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Selection Principle for Pointwise Bounded Sequences of Functions

Yu. V. Tret'yachenko[1*] and V. V. Chistyakov[2]**

[1] *Nizhnii Novgorod State University*

[2] *Higher School of Economics, Moscow State University (Nizhnii Novgorod Branch)*

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Abstract—For a number $\varepsilon > 0$ and a real function f on an interval $[a,b]$, denote by $N(\varepsilon, f, [a,b])$ the least upper bound of the set of indices n for which there is a family of disjoint intervals $[a_i, b_i]$, $i = 1, \dots, n$, on $[a,b]$ such that $|f(a_i) - f(b_i)| > \varepsilon$ for any $i = 1, \dots, n$ ($\sup \varnothing = 0$). The following theorem is proved: *if $\{f_j\}$ is a pointwise bounded sequence of real functions on the interval $[a,b]$ such that $n(\varepsilon) \equiv \limsup_{j\to\infty} N(\varepsilon, f_j, [a,b]) < \infty$ for any $\varepsilon > 0$, then the sequence $\{f_j\}$ contains a subsequence which converges, everywhere on $[a,b]$, to some function f such that $N(\varepsilon, f, [a,b]) \le n(\varepsilon)$ for any $\varepsilon > 0$.* It is proved that the main condition in this theorem related to the upper limit is necessary for any uniformly convergent sequence $\{f_j\}$ and is "almost" necessary for any everywhere convergent sequence of measurable functions, and many pointwise selection principles generalizing Helly's classical theorem are consequences of our theorem. Examples are presented which illustrate the sharpness of the theorem.

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1. MAIN RESULTS

The objective of the present note is to represent a new sufficient condition on a pointwise bounded sequence of real functions $\{f_j\} \equiv \{f_j\}_{j\in\mathbb{N}}$ on an interval $[a,b]$ of a real line $\mathbb{R}$ under which this sequence has a subsequence convergent everywhere on $[a,b]$. Our main result, Theorem 1, contains (as special cases) both the classical Helly's classical selection principles for monotone functions and for functions of (Jordan) bounded variation ([1, Chap. VIII, Sec. 4]) and, as will be shown in Sec. 4, also the majority of generalizations of these selection principles ([2, Part III, Sec. 2], [3]–[9], and the references therein). Note that Theorem 1 remains valid for a pointwise relatively compact sequence of functions $\{f_j\}$ acting from a nonempty subset of $\mathbb{R}$ to a metric space. However, to represent the ideas in the simplest form and to be able to compare the result with other selection principles, in this note we consider only real functions on an interval $[a,b]$.

For a number $\varepsilon > 0$ and a function $f\colon [a,b] \to \mathbb{R}$ on an interval $[a,b]$, introduce the quantity

$$N(\varepsilon, f, [a,b]) \in \{0\} \cup \mathbb{N} \cup \{\infty\}$$

as the least upper bound of the set of indices $n \in \mathbb{N}$ for which there is a family of disjoint intervals $[a_i, b_i]$, $i = 1, \dots, n$, on $[a,b]$ such that $|f(a_i) - f(b_i)| > \varepsilon$ for any $i = 1, \dots, n$ (with the agreement that $\sup \varnothing = 0$). One of the known properties of the quantity $N(\varepsilon, f, [a,b])$ is as follows (Theorem 2.1 in [2, Part III]): a function $f\colon [a,b] \to \mathbb{R}$ has one-sided finite left and right limits at all points of the interval $[a,b]$ if and only if $N(\varepsilon, f, [a,b]) < \infty$ for any $\varepsilon > 0$; in this case, the function f is bounded.

The main result of the paper, which is another application of the quantity $N(\varepsilon, f, [a,b])$ introduced above, is the following *pointwise selection principle*.

*E-mail: tretyachenko_y_v@mail.ru
**E-mail: czeslaw@mail.ru

Theorem 1. *Let $\{f_j\}$ be a pointwise bounded sequence of real functions on an interval $[a,b]$ such that*

$$\limsup_{j\to\infty} N(\varepsilon, f_j, [a,b]) < \infty \qquad \text{for any } \varepsilon > 0. \tag{1}$$

In this case, $\{f_j\}$ contains a subsequence which converges everywhere on $[a,b]$ to some function $f\colon [a,b] \to \mathbb{R}$ such that the number $N(\varepsilon, f, [a,b])$ does not exceed the upper limit in (1) *for any $\varepsilon > 0$.*

It is of interest to note that, as well as condition (9) presented in Sec. 4.4, which was found earlier in [7, Theorem 2] and [10, Lemma 4], assumption (1) of Theorem 1 is necessary for any uniformly convergent sequence $\{f_j\}$ and is "almost" necessary for any everywhere convergent sequence $\{f_j\}$ of measurable functions, as is proved in Theorem 2 below. However, the assumptions for the majority of known selection principles (cf. Secs. 4.1–4.3 and 4.5) are not necessary conditions.

Note that the quantity $N(\varepsilon, f, E)$ can be introduced on any set E, $\varnothing \neq E \subset [a,b]$, for any function $f\colon [a,b] \to \mathbb{R}$, if one assumes in addition that the ends a_i and b_i of the disjoint intervals $[a_i, b_i]$, $i = 1, \dots, n$, mentioned above belong to E.

