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CLASSIFICATION OF COMPACT LORENTZIAN 2-ORBIFOLDS WITH NONCOMPACT FULL ISOMETRY GROUPS

N. I. Zhukova and E. A. Rogozhina UDC 514.77

Abstract: Among closed Lorentzian surfaces, only flat tori can admit noncompact full isometry groups. Moreover, for every $n \geq 3$ the standard n-dimensional flat torus equipped with canonical metric has a noncompact full isometry Lie group. We show that this fails for $n = 2$ and classify the flat Lorentzian metrics on the torus with a noncompact full isometry Lie group. We also prove that every two-dimensional Lorentzian orbifold is very good. This implies the existence of a unique smooth compact 2-orbifold, called the pillow, admitting Lorentzian metrics with a noncompact full isometry group. We classify the metrics of this type and make some examples.

Keywords: Lorentzian orbifold, Lorentzian surface, isometry group, Anosov automorphism of the torus

Introduction

Lorentzian geometry is widely used in physics and differs considerably from the proper Riemannian geometry.

We can regard orbifolds as manifolds with singularities. Moreover, manifolds coincide with orbifolds whose all points are regular. The exact definitions and theorems of orbifold theory may be found in [1] for instance. Denote the category of orbifolds by $\mathcal{O}rb$.

The isometry group of a Lorentzian orbifold (M, g) is called *full* and denoted by $\mathfrak{I}(M, g)$. The results of Bagaev and Zhukova [2] (also see [3, 4]) imply that $\mathfrak{I}(M, g)$ is a Lie group of dimension at most $n(n+1)/2$, where n is the dimension of the orbifold M. In contrast to compact Riemannian orbifolds [5], the isometry group $\mathfrak{I}(M, g)$ of a compact Lorentzian orbifold is not compact in general.

Recall that the smooth action

$$G \times M \to M : (g, x) \mapsto g \cdot x \quad \forall (g, x) \in G \times M$$

of a Lie group G on an orbifold M is called *proper* if the convergence of two sequences $g_n \cdot x_n \to y$ and $x_n \to x$ in M, where $g_n \in G$ and $x_n, x, y \in M$, implies the existence of a converging subsequence $\{g_{n_k}\}$ in G.

Note that the isometry Lie group $\mathfrak{I}(M, g)$ of a compact Lorentzian orbifold (M, g) is not compact if and only if it acts improperly on M. The improper actions of isometry groups on compact Lorentzian manifolds were studied in [6–9] and also elsewhere (see the survey [10]). The full isometry group of Lorentzian metrics on the two-dimensional torus admitting a one-parameter isometry group was studied in [11, 12]. The goal of this article is to study the two-dimensional compact Lorentzian orbifolds admitting an improper action of the isometry group.

Denote by $\mathbb{Z}^n$ the integer-valued lattice on the n-dimensional flat $\mathbb{R}^n$. The torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is called *standard*. By a closed Lorentzian surface we understand a two-dimensional connected compact Lorentzian manifold (M, g) without boundary. It is shown in [13] that the isometry group of a closed Lorentzian surface (M, g) is not compact only if $(M, g) = (\mathbb{T}^2, g)$ is a flat Lorentzian torus. Recall

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that a pseudo-Riemannian manifold is called *geodesically complete* if every geodesic of the Levi-Civita connection may be extended so that the affine parameter on it runs over the whole real line. Henceforth by the completeness of Lorentzian manifolds we understand geodesic completeness. It is known that every compact flat Lorentzian manifold is complete.

A Lorentzian metric g_0 in $\mathbb{R}^n$, which in the Cartesian coordinate system has the matrix $\begin{pmatrix} -1 & 0 \\ 0 & E_{n-1} \end{pmatrix}$, where E_{n-1} is the $(n-1)$-dimensional identity matrix, is called *canonical.* Denote the pair $(\mathbb{R}^n, g_0)$ by E_1^n. The metric g_0 induces a Lorentzian metric of the same form on the torus $\mathbb{T}^n$, which we also call *canonical* and denote by g_0. It is shown in Section 2.2.1 of [14] that for $n \geq 3$ the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with the canonical metric g_0 has the noncompact isometry group $\mathfrak{I}(\mathbb{T}^n, g_0)$. We show below that this is false for $n = 2$ (Example 1). The full isometry groups of two-dimensional flat Lorentzian tori can both be compact or noncompact.

Henceforth we consider the standard two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and call the pair of vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ the *standard basis* of the tangent vector space $T_x\mathbb{T}^2$ with $x \in \mathbb{T}^2$. Denote by $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$ the quotient mapping, which is the universal covering of the torus. Denote by A the Anosov automorphism of the torus $\mathbb{T}^2$ determined by the matrix $A \in SL(2, \mathbb{Z})$, while by E, the identity 2×2 matrix.

We have found the following characterization of Lorentzian tori $(\mathbb{T}^2, g)$ with noncompact full isometry groups.

Theorem 1. *Let $(\mathbb{T}^2, g)$ be a Lorentzian torus and let $\mathfrak{I}(\mathbb{T}^2, g)$ be its full isometry group. The following are equivalent:*

(1) *$\mathfrak{I}(\mathbb{T}^2, g)$ is not compact;*

(2) *there exists an orientation-preserving Anosov automorphism of the torus which is an isometry;*

(3) *there exists an orientation-preserving Anosov automorphism A of the torus with the trace of A satisfying $\operatorname{tr} A > 2$ which is an isometry;*

(4) *the metric g in the standard basis of the form*

$$(g_{ij}) = \eta \begin{pmatrix} -2c & a-d \\ a-d & 2b \end{pmatrix}, \tag{1}$$

where η is a certain nonzero real number, and a, b, c, and d are integers satisfying

$$ad - bc = 1, \quad a + d > 2. \tag{2}$$

Applying Theorem 1 we prove the following statement.

Theorem 2. *If a compact Lorentzian surface (M, g) has a noncompact full isometry Lie group then $(M, g) = (\mathbb{T}^2, g)$ is a flat Lorentzian torus lacking closed isotropic geodesics. Furthermore, the full isometry Lie group $\mathfrak{I}(\mathbb{T}^2, g)$ is isomorphic to the semidirect product $\mathfrak{I}_0(\mathbb{T}^2, g) \ltimes T^2$ of the discrete noncompact stationary subgroup $\mathfrak{I}_0(\mathbb{T}^2, g)$ at $\Omega(0)$, with $0 \in \mathbb{R}^2$, and a compact abelian normal subgroup T^2. The Lie group of all orientation-preserving isometries of this Lorentzian torus $(\mathbb{T}^2, g)$ is isomorphic to the semidirect product $(\mathbb{Z}_2 \times \mathbb{Z}) \ltimes T^2$.*

Corollary 1. *If the full isometry group of a flat Lorentzian torus $(\mathbb{T}^2, g)$ is not compact then its isotropic foliations F_1 and F_2 consist of irrational cables of torus and each of their leaves is everywhere dense in the torus.*

The problem of classifying closed Lorentzian manifolds with noncompact full isometry Lie groups is posed in [15]. It is natural to state this problem for a larger class of objects: Lorentzian orbifolds.

In this article we solve this problem in dimension $n = 2$ for surfaces (Theorem 3) and for two-dimensional orbifolds (Theorems 5 and 6) which are not manifolds.

Denote by $\mathbb{K}$ the set of ordered quadruples of integers (a, b, c, d) satisfying (2) and enjoying the property

($\mathcal{S}$) there exists no common divisor k of the three numbers $a - d$, b, and c, distinct from ± 1, such that the integers a_1, b_1, c_1, and d_1, where

$$a - d = k(a_1 - d_1), \quad b = kb_1, \quad c = kc_1,$$

satisfy (2).

The equality $s(a, b, c, d) = (d, -b, -c, a)$ determines the inversion $s : \mathbb{K} \to \mathbb{K}$. Consider the group S isomorphic to $\mathbb{Z}_2$ with generator s. Let $\mathcal{K} = \mathbb{K}/S$ be the space of orbits of S and let $\mu : \mathbb{K} \to \mathcal{K}$ be the quotient mapping. Denote the orbit of $(a, b, c, d) \in \mathbb{K}$ by $[a, b, c, d]$. Then $\mathcal{K} = \{[a, b, c, d] \mid (a, b, c, d) \in \mathbb{K}\}$.

Recall that two Lorentzian metrics g_1 and g_2 on a manifold M are called *similar* if they differ by a constant factor; that is, if there exists a number $\eta \neq 0$ with $g_2 = \eta \cdot g_1$. Similarity is an equivalence relation on the set of all Lorentzian metrics on M. The equivalence class containing g is denoted by $[g]$. We emphasize that similar Lorentzian metrics have equal full isometry groups: $\mathfrak{I}(M, g) = \mathfrak{I}(M, h)$ for all $h \in [g]$.

