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A Banach algebra of functions of several variables of finite total variation and Lipschitzian superposition operators. II[☆]

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Abstract

We characterize superposition Nemytskii operators, which map the Banach algebra of functions of n real variables with finite total variation in the sense of Vitali, Hardy and Krause into itself and satisfy the global Lipschitz condition. Our results extend previous results in this direction by Matkowski and Miś in [Math. Nachr. 117 (1984) 155–159] for $n = 1$ and the author in [Monatsh. Math. 137(2) (2002) 99–114] for $n = 2$.

Keywords: Functions of several variables; Finite total variation; Banach algebra property; Superposition operator; Lipschitz condition

1. Main results of Part II

This paper is a continuation of [5, Part I]. In order to make the presentation as independent of Part I as possible, here we briefly recall definitions, lemmas and main theorems of [5] and present the main results of this paper.

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Let $\mathbb{N}$ be the set of positive integers, $n \in \mathbb{N}$ be fixed and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Given $x \in \mathbb{R}^n$, we write $x = (x_i \mid i \in \{1, \dots, n\}) = (x_1, \dots, x_n)$ and if $y \in \mathbb{R}^n$, we say that $x = y$, $x \leqslant y$, $y \geqslant x$ or $x < y$ (in $\mathbb{R}^n$) if and only if $x_i = y_i$, $x_i \leqslant y_i$, $y_i \geqslant x_i$ or $x_i < y_i$ for all $i \in \{1, \dots, n\}$, respectively. We set $x \pm y = (x_1 \pm y_1, \dots, x_n \pm y_n)$. If $x \leqslant y$ in $\mathbb{R}^n$, the n-dimensional rectangle I_x^y with the endpoints x and y is defined by $I_x^y = [x_1, y_1] \times \cdots \times [x_n, y_n] = \{z \in \mathbb{R}^n \mid x \leqslant z \leqslant y\}$.

In what follows we fix $a, b \in \mathbb{R}^n$ with $a < b$ and the rectangle I_a^b, called the *basic rectangle*, which will be the domain of most functions under consideration.

Greek letters will designate *multiindices*, i.e., elements of $\mathbb{N}_0^n = (\mathbb{N}_0)^n$. Given $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha x = (\alpha_1 x_1, \dots, \alpha_n x_n) \in \mathbb{R}^n$. We write 0 for $(0, \dots, 0) \in \mathbb{N}_0^n$ and 1 for $(1, \dots, 1) \in \mathbb{N}_0^n$; each time the dimension of the zero or unit multiindex will be clear from the context. We also set $\mathscr{A}_0(n) = \{\alpha \in \mathbb{N}_0^n \mid \alpha \leqslant 1\}$ and $\mathscr{A}(n) = \mathscr{A}_0(n) \setminus \{0\}$.

A summation over multiindices will be understood over *n-dimensional* multiindices, the range of the summation will be specified under the summation sign. Thus, the sum $\sum_{\alpha \in \mathscr{A}(n)}$ will be written as $\sum_{0 \neq \alpha \leqslant 1}$.

The *Vitali nth mixed difference of* $f : I_a^b \to \mathbb{R}$ on the subrectangle $I_x^y \subset I_a^b$, where $a \leqslant x < y \leqslant b$, is the quantity [10]:

$$\mathrm{md}_n(f, I_x^y) = \sum_{0 \leqslant \theta \leqslant 1} (-1)^{|\theta|} f(x + \theta(y - x)).$$

The *Vitali nth variation* [7,10] of $f : I_a^b \to \mathbb{R}$ is defined by

$$V_n(f, I_a^b) = \sup_{\mathscr{P}} \sum_{1 \leqslant \sigma \leqslant \kappa} |\mathrm{md}_n(f, I_{x[\sigma-1]}^{x[\sigma]})|, \tag{1.1}$$

where the supremum is taken over all multiindices $\kappa \in \mathbb{N}^n$ and all partitions $\mathscr{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ of I_a^b; here by a (net) partition $\mathscr{P}$ of I_a^b we understand a collection of points of the form $x[\sigma] \equiv (x_1(\sigma_1), \dots, x_n(\sigma_n))$ from I_a^b indexed by $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{N}_0^n$ with $\sigma \leqslant \kappa$, written as $\{x[\sigma]\}_{\sigma=0}^{\kappa}$, and satisfying $x[0] = a$, $x[\kappa] = b$ and $x[\sigma - 1] < x[\sigma]$ in $\mathbb{R}^n$ for all $\sigma \in \mathbb{N}^n$, $1 \leqslant \sigma \leqslant \kappa$.

The lower order (Hardy and Krause) variation of $f : I_a^b \to \mathbb{R}$ is introduced as follows. Let $\alpha \in \mathscr{A}(n)$. Given $x \in \mathbb{R}^n$, we define the *truncation of x by α* by $x \lfloor \alpha = (x_i \mid i \in \{1, \dots, n\}, \alpha_i = 1) \in \mathbb{R}^{|\alpha|}$ and $I_a^b \lfloor \alpha = I_{a \lfloor \alpha}^{b \lfloor \alpha}$. If $z \in I_a^b$, we define the *truncated function* $f_\alpha^z : I_a^b \lfloor \alpha \to \mathbb{R}$ *with the base at z* by

$$f_\alpha^z(x \lfloor \alpha) = f(z + \alpha(x - z)), \quad x \in I_a^b.$$

The *$|\alpha|$-variation of f*, denoted by $V_{|\alpha|}(f_\alpha^a, I_a^b \lfloor \alpha)$, is the value (1.1) where n on the left is replaced by $|\alpha|$, f — by f_α^a and I_a^b — by $I_a^b \lfloor \alpha$.