Theorem 2. (a) *If a sequence $\{f_j\}$ of real functions converges uniformly on $[a,b]$ to some function $f\colon [a,b] \to \mathbb{R}$ such that $N(\varepsilon, f, [a,b]) < \infty$ for any $\varepsilon > 0$, then condition* (1) *holds*; *more exactly,*

$$\limsup_{j\to\infty} N(\varepsilon, f_j, [a,b]) \le \lim_{\delta\to\varepsilon-0} N(\delta, f, [a,b]) \qquad \text{for any } \varepsilon > 0.$$

(b) *If a sequence of real measurable functions $\{f_j\}$ converges on $[a,b]$ everywhere* (*or almost everywhere*) *to some function $f\colon [a,b] \to \mathbb{R}$ satisfying the condition $N(\varepsilon, f, [a,b]) < \infty$ for any $\varepsilon > 0$, then, for any $\eta > 0$, there is a Lebesgue measurable set E_η on $[a,b]$ such that the measure of E_η does not exceed η and*

$$\limsup_{j\to\infty} N(\varepsilon, f_j, [a,b] \setminus E_\eta) < \infty \qquad \text{for any } \varepsilon > 0.$$

These theorems are proved in the next section. In Sec. 3, the sharpness of the conditions of Theorems 1 and 2 is illustrated by examples. In the last section, Sec. 4, Theorem 1 is compared with the most well-known (at present) pointwise selection principles generalizing Helly's theorem.

2. PROOF OF THE MAIN THEOREMS

Proof of Theorem 1. 1. Let us show that there is a subsequence of $\{f_j\}$, which we denote by $\{f_j\}$ again, and, for any $k \in \mathbb{N}$, there is a nondecreasing bounded function $n_k\colon [a,b] \to \mathbb{N}$ such that

$$\lim_{j\to\infty} N(1/k, f_j, [a,t]) = n_k(t) \qquad \text{for any } k \in \mathbb{N} \text{ and } t \in [a,b]. \tag{2}$$

Note first that, by condition (1), for any $\varepsilon > 0$, there are indices $M(\varepsilon), j_0(\varepsilon) \in \mathbb{N}$ such that $N(\varepsilon, f_j, [a,b]) \le M(\varepsilon)$ for any $j \ge j_0(\varepsilon)$. The sequence

$$\{t \mapsto N(1, f_j, [a,t])\}_{j \ge j_0(1)}$$

consisting of nondecreasing functions is uniformly bounded on $[a,b]$ by a constant $M(1)$. By Helly's selection principle for monotone functions, the sequence $\{f_j\}_{j\ge j_0(1)}$, and thus the original sequence $\{f_j\}$, has a subsequence $\{f_{J_1(j)}\}_{j=1}^{\infty}$ (here the sequence $J_1\colon \mathbb{N} \to \mathbb{N}$, as well as similar sequences J_k for $k \ge 2$ occurring below, is strictly increasing) such that $N(1, f_{J_1(j)}, [a,t])$ converges as $j \to \infty$ to $n_1(t)$ for any $t \in [a,b]$, where $n_1\colon [a,b] \to \mathbb{N}$ is a nondecreasing bounded function. Choose the least number $j_1 \in \mathbb{N}$ such that $J_1(j_1) \ge j_0(1/2)$. In this case, the sequence of nondecreasing functions

$$\{t \mapsto N(1/2, f_{J_1(j)}, [a,t])\}_{j \ge j_1}$$

is uniformly bounded on $[a,b]$ by a constant $M(1/2)$. Applying Helly's selection principle again, we find a subsequence $\{f_{J_2(j)}\}_{j=1}^{\infty}$ of $\{f_{J_1(j)}\}_{j\ge j_1}$ such that $N(1/2, f_{J_2(j)}, [a,t])$ converges as $j \to \infty$ to $n_2(t)$

for any $t \in [a,b]$, where $n_2\colon [a,b] \to \mathbb{N}$ is also a nondecreasing bounded function. Choose the minimal index $j_2 \in \mathbb{N}$ such that $J_2(j_2) \ge j_0(1/3)$. Let us now assume by induction that, for an index $k \ge 3$, a subsequence $\{f_{J_{k-1}(j)}\}_{j=1}^{\infty}$ of the original sequence $\{f_j\}$ and the least index $j_{k-1} \in \mathbb{N}$ such that

$$J_{k-1}(j_{k-1}) \ge j_0\left(\frac{1}{k}\right)$$

are already chosen. Applying Helly's theorem to the sequence of nondecreasing functions

$$\{t \mapsto N(1/k, f_{J_{k-1}(j)}, [a,t])\}_{j \ge j_{k-1}},$$

which is uniformly bounded on $[a,b]$ by a constant $M(1/k)$, shows that there are subsequences $\{f_{J_k(j)}\}_{j=1}^{\infty}$ of the sequence $\{f_{J_{k-1}(j)}\}_{j \ge j_{k-1}}$ and a nondecreasing bounded function $n_k\colon [a,b] \to \mathbb{N}$ such that $N(1/k, f_{J_k(j)}, [a,t])$ converges as $j \to \infty$ to $n_k(t)$ for any $t \in [a,b]$. Noticing that, for any $k \in \mathbb{N}$, the sequence $\{f_{J_j(j)}\}_{j \ge k}$ is a subsequence of $\{f_{J_k(j)}\}_{j=1}^{\infty}$, we see by the above considerations that the diagonal sequence $\{f_{J_j(j)}\}_{j=1}^{\infty}$, which we denote by $\{f_j\}$ again, satisfies condition (2).

2. For any $k \in \mathbb{N}$, the set $S_k \subset [a,b]$ of discontinuity points of the monotone function n_k is at most countable. Write

$$S = ([a,b] \cap \mathbb{Q}) \cup \bigcup_{k=1}^{\infty} S_k,$$

where $\mathbb{Q}$ is the set of all rational numbers. In this case, S is a countable dense subset of $[a,b]$ such that

$$\text{the function } n_k \text{ is continuous on } [a,b] \setminus S \text{ for any } k \in \mathbb{N}. \tag{3}$$

Since the sequence $\{f_j\}$ is pointwise bounded and S is countable, we can assume without loss of generality (passing to a subsequence of $\{f_j\}$ by using the standard diagonal process if necessary) that, for any $s \in S$, the sequence $\{f_j(s)\}$ converges on $\mathbb{R}$ as $j \to \infty$ to some point denoted by $f(s)$.