We obtain the following classification of Lorentzian metrics on the two-dimensional torus with a noncompact full isometry group.

Theorem 3. *Let $\mathcal{M} = \{[g]\}$ be a set of classes of similar flat Lorentzian metrics g on the torus $\mathbb{T}^2$ with a noncompact full isometry group. Then we have the bijection*

$$\alpha : \mathcal{K} \to \mathcal{M} : [a, b, c, d] \mapsto [g],$$

where the metric g in the standard basis is determined by the matrix $(g_{ij}) = \begin{pmatrix} -2c & a-d \\ a-d & 2b \end{pmatrix}$.

Corollary 2. *There exists a countable family of classes of similar flat Lorentzian metrics on the torus with a noncompact full isometry group.*

Corollary 3. *If $[g] = \alpha([a, b, c, d])$ in the notation of Theorem 3 then the Lie group of all orientation-preserving isometries of the Lorentzian torus $(\mathbb{T}^2, g)$ is equal to $\mathfrak{I}_0^+(\mathbb{T}^2, g) \ltimes T^2$, where $\mathfrak{I}_0^+(\mathbb{T}^2, g)$ is the group generated by* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $-E$.

Consider pseudo-Riemannian orbifolds (M, g) and $(\mathcal{N}, g_{\mathcal{N}})$. A covering $r : M \to \mathcal{N}$ is called *pseudo-Riemannian* whenever $g = r^* g_{\mathcal{N}}$, where r^* is the codifferential of r.

Since we can regard manifolds as orbifolds, to avoid confusion, we refer henceforth as *orbifolds* only to those with orbifold points, that is, not manifolds.

Recall that an orbifold is called *good* whenever it can be expressed as the space of orbits M/Ψ of a manifold M for a certain diffeomorphism group Ψ. Furthermore, if Ψ is a finite group then the orbifold $\mathcal{N}$ is called *very good*.

Theorem 4. *Every two-dimensional Lorentzian orbifold $(\mathcal{N}, g_{\mathcal{N}})$ is very good and can be expressed as the quotient space $\mathcal{N} = M/\Psi$ of a Lorentzian manifold (M, g) for an isometry group Ψ isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, the quotient mapping $r : M \to \mathcal{N}$ is a pseudo-Riemannian covering.*

Denote the linear transformation $z \mapsto Dz + \delta$ of the plane determined by a matrix $D \in GL(2, \mathbb{R})$ and a vector $\delta = \begin{pmatrix} \delta^1 \\ \delta^2 \end{pmatrix}$, by $\langle D, \delta \rangle$. If $D \in GL(2, \mathbb{Z})$ then the linear transformation $\langle D, \delta \rangle$ induces a transformation of $\mathbb{T}^2$, which we denote by $\langle D, \{\delta\} \rangle$, where $\{\delta\} = \begin{pmatrix} \{\delta^1\} \\ \{\delta^2\} \end{pmatrix}$ with $\{\delta^i\}$ the fractional part of δ^i for $i = \overline{1, 2}$. The composition of transformations $\langle D, \{\delta\} \rangle$ and $\langle D', \{\delta'\} \rangle$ corresponds to the equality

$$\langle D', \{\delta'\} \rangle \circ \langle D, \{\delta\} \rangle = \langle D'D, \{D'\delta + \delta'\} \rangle,$$

where $D'D$ is the matrix product of D' and D.

By Theorem 4, the study of two-dimensional Lorentzian orbifolds with improper action of the full isometry groups is reduced to Lorentzian surfaces with improper action of the full isometry groups (Proposition 1). We have

Theorem 5. *Let $(\mathcal{N}, g_{\mathcal{N}})$ be a two-dimensional compact Lorentzian orbifold with a noncompact full isometry Lie group $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$. Then*

(1) The orbifold $\mathcal{N}$ is isomorphic in the category $\mathcal{O}rb$ to the standard pillow, that is, $\mathcal{N} = \mathbb{T}^2/\Psi_0$, where $\Psi_0 = \langle \psi_0 \rangle$ and $\psi_0 = \langle -E, \{0\} \rangle$, has a stratification $\Delta = \{\Delta^2, \Delta^0\}$, where $\Delta^0 = \{w_i \mid i = \overline{1,4}\}$ is a zero-dimensional stratum and Δ^2 is a two-dimensional stratum.

(2) Assume that $r : \mathbb{T}^2 \to \mathcal{N}$ is the quotient mapping. The induced Lorentzian metric $g := r^* g_{\mathcal{N}}$ on $\mathbb{T}^2$ is complete and flat, and in the standard basis it is of the form (1). The Lorentzian metric $g_{\mathcal{N}}$ is also complete and flat.

(3) For all points $w_i \in \Delta^0$, $i = \overline{1,4}$, the stationary subgroup $\mathfrak{I}_{w_i}(\mathcal{N}, g_{\mathcal{N}})$ of the isometry group $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ is isomorphic to the quotient group $\mathfrak{I}_0(\mathbb{T}^2, g)/\Psi_0$, where $\mathfrak{I}_0(\mathbb{T}^2, g)$ is the stationary subgroup at $\Omega(0)$, $0 \in \mathbb{R}^2$, of the full isometry group of the torus. The full isometry group $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ is discrete and equals the semidirect product $\mathfrak{I}_{w_i}(\mathcal{N}, g_{\mathcal{N}}) \ltimes \Xi$, where $\Xi = \langle \xi_1, \xi_2 \mid (\xi_1)^2, (\xi_2)^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$,

$$\xi_1 = \left\langle E, \left\{ \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right\} \right\rangle, \quad \xi_2 = \left\langle E, \left\{ \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \right\} \right\rangle.$$

(4) The orientation-preserving isometry group of the Lorentzian orbifold $(\mathcal{N}, g_{\mathcal{N}})$ is isomorphic to the semidirect product $\mathbb{Z} \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

The next theorem establishes an equivalence of the classifications of compact Lorentzian surfaces and 2-orbifolds with noncompact full isometry groups.

Theorem 6. *Consider the orbifold $\mathcal{N} = \mathbb{T}^2/\Psi_0$, where $\Psi_0 = \langle \psi_0 \rangle$ and $\psi_0 = \langle -E, \{0\} \rangle$, called the standard pillow, and the quotient mapping $r : M \to \mathcal{N}$. Let $\mathcal{M} = \{[g]\}$ be the set of classes of similar flat Lorentzian metrics g on the torus $\mathbb{T}^2$ with a noncompact full isometry group. Denote by $\mathcal{L} = \{[g_{\mathcal{N}}]\}$ the set of classes of similar Lorentzian metrics on $\mathcal{N}$ with a noncompact full isometry group. Then $\Psi_0 \subset \mathfrak{I}(\mathbb{T}^2, g)$ for every metric $g \in [g] \in \mathcal{M}$. Therefore, there exists a Lorentzian metric $g_{\mathcal{N}}$ on $\mathcal{N}$ such that $g = r^* g_{\mathcal{N}}$ and we have the bijection $\kappa : \mathcal{M} \to \mathcal{L} : [g] \mapsto [g_{\mathcal{N}}]$.*

Corollary 4. *If $\alpha : \mathcal{K} \to \mathcal{M}$ is a bijection satisfying Theorem 3 then the mapping $\beta := \kappa \circ \alpha : \mathcal{K} \to \mathcal{L}$ is a bijection as well.*

1. The Main Concepts and Notation

1.1. The pseudo-orthogonal group. Consider the pseudo-orthogonal group $O(1,1)$. Denote its elements as follows: $A_t^{++} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, $E^{+-} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $A_t^{+-} = A_t^{++} E^{+-}$, $A_t^{-+} = -A_t^{+-}$, and $A_t^{--} = -A_t^{++}$ for all $t \in \mathbb{R}$.

Observe that $A \in O(1,1)$ is an Anosov automorphism of the pseudo-Euclidean flat E_1^2 if and only if $|\operatorname{tr} A| > 2$. The identity component $O_e(1,1)$ consists of the matrices A_t^{++} with trace $\operatorname{tr} A_t^{++} \geq 2$.

The mapping $\mathbb{R}^1 \to O_e(1,1) : t \mapsto A_t^{++}$ for $t \in \mathbb{R}^1$ is an isomorphism of the additive group $\mathbb{R}^1$ of the real numbers with $O_e(1,1)$. Therefore, the group $O_e(1,1)$ has no nontrivial finite subgroups, and every discrete subgroup of it is isomorphic to the group $\mathbb{Z}$ of integers. The following lemma is easy.

