_1, in other words over $\mathbb{C}(\varepsilon_1)$. Therefore it must be injective also over $\mathbf{B}_1$. We thus get

Lemma B.5.13. —

$$\begin{aligned} &\widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g}) \cong \mathbf{B}_1[\widetilde{P}^i_n]_{i\in I, n<0}|0\rangle, \\ &\widetilde{H}^{1,0}_{\mathbf{B}_1}(\mathfrak{g}) \cong \bigoplus_i \mathbf{B}_1[\widetilde{P}^j_n]_{j\in I, n<0}\psi^*_{\alpha_i,0}|0\rangle. \end{aligned} \tag{B.5.14}$$

The substitution $\varepsilon_2 = -1$ makes the vertex operator (B.5.5) well-defined: We replace λb_n by (B.5.6), hence

$$\lambda b_n = -\widetilde{P}^i_n. \tag{B.5.15}$$

The vertex operator is a homomorphism between $\mathbf{B}_1$-modules.

Now we let $\varepsilon_1 = 0$. We have the Künneth theorem

$$0 \to \widetilde{H}^n_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C} \to H^n(C^\bullet_{\mathbf{B}_1}(\mathfrak{g})_0 \otimes_{\mathbf{B}_1} \mathbb{C}) \to \operatorname{Tor}^{\mathbf{B}_1}_1(\widetilde{H}^{n+1}_{\mathbf{B}_1}(\mathfrak{g}), \mathbb{C}) \to 0, \tag{B.5.16}$$

where $\mathbb{C} = \mathbf{B}_1/\varepsilon_1\mathbf{B}_1$. The middle term is the cohomology at the classical limit, and is known (see [**30**, §15.4.8]). In particular, we get

$$\begin{aligned} \mathbb{C}[\widetilde{P}^i_n]|0\rangle = \widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C} &\cong H^0(C^\bullet_{\mathbf{B}_1}(\mathfrak{g})_0 \otimes_{\mathbf{B}_1} \mathbb{C}), \\ \bigoplus \mathbb{C}[\widetilde{P}^j_n]\psi^*_{\alpha_i,0}|0\rangle = \widetilde{H}^{1,0}_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C} &\cong H^1(C^{1,\bullet}_{\mathbf{B}_1}(\mathfrak{g})_0 \otimes_{\mathbf{B}_1} \mathbb{C}), \\ \widetilde{H}^{p+1,-p}_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C} = H^1(C^{p+1,\bullet}_{\mathbf{B}_1}&(\mathfrak{g})_0 \otimes_{\mathbf{B}_1} \mathbb{C}) = 0 \quad \text{for } p > 0. \end{aligned} \tag{B.5.17}$$

Next we study $\widetilde{\chi}$ at $\varepsilon_1 = 0$. Recall that $\widetilde{\chi} = \chi/\varepsilon_1$, so we need to divide $\int V_\lambda(z)$ in (B.5.5) by ε_1. We see that the induced operator

$$\text{(B.5.18)}\qquad \widetilde{\chi}|_{\substack{\varepsilon_1=0\\ \varepsilon_2=-1}} : \widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C} = \mathbb{C}[\widetilde{P}^i_n]|0\rangle \\ \to \widetilde{H}^1_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C} = \bigoplus \mathbb{C}[\widetilde{P}^j_n]\psi^*_{\alpha_i,0}|0\rangle$$

is given by the formula

$$\text{(B.5.19)}\qquad \sum_i \sum_{j=1}^{\ell} (\alpha_i, \alpha_j) \sum_{m\le 0} \mathbf{V}_i[m] \frac{\partial}{\partial \widetilde{P}^j_{m-1}},$$

with

$$\text{(B.5.20)}\qquad \sum_{n\le 0} \mathbf{V}_i[n] z^{-n} = S_i \exp\left(\sum_{n<0} \frac{\widetilde{P}^i_n}{n} z^{-n} \right).$$

Here the operator S_i sends the highest weight vector $|0\rangle$ to $\psi^*_{\alpha_i,0}|0\rangle$. The point here is the commutation relation $[\widetilde{P}^i_m, \widetilde{P}^j_n] = m\varepsilon_1(\alpha_i,\alpha_j)\delta_{m,-n}$ at $\varepsilon_2 = -1$. This vanishes at $\varepsilon_1 = 0$, and hence only linear terms in the expansion of the second exponential in (B.5.5) survive.