Let us now show that the sequence $\{f_j(t)\}$ is a Cauchy sequence at any point $t \in [a,b] \setminus S$. Let $\varepsilon > 0$ be arbitrary. Choose and fix an index $k = k(\varepsilon) \in \mathbb{N}$ such that $1/k \le \varepsilon/3$. Since by (3), t is a continuity point for the function n_k and the set S is dense in $[a,b]$, there is a point $s = s(k,t) \in S$, which thus depends on ε only, such that $n_k(t) = n_k(s)$. To be definite, let $s < t$ (the case of $t < s$ is similar). Using (2), we choose indices $J_1 = J_1(k,t)$, $J_2 = J_2(k,s) \in \mathbb{N}$ also depending on ε only and such that

$$N\left(\frac{1}{k}, f_j, [a,t]\right) = n_k(t) \quad \text{for any } j \ge J_1 \qquad \text{and} \qquad N\left(\frac{1}{k}, f_j, [a,s]\right) = n_k(s) \quad \text{for any } j \ge J_2.$$

In this case, for $j \ge \max\{J_1, J_2\}$, we see that

$$N\left(\frac{1}{k}, f_j, [s,t]\right) \le N\left(\frac{1}{k}, f_j, [a,t]\right) - N\left(\frac{1}{k}, f_j, [a,s]\right) = n_k(t) - n_k(s) = 0,$$

and hence $N(1/k, f_j, [s,t]) = 0$, which means (by the definition of the quantity on the left-hand side of the last equality) that, in particular,

$$|f_j(s) - f_j(t)| \le \frac{1}{k} \le \frac{\varepsilon}{3}.$$

Since the sequence $\{f_j(s)\}$ converges, it is a Cauchy sequence, and therefore there is an index

$$J_3 = J_3(\varepsilon, s) \in \mathbb{N}$$

such that

$$|f_j(s) - f_l(s)| \le \frac{\varepsilon}{3} \qquad \text{for any } j, l \ge J_3.$$

Hence the index $J = \max\{J_1, J_2, J_3\}$ depends on ε only, and, for any $j, l \ge J$, we obtain

$$|f_j(t) - f_l(t)| \le |f_j(t) - f_j(s)| + |f_j(s) - f_l(s)| + |f_l(s) - f_l(t)| \le \varepsilon.$$

The sequence $\{f_j(t)\}$ converges because it is a Cauchy sequence. Denote by $f(t)$ the limit of this sequence. This defines a function $f\colon [a,b]=S\cup([a,b]\setminus S)\to\mathbb{R}$, and the sequence $\{f_j\}$, which is a subsequence of the original sequence $\{f_j\}$, converges to the function f everywhere on $[a,b]$. It remains to note that

$$N(\varepsilon,f,[a,b])\le\liminf_{j\to\infty}N(\varepsilon,f_j,[a,b])\le\limsup_{j\to\infty}N(\varepsilon,f_j,[a,b]) \tag{4}$$

for any $\varepsilon>0$ and to establish here the left inequality.

To this end, without loss of generality, assume that $N(\varepsilon,f,[a,b])>0$. It is sufficient to show that, if, for an index $n\in\mathbb{N}$, there are disjoint intervals $[a_i,b_i]$, $i=1,\dots,n$, on $[a,b]$ such that $|f(a_i)-f(b_i)|>\varepsilon$ for any $i=1,\dots,n$, then

$$n\le\liminf_{j\to\infty}N(\varepsilon,f_j,[a,b]).$$

For any number $\varepsilon'=\varepsilon'(n)$ such that

$$\min_{1\le i\le n}|f(a_i)-f(b_i)|>\varepsilon'>\varepsilon,$$

since f_j converges to f everywhere on $[a,b]$, there is an index $J=J(n)\in\mathbb{N}$ such that the values $|f(a_i)-f_j(a_i)|$ and $|f_j(b_i)-f(b_i)|$ do not exceed $(\varepsilon'-\varepsilon)/2$ for any $i=1,\dots,n$ and $j\ge J$. Consequently, for the same i and j, we see that

$$\begin{aligned}\varepsilon'<|f(a_i)-f(b_i)|&\le|f(a_i)-f_j(a_i)|+|f_j(a_i)-f_j(b_i)|+|f_j(b_i)-f(b_i)|\\&\le\frac{\varepsilon'-\varepsilon}{2}+|f_j(a_i)-f_j(b_i)|+\frac{\varepsilon'-\varepsilon}{2}=|f_j(a_i)-f_j(b_i)|+\varepsilon'-\varepsilon,\end{aligned}$$

and therefore $|f_j(a_i)-f_j(b_i)|>\varepsilon$, $i=1,\dots,n$. By definition of $N(\varepsilon,f_j,[a,b])$, we then see that $n\le N(\varepsilon,f_j,[a,b])$ for any $j\ge J$, and hence

$$n\le\inf_{j\ge J}N(\varepsilon,f_j,[a,b])\le\liminf_{j\to\infty}N(\varepsilon,f_j,[a,b]),$$

which completes the proof of the inequality, and hence of Theorem 1 as well. □

Proof of Theorem 2. (a) By assumption, the function f is bounded (see Sec. 1), and it follows from the uniform convergence of $\{f_j\}$ to f that

$$\sup_{s,t\in[a,b]}|(f_j-f)(s)-(f_j-f)(t)|\le2\sup_{t\in[a,b]}|f_j(t)-f(t)|\to0\qquad\text{as}\quad j\to\infty.$$