The *total variation* of $f : I_a^b \to \mathbb{R}$ in the sense of Hildebrandt [6, III.6.3] and Leonov [7] (see also [5]) is defined by

$$\mathrm{TV}(f, I_a^b) = \sum_{0 \neq \alpha \leqslant 1} V_{|\alpha|}(f_\alpha^a, I_a^b \lfloor \alpha). \tag{1.2}$$

We set $\mathrm{BV}(I_a^b;\mathbb{R})=\{f:I_a^b\to\mathbb{R}\,|\,\mathrm{TV}(f,I_a^b)<\infty\}$ and equip it with the norm ([6, III.6.3] if $n=2$, [7] for all $n\in\mathbb{N}$):

$$\|f\|=|f(a)|+\mathrm{TV}(f,I_a^b),\qquad f\in\mathrm{BV}(I_a^b;\mathbb{R}). \tag{1.3}$$

The main result of Part I is the following:

Theorem 1 (*Chistyakov [5, Theorem 1]*). *The space* $\mathrm{BV}(I_a^b;\mathbb{R})$ *is a Banach algebra with respect to the usual pointwise operations and norm* (1.3), *and the following inequality holds*: $\|f\cdot g\|\leqslant 2^n\,\|f\|\cdot\|g\|$ *for all* $f,\ g\in\mathrm{BV}(I_a^b;\mathbb{R})$.

Let $\mathbb{R}^I$ be the algebra of all functions $f:I\to\mathbb{R}$ from $I=I_a^b$ into $\mathbb{R}$ equipped with the pointwise operations and $h:I\times\mathbb{R}\to\mathbb{R}$ be a given function. The *superposition operator* $H=H_h:\mathbb{R}^I\to\mathbb{R}^I$ generated by h is defined by

$$Hf(x)\equiv H(f)(x)=h(x,f(x)),\qquad x\in I,\quad f\in\mathbb{R}^I. \tag{1.4}$$

As a consequence of Theorem 1 we get: if $h:I\times\mathbb{R}\to\mathbb{R}$ is given by $h(x,u)=h_1(x)u+h_0(x)$, $x\in I_a^b$, $u\in\mathbb{R}$, for some functions $h_0,h_1\in\mathrm{BV}(I_a^b;\mathbb{R})$, then the superposition operator H, generated by h, maps $\mathrm{BV}(I_a^b;\mathbb{R})$ into itself and is Lipschitzian: there exists a nonnegative constant L (actually, $L=2^n\|h_1\|$) such that

$$\|Hf_1-Hf_2\|\leqslant L\,\|f_1-f_2\|\quad\text{for all } f_1,\ f_2\in\mathrm{BV}(I_a^b;\mathbb{R}). \tag{1.5}$$

It is the aim of this paper to show that the last assertion can be (at least partially) reversed, so that it is almost a characterization of Lipschitzian superposition operators H on $\mathrm{BV}(I_a^b;\mathbb{R})$ (see Theorem 3 and Corollary 4 below).

Given $f\in\mathrm{BV}(I_a^b;\mathbb{R})$, we define the *left regularization* $f^*:I_a^b\to\mathbb{R}$ of f as follows: for any $y\in I_a^b$, $a<y\leqslant b$, and any $\gamma\in\mathscr{A}_0(n)$ we set

$$f^*(a+\gamma(y-a))=\lim_{x\to(a+0)+\gamma((y-0)-(a+0))} f(x). \tag{1.6}$$

It is to be noted that "$x\to\cdots$" under the limit sign in (1.6) means that $x=(x_1,\ldots,x_n)\in I_a^b$, $x_i<y_i$ for those $i\in\{1,\ldots,n\}$ for which $\gamma_i=1$, $a_i<x_i$ for those i for which $\gamma_i=0$ and x tends to $a+\gamma(y-a)$ in $\mathbb{R}^n$ (that is, $x_i\to y_i-0$ if $\gamma_i=1$ and $x_i\to a_i+0$ if $\gamma_i=0$, $i\in\{1,\ldots,n\}$). The existence of all the limits in (1.6) will be proved in Lemma 12 in Section 4.

A function $f:I_a^b\to\mathbb{R}$ is said to be *left continuous* if $f^*(y)=f(y)$ for all $y\in I_a^b$ with $a<y\leqslant b$. We denote by $\mathrm{BV}^*(I_a^b;\mathbb{R})$ the subset of $\mathrm{BV}(I_a^b;\mathbb{R})$ of those functions which are left continuous.

The main results of Part II are the following two theorems.

Theorem 2. *If* $f\in\mathrm{BV}(I_a^b;\mathbb{R})$, *then* $f^*\in\mathrm{BV}^*(I_a^b;\mathbb{R})$ *and we have*

$$V_{|\alpha|}((f^*)^a_\alpha,I_a^b\lfloor\alpha)\leqslant\sum_{\alpha\leqslant\beta\leqslant 1}V_{|\beta|}(f^a_\beta,I_a^b\lfloor\beta),\qquad \alpha\in\mathscr{A}(n) \tag{1.7}$$

and, in particular,

$$V_n(f^*, I_a^b) \leqslant V_n(f, I_a^b) \quad \text{and} \quad \mathrm{TV}(f^*, I_a^b) \leqslant (2^n - 1)\mathrm{TV}(f, I_a^b). \tag{1.8}$$