This computation appears in the study of the classical limit of the $\mathscr{W}$-algebra [**29**, Chap. 8]. In particular, the followings were shown there:

- $\widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C}$ is isomorphic to the ring of functions on the space $\mathrm{MOp}_G(D)_{\mathrm{gen}}$ of generic Miura opers on the formal disk D.
- Each generic Miura oper can be uniquely transformed into the following form

$$\text{(B.5.21)}\qquad \nabla = \partial_t + p_- + \mathbf{u}(t), \qquad \mathbf{u}(t) \in \mathfrak{h}[[t]].$$

- The kernel of $\widetilde{\chi}|_{\substack{\varepsilon_1=0\\ \varepsilon_2=-1}}$ is isomorphic to the ring of functions on the space $\mathrm{Op}_G(D)$ of opers. The inclusion $\mathrm{Ker}(\widetilde{\chi}|_{\substack{\varepsilon_1=0\\ \varepsilon_2=-1}}) \to \widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C}$ is given by the forgetting morphism $\mathrm{MOp}_G(D)_{\mathrm{gen}} \to \mathrm{Op}_G(D)$.

We do not recall the definition of generic Miura opers here, as it is enough to consider the space of connections of the form (B.5.21). The morphism $\mathrm{MOp}_G(D)_{\mathrm{gen}} \to \mathrm{Op}_G(D)$ is given just by considering a connection in (B.5.21) as a G-oper. As we have already known that $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ at $\varepsilon_1 = 0$, $\varepsilon_2 = -1$ is the ring of functions on $\mathrm{Op}_G(D)$ in §B.4, we get

$$\text{(B.5.22)}\qquad \mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C} = \mathrm{Ker}(\widetilde{\chi}|_{\substack{\varepsilon_1=0\\ \varepsilon_2=-1}}).$$

Finally we study the filtration in the both sides of (B.5.22). The left hand side has a filtration as it comes from the specialization of the grading on $\mathscr{W}_{\mathbf{A}}(\mathfrak{g})$ at $\varepsilon_1 = 0$, $\varepsilon_2 = -1$. On the other hand, we have filtration on $\widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g})$ and $\widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C}$ given by ${}^{\mathrm{c}}\!\deg \widetilde{P}^i_n = 1$, as they are polynomial rings (see Lemma B.5.13 and (B.5.17).) Since $\widetilde{H}^0_{\mathbf{A}}(\mathfrak{g})$ is also free by Lemma B.5.7, the filtrations come from the specialization. We give an induced filtration on $\operatorname{Ker}(\widetilde{\chi}|_{\substack{\varepsilon_1=0\\ \varepsilon_2=-1}})$ as a subspace of $\widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g})\otimes_{\mathbf{B}_1}\mathbb{C}$. Then (B.5.22) respects the filtration as the inclusion $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})\to\widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g})$ does.

On the ring of functions on $\operatorname{Op}_G(D)$, the filtration can be understood by considering $(-\varepsilon_2)$-opers [**8**, §3.1.14] as follows. A filtration on an algebra can be identified with a graded flat $\mathbb{C}[\varepsilon_2]$-algebra with $\deg\varepsilon_2 = 1$. The latter is considered as the ring of functions on a flat affine scheme X over $\mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[\varepsilon_2]$ with a $\mathbb{G}_m$-action compatible with the action by homotheties on $\mathbb{A}^1$. The space of $(-\varepsilon_2)$-opers provides such a scheme, where the $\mathbb{G}_m$-action is given by $\nabla\mapsto\lambda\nabla$ for $\lambda\in\mathbb{G}_m$. More precisely, we need to compose it with a gauge transformation so that the form (B.4.3) is preserved. Since $(-\varepsilon_2)$-opers appear at the specialization at $\varepsilon_1 = 0$ in §B.4, our filtration is given in this way.