Moreover, for any $\varepsilon>0$ and $j\in\mathbb{N}$, the following inequality holds:

$$N(\varepsilon,f_j,[a,b])\le N(\varepsilon-\delta,f_j-f,[a,b])+N(\delta,f,[a,b])\qquad\text{for any }\ 0<\delta<\varepsilon. \tag{5}$$

Indeed, without loss of generality, assuming that the left-hand side of the inequality in (5) is greater than zero, we suppose that the index $n\in\mathbb{N}$ has the following property: there is a family $[a_i,b_i]$, $i=1,\dots,n$, of disjoint intervals on $[a,b]$ such that

$$|f_j(a_i)-f_j(b_i)|>\varepsilon\qquad\text{for any }\ i=1,\dots,n.$$

Since, for these i, we have

$$\varepsilon<|f_j(a_i)-f_j(b_i)|\le|(f_j-f)(a_i)-(f_j-f)(b_i)|+|f(a_i)-f(b_i)|,$$

it follows that either

$$|(f_j-f)(a_i)-(f_j-f)(b_i)|>\varepsilon-\delta\qquad\text{or}\qquad|f(a_i)-f(b_i)|>\delta;$$

we refer the index i to the set $I_1\subset\{1,\dots,n\}$ in the first case and to the set $I_2\subset\{1,\dots,n\}$ in the other case and denote by $|I_1|$ and $|I_2|$ the number (which can be zero) of elements in I_1 and I_2, respectively. It is clear that $|I_1|+|I_2|\ge n$. On the other hand, it follows from the definition of the quantity $N(\cdots)$ in Sec. 1 that

$$|I_1|\le N(\varepsilon-\delta,f_j-f,[a,b])\qquad\text{and}\qquad|I_2|\le N(\delta,f,[a,b]).$$

Thus, we have

$$n \le N(\varepsilon - \delta, f_j - f, [a,b]) + N(\delta, f, [a,b]),$$

which implies (5) because the above indices n are arbitrary.

Suppose now that ε, $\varepsilon > 0$, and δ, $0 < \delta < \varepsilon$, are arbitrary. There is an index $J \in \mathbb{N}$ depending on ε and δ such that

$$|(f_j - f)(s) - (f_j - f)(t)| \le \varepsilon - \delta \quad \text{for any } s, t \in [a,b] \text{ and } j \ge J.$$

Hence, by the definition of $N(\,\cdot\,)$, the first summand on the right-hand side in inequality (5) vanishes, and thus $N(\varepsilon, f_j, [a,b]) \le N(\delta, f, [a,b])$ for any $j \ge J$. Therefore

$$\limsup_{j\to\infty} N(\varepsilon, f_j, [a,b]) \le \sup_{j \ge J} N(\varepsilon, f_j, [a,b]) \le N(\delta, f, [a,b]).$$

It remains to take into account the fact that the function $\delta \mapsto N(\delta, f, [a,b])$ is nonincreasing and to pass to the limit in the last inequality for $\delta \to \varepsilon - 0$.

(b) By the Egorov theorem [1, Chap. IV, Sec. 3], it follows from the (almost) everywhere convergence of $\{f_j\}$ to f that, for any $\eta > 0$, there is a measurable set E_η on $[a,b]$ of measure $\le \eta$ such that $\{f_j\}$ uniformly converges to f on $[a,b] \setminus E_\eta$. Replacing the interval $[a,b]$ by $[a,b] \setminus E_\eta$ in the considerations in (a), we arrive at the bound

$$\limsup_{j\to\infty} N(\varepsilon, f_j, [a,b] \setminus E_\eta) \le \lim_{\delta\to\varepsilon-0} N(\delta, f, [a,b] \setminus E_\eta) \le \lim_{\delta\to\varepsilon-0} N(\delta, f, [a,b]) < \infty$$

for any $\varepsilon > 0$, as was to be established. □

3. EXAMPLES

In this section, we collect some examples of function sequences $\{f_j\}$ on the interval $[0,1]$ (except for Secs. 3.2 and 3.4, where $[a,b] = [0, 2\pi]$) showing the sharpness of the assumptions and conclusions of Theorems 1 and 2.

Since the oscillation

$$\operatorname{osc}(f, [a,b]) = \sup_{s,t \in [a,b]} |f(s) - f(t)|$$

of any bounded function $f\colon [a,b] \to \mathbb{R}$ is finite, it follows from the condition $\operatorname{osc}(f, [a,b]) = 0$ that f is constant, and hence $N(\varepsilon, f, [a,b]) = 0$ for any $\varepsilon > 0$; moreover, if $\varepsilon \ge \operatorname{osc}(f, [a,b]) > 0$, then $N(\varepsilon, f, [a,b]) = 0$ by definition. Therefore, below we write out estimates for $N(\varepsilon, f, [a,b])$ only for the case in which $0 < \varepsilon < \operatorname{osc}(f, [a,b])$. For instance, if $V_a^b(f)$ stands for the ordinary Jordan variation of a function f on an interval $[a,b]$ (see also Sec. 4.1 for $\varphi(t,u) = u$), then

$$\operatorname{osc}(f, [a,b]) \le V_a^b(f) \quad \text{and} \quad N(\varepsilon, f, [a,b]) \le \frac{1}{\varepsilon} V_a^b(f) \qquad \text{for} \quad 0 < \varepsilon < \operatorname{osc}(f, [a,b]). \tag{6}$$

3.1. The condition of pointwise boundedness of $\{f_j\}$ is essential in Theorem 1. Indeed, the function sequence with $f_j(t) = 0$ for $0 \le t < 1$ and $f_j(1) = j$ has no everywhere convergent subsequence; however, $N(\varepsilon, f_j, [0,1]) = 1$ for $0 < \varepsilon < j$, and therefore

$$\limsup_{j\to\infty} N(\varepsilon, f_j, [0,1]) = 1 \qquad \text{for any } \varepsilon > 0.$$