Theorem 3. *Let $H : \mathbb{R}^I \to \mathbb{R}^I$ be the superposition operator generated by a function $h : I \times \mathbb{R} \to \mathbb{R}$ according to* (1.4) *with $I = I_a^b$. If H maps $\mathrm{BV}(I_a^b; \mathbb{R})$ into itself and is Lipschitzian* (*in the sense of* (1.5)), *then the family $\{h(x, \cdot)\}_{x \in I_a^b} : \mathbb{R} \to \mathbb{R}$ is uniformly Lipschitzian and there exist two functions $h_0, h_1 \in \mathrm{BV}^*(I_a^b; \mathbb{R})$ such that*

$$h^*(x, u) = h_1(x)u + h_0(x), \quad x \in I_a^b, \quad u \in \mathbb{R}, \tag{1.9}$$

where $h^*(x, u)$ is the left regularization of the function $y \mapsto h(y, u)$ at the point $x \in I_a^b$ for each fixed $u \in \mathbb{R}$.

This theorem contains as particular cases the results of [9] (for $n = 1$) and [2] (for $n = 2$). As a corollary, we get:

Corollary 4. *Suppose the generator $h : I_a^b \times \mathbb{R} \to \mathbb{R}$ of a superposition operator H is such that $h^* = h$ on $I_a^b \times \mathbb{R}$. Then the following two conditions* (i) *and* (ii) *are equivalent*: (i) *H maps the space $\mathrm{BV}(I_a^b; \mathbb{R})$ into itself and is Lipschitzian*; (ii) *there exist two functions $h_0, h_1 \in \mathrm{BV}^*(I_a^b; \mathbb{R})$ such that $h(x, u) = h_1(x)u + h_0(x)$ for all $x \in I_a^b$ and $u \in \mathbb{R}$.*

This paper (Part II) is organized as follows. In Section 2, we study properties of the total variation. In Section 3, we establish properties of totally monotone functions. Theorem 2 will be proved in Section 4. In Section 5, we prove Theorem 3 and construct an example showing the sharpness of Theorem 3 in the sense that in general one cannot replace h^* by h in the representation (1.9). Finally, Section 6 is devoted to certain generalizations of the main results when functions under consideration have their values in normed linear spaces.

The main results of this paper were announced in [3,4].

To end this section, we recall some facts concerning properties of mixed differences of all orders established in [5] and needed below. Lemmas 5–7 below are respective Lemmas 5–7 from [5].

Lemma 5. *If $f : I_a^b \to \mathbb{R}$, $x, y \in I_a^b$, $x \leqslant y$, $z \in I_a^b$ and $\alpha \in \mathscr{A}(n)$, then*

$$\mathrm{md}_{|\alpha|}(f_\alpha^z, I_x^y \lfloor \alpha) = \sum_{0 \leqslant \theta \leqslant \alpha} (-1)^{|\theta|} f(z + \alpha(x - z) + \theta(y - x)). \tag{1.10}$$

In particular, if $z = a$ or $z = x$, we have, respectively,

$$\mathrm{md}_{|\alpha|}(f_\alpha^a, I_x^y \lfloor \alpha) = \mathrm{md}_{|\alpha|}(f_\alpha^{a+\alpha(x-a)}, I_{a+\alpha(x-a)}^y \lfloor \alpha), \tag{1.11}$$

$$\mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y \lfloor \alpha) = \sum_{0 \leqslant \theta \leqslant \alpha} (-1)^{|\theta|} f(x + \theta(y - x)). \tag{1.12}$$

Lemma 6. *Given* $f : I_a^b \to \mathbb{R}$, $x, y \in I_a^b$, $x < y$, *and* $\gamma \in \mathscr{A}(n)$, *we have*

$$f(x + \gamma(y - x)) - f(x) = \sum_{0 \neq \alpha \leqslant \gamma} (-1)^{|\alpha|} \mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y \lfloor \alpha).$$

Lemma 7. *If* $f : I_a^b \to \mathbb{R}$, $x,\ y \in I_a^b$, $x < y$, *and* $\alpha \in \mathscr{A}(n) \backslash \{1\}$, *then*

$$\mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y \lfloor \alpha) = (-1)^{|\alpha|} \sum_{\alpha \leqslant \beta \leqslant 1} (-1)^{|\beta|} \mathrm{md}_{|\beta|}(f_\beta^a, I_{a+\alpha(x-a)}^{x+\alpha(y-x)} \lfloor \beta).$$

2. Properties of the total variation

The fundamental well-known property of md_n and V_n (and consequently, of $V_{|\alpha|}$ for each $\alpha \in \mathscr{A}(n)$, taking into account obvious modifications) is the *additivity*, i.e., if $\mathscr{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ is a partition of I_a^b and $f \in \mathrm{BV}(I_a^b; \mathbb{R})$, then

$$\mathrm{md}_n(f, I_a^b) = \sum_{1 \leqslant \sigma \leqslant \kappa} \mathrm{md}_n(f, I_{x[\sigma-1]}^{x[\sigma]}) \quad \text{and} \quad V_n(f, I_a^b) = \sum_{1 \leqslant \sigma \leqslant \kappa} V_n(f, I_{x[\sigma-1]}^{x[\sigma]}).$$

From Lemma 6 and definition (1.2) we get the following inequality due to Leonov ([7, Corollary 5]):

$$|f(y) - f(x)| \leqslant \mathrm{TV}(f, I_x^y), \quad f \in \mathrm{BV}(I_a^b; \mathbb{R}), \quad x,\ y \in I_a^b, \ \ x \leqslant y, \tag{2.1}$$

which generalizes the well-known property of functions of bounded variation of one variable onto BV functions of several variables. In the following lemma we present one more property of this type, which is classical for $n = 1$ and known for $n = 2$ (cf. inequality (22) from [2] if $n = 2$).