Lemma 1. *A discrete subgroup Φ of $O(1,1)$ is noncompact if and only if the subgroup $\Phi_e := \Phi \cap O_e(1,1)$ is isomorphic to the group of integers.*

1.2. A pseudo-Euclidean metric on $\mathbb{R}^n$. The pseudo-Riemannian metric g of signature $(k, n-k)$ on $\mathbb{R}^n$ is called *pseudo-Euclidean* whenever $\mathbb{R}^n$ admits a global coordinate system $0, x^1, \dots, x^n$ in which g is of the form

$$g = -dx^1 \otimes dx^1 - \dots - dx^k \otimes dx^k + dx^{k+1} \otimes dx^{k+1} + \dots + dx^n \otimes dx^n.$$

Below we will use another easy lemma.

Lemma 2. *Every complete flat pseudo-Riemannian metric on $\mathbb{R}^n$ is pseudo-Euclidean.*

Consider a flat Lorentzian torus $(\mathbb{T}^2, g)$ with a universal covering $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$. Since $\mathbb{T}^2$ is compact, the metric g is complete. Therefore, the induced metric $\Omega^* g$ on $\mathbb{R}^2$ is complete and flat, and so by Lemma 2 it is pseudo-Euclidean. Denote this metric also by g. We will consider $O(1,1)$ as the stationary subgroup of $\mathfrak{I}(E_1^2)$ at the origin and the stationary subgroup $\mathfrak{I}_0(\mathbb{R}^2, g)$ of $\mathfrak{I}(\mathbb{R}^2, g)$ at the origin. Then $O(1,1)$ and $\mathfrak{I}_0(\mathbb{R}^2, g)$ are conjugate subgroups in $GL(2,\mathbb{R})$. It is known that the full isometry Lie group $\mathfrak{I}(\mathbb{R}^2, g)$ is equal to the semidirect product $\mathfrak{I}_0(\mathbb{R}^2, g) \ltimes \mathbb{R}^2$ of Lie groups.

The normalizer $\mathbf{N}(\mathbb{Z}^2)$ of the subgroup $\mathbb{Z}^2$ in $\mathfrak{I}(\mathbb{R}^2, g)$ is equal to the semidirect product $(\mathfrak{I}_0(\mathbb{R}^2, g) \cap GL(2,\mathbb{Z})) \ltimes \mathbb{R}^2$. Since $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the Lie group $\mathfrak{I}(\mathbb{T}^2, g)$ is isomorphic to the semidirect product $\mathfrak{I}_0(\mathbb{T}^2, g) \ltimes T^2$, where $\mathfrak{I}_0(\mathbb{T}^2, g) = \mathfrak{I}_0(\mathbb{R}^2, g) \cap GL(2,\mathbb{Z})$, while T^2 is the torus as a compact abelian Lie group.

Put $\mathfrak{I}_0^{++}(\mathbb{T}^2, g) := \{A \in \mathfrak{I}_0(\mathbb{T}^2, g) \mid \operatorname{tr} A \geq 2\}$. Then $\mathfrak{I}_0^{++}(\mathbb{T}^2, g)$ is a Lie subgroup of $\mathfrak{I}_0(\mathbb{T}^2, g)$.

2. Proof of Theorem 1

(1)⇔(2)⇔(3) Assume that the full isometry group $\mathfrak{I}(\mathbb{T}^2, g)$ of the Lorentzian torus $(\mathbb{T}^2, g)$ is not compact. It is shown in [13] that this is possible only when the metric g is flat; therefore, as we showed above, it is locally pseudo-Euclidean, and the Lie group $\mathfrak{I}_0(\mathbb{T}^2, g)$ is not compact. Lemma 1 implies that the subgroup $\mathfrak{I}_0^{++}(\mathbb{T}^2, g)$ of $\mathfrak{I}_0(\mathbb{T}^2, g)$ is conjugate to the group $\Phi_e = \Phi \cap O_e(1,1)$, and so it is not compact. Consequently, there exists an Anosov automorphism $A \in \mathfrak{I}_0^{++}(\mathbb{T}^2, g)$; that is, (3) holds.

The implications (3)⇒(2) and (2)⇒(1) are obvious. Thus, (1), (2), and (3) are equivalent.

(3)⇒(4) Suppose that the Lie group $\mathfrak{I}(\mathbb{T}^2, g)$ contains an orientation-preserving Anosov automorphism. Since (3) implies (1), the Lie group $\mathfrak{I}(\mathbb{T}^2, g)$ is not compact. Consequently, the Lorentzian metric g is complete and flat. Furthermore (see Section 1.2), the Lie group $\mathfrak{I}(\mathbb{T}^2, g)$ amounts to the semidirect product $\mathfrak{I}_0(\mathbb{T}^2, g) \ltimes T^2$, while $\mathfrak{I}_0(\mathbb{T}^2, g)$ contains an orientation-preserving Anosov automorphism of the torus $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, and d are integers with $a + d = \operatorname{tr} A > 2$. Thus, these integers satisfy (2), which implies that

$$bc \neq 0. \tag{3}$$

Since $(\mathbb{R}^2, g)$ is a complete pseudo-Euclidean space, the matrix $G = (g_{ij})$, $i, j = 1, 2$, of its metric tensor in the standard basis has constant coefficients. By the definition of isometry, A satisfies $A^t G A = G$. Since

$$\begin{aligned} A^t G A &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a g_{11} + c g_{12} & a g_{12} + c g_{22} \\ b g_{11} + d g_{12} & b g_{12} + d g_{22} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2 g_{11} + 2ac g_{12} + c^2 g_{22} & ab g_{11} + (ad + bc) g_{12} + cd g_{22} \\ ab g_{11} + (ad + bc) g_{12} + cd g_{22} & b^2 g_{11} + 2bd g_{12} + d^2 g_{22} \end{pmatrix}, \end{aligned}$$

inserting the last expression into $A^t G A = G$, we obtain

$$\begin{pmatrix} a^2 g_{11} + 2ac g_{12} + c^2 g_{22} & ab g_{11} + (ad + bc) g_{12} + cd g_{22} \\ ab g_{11} + (ad + bc) g_{12} + cd g_{22} & b^2 g_{11} + 2bd g_{12} + d^2 g_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix},$$

which we can regard as the homogeneous system of linear equations

$$WX = 0, \tag{4}$$

where

$$W = \begin{pmatrix} a^2 - 1 & 2ac & c^2 \\ ab & ad + bc - 1 & cd \\ b^2 & 2bd & d^2 - 1 \end{pmatrix}, \quad X = \begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix}.$$

Since

$$\det A = ad - bc = 1, \tag{5}$$

we can express W as

$$W = \begin{pmatrix} a^2-1 & 2ac & c^2 \\ ab & 2bc & cd \\ b^2 & 2bd & d^2-1 \end{pmatrix}.$$

We can easily verify that the determinant of W is equal to zero. Therefore, the homogeneous linear system (4) has a nonzero solution.

Since (3) and (5) hold, the minor of W at the intersections of columns 1 and 2 and rows 2 and 3 is equal to $2b^2 \neq 0$. Therefore, the rank of W is equal to 2, and (4) is equivalent to

$$\begin{cases} abg_{11} + 2bcg_{12} = -cdg_{22}, \\ b^2 g_{11} + 2bdg_{12} = (1-d^2)g_{22}. \end{cases}$$

Solving this system, we obtain $g_{11} = -2c\eta$, $g_{12} = (a-d)\eta$, $g_{22} = 2b\eta$, $\eta \in \mathbb{R}^1 \backslash \{0\}$.

Thus, if the full isometry Lie group contains an orientation-preserving Anosov automorphism then there exists a number $\eta \neq 0$ such that the matrix G of the metric g in the standard basis is of the form (1), and moreover (2) holds.