The action is induced from the action $\lambda\operatorname{Ad}(\lambda)$ on $\mathfrak{a}_+$ under $\operatorname{Op}_G(D)\cong\mathfrak{a}_+[[t]]$, where $\operatorname{Ad}(\lambda)$ is given by the SL_2 embedding associated with the nilpotent element p_-. It is known that the degrees of the $\mathbb{G}_m$-action on $\mathfrak{a}_+$ are given by $d_\kappa + 1$ ($\kappa = 1,\dots,\ell$), hence are the same as our '${}^c\deg$' by §B.2. This is another reason why we define the degree in that way.

We can define the $\mathbb{G}_m$-action on $\operatorname{MOp}_G(D)_{\mathrm{gen}}$ in the same way so that the morphism $\operatorname{MOp}_G(D)_{\mathrm{gen}}\to\operatorname{Op}_G(D)$ is $\mathbb{G}_m$-equivariant. Under $\operatorname{MOp}_G(D)_{\mathrm{gen}}\cong\mathfrak{h}[[t]]$, it is just homotheties on $\mathfrak{h}$. The corresponding filtration is the same as ours.

The homomorphism between the associated graded of $\operatorname{Ker}(\widetilde{\chi}|_{\substack{\varepsilon_1=0\\ \varepsilon_2=-1}})$ and $\widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g})\otimes_{\mathbf{B}_1}\mathbb{C}$ is induced by the morphism

(B.5.23)
$$\{\nabla = p_- + \mathbf{u}(t)\mid\mathbf{u}(t)\in\mathfrak{h}[[t]]\}\\ \to\{\nabla = p_- + A(t)\mid A(t)\in\mathfrak{b}_+[[t]]\}/N_+[[t]]$$

of 0-opers.

Let us write down the embedding of the $\mathscr{W}$-algebra into the Heisenberg algebra at $\varepsilon_1 = \varepsilon_2 = 0$ induced from the morphism (B.5.23) of 0-opers explicitly. It is given in [**29**, §3.3.4]. Let $F^{(\kappa)}\in S(\mathfrak{h})^W$ ($\kappa = 1,\dots,\ell$) be generators of degree $d_\kappa + 1$, corresponding to $p_-^{(\kappa)}$ in §B.2. We regard it as a polynomial in h^i, i.e., $F^{(\kappa)}(h^i) = F^{(\kappa)}(h^1,\dots,h^\ell)$. Then $\widetilde{W}^{(\kappa)}_n$ (at ε_1, $\varepsilon_2 = 0$) is given by the formula

(B.5.24)
$$F^{(\kappa)}\left(\sum_{n<0}\widetilde{P}^{(i)}_n z^{-n-1}\right) = \sum_{n<0}\widetilde{W}^{(\kappa)}_n z^{-n-d_\kappa-1}.$$

For example, we have

(B.5.25)
$$\widetilde{L}_n = -\frac{1}{4}\sum_{n<l<0}\widetilde{P}_l\widetilde{P}_{n-l}$$

for $\mathfrak{sl}_2$.

B.6. Kernel of the screening operator

Recall that we have a natural inclusion $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \subset \operatorname{Ker}(\widetilde{\chi}|_{\varepsilon_2=-1})$ from the construction. They coincide for generic ε_1. We prove a stronger result.