Lemma 8. *If* $f \in \mathrm{BV}(I_a^b; \mathbb{R})$ *and* $x,\ y \in I_a^b$, $x \leqslant y$, *then*

$$\mathrm{TV}(f, I_x^y) \leqslant \mathrm{TV}(f, I_a^y) - \mathrm{TV}(f, I_a^x).$$

Proof. 1. We start with the following observation: given $f : I_a^b \to \mathbb{R}$, $x,\ y \in I_a^b$, $x \leqslant y$, and $\alpha \in \mathscr{A}(n)$,

$$\text{if } x_i = y_i \text{ and } \alpha_i = 1 \text{ for some } i \in \{1, \ldots, n\}, \text{ then } V_{|\alpha|}(f_\alpha^a, I_x^y \lfloor \alpha) = 0. \tag{2.2}$$

In fact, if $\theta \in \mathscr{A}_0(n)$, $\theta \leqslant \alpha$, we set $\overline{\theta} = (\theta_1, \ldots, \theta_{i-1}, 1 - \theta_i, \theta_{i+1}, \ldots, \theta_n)$ and employ formula (1.10) with $z = a$. Since $\alpha_i = 1$, we have $\overline{\theta} \leqslant \alpha$. The multiindices θ and $\overline{\theta}$ are of different evenness and, moreover, because

$$a_i + \alpha_i(x_i - a_i) + \theta_i(y_i - x_i) = x_i = a_i + \alpha_i(x_i - a_i) + \overline{\theta}_i(y_i - x_i),$$

we have $a + \alpha(x - a) + \theta(y - x) = a + \alpha(x - a) + \overline{\theta}(y - x)$. It follows that the sum of two terms from (1.10) corresponding to θ and $\overline{\theta}$ vanishes, and since such pairs θ and $\overline{\theta}$ exhaust the set $\{\theta \in \mathscr{A}_0(n) \mid \theta \leqslant \alpha\}$, we infer that $\mathrm{md}_{|\alpha|}(f_\alpha^a, I_x^y \lfloor \alpha) = 0$. Now if $x',\ y' \in I_x^y$, $x' \leqslant y'$, are arbitrary, since $x_i' = x_i = y_i = y_i'$, similar arguments apply to show that $\mathrm{md}_{|\alpha|}(f_\alpha^a, I_{x'}^{y'} \lfloor \alpha) = 0$. Then the definition of $|\alpha|$-variation implies $V_{|\alpha|}(f_\alpha^a, I_x^y \lfloor \alpha) = 0$, which ends the proof of (2.2).

In what follows we will prove the following assertion equivalent to the inequality in our lemma: if $f \in \mathrm{BV}(I_a^b; \mathbb{R})$, $x, y \in I_a^b$, $x < y$, and $\gamma \in \mathscr{A}(n)$, then

$$\mathrm{TV}(f, I_x^{x+\gamma(y-x)}) \leqslant \mathrm{TV}(f, I_a^{x+\gamma(y-x)}) - \mathrm{TV}(f, I_a^x).$$

2. Let us show that if $\alpha \in \mathscr{A}(n)$, then

$$V_{|\alpha|}(f_\alpha^x, I_x^y \lfloor \alpha) \leqslant \sum_{\alpha \leqslant \beta \leqslant 1} V_{|\beta|}(f_\beta^a, I_{a+\alpha(x-a)}^{x+\alpha(y-x)} \lfloor \beta). \tag{2.3}$$

Note that if $x \leqslant \overline{x} < y$, then (1.11) with a replaced by x and x—by $\overline{x}$ and Lemma 7 with x replaced by $x + \alpha(\overline{x} - x)$ imply

$$\begin{aligned} \mathrm{md}_{|\alpha|}(f_\alpha^x, I_{\overline{x}}^y \lfloor \alpha) &= \mathrm{md}_{|\alpha|}(f_\alpha^{x+\alpha(\overline{x}-x)}, I_{x+\alpha(\overline{x}-x)}^y \lfloor \alpha) \\ &= (-1)^{|\alpha|} \sum_{\alpha \leqslant \beta \leqslant 1} (-1)^{|\beta|} \mathrm{md}_{|\beta|}(f_\beta^a, I_{a+\alpha(\overline{x}-a)}^{x+\alpha(y-x)} \lfloor \beta). \end{aligned} \tag{2.4}$$

Let $\mathscr{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ be a partition of I_x^y, so that $\kappa \in \mathbb{N}^n$, $x[0] = x$, $x[\kappa] = y$ and $x[\sigma-1] < x[\sigma]$ for all $\sigma \in \mathbb{N}^n$, $\sigma \leqslant \kappa$. We have

$$\bigcup_{\sigma \lfloor \alpha} I_{a+\alpha(x[\sigma-1]-a)}^{x+\alpha(x[\sigma]-x)} = I_{a+\alpha(x-a)}^{x+\alpha(y-x)}$$

is the union of nonoverlapping rectangles taken only over those σ_i in the range $1 \leqslant \sigma_i \leqslant \kappa_i$, for which $\alpha_i = 1$, $i \in \{1, \ldots, n\}$. Setting $\overline{x} = x[\sigma - 1]$ and $y = x[\sigma]$ in (2.4) and taking into account the additivity property of $V_{|\beta|}$ we get