(4)$\Rightarrow$(3) Suppose that the metric g on $\mathbb{T}^2$ in the standard basis e_1, e_2 is determined by a matrix G of the form (1). Since $a + d > 2$, the determinant of G satisfies

$$\det G = \eta^2(4 - (a+d)^2) < 0.$$

Therefore, the metric g is Lorentzian. We can easily verify that if (2) holds then the transformation $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an orientation-preserving isometry of the Lorentzian torus $(\mathbb{T}^2, g)$. Since $\operatorname{tr} A = a+d > 2$, it follows that A is an Anosov automorphism of the torus. □

Lemma 3. *Consider a matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ *in* $\mathfrak{I}_0^{++}(\mathbb{T}^2, g) \setminus \{E\}$. *If* $A = B^n$ *for some integer* n *distinct from* ± 1 *then*

($\mathcal{S}^*$) *there exists a common divisor* k *of three numbers* $(a-d)$, b, *and* c, *distinct from* ± 1 *and satisfying*

$$a - d = k(a_1 - d_1), \quad b = kb_1, \quad c = kc_1. \tag{6}$$

PROOF. Induct on the positive integer n. For $n = 2$, considering that $\det B = 1$, we have

$$A = B^2 = \begin{pmatrix} a_1(a_1+d_1) - 1 & b_1(a_1+d_1) \\ c_1(a_1+d_1) & d_1(a_1+d_1) - 1 \end{pmatrix}.$$

Therefore, (6) holds. Assume that the statement of this lemma holds for some integer $n > 2$, that is, if $\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = B^n$ then there exists a common divisor k of three numbers $(\tilde{a} - \tilde{d})$, $\tilde{b}$, and $\tilde{c}$, distinct from ± 1 and satisfying

$$\tilde{a} - \tilde{d} = k(a_1 - d_1), \quad \tilde{b} = kb_1, \quad \tilde{c} = kc_1. \tag{7}$$

Applying (7), we obtain

$$A = B^{n+1} = \begin{pmatrix} \tilde{d} + k(a_1 - d_1) & kb_1 \\ kc_1 & \tilde{d} \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_1(ka_1 + \tilde{d}) - k & b_1(ka_1 + \tilde{d}) \\ c_1(ka_1 + \tilde{d}) & d_1(ka_1 + \tilde{d}) - k \end{pmatrix}.$$

Consequently, (6) holds for the factor $ka_1 + \tilde{d}$.

Suppose that $ka_1 + \tilde{d} = 1$. Then $\det A = 1 + k^2 - k(a_1 + d_1)$, and $\det A = 1$ yields $k = a_1 + d_1$. Hence,

$$\operatorname{tr} A = a_1 + d_1 - 2k = -(a_1 + d_1) = -\operatorname{tr} B,$$

which contradicts the choice of $A, B \in \mathfrak{I}_0^{++}(\mathbb{T}^2, g)\backslash\{E\}$.

Suppose now that $ka_1 + \tilde{d} = -1$. Then $\det A = 1 + k^2 + k(a_1 + d_1) = 1$, whence $k = -(a_1 + d_1)$. Therefore,

$$A = B^{n+1} = \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} = B^{-1},$$

which contradicts the condition of the lemma. Thus, $(\mathcal{S}^*)$ is established. By induction, the statement of the lemma holds for all integers $n \geq 2$.

Suppose now that $n < 0$ and put $m := -n$. Then the equality $A = B^n$ becomes $A^{-1} = B^m$, where $m \in \mathbb{N}$. By the argument above, (6) holds for A^{-1}. Since $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, the matrix A^{-1} enjoys the property $(\mathcal{S}^*)$ if and only if so does A. $\square$

Corollary 5. *A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a generator of $\mathfrak{I}_0^{++}(\mathbb{T}^2, g)$ if and only if $(a, b, c, d) \in \mathbb{K}$.*

3. Proof of Theorem 2

Lemma 4. *The square root of every positive integer is either an integer or irrational number.*

Proof. Assume that there exists a positive integer n for which $\sqrt{n} = \dfrac{p}{q}$ is a fractional rational number, where p and q are coprime positive integers. Decompose n into prime factors. Then $\sqrt{n} = m\sqrt{k}$, where $k = k_1 \cdot \ldots \cdot k_s$ is a product of distinct primes, $k \neq 1$. Furthermore, $\sqrt{k_1 \cdot \ldots \cdot k_s} = \dfrac{p}{qm}$. Divide both the numerator and the denominator of this fraction by the greatest common divisor of p and m. Then $\sqrt{k_1 \cdot \ldots \cdot k_s} = \dfrac{a}{b}$ is an irreducible fraction, and consequently $k_1 \cdot \ldots \cdot k_s = \dfrac{a^2}{b^2} \Rightarrow (k_1 \cdot \ldots \cdot k_s)b^2 = a^2 \Rightarrow a \vdots (k_1 \cdot \ldots \cdot k_s) \Rightarrow a^2 = k_1^2 \cdot \ldots \cdot k_s^2 \Rightarrow b \vdots (k_1 \cdot \ldots \cdot k_s) \Rightarrow k_1 \cdot \ldots \cdot k_s$ is a common factor of a and b, which contradicts the irreducibility of $\dfrac{a}{b}$. $\square$

Assume that the isometry group $\mathfrak{I}(M, g)$ of a compact Lorentzian surface is not compact. Then according to [13] $(M, g) = (\mathbb{T}^2, g)$ is a flat torus. By Theorem 1 the metric g is of the form (1) in the standard basis, where the numbers a, b, c, and d satisfy (2).

Assume also that there exists a closed isotropic geodesic γ on $(\mathbb{T}^2, g)$ and take $x = \gamma(0) = \gamma(s_0) \in \mathbb{T}^2$. Then an isotropic geodesic $\hat{\gamma}$ of the pseudo-Euclidean space $(\mathbb{R}^2, g)$ covering γ passes through the point $z \in \Omega^{-1}(x)$; moreover, $\hat{\gamma}(0) = z$ and $\hat{\gamma}(s_0) = z + z_0$, where $z_0 = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$ is a nonzero isotropic vector in the plane with integer-valued coordinates. Therefore, $g(z_0, z_0) = 0$ or

$$-2c(z^1)^2 + 2(a - d)z^1 z^2 + 2b(z^2)^2 = 0.$$

As we showed above (see (3)), $bc \neq 0$, so that

$$z^1 = \frac{d - a \pm \sqrt{(a + d)^2 - 4}}{-2c} z^2.$$

In order for the roots z^1 and z^2 to be integers not vanishing simultaneously, by Lemma 4 there exists an integer m satisfying $\sqrt{(a + d)^2 - 4} = m$, which yields $(a + d)^2 - m^2 = 4 \Leftrightarrow (a + d - m)(a + d + m) = 4$. Then $a + d - m$ and $a + d + m$ are of the same parity, and therefore the last equality holds only in the two cases:

$$\begin{cases} a + d + m = 2, \\ a + d - m = 2, \end{cases} \quad \begin{cases} a + d + m = -2, \\ a + d - m = -2. \end{cases}$$

Hence, $|a+d| = 2$, which contradicts (2). The resulting contradiction shows that the assumption of existence of a closed isotropic geodesic is false.

The remaining claims of Theorem 2 follow from Subsection 1.2.

4. Proof of Theorem 3

Let $[g] = \alpha([a,b,c,d])$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By the definition of $\mathcal{K}$ we have $A \in \mathfrak{I}_0^{++}(\mathbb{T}^2, g)$. Corollary 5 implies that A is a generator of $\mathfrak{I}_0^{++}(\mathbb{T}^2, g)$. The group $\mathfrak{I}_0^{++}(\mathbb{T}^2, g)$ is isomorphic to $\mathbb{Z}$ and has two generators, A and A^{-1}. Since $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and $s(a,b,c,d) = (d,-b,-c,a)$, the mapping $\alpha : \mathcal{K} \to \mathcal{M}$ is well defined.

Since every element of $[g] \in \mathcal{M}$ uniquely determines an isometry subgroup $\mathfrak{I}_0^{++}(\mathbb{T}^2, g)$, and so the pair of its generators, the mapping α is both injective and surjective. $\square$

5. Proof of Theorem 4

A Lorentzian metric g on an orbifold $\mathcal{N}$ determines a principal $O(1,1)$-bundle $q : \mathcal{P} \to \mathcal{N}$ with a proper right action of $O(1,1)$ on $\mathcal{P}$ whose space of orbits coincides with $\mathcal{N}$. It is shown in [2] that $\mathcal{P}$ amounts to a smooth manifold. It carries a smooth proper action of the normal subgroup $O_e(1,1)$ of $O(1,1)$; moreover, $M := \mathcal{P}/O_e(1,1)$ is a smooth orbifold.

Denote by $p : \mathcal{P} \to M = \mathcal{P}/O_e(1,1)$ the projection onto the space of orbits. There is an induced properly discontinuous action of the quotient group $\Psi = O(1,1)/O_e(1,1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ on M; moreover, the space of orbits M/Ψ coincides with $\mathcal{N}$ and $q = r \circ p$, where $r : M \to \mathcal{N} = M/\Psi$ is the quotient mapping.