At the same time, the condition of pointwise boundedness is not necessary, because the function sequence with $f_j(t) = 0$ for $0 \le t < 1$ and $f_j(1) = j^{(-1)^j}$ contains an everywhere convergent subsequence corresponding to the odd indices j and is unbounded at the point $t = 1$, and, as above,

$$\limsup_{j\to\infty} N(\varepsilon, f_j, [0,1]) = 1 \qquad \text{for any } \varepsilon > 0.$$

3.2. Without assumption (1), Theorem 1 fails. Indeed, as is known, the sequence $f_j(t) = \sin(jt)$, $0 \le t \le 2\pi$, has no subsequence convergent everywhere on $[0, 2\pi]$. On the other hand, we claim that

$$4j \le N(\varepsilon, f_j, [0, 2\pi]) \le \frac{4j}{\varepsilon} \qquad \text{for any } 0 < \varepsilon < 1. \tag{7}$$

Since

$$\operatorname{osc}(f_j, [0, 2\pi]) = 2 \qquad \text{and} \qquad V_0^{2\pi}(f_j) = 4j,$$

the right inequality follows from (6) for $0 < \varepsilon < 2$. Suppose now that $0 < \varepsilon < 1$. For $j \in \mathbb{N}$, write

$$t_i^j = \frac{\pi i}{2j}, \quad i = 0, 1, \dots, 4j, \qquad \text{and} \qquad [a_i, b_i] = [t_{i-1}^j, t_i^j], \quad i = 1, \dots, 4j.$$

In this case, for any $i = 1, \dots, 4j$, we obtain

$$|f_j(a_i) - f_j(b_i)| = \left|\sin\left(\frac{\pi i}{2}\right) - \sin\left(\frac{\pi(i-1)}{2}\right)\right| = 1 > \varepsilon,$$

which implies the left inequality in (7) by the definition of the quantity $N(\,\cdot\,)$, and it follows from this inequality that

$$\limsup_{j\to\infty} N(\varepsilon, f_j, [0, 2\pi]) = \infty \qquad \text{for any } 0 < \varepsilon < 1.$$

3.3. Condition (1) is not necessary, and thus Theorem 2, (a) is violated for some everywhere convergent sequence $\{f_j\}$. Let $\mathscr{D}$ be the Dirichlet function,

$$\mathscr{D}(t) = 1 \quad \text{for } t \in [0,1] \cap \mathbb{Q} \qquad \text{and} \qquad \mathscr{D}(t) = 0 \quad \text{for } t \in [0,1] \setminus \mathbb{Q}.$$

In this case, $N(\varepsilon, \mathscr{D}, [0, 1]) = \infty$ for $0 < \varepsilon < 1 = \operatorname{osc}(\mathscr{D}, [0, 1])$. Consider the sequence

$$f_j(t) = \lim_{m\to\infty} (\cos(j!\,\pi t))^{2m}$$

on $[0, 1]$; the function f_j takes the value 1 if $j!\,t$ is an integer and the value 0 otherwise, and therefore the sequence converges to the Dirichlet function $\mathscr{D}$ everywhere on $[0, 1]$. Since $\operatorname{osc}(f_j, [0, 1]) = 1$ and $V_0^1(f_j) = 2 \cdot j!$, it follows that $N(\varepsilon, f_j, [0, 1]) = 2 \cdot j!$ for $0 < \varepsilon < 1$, and thus

$$\limsup_{j\to\infty} N(\varepsilon, f_j, [0, 1]) = \infty \qquad \text{for any } 0 < \varepsilon < 1.$$

Note that, as is also proved here, for any everywhere convergent sequence $\{f_j\}$, the condition that $N(\varepsilon, f_j, [0, 1]) < \infty$ for any $\varepsilon > 0$ and $j \in \mathbb{N}$ does not imply the relation

$$\limsup_{j\to\infty} N(\varepsilon, f_j, [0, 1]) < \infty \qquad \text{for any } \varepsilon > 0.$$

Moreover, it can also happen that all functions f_j in a sequence $\{f_j\}$ do not satisfy the condition $N(\varepsilon, f_j, [a, b]) < \infty$ for any $\varepsilon > 0$, whereas the limit function f in Theorem 1 always satisfies this condition, for instance, the sequence $f_j(t) = \mathscr{D}(t)/j$ converges to the zero function uniformly on $[0, 1]$, whereas $N(\varepsilon, f_j, [0, 1]) = \infty$ for $0 < \varepsilon < 1/j = \operatorname{osc}(f_j, [0, 1])$ and $N(\varepsilon, f_j, [0, 1]) = 0$ for any $\varepsilon > 0$ and $j \ge 1/\varepsilon$.

3.4. Another example of a convergent sequence $\{f_j\}$ (tending to the zero function) for which condition (1) is violated is the example in [10, Example 4], namely, by setting

$$f_j(t) = \sin(j^2 t) \quad \text{for } 0 \le t \le 2\pi/j \qquad \text{and} \qquad f_j(t) = 0 \quad \text{for } 2\pi/j \le t \le 2\pi,$$

we see that inequalities (7) are satisfied.