$$\begin{aligned} \sum_{\sigma \lfloor \alpha} |\mathrm{md}_{|\alpha|}(f_\alpha^x, I_{x[\sigma-1]}^{x[\sigma]} \lfloor \alpha)| &\leqslant \sum_{\sigma \lfloor \alpha} \sum_{\alpha \leqslant \beta \leqslant 1} |\mathrm{md}_{|\beta|}(f_\beta^a, I_{a+\alpha(x[\sigma-1]-a)}^{x+\alpha(x[\sigma]-x)} \lfloor \beta)| \\ &\leqslant \sum_{\alpha \leqslant \beta \leqslant 1} \sum_{\sigma \lfloor \alpha} V_{|\beta|}(f_\beta^a, I_{a+\alpha(x[\sigma-1]-a)}^{x+\alpha(x[\sigma]-x)} \lfloor \beta) \\ &= \sum_{\alpha \leqslant \beta \leqslant 1} V_{|\beta|}(f_\beta^a, I_{a+\alpha(x-a)}^{x+\alpha(y-x)} \lfloor \beta) \end{aligned}$$

from which (2.3) follows.

3. Taking into account that

$$I_{a_i}^{y_i} = [a_i, y_i] = [a_i, x_i] \cup [x_i, y_i] = \bigcup_{\alpha_i = 0,1} I_{a_i+\alpha_i(x_i-a_i)}^{x_i+\alpha_i(y_i-x_i)}, \quad i = 1, \ldots, n$$

for each $\beta \in \mathscr{A}(n)$ we have the following union of nonoverlapping rectangles:

$$I_a^{x+\gamma(y-x)} \lfloor \beta = \bigcup_{0 \leqslant \alpha \leqslant \beta\gamma} (I_{a+\alpha(x-a)}^{x+\alpha(y-x)} \lfloor \beta) = (I_a^x \lfloor \beta) \cup \bigcup_{0 \neq \alpha \leqslant \beta\gamma} (I_{a+\alpha(x-a)}^{x+\alpha(y-x)} \lfloor \beta). \tag{2.5}$$

This and the additivity property of $V_{|\beta|}$ imply

$$V_{|\beta|}(f^a_\beta, I_a^{x+\gamma(y-x)} \lfloor \beta) = V_{|\beta|}(f^a_\beta, I_a^x \lfloor \beta) + \sum_{0 \neq \alpha \leqslant \beta\gamma} V_{|\beta|}(f^a_\beta, I_{a+\alpha(x-a)}^{x+\alpha(y-x)} \lfloor \beta). \tag{2.6}$$

Now we estimate the total variation of f over $I_x^{x+\gamma(y-x)}$. If $\alpha \in \mathscr{A}(n)$ and $\alpha \nleqslant \gamma$, then $\alpha_i = 1$ and $\gamma_i = 0$ for some $i \in \{1, \ldots, n\}$ or, $x_i + \gamma_i(y_i - x_i) = x_i$ and $\alpha_i = 1$, and so, by (2.2) with $a = x$, $V_{|\alpha|}(f^x_\alpha, I_x^{x+\gamma(y-x)} \lfloor \alpha) = 0$. Moreover, we note that if $\alpha \leqslant \gamma$, then $(x + \gamma(y-x)) \lfloor \alpha = y \lfloor \alpha$. These remarks together with (1.2), (2.3) and (2.6) yield:

$$\begin{aligned}
\mathrm{TV}(f, I_x^{x+\gamma(y-x)}) &= \sum_{0 \neq \alpha \leqslant \gamma} V_{|\alpha|}(f^x_\alpha, I_x^{x+\gamma(y-x)} \lfloor \alpha) \\
&= \sum_{0 \neq \alpha \leqslant \gamma} V_{|\alpha|}(f^x_\alpha, I_x^y \lfloor \alpha) \\
&\leqslant \sum_{0 \neq \alpha \leqslant \gamma} \sum_{\alpha \leqslant \beta \leqslant 1} V_{|\beta|}(f^a_\beta, I_{a+\alpha(x-a)}^{x+\alpha(y-x)} \lfloor \beta) \qquad (2.7) \\
&= \sum_{0 \neq \beta \leqslant 1} \sum_{0 \neq \alpha \leqslant \beta\gamma} V_{|\beta|}(f^a_\beta, I_{a+\alpha(x-a)}^{x+\alpha(y-x)} \lfloor \beta) \\
&= \sum_{0 \neq \beta \leqslant 1} (V_{|\beta|}(f^a_\beta, I_a^{x+\gamma(y-x)} \lfloor \beta) - V_{|\beta|}(f^a_\beta, I_a^x \lfloor \beta)) \\
&= \mathrm{TV}(f, I_a^{x+\gamma(y-x)}) - \mathrm{TV}(f, I_a^x) \qquad (2.8)
\end{aligned}$$

and the desired inequality follows. □

3. On totally monotone functions

Recall that a function $f : I_a^b \to \mathbb{R}$ is said to be *totally monotone* (e.g., [7,11]) if $(-1)^{|\alpha|} \mathrm{md}_{|\alpha|}(f^x_\alpha, I_x^y \lfloor \alpha) \geqslant 0$ for all $\alpha \in \mathscr{A}(n)$ and $x, y \in I_a^b$, $x \leqslant y$. In this case $x \leqslant y$ implies $V_{|\alpha|}(f^x_\alpha, I_x^y \lfloor \alpha) = (-1)^{|\alpha|} \mathrm{md}_{|\alpha|}(f^x_\alpha, I_x^y \lfloor \alpha)$, so that, by Lemma 6 with $\gamma = 1$, $f(x) \leqslant f(y)$ and $\mathrm{TV}(f, I_x^y) = f(y) - f(x)$.