Theorem B.6.1. — *We have isomorphisms*

$$\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \cong \operatorname{Ker}(\widetilde{\chi}|_{\varepsilon_2=-1}), \tag{B.6.2}$$

$$\mathscr{W}_{\mathbf{A}}(\mathfrak{g}) \cong \bigcap_i \mathfrak{Vir}_{i,\mathbf{A}}|_{\varepsilon_1\to\varepsilon_1'} \otimes_{\mathbf{A}} \mathfrak{Heis}_{\mathbf{A}}(\alpha_i^\perp), \tag{B.6.3}$$

where $\mathfrak{Vir}_{i,\mathbf{A}}|_{\varepsilon_1\to\varepsilon_1'}$ *is the* $\mathbf{A}$*-form of the Virasoro algebra with* ε_1 *replaced by* $\varepsilon_1' = \frac{\varepsilon_1(\alpha_i,\alpha_i)}{2}$. *Moreover* (B.6.2) *preserves filtrations.*

Proof. — Let us first consider (B.6.2) and denote $\widetilde{\chi}$ at $\varepsilon_2 = -1$ also by $\widetilde{\chi}$ for brevity:

$$\widetilde{\chi}\colon \widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g}) \to \widetilde{H}^{1,0}_{\mathbf{B}_1}(\mathfrak{g}). \tag{B.6.4}$$

We know that both $\widetilde{H}^0_{\mathbf{B}_1}(\mathfrak{g})$ and $\widetilde{H}^{1,0}_{\mathbf{B}_1}(\mathfrak{g})$ are free over $\mathbf{B}_1$ (see Lemma B.5.13). We also know that their specialization is the cohomology group at $\varepsilon_1 = 0$, $\varepsilon_2 = -1$ (see (B.5.17)). Therefore we have an exact sequence

$$0 \to \operatorname{Ker}\widetilde{\chi} \otimes_{\mathbf{B}_1} \mathbb{C} \to \operatorname{Ker}(\widetilde{\chi}|_{\substack{\varepsilon_1=0\\ \varepsilon_2=-1}}) \to \operatorname{Tor}_1^{\mathbf{B}_1}(\operatorname{Cok}\widetilde{\chi}, \mathbb{C}) \to 0. \tag{B.6.5}$$

We have a homomorphism from $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C}$ to the first term $\operatorname{Ker}\widetilde{\chi} \otimes_{\mathbf{B}_1} \mathbb{C}$, and its composition to the middle term is an isomorphism by (B.5.22). Therefore we have

$$\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C} \cong \operatorname{Ker}\widetilde{\chi} \otimes_{\mathbf{B}_1} \mathbb{C} \cong \operatorname{Ker}(\widetilde{\chi}|_{\substack{\varepsilon_1=0\\ \varepsilon_2=-1}}). \tag{B.6.6}$$

Since (B.5.22) preserves the filtration, we have an induced isomorphism between the associated graded

$$\operatorname{gr}(\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C}) \cong \operatorname{gr}(\operatorname{Ker}\widetilde{\chi} \otimes_{\mathbf{B}_1} \mathbb{C}). \tag{B.6.7}$$

Let $0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots$ be the filtration on $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})$ as before. Then the filtration on $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C}$ is given by

$$0 \subset {}^{F_0}/_{\varepsilon_1\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})\cap F_0} \subset {}^{F_1}/_{\varepsilon_1\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})\cap F_1} \subset \cdots, \tag{B.6.8}$$

as $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C} \cong \mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})/\varepsilon_1\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})$. From the definition of F_p, we have $\varepsilon_1\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \cap F_p = \varepsilon_1 F_{p-1}$. Therefore

$$\operatorname{gr}(\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \otimes_{\mathbf{B}_1} \mathbb{C}) = \bigoplus_{p>0} {}^{F_p}/_{\varepsilon_1 F_{p-1}+F_{p-1}} \cong {}^{\operatorname{gr}\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})}/_{\varepsilon_1\operatorname{gr}\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g})}. \tag{B.6.9}$$

(Here we have used $\operatorname{gr} W/\varepsilon_1 \operatorname{gr} W = \bigoplus(F_p/F_{p-1})/\varepsilon_1(F_{p-1}/F_{p-2})$ as ε_1 shift the grading by 1). The same is true for $\operatorname{gr}(\operatorname{Ker}\widetilde{\chi} \otimes_{\mathbf{B}_1} \mathbb{C})$.