3.5. One cannot replace the left limit as $\delta \to \varepsilon - 0$ in the inequality of Theorem 2, (a) by the expression $N(\varepsilon, f, [a,b])$. Namely, the sequence given by

$$f_j(t) = 0 \quad \text{for} \quad 0 \le t < 1 \qquad \text{and} \qquad f_j(1) = 1 + \frac{1}{j}$$

converges uniformly on $[0,1]$ to the function f with $f(t) = 0$ for $0 \le t < 1$ and $f(1) = 1$, and thus, for $\varepsilon = 1$, we have

$$N(\varepsilon, f_j, [0,1]) = 1 \quad \text{for any } j \in \mathbb{N} \qquad \text{and} \qquad N(\delta, f, [0,1]) = 1 \quad \text{for any } 0 < \delta < \varepsilon,$$

whereas $N(\varepsilon, f, [0,1]) = 0$.

3.6. It can happen that, for some everywhere convergent sequence $\{f_j\}$, condition (1) holds, whereas the inequality in Theorem 2, (a) fails to hold. For instance, the sequence given by

$$f_j(t) = 1 \quad \text{for} \quad t = \frac{1}{j+1} \qquad \text{and} \qquad f_j(t) = 0 \quad \text{for} \quad t \neq \frac{1}{j+1}$$

converges everywhere on $[0,1]$ to the function with $f(0) = 1$ and $f(t) = 0$ for $0 < t \le 1$; moreover, if $0 < \varepsilon < 1$, then

$$N(\varepsilon, f_j, [0,1]) = 2 \qquad \text{for any } j \in \mathbb{N}, \text{ and } N(\varepsilon, f, [0,1]) = 1.$$

3.7. We present here an example of a sequence $\{f_j\}$ on $[0,1]$ which is not uniformly bounded and to which Theorem 1 can be applied. Set

$$f_j(t) = j \quad \text{for} \quad t = \frac{1}{j+1} \qquad \text{and} \qquad f_j(t) = 0 \quad \text{for} \quad t \neq \frac{1}{j+1}.$$

In this case, $\{f_j\}$ converges everywhere to $f \equiv 0$, and therefore $\{f_j\}$ is pointwise bounded; moreover,

$$\{f_j(t)\} = \{0\} \quad \text{for} \quad t \notin \{1/(k+1)\}_{k \in \mathbb{N}} \qquad \text{and} \qquad \{f_j(t)\} = \{0, k\} \quad \text{for} \quad t = 1/(k+1),\ k \in \mathbb{N}.$$

Further, $\operatorname{osc}(f_j, [0,1]) = j$ (this value tends to infinity as $j \to \infty$) and, by (6), we have

$$N(\varepsilon, f_j, [0,1]) = 2 \qquad \text{for} \quad 0 < \varepsilon < j,$$

which yields

$$\limsup_{j \to \infty} N(\varepsilon, f_j, [0,1]) = 2 \qquad \text{for any } \varepsilon > 0.$$

3.8. There are sequences $\{f_j\}$ which converge to zero everywhere (and hence are pointwise bounded) for which condition (1) is violated and

$$\limsup_{j \to \infty} \operatorname{osc}(f_j, [0,1]) = \infty.$$

Set $f_j(t) = jt^j$ for $0 \le t < 1$ and $f_j(1) = 0$. In this case, $\operatorname{osc}(f_j, [0,1]) = j$ and $V_0^1(f_j) = 2j$, and therefore, by (6), we have

$$N(\varepsilon, f_j, [0,1]) \le 2j/\varepsilon \qquad \text{for} \quad 0 < \varepsilon < j.$$

Let us show that

$$N(\varepsilon, f_j, [0,1]) \ge [j/(2\varepsilon)] \qquad \text{for} \quad j > 2\varepsilon$$

(here the square brackets stand for the integral part of a number), which will imply that

$$\limsup_{j \to \infty} N(\varepsilon, f_j, [0,1]) = \infty \qquad \text{for any } \varepsilon > 0.$$

Indeed, for $j > 2\varepsilon$, write

$$t_i = \left(\frac{2i\varepsilon}{j}\right)^{1/j}, \quad i = 0, 1, \dots, \left[\frac{j}{2\varepsilon}\right], \qquad \text{and} \qquad [a_i, b_i] = [t_{i-1}, t_i], \quad i = 1, \dots, \left[\frac{j}{2\varepsilon}\right].$$

Then $0 = t_0 < t_1 < \cdots < t_{[j/(2\varepsilon)]} \le 1$ and

$$|f_j(a_i) - f_j(b_i)| = j(t_i)^j - j(t_{i-1})^j = j\frac{2i\varepsilon}{j} - j\frac{2(i-1)\varepsilon}{j} = 2\varepsilon > \varepsilon$$

for any $i = 1, \dots, [j/(2\varepsilon)]$, which implies the desired inequality.

3.9. In the definition of the quantity $N(\varepsilon, f, [a, b])$ in Sec. 1 it is assumed that the disjoint intervals $[a_i, b_i]$ have the following property: $|f(a_i) - f(b_i)| > \varepsilon$ for any $i = 1, \dots, n$. If we replace here the inequality $> \varepsilon$ by $\ge \varepsilon$ and denote the new quantity by $N_{\ge}(\varepsilon, f, [a, b])$, then the left inequality in (4) can fail to hold for the new quantity. For instance, the sequence given by $f_j(t) = 1$ for $t = 0$, $f_j(t) = 0$ for $0 < t < 1$, and $f_j(t) = 1 - (1/j)$ for $t = 1$ converges uniformly for $[0, 1]$ to the function given by $f(0) = f(1) = 1$ and $f(t) = 0$ for $0 < t < 1$. For $\varepsilon = 1$, we see that

$$N_{\ge}(\varepsilon, f_j, [0, 1]) = 1 \qquad \text{for any } j \in \mathbb{N} \text{ and } N_{\ge}(\varepsilon, f, [0, 1]) = 2.$$

4. COMPARISON WITH KNOWN SELECTION PRINCIPLES

4.1. Let a function $\varphi\colon [a, b] \times \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$, satisfy the following two conditions:

(i) for any $t \in [a, b]$, the function $\varphi(t, \,\cdot\,)$ of the second argument is nondecreasing and continuous on $\mathbb{R}^+$, and $\varphi(t, u) \to \infty$ as $u \to \infty$;

(ii) $\varphi(t, 0) = 0$ for any $t \in [a, b]$, and $\inf_{t\in[a,b]} \varphi(t, u) > 0$ and any $u > 0$.