Given $f \in \mathrm{BV}(I_a^b; \mathbb{R})$, the function $v(x) \equiv v_f(x) = \mathrm{TV}(f, I_a^x)$, $x \in I_a^b$, is called the *total variation function* of f. We have

Lemma 9. *If* $f \in \mathrm{BV}(I_a^b; \mathbb{R})$, $\alpha \in \mathscr{A}(n)$, *and* $x, y \in I_a^b$, $x \leqslant y$, *then*

$$(-1)^{|\alpha|} \mathrm{md}_{|\alpha|}(v^x_\alpha, I_x^y \lfloor \alpha) = \sum_{\alpha \leqslant \beta \leqslant 1} V_{|\beta|}(f^a_\beta, I_{a+\alpha(x-a)}^{x+\alpha(y-x)} \lfloor \beta).$$

Proof. By (1.12) and definitions of v and TV, we get

$$\begin{aligned}\mathrm{md}_{|\alpha|}(v_\alpha^x, I_x^y \lfloor \alpha) &= \sum_{0\leqslant\theta\leqslant\alpha} (-1)^{|\theta|} v(x+\theta(y-x)) \\ &= \sum_{0\neq\beta\leqslant 1}\sum_{0\leqslant\theta\leqslant\alpha} (-1)^{|\theta|} V_{|\beta|}(f_\beta^a, I_a^{x+\theta(y-x)}\lfloor\beta). \end{aligned} \tag{3.1}$$

Let us show that the sum over $\beta \in \mathscr{A}(n)$ can be replaced by the sum over $\beta \in \mathscr{A}(n)$ with $\beta \geqslant \alpha$ or, in other words, if $\beta \in \mathscr{A}(n)$ and $\beta \not\geqslant \alpha$, then

$$\sum_{0\leqslant\theta\leqslant\alpha} (-1)^{|\theta|} V_{|\beta|}(f_\beta^a, I_a^{x+\theta(y-x)}\lfloor\beta) = 0. \tag{3.2}$$

In fact, $\#\{\theta \in \mathscr{A}_0(n) \mid \theta \leqslant \alpha\} = 2^{|\alpha|}$ is an even number, and since $\beta \not\geqslant \alpha$, there exists $i \in \{1, \ldots, n\}$ such that $\beta_i = 0$ and $\alpha_i = 1$, and so, in the union

$$\{\theta \in \mathscr{A}_0(n) \mid \theta\leqslant\alpha\} = \{\theta \in \mathscr{A}_0(n) \mid \theta\leqslant\alpha,\ \theta_i = 1\} \cup \{\theta \in \mathscr{A}_0(n) \mid \theta\leqslant\alpha,\ \theta_i = 0\}$$

the sets on the right are disjoint and have the same number of elements equal to $2^{|\alpha|-1}$. If $\theta \in \mathscr{A}_0(n)$, $\theta\leqslant\alpha$, we set $\overline{\theta} = (\theta_1, \ldots, \theta_{i-1}, 1-\theta_i, \theta_{i+1}, \ldots, \theta_n)$, so that $\overline{\theta} \in \mathscr{A}_0(n)$, $\overline{\theta}\leqslant\alpha$ and $||\theta| - |\overline{\theta}|| = 1$. Since $\beta_i = 0$, we have $(x+\theta(y-x))\lfloor\beta = (x+\overline{\theta}(y-x))\lfloor\beta$, and so, the sum of the two terms on the left in (3.2) corresponding to θ and $\overline{\theta}$ vanishes, and equality (3.2) follows.

By virtue of (2.6) (with γ replaced by $\theta \in \mathscr{A}_0(n)$),

$$V_{|\beta|}(f_\beta^a, I_a^{x+\theta(y-x)}\lfloor\beta) = \sum_{0\leqslant\gamma\leqslant\beta\theta} V_{|\beta|}(f_\beta^a, I_{a+\gamma(x-a)}^{x+\gamma(y-x)}\lfloor\beta)$$

and so, (3.1) yields (note that in the third sum below conditions $\gamma\leqslant\theta$, $\theta\leqslant\alpha$ and $\alpha\leqslant\beta$ imply $\gamma\leqslant\beta$)

$$\begin{aligned}\mathrm{md}_{|\alpha|}(v_\alpha^x, I_x^y \lfloor \alpha) &= \sum_{\alpha\leqslant\beta\leqslant 1}\sum_{0\leqslant\theta\leqslant\alpha} (-1)^{|\theta|} \sum_{0\leqslant\gamma\leqslant\theta} V_{|\beta|}(f_\beta^a, I_{a+\gamma(x-a)}^{x+\gamma(y-x)}\lfloor\beta) \\ &= \sum_{\alpha\leqslant\beta\leqslant 1}\sum_{0\leqslant\gamma\leqslant\alpha}\left[\sum_{\gamma\leqslant\theta\leqslant\alpha}(-1)^{|\theta|}\right] V_{|\beta|}(f_\beta^a, I_{a+\gamma(x-a)}^{x+\gamma(y-x)}\lfloor\beta).\end{aligned}$$

Let us denote the quantity in square brackets by c_γ. If $\gamma\leqslant\theta\leqslant\alpha$ and $j = |\theta|$, then $|\gamma|\leqslant j\leqslant|\alpha|$, and since

$$\#\{\theta \in \mathscr{A}_0(n) \mid \gamma\leqslant\theta\leqslant\alpha,\ |\theta| = j\} = C_{|\alpha|-|\gamma|}^{j-|\gamma|}, \quad \text{where } C_m^i = \frac{m!}{i!(m-i)!} \tag{3.3}$$