By graded Nakayama's lemma, we conclude $\operatorname{gr}\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \cong \operatorname{gr}\operatorname{Ker}\widetilde{\chi}$. Using it again, we get (B.6.2).

Next consider (B.6.3). Since both sides are Rees algebras of the corresponding vertex algebras at $\varepsilon_2 = -1$ with the induced filtration, it is enough to show that we have a filtration preserving isomorphism at $\varepsilon_2 = -1$:

$$\mathscr{W}_{\mathbf{B}_1}(\mathfrak{g}) \cong \bigcap_i \mathfrak{Vir}_{i,\mathbf{B}_1}|_{\varepsilon_1 \to \varepsilon_1'} \otimes_{\mathbf{B}_1} \mathfrak{Heis}_{\mathbf{B}_1}(\alpha_i^\perp), \tag{B.6.10}$$

where $\mathfrak{Vir}_{i,\mathbf{B}_1}$, $\mathfrak{Heis}_{\mathbf{B}_1}(\alpha_i^\perp)$ are defined in an obvious manner.

We use (B.6.2) $\mathscr{W}_{\mathbf{B}_1}(\mathfrak{sl}_2) = \mathfrak{Vir}_{\mathbf{B}_1} \cong \operatorname{Ker}(\widetilde{\chi}|_{\varepsilon_2=-1})$ for $\mathfrak{g} = \mathfrak{sl}_2$ and the observation that $\widetilde{\chi}$ is the sum of operators over $i \in I$, we see that the right hand side is $\operatorname{Ker}(\widetilde{\chi}|_{\varepsilon_2=-1})$. The substitution $\varepsilon_1 \to \varepsilon_1' = \frac{(\alpha_i,\alpha_i)\varepsilon_1}{2}$ is necessary, as the Heisenberg commutation (6.3.12) involves (α_i, α_j). Now we use (B.6.2) for the original $\mathfrak{g}$ and deduce (B.6.10). □

From this result, we extend the duality for the $\mathscr{W}$-algebra in [**30**, Prop. 15.4.16] from generic to arbitrary level.

Corollary B.6.11. — *Let ${}^L\mathfrak{g}$ be the Langlands dual of $\mathfrak{g}$. Then we have*

$$\mathscr{W}_{\mathbf{A}}(\mathfrak{g}) \cong \mathscr{W}_{\mathbf{A}}({}^L\mathfrak{g})|_{\substack{\varepsilon_1 \to r^\vee \varepsilon_2 \\ \varepsilon_2 \to \varepsilon_1}}, \tag{B.6.12}$$

where $r^\vee$ is the maximal number of edges connecting two vertices of the Dynkin diagram of $\mathfrak{g}$ (the lacing number).

This is because $\mathfrak{Vir}_{i,\mathbf{A}}$ is invariant under $\varepsilon_1 \leftrightarrow \varepsilon_2$ and $(\varepsilon_1, \varepsilon_2) \to (c\varepsilon_1, c\varepsilon_2)$ $(c \in \mathbb{C}^*)$.

B.7. The embedding $\mathscr{W}_{\mathbf{A}}(\mathfrak{g}) \to \mathscr{W}_{\mathbf{A}}(\mathfrak{l})$

The result in this subsection will not be used elsewhere, but shows that the hyperbolic restriction functor $\Phi_{L,G}$ for general L corresponds to in the $\mathscr{W}$-algebra side.

Let L be a standard Levi subgroup of G with Lie algebra $\mathfrak{l}$. We can write $\mathfrak{l}$ as $[\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{z}(\mathfrak{l})$, where $\mathfrak{z}(\mathfrak{l})$ denotes the center of $\mathfrak{l}$. The above discussion can be applied to the Lie algebra $\mathfrak{l}$ instead of $\mathfrak{g}$ and we get a well-defined vertex operator algebra $\mathscr{W}_{\mathbf{A}}(\mathfrak{l})$ over $\mathbf{A}$ and we have an embedding $\mathscr{W}_{\mathbf{A}}(\mathfrak{l}) \hookrightarrow \mathfrak{Heis}(\mathfrak{h})$. It is also clear that $\mathscr{W}_{\mathbf{A}}(\mathfrak{l})$ is isomorphic to $\mathscr{W}_{\mathbf{A}}([\mathfrak{l}, \mathfrak{l}]) \underset{\mathbf{A}}{\otimes} \mathfrak{Heis}_{\mathbf{A}}(\mathfrak{z}(\mathfrak{l}))$.