Following [3], for a function $f\colon [a, b] \to \mathbb{R}$, we set

$$V_\varphi(f, [a, b]) = \sup \sum_{i=1}^{n} \varphi(t_i, |f(a_i) - f(b_i)|),$$

where the supremum is taken over all $n \in \mathbb{N}$, all families $[a_i, b_i]$, $i = 1, \dots, n$, of disjoint intervals on $[a, b]$, and all points $t_i \in [a_i, b_i]$, $i = 1, \dots, n$. This generalized variation of the function f corresponds to the Wiener–Young φ-variation for $\varphi(t, u) = \varphi(u)$ and to the ordinary Jordan variation $V_a^b(f)$ for $\varphi(t, u) = u$. In [3], the following generalization of Helly's theorem was established: *if a sequence of functions $\{f_j\}$ on an interval $[a, b]$ is bounded at some point in $[a, b]$ and if*

$$\sup_{j\in\mathbb{N}} V_\varphi(f_j, [a, b]) = C < \infty,$$

then $\{f_j\}$ contains a subsequence which converges to some function f everywhere on $[a, b]$ for which $V_\varphi(f, [a, b]) \le C$. This result immediately follows from Theorem 1 (together with Helly's theorem for $\varphi(t, u) = u$ and its generalization in [4, Theorem 1.3] for $\varphi(t, u) = \varphi(u)$) by taking into account the bounds (cf. the bounds in the proof of Theorem 9 in [12])

$$\operatorname{osc}(f_j, [a, b]) \le 2 \max\{u \in \mathbb{R}^+ \mid \varphi(a, u) \le V_\varphi(f_j, [a, b])\},$$
$$N(\varepsilon, f_j, [a, b]) \le \frac{V_\varphi(f_j, [a, b])}{\inf_{t\in[a,b]} \varphi(t, \varepsilon)} \qquad \text{for} \quad 0 < \varepsilon < \operatorname{osc}(f_j, [a, b]),$$

and also the sequential lower semicontinuity of the functional V_φ meaning that, if a sequence $\{f_j\}$ converges everywhere on $[a, b]$ to a function f, then

$$V_\varphi(f, [a, b]) \le \liminf_{j\to\infty} V_\varphi(f_j, [a, b]) \le C.$$

4.2. Let $\Phi = \{\varphi_i\}_{i\in\mathbb{N}}$ be a sequence consisting of continuous nondecreasing unbounded functions $\varphi_i\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi_i(u) = 0$ for $u = 0$ only and the following two conditions are satisfied:

(a) $\varphi_{i+1}(u) \le \varphi_i(u)$ for any $i \in \mathbb{N}$ and $u \in \mathbb{R}^+$;

(b) $\sum_{i=1}^{\infty} \varphi_i(u) = \infty$ for any $u > 0$.

Following [8], for any function $f\colon [a,b] \to \mathbb{R}$, write

$$V_\Phi(f,[a,b]) = \sup \sum_{i=1}^{n} \varphi_i(|f(a_i) - f(b_i)|),$$

where the supremum is taken over all $n \in \mathbb{N}$ and all unordered families $[a_i, b_i]$, $i = 1,\dots,n$, of disjoint intervals on $[a,b]$. This generalized variation of the function f corresponds to the Jordan variation $V_a^b(f)$ for $\varphi_i(u) = u$, to the Wiener–Young φ-variation for $\varphi_i(u) = \varphi(u)$, and to the Waterman Λ-variation [9] for $\varphi_i(u) = u/\lambda_i$ for any $i \in \mathbb{N}$ and $u \in \mathbb{R}^+$, where $\Lambda = \{\lambda_i\}_{i\in\mathbb{N}}$ is a nondecreasing sequence of positive numbers for which $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. A generalization of Helly's theorem in [8, Theorem 2.8] (and its special case in [9, Theorem 5]) is formulated in just the same way as the corresponding assertion in Sec. 4.1 in which V_φ is replaced by V_Φ. This result follows from Theorem 1 thanks to the following bounds (cf. the bounds in the proof of Theorem 10 in [12]):

$$\operatorname{osc}(f_j,[a,b]) \le \max\{u \in \mathbb{R}^+ \mid \varphi_1(u) \le V_\Phi(f_j,[a,b])\},$$

$$N(\varepsilon, f_j,[a,b]) \le \max\biggl\{n \in \mathbb{N} \biggm| \sum_{i=1}^{n} \varphi_i(\varepsilon) \le V_\Phi(f_j,[a,b])\biggr\} \qquad \text{for} \quad 0 < \varepsilon < \operatorname{osc}(f_j,[a,b]),$$

and also to the property of sequential lower semicontinuity of the functional V_Φ.