(with the usual convention that $0! = 1$), we have:

$$c_\gamma = \sum_{j=|\gamma|}^{|\alpha|} (-1)^j C_{|\alpha|-|\gamma|}^{j-|\gamma|} = \begin{cases} (-1)^{|\gamma|}(1-1)^{|\alpha|-|\gamma|} = 0 & \text{if } \alpha \neq \gamma\leqslant\alpha, \\ (-1)^{|\alpha|} & \text{if } \gamma = \alpha, \end{cases}$$

which completes the proof of Lemma 9. □

As a corollary, we get the following result due to Leonov [7, Theorem 3]:

Corollary 10. *If* $f \in \mathrm{BV}(I_a^b; \mathbb{R})$, *then the total variation function* $v = v_f$ *and the function* $\pi = v - f$ *are totally monotone.*

Proof. That v is totally monotone is a consequence of Lemma 9 and the definition of monotonicity. The total monotonicity of π is a consequence of (1.12) and Lemmas 9 and 7: if $\alpha \in \mathscr{A}(n)$ and $x, \ y \in I_a^b$, $x \leqslant y$, then

$$(-1)^{|\alpha|}\mathrm{md}_{|\alpha|}(\pi_\alpha^x, I_x^y \lfloor \alpha) = (-1)^{|\alpha|}\mathrm{md}_{|\alpha|}(v_\alpha^x, I_x^y \lfloor \alpha) - (-1)^{|\alpha|}\mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y \lfloor \alpha)$$
$$= \sum_{\alpha \leqslant \beta \leqslant 1} [V_{|\beta|}(f_\beta^a, I_{a+\alpha(x-a)}^{x+\alpha(y-x)} \lfloor \beta) - (-1)^{|\beta|}$$
$$\times \,\mathrm{md}_{|\beta|}(f_\beta^a, I_{a+\alpha(x-a)}^{x+\alpha(y-x)} \lfloor \beta)] \geqslant 0. \qquad \square$$

Corollary 11. *If* $f \in \mathrm{BV}(I_a^b; \mathbb{R})$ *and* $\alpha \in \mathscr{A}(n)$, *then*

$$V_{|\alpha|}(v_\alpha^a, I_a^b \lfloor \alpha) = \sum_{\alpha \leqslant \beta \leqslant 1} V_{|\beta|}(f_\beta^a, I_a^{a+\alpha(b-a)} \lfloor \beta)$$

and, in particular, $V_n(v, I_a^b) = V_n(f, I_a^b)$ *and* $\mathrm{TV}(v, I_a^b) = \mathrm{TV}(f, I_a^b)$.

Proof. Given $x, \ y \in I_a^b$, $x \leqslant y$, applying (1.11) and Lemma 9 with x replaced by $a+\alpha(x-a)$, we get

$$(-1)^{|\alpha|}\mathrm{md}_{|\alpha|}(v_\alpha^a, I_x^y \lfloor \alpha) = (-1)^{|\alpha|}\mathrm{md}_{|\alpha|}(v_\alpha^{a+\alpha(x-a)}, I_{a+\alpha(x-a)}^y \lfloor \alpha)$$
$$= \sum_{\alpha \leqslant \beta \leqslant 1} V_{|\beta|}(f_\beta^a, I_{a+\alpha(x-a)}^{a+\alpha(y-a)} \lfloor \beta).$$

Let $\mathscr{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ be a partition of I_a^b. The additivity of $V_{|\beta|}$ gives:

$$\sum_{\sigma \lfloor \alpha} |\mathrm{md}_{|\alpha|}(v_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]} \lfloor \alpha)| = \sum_{\sigma \lfloor \alpha} (-1)^{|\alpha|}\mathrm{md}_{|\alpha|}(v_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]} \lfloor \alpha)$$
$$= \sum_{\alpha \leqslant \beta \leqslant 1} \sum_{\sigma \lfloor \alpha} V_{|\beta|}(f_\beta^a, I_{a+\alpha(x[\sigma-1]-a)}^{a+\alpha(x[\sigma]-a)} \lfloor \beta)$$
$$= \sum_{\alpha \leqslant \beta \leqslant 1} V_{|\beta|}(f_\beta^a, I_a^{a+\alpha(b-a)} \lfloor \beta).$$

It remains to note that, by virtue of (2.7)–(2.8) with $\gamma = 1$, $x = a$ and $y = b$,

$$\mathrm{TV}(v, I_a^b) = \sum_{0 \neq \alpha \leqslant 1} V_{|\alpha|}(v_\alpha^a, I_a^b \lfloor \alpha)$$
$$= \sum_{0 \neq \alpha \leqslant 1} \sum_{\alpha \leqslant \beta \leqslant 1} V_{|\beta|}(f_\beta^a, I_a^{a+\alpha(b-a)} \lfloor \beta) = \mathrm{TV}(f, I_a^b). \qquad \square$$

4. The left regularization

In order to prove the first main result of Part II (Theorem 2 in Section 1), we first show that the limits on the right in (1.6), defining the left regularization f^* of a function $f \in \mathrm{BV}(I_a^b;\mathbb{R})$, do exist.

Lemma 12. *If $f \in \mathrm{BV}(I_a^b;\mathbb{R})$, the function $f^* : I_a^b \to \mathbb{R}$ given by* (1.6) *is well defined.*

Proof. For the sake of clarity we divide the proof into four steps.