Theorem B.7.1. — *There exists an embedding $\mathscr{W}_{\mathbf{A}}(\mathfrak{g}) \to \mathscr{W}_{\mathbf{A}}(\mathfrak{l})$ compatible with the embedding of both algebras into $\mathfrak{Heis}(\mathfrak{h})$.*

Proof. — Clearly, it is enough to construct any map $\mathscr{W}_{\mathbf{A}}(\mathfrak{g}) \to \mathscr{W}_{\mathbf{A}}(\mathfrak{l})$ whose composition with the embedding $\mathscr{W}_{\mathbf{A}}(\mathfrak{l}) \hookrightarrow \mathfrak{Heis}(\mathfrak{h})$ gives the map $\mathscr{W}_{\mathbf{A}}(\mathfrak{g}) \hookrightarrow \mathfrak{Heis}(\mathfrak{h})$ constructed before. To this end, we are going to construct another double complex structure on $C^\bullet_{\mathbf{A}}(\mathfrak{g})_0$ (with the same total complex).

Let $\mathfrak{p}$ be the parabolic subalgebra containing $\mathfrak{l}$ and $\mathfrak{n}_+$ and let $\mathfrak{n}(\mathfrak{p})$ be its nilpotent radical. We can write $\mathfrak{n}_+ = \mathfrak{n}_+(\mathfrak{l}) \oplus \mathfrak{n}(\mathfrak{p})$. Accordingly, we can decompose $\chi = \chi_1 + \chi_2$ where $\chi_1 \in \mathfrak{n}_+(\mathfrak{l})^*$ and $\chi_2 \in \mathfrak{n}(\mathfrak{p})^*$. Let $h_{\mathfrak{l}} \in \mathfrak{z}(\mathfrak{l})$ denote the (unique) element such that for every simple root α_i we have $\operatorname{ad}_{h_{\mathfrak{l}}}(e_i) = e_i$ if e_i is not in $\mathfrak{l}$ and $\operatorname{ad}_{h_{\mathfrak{l}}}(e_i) = 0$

otherwise. Now define a new grading on $C^{\bullet}_{\mathbf{A}}(\mathfrak{g})_0$ in a way similar to (B.1.4) but where instead of the principal gradation and the root height we use the eigenvalue with respect to $ad_{h_{\mathfrak{l}}}$. Then the action of χ_2 has bidegree $(1,0)$ and the action of $d_{st}+\chi_1$ has bidegree $(0,1)$. In this way we get a new bicomplex structure on $C^{\bullet}_{\mathbf{A}}(\mathfrak{g})_0$ with the same total differential and total degree.

It is easy to see that we have $C^{p,q}_{\mathbf{A}}(\mathfrak{g})_0 = 0$ unless $p \geq 0$ and $p+q \geq 0$. Note that it is no longer true that for $p=0$ the complex $C^{0,q}_{\mathbf{A}}(\mathfrak{g})_0$ vanishes unless $q=0$; moreover, the complex $C^{0,\bullet}_{\mathbf{A}}(\mathfrak{g})_0$ (with respect to the differential $d_{st}+\chi_1$) is just $C^{\bullet}_{\mathbf{A}}(\mathfrak{l})_0$. Thus we get a morphism $H^0(C^{\bullet}_{\mathbf{A}}(\mathfrak{g})_0) \to H^0(C^{\bullet}_{\mathbf{A}}(\mathfrak{l})_0)$ by mapping every cocycle to its degree $(0,0)$-component with respect to the above grading. ☐

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