Bagaev and Zhukova showed [4, Proposition 3] that every orbifold group Γ of the orbifold M is isomorphic to a certain subgroup of the structure group $O_e(1,1)$ of the principal bundle $p : \mathcal{P} \to M$ of pseudo-orthogonal frames. Since every group Γ is finite, while the group $O_e(1,1)$ has no nontrivial finite subgroups, the triviality of Γ follows. This means that M is a smooth manifold and the action of $O_e(1,1)$ on $\mathcal{P}$ is free.

We will consider $u \in \mathcal{P}$ as a linear isomorphism $u : \mathbb{R}^2 \to T_xM$, where $x = p(u)$. If e_1, e_2 is the standard basis in $\mathbb{R}^2$, while ∂_1, ∂_2 is the basis u at x, then $u(X^i e_i) := X^i \partial_i$, $i = \overline{1,2}$. Denote by $(\cdot,\cdot)$ the inner product on the pseudo-Euclidean space E_1^2. Put

$$g(X,Y) := (u^{-1}X, u^{-1}Y) \quad \forall X, Y \in T_xM. \tag{8}$$

Since $(\cdot,\cdot)$ is invariant under $O(1,1)$, it follows that (8) is independent of the choice of $u \in p^{-1}(x)$. Thus, g is a Lorentzian metric on M; moreover, Ψ is an isometry group of the Lorentzian manifold (M,g). Consequently, $r : M \to \mathcal{N}$ is a pseudo-Riemannian covering for the Lorentzian orbifold $(\mathcal{N}, g_{\mathcal{N}})$.

We emphasize that the total space $\mathcal{P}$ of the bundle of pseudo-orthogonal frames $q : \mathcal{P} \to \mathcal{N}$ is disconnected in general. One connected component of $\mathcal{P}$ maps onto another via a certain translation R_a, $a \in O(1,1)$; therefore, the distinct connected components are diffeomorphic.

Since $\mathcal{N}$ is an orbifold that is not a manifold there exists at least one orbifold point x_0 with a nontrivial finite orbifold group Γ_0. Furthermore, Γ_0 is isomorphic to a certain finite subgroup Φ_0 of $O(1,1)$. Since Φ_0 is not a subgroup of $O_e(1,1)$, it intersects at least two connected components of the Lie group $O(1,1)$. Therefore, the fiber $q^{-1}(x_0)$ above x_0 is either connected or has two connected components, accordingly, the group Φ_0 is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2$. The fiber $q^{-1}(x_0)$ is connected if and only if Φ_0 intersects all connected components of $O(1,1)$, that is, when $\Phi_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

For the orbifold $\mathcal{N}$ the total space of the bundle $\mathcal{P}$ is either connected, M is connected and $\Psi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathcal{P}$ has two connected components and $\Psi \cong \mathbb{Z}_2$, implying that instead of M we consider its connected component.

Thus, given a Lorentzian orbifold $(\mathcal{N}, g_{\mathcal{N}})$, there exists a connected Lorentzian manifold (M,g) such that $\mathcal{N} = M/\Psi$, where Ψ is a subgroup of the isometry group $\mathfrak{I}(M,g)$ isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. This means that the quotient mapping $r : M \to \mathcal{N}$ is a pseudo-Riemannian covering, and $g = r^* g_{\mathcal{N}}$. $\square$

6. Pseudo-Euclidicity of a Metric on an Orbifold

Definition 1. Let $r : M \to \mathcal{N}$ be the covering mapping for an orbifold $\mathcal{N}$. Say that $\hat{f} \in \mathrm{Diff}(M)$ *lies over* $f \in \mathrm{Diff}(\mathcal{N})$ whenever $r \circ \hat{f} = f \circ r$.

Consider pseudo-Riemannian manifolds (M_1, g_1) and (M_2, g_2) and a pseudo-Riemannian covering $r : M_1 \to M_2$. It is known that, in general there exists no isometry of (M_1, g_1) lying over this isometry of (M_2, g_2). But for particular coverings of Lorentzian orbifolds we have the following statement.

Proposition 1. *Let $(\mathcal{N}, g_{\mathcal{N}})$ be a Lorentzian orbifold and let $r : M \to \mathcal{N}$ of $(\mathcal{N}, g_{\mathcal{N}})$ be a pseudo-Riemannian covering by a manifold (M, g) satisfying Theorem 4. Then*

(1) *for every $f \in \mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ there exists $\hat{f} \in \mathfrak{I}(M, g)$ lying over f;*

(2) *$\hat{f} \in \mathfrak{I}(M, g)$ lies over some $f \in \mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ if and only if $\Psi \circ \hat{f} = \hat{f} \circ \Psi$;*

(3) *the set of all $\hat{f} \in \mathfrak{I}(M, g)$ lying over the isometries in $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ coincides with the normalizer $\mathbf{N}(\Psi)$ of Ψ in $\mathfrak{I}(M, g)$, and moreover $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}}) \cong \mathbf{N}(\Psi)/\Psi$;*

(4) *if $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ is not compact then $(M, g) = (\mathbb{T}^2, g)$ is a flat torus with the noncompact isometry group $\mathfrak{I}(\mathbb{T}^2, g)$.*

Proof. Keep the above notation. Consider the principal $O(1,1)$-bundle $q : \mathcal{P} \to \mathcal{N}$ above the Lorentzian orbifold $(\mathcal{N}, g_{\mathcal{N}})$. The existence of the Levi-Civita connection $\nabla^{g_{\mathcal{N}}}$ on $\mathcal{N}$ corresponding to $g_{\mathcal{N}}$ implies that we are given some $\mathfrak{o}(1,1)$-valued 1-form $\widetilde{\omega}$ on $\mathcal{P}$, where $\mathfrak{o}(1,1) = \mathbb{R}^1$ is the Lie algebra of the Lie group $O(1,1)$. Denote by θ the canonical 1-form on $\mathcal{P}$, which is an $\mathbb{R}^2$-valued 1-form [2] (for manifolds see [16]).

Denote by $\mathfrak{g}$ the Lie algebra of the isometry Lie group G of the pseudo-Euclidean space E^2_1, which amounts to the semidirect product $O(1,1) \ltimes \mathbb{R}^2$, and moreover $\mathbb{R}^2$ is the group of translations by the vectors in $\mathbb{R}^2$, which is a normal subgroup of G. The equality

$$\omega(X) = \widetilde{\omega}(X) + \theta(X) \in \mathfrak{g},$$

where X is a smooth vector field on $\mathcal{P}$, determines a $\mathfrak{g}$-valued 1-form ω which is a Cartan connection. Thus, the pair $\xi = (\mathcal{P}(M, O(1,1)), \omega)$ is an induced Cartan geometry on the orbifold $(\mathcal{N}, g_{\mathcal{N}})$ [3]. Since the homogeneous space $G/O(1,1)$ is reductive, and moreover carries an effective action of G by left translations, it follows that ξ is an effective reductive Cartan geometry.

Observe that, furthermore, the principal $O_e(1,1)$-bundle $p : \mathcal{P} \to M$ with $\mathfrak{g}$-valued 1-form ω determines the Cartan geometry $\xi_0 = (\mathcal{P}(M, O_e(1,1)), \omega)$ on M. Consider the automorphism groups $\widehat{\mathfrak{I}}(\mathcal{P}, \xi)$ and $\widehat{\mathfrak{I}}(\mathcal{P}, \xi_0)$ of the Cartan geometries ξ and ξ_0. Then

$$\widehat{\mathfrak{I}}(\mathcal{P}, \xi) = \{\hat{f} \in \mathrm{Diff}(\mathcal{P}) \mid \hat{f}^*\omega = \omega,\ \hat{f} \circ R_a = R_a \circ \hat{f}\ \forall a \in O(1,1)\},$$
$$\widehat{\mathfrak{I}}(\mathcal{P}, \xi_0) = \{\hat{f} \in \mathrm{Diff}(\mathcal{P}) \mid \hat{f}^*\omega = \omega,\ \hat{f} \circ R_a = R_a \circ \hat{f}\ \forall a \in O_e(1,1)\}.$$

Consequently, $\widehat{\mathfrak{I}}(\mathcal{P}, \xi) \subset \widehat{\mathfrak{I}}(\mathcal{P}, \xi_0)$.

Consider an arbitrary isometry $f \in \mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$. For it we have a transformation $\hat{f} \in \widehat{\mathfrak{I}}(\mathcal{P}, \xi_0)$ carrying $u \in \mathcal{P}$ to $f_{*x} \circ u$, where $x = q(u) \in \mathcal{N}$, and f_{*x} is the differential of f at x. Since $\hat{f} \circ R_a = R_a \circ \hat{f}$ for all $a \in O_e(1,1)$, it follows that $\hat{f}$ lies over some diffeomorphism h of M with respect to $p : \mathcal{P} \to M$. Verify that $h \in \mathfrak{I}(M, g)$.