4.3. Theorem 1 implies a Helly-type selection principle established in another way in [2, Part III, Sec. 2, Proposition 2.8] in which, for the validity of the conclusion of Theorem 1, it is assumed much more, namely, that the sequence $\{f_j\}$ is uniformly bounded,

$$\sup_{j\in\mathbb{N}} \operatorname{osc}(f_j,[a,b]) < \infty, \quad \text{and} \quad \sup_{j\in\mathbb{N}} N(\varepsilon, f_j,[a,b]) < \infty \qquad \text{for any } \varepsilon > 0. \tag{8}$$

4.4. Recall that by the *Chanturiya modulus of variation* [13] of a function $f\colon [a,b] \to \mathbb{R}$ one means the sequence $\{\nu(n,f,[a,b])\}_{n\in\mathbb{N}}$ defined by the rule

$$\nu(n,f,[a,b]) = \sup \sum_{i=1}^{n} |f(a_i) - f(b_i)|,$$

where, for a chosen $n \in \mathbb{N}$, the supremum is taken over all families $[a_i,b_i]$, $i = 1,\dots,n$, of disjoint intervals on $[a,b]$. As was noted in [13, Theorem 5] (see also [7, Theorem 3]), a function $f\colon [a,b] \to \mathbb{R}$ has finite one-sided left and right limits at all points of $[a,b]$ if and only if $\nu(n,f,[a,b]) = o(n)$ (i.e., $\nu(n,f,[a,b])/n \to 0$ as $n \to \infty$). Note that [2, Part III, Sec. 2, Theorem 2.2], for any sequence of functions $\{f_j\}$, conditions (8) are equivalent to the single condition

$$\sup_{j\in\mathbb{N}} \nu(n,f_j,[a,b]) = o(n).$$

In [7, Theorem 1] (and in its generalizations in [10]–[12]), the following selection principle generalizing Helly's theorem was established: *if a sequence of functions* $\{f_j\}$ *on* $[a,b]$ *is bounded at some point of* $[a,b]$ *and if*

$$\limsup_{j\to\infty} \nu(n,f_j,[a,b]) = o(n), \tag{9}$$

then $\{f_j\}$ *contains a subsequence which converges, everywhere on* $[a,b]$, *to some function* f *such that* $\nu(n,f,[a,b]) = o(n)$. This result follows from Theorem 1 if one takes into account that condition (9) is equivalent to each of the following two conditions simultaneously, namely, to condition (1) and to the condition $\limsup_{j\to\infty} \operatorname{osc}(f_j,[a,b]) < \infty$. Note that, in all above generalizations of Helly's theorem except for Theorem 1, the assumptions of these theorems imply that the original sequence $\{f_j\}$ is uniformly bounded.

4.5. For a function $f\colon [a,b] \to \mathbb{R}$ of alternating sign, denote by $\mathscr{P}(f)$ the set of all families of points $\{t_1, \dots, t_n\} \subset [a,b]$, where $n \in \mathbb{N}$, such that $t_1 < \dots < t_n$ and one of the following three conditions holds: $(-1)^i f(t_i) > 0$ for any $i = 1, \dots, n$, $(-1)^i f(t_i) < 0$ for any $i = 1, \dots, n$, or $(-1)^i f(t_i) = 0$ for any $i = 1, \dots, n$. Following [14], write

$$T(f,[a,b]) = \sup\biggl\{\sum_{i=1}^{n} |f(t_i)| \biggm| n \in \mathbb{N} \text{ and } \{t_i\}_{i=1}^{n} \in \mathscr{P}(f)\biggr\}.$$

If the function f is either nonnegative everywhere on $[a,b]$ or nonpositive everywhere on $[a,b]$, then we set $T(f,[a,b]) = \sup_{t\in[a,b]} |f(t)|$. The quantity $T(f,[a,b])$ thus defined is referred to as the *oscillation of f on $[a,b]$ in the sense of Schrader*.

The following generalization of Helly's theorem was established in [14, Theorem 1.2]: *if a sequence of functions $\{f_j\}$ on an interval $[a,b]$ is such that*

$$\sup_{j,k\in\mathbb{N}} T(f_j - f_k, [a,b]) < \infty,$$

then it contains a subsequence which converges everywhere on $[a,b]$. Such a sequence $\{f_j\}$ is pointwise bounded; however, no "regularity" properties of the limit function are claimed, as well as in the previous generalizations in Secs. 4.1–4.4. Let us present examples showing that this result and Theorem 1 are independent (a development of the Schrader oscillation was presented in [15], where the corresponding generalization of Helly's theorem was established (Theorem 2.1 in [15]; this result and Theorem 1 are also independent [16]), namely, the sequence in Example 3.7 satisfies the assumptions of Theorem 1 but does not satisfy any generalization of Helly's theorem presented above, whereas the sequence $f_j(t) = (-1)^j \mathscr{D}(t)$ on $[0,1]$ satisfies the Schrader selection principle but does not satisfy the conditions of any other selection principle presented above.

4.6. As usual, the modulus of continuity

$$\omega(\,\cdot\,, f)\colon [0, b-a] \to \mathbb{R}^+$$

of a continuous function $f\colon [a,b] \to \mathbb{R}$ is defined by the rule

$$\omega(\rho, f) = \sup\{|f(s) - f(t)| : s,t \in [a,b],\ |s-t| \le \rho\} \qquad \text{for} \quad 0 < \rho \le b-a,$$
$$\omega(0,f) = \lim_{\rho\to+0} \omega(\rho, f) = 0.$$

By the Weierstrass theorem, the oscillation $\operatorname{osc}(f,[a,b])$ is finite. A sequence of functions $\{f_j\}$ on $[a,b]$ is said to be equicontinuous if

$$\lim_{\rho\to+0} \sup_{j\in\mathbb{N}} \omega(\rho, f_j) = 0.$$

The following well-known Ascoli theorem implies that *every pointwise bounded equicontinuous sequence of real functions $\{f_j\}$ on $[a,b]$ contains an everywhere convergent subsequence.* This statement follows from Theorem 1 if one takes into account the relation

$$N(\varepsilon, f_j, [a,b]) \le \frac{b-a}{\min\{\rho \in \mathbb{R}^+ \mid \omega(\rho, f_j) = \varepsilon\}} \qquad \text{for} \quad 0 < \varepsilon < \operatorname{osc}(f_j, [a,b]).$$

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