Consider the $O(1,1)$-bundle $\tilde{p} : P \to M$ of pseudo-orthonormal frames for the Lorentzian manifold (M, g). Then P contains a closed submanifold $\mathcal{P}$. Furthermore, the equality $P = \mathcal{P}$ is not excluded. Take the Cartan geometry $\hat{\xi} = (P(M, O(1,1), \widehat{\omega}))$ corresponding to the Lorentzian geometry (M, g).

Put $\hat{h}|_{\mathcal{P}} := \hat{f}$ and define an extension $\hat{h}$ to all P. Assume that $v \in P$ is obtained from $u \in \mathcal{P}$ using the translation by R_a, where $a \in O(1,1)$, that is, $v = u \cdot a$. Put $\hat{h}(v) = \hat{h}(u \cdot a) := R_a(\hat{h}(u))$.

A straightforward verification shows that $\hat{h}^*\widehat{\omega} = \widehat{\omega}$, where $\widehat{\omega}$ is a reductive Cartan connection on P corresponding to the Levi-Civita connection ∇^g of the Lorentzian metric g on M. The definition of $\hat{h}$

implies that $R_a \circ \hat{h} = \hat{h} \circ R_a$ for all $a \in O(1,1)$. This means that $\hat{h} \in \widehat{\mathfrak{I}}(P, \hat{\xi})$, and in addition $\hat{h}$ lies over some isometry $h \in \mathfrak{I}(M, g)$ with respect to $\tilde{p}$. Observe that h lies with respect to r over an isometry $f \in \mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$, which is the projection of the transformation $\hat{f} \in \widehat{\mathfrak{I}}(\mathcal{P}, \xi)$ inducing $\hat{h}$.

Thus, above every isometry $f \in \mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ there lies some isometry $h \in \mathfrak{I}(M, g)$ of (M, g) with respect to the projection $r : M \to \mathcal{N}$.

We can easily verify that the set Φ of isometries of (M, g) lying over $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ constitutes a Lie group, while the mapping $\chi : \Phi \to \mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ associating to an isometry $h \in \mathfrak{I}(M, g)$ the isometry $f \in \mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ above which h lies is a Lie group homomorphism. Its kernel $\operatorname{Ker}\chi$ consists of all transformations in $\mathfrak{I}(M, g)$ lying over $\mathrm{Id}_{\mathcal{N}}$. Consequently, $\operatorname{Ker}\chi = \Psi$.

Since $\operatorname{Ker}\chi$ is a normal subgroup of Φ, for every $\tilde{h} \in \Phi$ we have

$$\tilde{h} \circ \Psi = \Psi \circ \tilde{h}. \tag{9}$$

Verify that the validity of this equality suffices for the isometry $\tilde{h}$ to lie over some isometry in $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$. For all $x \in \mathcal{N}$ and $z \in r^{-1}(x)$ the equality $\varphi(x) = r(\tilde{h}(z))$ determines a transformation φ of the orbifold $\mathcal{N}$. Indeed, if z' is another point of $r^{-1}(x)$ then there is $\psi \in \Psi$ with $z' = \psi(z)$. Since Ψ lies over $\mathrm{Id}_{\mathcal{N}}$, using (9) we infer that $r(\tilde{h}(z')) = r(\tilde{h}(\psi(z))) = r(\psi'(\tilde{h}(z))) = r(\tilde{h}(z))$, where $\psi' \in \Psi$. Since $r \circ \tilde{h} = \varphi \circ r$ and the covering $r : M \to \mathcal{N}$ is pseudo-Riemannian, φ is an isometry of $(\mathcal{N}, g_{\mathcal{N}})$.

Thus, $\Phi = \mathbf{N}(\Psi)$ is the normalizer of Ψ in $\mathfrak{I}(M, g)$, and so $\mathbf{N}(\Psi)$ is a Lie subgroup of $\mathfrak{I}(M, g)$. We have the Lie group isomorphism $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}}) \cong \mathbf{N}(\Psi)/\Psi$, and the noncompactness of $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ implies the noncompactness of $\mathfrak{I}(M, g)$. Therefore, $(M, g) = (\mathbb{T}^2, g)$ is a flat Lorentzian torus with noncompact full isometry group. □

7. Isometry Groups of Flat Lorentzian 2-Orbifolds

7.1. Orientable orbifolds.

Definition 2. Consider an arbitrary good 2-orbifold $\mathcal{N}$ and a covering $r : M \to \mathcal{N}$ for $\mathcal{N}$ by a manifold M. Refer as the *fundamental polygon* of $\mathcal{N}$ to a curviliner polygon Σ on M such that $\mathcal{N} = r(\Sigma)$ and, moreover, the restriction $r|_{\Sigma^0}$ to the interior of Σ is a homeomorphism onto the two-dimensional stratum Δ^2.

Proposition 2. *Let $(\mathcal{N}, g_{\mathcal{N}})$ be a Lorentzian orbifold. If there exist a flat torus $(\mathbb{T}^2, g)$ and an isometry subgroup $\Psi = \langle\psi\rangle \cong \mathbb{Z}_2$, where ψ preserves the orientation, such that $\mathcal{N} = M/\Psi$ and the quotient mapping $r : M \to \mathcal{N}$ is a pseudo-Riemannian covering then $\mathcal{N}$ is diffeomorphic in the category $\mathfrak{Orb}$ to the standard pillow, the full isometry Lie group $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ is discrete and*

$$\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}}) \cong (\mathfrak{I}_0(\mathbb{T}^2, g)/\Psi_0) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2),$$

where $\Psi_0 = \langle\psi_0\rangle$, $\psi_0 = \langle -E, \{0\}\rangle$.

Moreover, if $\mathfrak{I}(\mathbb{T}^2, g)$ is not compact and $\mathfrak{I}^+(\mathcal{N}, g_{\mathcal{N}})$ is the Lie group of all isometries preserving the orientation of the orbifold then

$$\mathfrak{I}^+(\mathcal{N}, g_{\mathcal{N}}) \cong \mathbb{Z} \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2).$$

Proof. Since ψ preserves the orientation of the torus, $\psi^2 = \langle E, \{0\}\rangle$ implies that $\psi = \langle -E, \{\sigma\}\rangle$.

Assume firstly that $\psi_0 = \langle -E, \{0\}\rangle$, i.e., $\sigma = 0$. As above, take a universal covering $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$ for $\mathbb{T}^2$ and the quotient mapping $r : \mathbb{T}^2 \to \mathcal{N} = \mathbb{T}^2/\Psi_0$, where $\Psi_0 = \langle\psi_0\rangle$. Furthermore, $r \circ \Omega : \mathbb{R}^2 \to \mathcal{N}$ is a universal covering for $\mathcal{N}$.

Represent $\mathbb{T}^2$ as the square $[0,1] \times [0,1]$ with its opposite sides glued in the corresponding directions. It is easy to verify that we can take the rectangle $[0,1] \times [0,1/2]$ as the fundamental polygon Σ_0 on $\mathbb{T}^2$ for $\mathcal{N}$. Fig. 1 shows the gluing rule for the sides.

Consequently, $\mathcal{N}$ has a zero-dimensional stratum Δ^0 consisting of four points w_i, $i = \overline{1,4}$, where $w_i = r(z_i)$, $z_1 = (0,0)$, $z_2 = (1/2, 0)$, $z_3 = (0, 1/2)$, and $z_4 = (1/2, 1/2) \in \Sigma_0$. Therefore, $\mathcal{N}$ amounts to the standard pillow. Observe that the normalizer $\mathbf{N}(\Psi_0)$ of Ψ_0 in $\mathfrak{I}(\mathbb{T}^2, g)$ consists of the isometries $\langle A, \{\delta\}\rangle$ of the torus satisfying $\{\delta\} = \{-\delta\}$, which holds only for four vectors $\{\delta\}_{(i)}$:

$$\{\delta\}_{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \{\delta\}_{(2)} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \quad \{\delta\}_{(3)} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \quad \{\delta\}_{(4)} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Thus from the group $\{E\} \ltimes T^2$ of translations of the torus only the isometries $\langle E, \{\delta\}_{(i)}\rangle$ for $i = \overline{1,4}$ project to $\mathcal{N}$. Consequently, $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ is transitive on Δ^0.

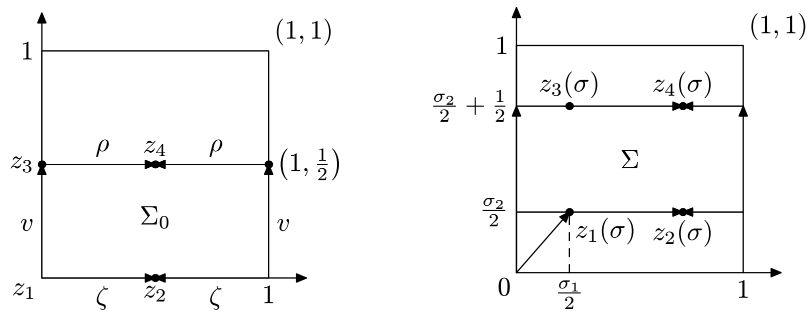

Fig. 1. Pillow.

Since $\Psi_0 = \langle\psi_0\rangle$ and $\psi_0 = \langle -E, \{0\}\rangle$, the normalizer $\mathbf{N}_0(\Psi_0)$ in the stationary subgroup $\mathfrak{I}_0(\mathbb{T}^2, g)$ at the point $w := r \circ \Omega(0)$, $0 \in \mathbb{R}^2$, coincides with $\mathfrak{I}_0(\mathbb{T}^2, g)$. Therefore, the stationary subgroup $\mathfrak{I}_w(\mathcal{N}, g_{\mathcal{N}})$ at w is isomorphic to $\mathfrak{I}_0(\mathbb{T}^2, g)/\Psi_0$.

We have the isomorphism of discrete Lie groups

$$\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}}) \cong (\mathfrak{I}_0(\mathbb{T}^2, g)/\Psi_0) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2).$$

If $\mathfrak{I}(\mathbb{T}^2, g)$ is not compact then Theorem 2 implies that the subgroup $\mathfrak{I}^+(\mathcal{N}, g_{\mathcal{N}})$ consisting of the orientation-preserving isometries is discrete and isomorphic to $\mathbb{Z} \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Suppose now that $\Psi = \langle\psi\rangle \cong \mathbb{Z}_2$, where $\psi = \langle -E, \{\sigma\}\rangle$ with $\sigma \neq 0$. Put $\mu = \langle E, \{\sigma/2\}\rangle$. Then $\mu^{-1} = \langle E, \{-\sigma/2\}\rangle$ and, consequently,

$$\psi = \mu \circ \psi_0 \circ \mu^{-1} \Leftrightarrow \Psi = \mu \circ \Psi_0 \circ \mu^{-1}.$$

Thus, Ψ and Ψ_0 are conjugate in $\mathfrak{I}(\mathbb{T}^2, g)$. Therefore, if Σ_0 is a fundamental polygon of the orbifold $\mathcal{N}_0 = \mathbb{T}^2/\Psi_0$ on $\mathbb{T}^2$ then

$$\Psi_0 \cdot x \cap \Sigma_0 \neq \varnothing \Leftrightarrow \Psi \cdot \mu(x) \cap \mu(\Sigma_0) \neq \varnothing, \tag{10}$$

where $\Psi_0 \cdot x$ is the orbit of x under Ψ_0, while $\Psi \cdot \mu(x)$ is the orbit of $\mu(x)$ under Ψ. It follows from (10) that $\Sigma = \mu(\Sigma_0)$ is a fundamental polygon of $\mathcal{N} = \mathbb{T}^2/\Psi$ on the same torus. We obtain it from Σ_0 using the translation $\mu = \langle E, \{\sigma/2\}\rangle$ (see Fig. 1).

Let $r_0 : \mathbb{T}^2 \to \mathcal{N}_0 = \mathbb{T}^2/\Psi_0$ be the quotient mapping. Since $\Psi = \mu \circ \Psi_0 \circ \mu^{-1}$, the transformation μ of the torus induces the mapping $\mu_0 : \mathcal{N}_0 \to \mathcal{N}$ of orbifolds satisfying $r \circ \mu = \mu_0 \circ r_0$.

Since r and r_0 are pseudo-Riemannian covering mappings, while μ is an isometry of $(\mathbb{T}^2, g)$, it follows that μ_0 is an isometry between $(\mathcal{N}, g_{\mathcal{N}})$ and $(\mathcal{N}_0, g_{\mathcal{N}_0})$. Therefore, the isometry groups $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ and $\mathfrak{I}(\mathcal{N}_0, g_{\mathcal{N}_0})$ are isomorphic Lie groups. Consequently, the claims of Proposition 2 proved above for $\mathfrak{I}(\mathcal{N}_0, g_{\mathcal{N}_0})$ hold for $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ as well. $\square$

7.2. Proof of Theorem 5.

Notation. For every group Ψ of diffeomorphisms of the manifold M put $\operatorname{fix}(\Psi) = \{x \in M \mid \psi(x) = x\}$. Denote by Ψ_{*x} the group consisting of the differentials $\{\psi_{*x}\}$ at $x \in \operatorname{fix}(\Psi)$ of all diffeomorphisms ψ in Ψ. The following lemma is easy.

Lemma 5. *Let $\Psi = \langle\psi\rangle$ be an isometry group of a pseudo-Euclidean flat E_1^2, where $\psi = \langle A_t^{+-}, 0\rangle$ for $t \neq 0$. Then*

(1) *the normalizer $N(\Psi)$ of Ψ in $O(1,1)$ is equal to $\Psi \times \Phi$, where $\Phi = \langle -E\rangle$, and $N(\Psi)/\Psi \cong \Phi \cong \mathbb{Z}_2$;*

(2) *the set of vectors fixed under the group Ψ_{*x}, where $x \in \mathrm{fix}(\Psi)$, constitutes a one-dimensional vector subspace of the tangent space $T_x\mathbb{T}^2$.*

Keep the above notation. Suppose that the isometry group $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ of a compact Lorentzian orbifold $(\mathcal{N}, g_{\mathcal{N}})$ is not compact. By Proposition 1 there exist a complete flat Lorentzian torus $(\mathbb{T}^2, g)$ and an isometry group $\Psi \subset \mathfrak{I}(\mathbb{T}^2, g)$ such that $\mathcal{N} = \mathbb{T}^2/\Psi$, while the quotient mapping $r : \mathbb{T}^2 \to \mathcal{N} = \mathbb{T}^2/\Psi$ is a pseudo-Riemannian covering and, moreover, Ψ is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2$. This means that $g = r^* g_{\mathcal{N}}$. Furthermore, the metric $g_{\mathcal{N}}$ is also complete and flat.

Case 1. Suppose that $\Psi \cong \mathbb{Z}_2$ and, moreover, $\Psi = \langle\psi\rangle$, where ψ reverses the orientation of the torus. Assume that $\psi = \langle B, \delta\rangle$. Then $B \in GL(2,\mathbb{Z})$ and $\det(B) = -1$. Since the orbifold $\mathcal{N} = \mathbb{T}^2/\Psi$ has an orbifold point, there exists a point $v \in \mathbb{T}^2$ fixed by ψ. In the standard basis the differential ψ_{*v} of ψ at v is determined by some matrix $B = (b_{ij})$, $i, j = 1, 2$. Therefore, the equation $\psi_{*v}(X) = X$, where $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, becomes $(B - E)X = 0$.

According to Lemma 5, every orientation-reversing isometry of the pseudo-Euclidean vector space $(T_v\mathbb{T}^2, g_v)$, where g_v is the restriction of g onto $T_v\mathbb{T}^2$, has a one-dimensional eigensubspace associated to the eigenvalue 1. Therefore, $\det(B - E) = 0$. Hence, considering that $\det(B) = b_{11}b_{22} - b_{12}b_{21} = -1$, we obtain $b_{11} + b_{22} = 0$. Consequently,

$$X = \begin{cases} \lambda \begin{pmatrix} b_{12} \\ 1 - b_{11} \end{pmatrix}, & \text{if } b_{12} \neq 0, \\ \lambda \begin{pmatrix} 2 \\ b_{21} \end{pmatrix}, & \text{if } b_{12} = 0,\ b_{11} = 1, \end{cases} \qquad \lambda \in \mathbb{R}^1 \backslash 0. \tag{11}$$

Since $(\mathbb{T}^2, g)$ is a complete Lorentzian manifold, the exponential mapping Exp_v is defined on the whole tangent space $T_v\mathbb{T}^2$. By the isometry property of pseudo-Riemannian manifolds, ψ satisfies

$$\mathrm{Exp}_v \circ \psi_{*v} = \psi \circ \mathrm{Exp}_v\,. \tag{12}$$

For the vectors X satisfying (11) this yields

$$\mathrm{Exp}_v(sX) = \psi(\mathrm{Exp}_v(sX)) \quad \forall s \in \mathbb{R}^1.$$

Consequently, ψ fixes every point of the geodesic $\gamma_X(s) = \mathrm{Exp}_v(sX)$. This geodesic is the image of the line in $T_v\mathbb{T}^2$ defined in the coordinate system O, e_1, e_2 by the equation $y = kx + b$, where

$$k = \begin{cases} (1 - b_{11})/b_{12} & \text{if } b_{12} \neq 0, \\ b_{21}/2 & \text{if } b_{12} = 0,\ b_{11} = 1, \\ \infty & \text{if } b_{12} = 0,\ b_{11} = -1. \end{cases}$$

For $k = \infty$ the geodesic γ is closed. If $k \neq \infty$ then k is a rational number; therefore, $\gamma = \gamma_X(s)$ is also a closed geodesic of the torus, while $r(\gamma)$, which is a closed geodesic of the orbifold $(\mathcal{N}, g_{\mathcal{N}})$, is a connected component of the one-dimensional stratum Δ^1.

Observe that $\phi_\tau = \{\langle E, \{\tau X\}\rangle \mid \tau \in \mathbb{R}^1\}$ amounts to a compact one-parameter isometry group of the torus $(\mathbb{T}^2, g)$; moreover, $\phi_\tau \circ \psi = \psi \circ \phi_\tau$ for every $\tau \in \mathbb{R}^1$. Consequently, ϕ_τ induces a one-parameter isometry group of $(\mathcal{N}, g_{\mathcal{N}})$ which is transitive on $r(\gamma)$ and isomorphic to S^1.

The orbifold $\mathcal{N}$ has the stratification $\Delta = \{\Delta^2, \Delta^1\}$ and, moreover, every connected component of Δ^1 is diffeomorphic to S^1. Theorem 1.1 of [2] implies that the stratum of the lowest dimension is a closed submanifold of $\mathcal{N}$. Therefore, Δ^1 amounts to a union of finitely many circles. Lemma 5 implies that every stationary subgroup $\mathfrak{I}_w(\mathcal{N}, g_{\mathcal{N}})$ is isomorphic to a subgroup of $\mathbb{Z}_2$. Hence, the invariance of Δ^1

under $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ and the transitivity of $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ on every connected component of Δ^1 imply that the group $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ is compact.

CASE 2. Suppose that $\Psi \cong \mathbb{Z}_2$ and, moreover, $\Psi = \langle \psi \rangle$, where ψ preserves the orientation of the torus. By Proposition 2 $\mathcal{N}$ is isomorphic in the category $\mathcal{O}rb$ to the standard pillow, while the full isometry group $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ of $\mathcal{N}$ is not compact. Therefore, without loss of generality, we can assume that $\mathcal{N} = \mathbb{T}^2/\Psi_0$, where $\Psi_0 = \langle \psi_0 \rangle$ and $\psi_0 = \langle -E, \{0\} \rangle$. The claims (3) and (4) of Theorem 5 follow from the corresponding claims of Proposition 2.

CASE 3. Suppose that $\Psi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, there are distinct orientation-reversing isometries $\phi, \gamma \in \Psi$ of the torus isometry with $\Psi \cong \Gamma \times \Phi$, where $\Gamma = \langle \gamma \rangle \cong \mathbb{Z}_2$ and $\Phi = \langle \phi \rangle \cong \mathbb{Z}_2$. Observe that $\phi = -\gamma$ and $\phi^2 = \mathrm{Id}_{\mathbb{T}^2}$. Using this, we can easily show that $\mathrm{fix}(\Psi) \neq \varnothing$.

The composition $\psi_1 = \phi \circ \gamma$ is a nontrivial orientation-preserving isometry in Ψ; moreover, the group $\Psi_1 = \langle \psi_1 \rangle$ is isomorphic to $\mathbb{Z}_2$. As we showed in the proof of Proposition 2, the subset $\mathrm{fix}(\Psi_1)$ consists of four points. Since $\mathrm{fix}(\Psi) \subset \mathrm{fix}(\Psi_1)$, the subset $\mathrm{fix}(\Psi)$ is finite. Consequently, the subset Δ^0 of $\mathcal{N}$ consisting of the orbifold points with the orbifold group Ψ is finite as well. Lemma 5 implies that the stationary subgroup $\mathfrak{I}_w(\mathcal{N}, g_{\mathcal{N}})$ of the isometry group $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ at $w = r(x)$, where $x \in \mathrm{fix}(\Psi)$, is finite. Hence, the invariance of Δ^0 under $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ implies the finiteness of the orbit of $w \cdot \mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$. Therefore, $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ is a finite group.

Thus, $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}})$ is not compact only in case 2. $\square$

8. Proof of Theorem 6

Since Ψ_0 is an isometry group for every flat Lorentzian metric g on the standard torus $\mathbb{T}^2$; therefore, g uniquely determines the metric $g_{\mathcal{N}}$ on $\mathcal{N}$ with respect to which the quotient mapping $r : \mathbb{T}^2 \to \mathcal{N} = \mathbb{T}^2/\Psi_0$ is a pseudo-Riemannian covering. Furthermore, the class of similar Lorentzian metrics $[g] \in \mathcal{M}$ on $\mathbb{T}^2$ corresponds the class of similar Lorentzian metrics $[g_{\mathcal{N}}] \in \mathcal{L}$ on $\mathcal{N}$. In addition, if $[g_{\mathcal{N}}] \in \mathcal{L}$ then by Proposition 1 the Lorentzian metric $g = r^* g_{\mathcal{N}}$ determines the class $[g] \in \mathcal{M}$. Thus, $\kappa : \mathcal{M} \to \mathcal{L}$ is a bijection. $\square$

9. Examples

In all examples $(\mathbb{T}^2, g)$ is a flat Lorentzian torus and $(\mathcal{N} = \mathbb{T}^2/\Psi_0, g_{\mathcal{N}})$ is the standard pillow with induced Lorentzian metric, that is, $r^* g_{\mathcal{N}} = g$.

EXAMPLE 1. Let the Lorentzian metric g of the torus have the canonical form $G = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in the standard basis. Then $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$ maps the isotropic geodesic of the pseudo-Euclidean flat $(\mathbb{R}^2, g)$ defined by the equation $y = x$ into a closed geodesic on the torus. Theorem 2 implies that $\mathfrak{I}(\mathbb{T}^2, g)$ is a compact group. Its subgroup $\mathfrak{I}_0(\mathbb{T}^2, g)$ is generated by the transformations E^{+-} and $-E$. By Proposition 2 the full isometry group of the orbifold $(\mathcal{N}, g_{\mathcal{N}})$ is finite and isomorphic to the semidirect product $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

EXAMPLE 2. Let g be the flat Lorentzian metric on $\mathbb{T}^2$ defined by the matrix $G = \eta \begin{pmatrix} 1 - k^2 & 0 \\ 0 & 1 \end{pmatrix}$ in the standard basis, where η is an arbitrary nonzero real number and k is an integer distinct from ± 1. The group $\mathfrak{I}_0(\mathbb{T}^2, g)$ is generated by the transformations $A = \begin{pmatrix} k & 1 \\ k^2 - 1 & k \end{pmatrix}$, E^{+-}, and $-E$. Consequently, $\mathfrak{I}(\mathbb{T}^2, g) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}) \ltimes T^2$ and $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}}) \cong (\mathbb{Z}_2 \times \mathbb{Z}) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a noncompact discrete group.

EXAMPLE 3. Let g be a flat Lorentzian metric on $\mathbb{T}^2$ defined by the matrix $G = \eta \begin{pmatrix} 2 & m \\ m & 2 \end{pmatrix}$ in the standard basis, where η is a nonzero real number and m is an integer with $|m| > 2$. The group $\mathfrak{I}_0(\mathbb{T}^2, g)$ is generated by the transformations $A = \begin{pmatrix} m & 1 \\ -1 & 0 \end{pmatrix}$, $\widetilde{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $-E$. Therefore, $\mathfrak{I}(\mathbb{T}^2, g) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}) \ltimes T^2$ and $\mathfrak{I}(\mathcal{N}, g_{\mathcal{N}}) \cong (\mathbb{Z}_2 \times \mathbb{Z}) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a noncompact discrete group.

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N. I. Zhukova; E. A. Rogozhina
Nizhniĭ Novgorod State University, Nizhniĭ Novgorod, Russia
E-mail address: n.i.zhukova@rambler.ru; earogozhina@yandex.ru