Let $y \in I_a^b$, $a < y \leqslant b$, $\gamma \in \mathscr{A}_0(n)$ and $x \in I_a^b$ be such that $x_i < y_i$ for those $i \in \{1,\dots,n\}$ for which $\gamma_i = 1$ and $a_i < x_i$ for the remaining i's.

1. Given $\alpha \in \mathscr{A}(n)$, (1.10) and the change of variables $\theta \mapsto \alpha - \theta$ yield:

$$\begin{aligned}
&\mathrm{md}_{|\alpha|}(f_\alpha^{a+\gamma(y-a)}, I_{a+\gamma(x-a)}^{x+\gamma(y-x)} \lfloor \alpha)\\
&\quad = \sum_{0\leqslant\theta\leqslant\alpha} (-1)^{|\theta|} f(a+\gamma(y-a)+(\alpha-\theta)\gamma(x-y)+\theta(1-\gamma)(x-a))\\
&\quad = (-1)^{|\alpha|} \sum_{0\leqslant\theta\leqslant\alpha} (-1)^{|\theta|} f(a+\gamma(y-a)+\theta\gamma(x-y)\\
&\qquad + (\alpha-\theta)(1-\gamma)(x-a)).
\end{aligned} \tag{4.1}$$

2. We have the following counterpart of Lemma 6:

$$f(x)-f(a+\gamma(y-a)) = \sum_{0\neq\alpha\leqslant 1} (-1)^{|\alpha(1-\gamma)|} \mathrm{md}_{|\alpha|}(f_\alpha^{a+\gamma(y-a)}, I_{a+\gamma(x-a)}^{x+\gamma(y-x)} \lfloor \alpha). \tag{4.2}$$

In order to prove it, we note (see (4.1)) that, given $\delta, \eta \in \mathscr{A}_0(n)$, $0 \leqslant \delta \leqslant \gamma$, $0 \leqslant \eta \leqslant 1-\gamma$ and $\alpha \in \mathscr{A}(n)$, the system of equations $\{\theta\gamma = \delta,\ (\alpha-\theta)(1-\gamma) = \eta\}$ has a unique solution $\theta \in \mathscr{A}_0(n)$, $\theta \leqslant \alpha$, if and only if $\alpha \geqslant \delta \vee \eta \equiv \max\{\delta,\eta\}$; moreover, this solution is given by $\theta = \alpha(1-\gamma)+\delta-\eta = \alpha(1-\gamma+\delta-\eta)$ and we have $|\theta| = |\alpha(1-\gamma)| + |\delta| - |\eta|$. Consequently, the right-hand side of (4.2) can be rewritten, by virtue of (4.1), as the sum of terms of the form

$$c(\delta,\eta)\cdot f(a+\gamma(y-a)+\delta(x-y)+\eta(x-a))$$

over all multiindices $0 \leqslant \delta \leqslant \gamma$ and $0 \leqslant \eta \leqslant 1-\gamma$. To evaluate the factor $c(\delta,\eta)$, we set $\lambda = \delta \vee \eta$ and note that (cf. (3.3))

$$\#\{\alpha \in \mathscr{A}(n) \mid \alpha \geqslant \lambda,\ |\alpha| = i\} = C_{n-|\lambda|}^{i-|\lambda|} \quad \text{if} \quad \max\{1,|\lambda|\} \leqslant i \leqslant n$$

and so, the right-hand side of (4.2) and (4.1) imply

$$c(\delta,\eta) = (-1)^{|\delta|+|\eta|} \sum_{i=\max\{1,|\lambda|\}}^{n} (-1)^i C_{n-|\lambda|}^{i-|\lambda|}.$$

The binomial formula gives: $c(0,0) = -1$ (which corresponds to $f(a+\gamma(y-a))$), $c(\gamma, 1-\gamma) = 1$ (which corresponds to $f(x)$), and $c(\delta,\eta) = 0$ otherwise.

3. The following equality is a variant of Lemma 7: if $\overline{x}, \overline{y} \in I_a^b$, $\overline{x} < \overline{y}$, and $\alpha \in \mathscr{A}(n)$, we have

$$(-1)^{|\alpha|}\mathrm{md}_{|\alpha|}(f_\alpha^{a+\gamma(y-a)}, I_{\overline{x}}^{\overline{y}}\lfloor\alpha) = \sum_{\alpha\leqslant\beta\leqslant\alpha\vee\gamma}(-1)^{|\beta|}\mathrm{md}_{|\beta|}(f_\beta^{a+\alpha(\overline{x}-a)}, I_{a+\alpha(\overline{x}-a)}^{y+\alpha(\overline{y}-y)}\lfloor\beta), \tag{4.3}$$

where $\alpha \vee \gamma \equiv \max\{\alpha, \gamma\} = \alpha + \gamma - \alpha\gamma$.

In fact, according to (1.10) the left-hand side of (4.3) is equal to

$$(-1)^{|\alpha|}\sum_{0\leqslant\theta\leqslant\alpha}(-1)^{|\theta|}f(a+\gamma(1-\alpha)(y-a)+\alpha(\overline{x}-a)+\theta(\overline{y}-\overline{x})) \tag{4.4}$$

and the right-hand side of (4.3) is the expression

$$\sum_{\alpha\leqslant\beta\leqslant\alpha\vee\gamma}(-1)^{|\beta|}\sum_{0\leqslant\eta\leqslant\beta}(-1)^{|\eta|}\underbrace{f(a+\alpha(\overline{x}-a)+\eta[y-a+\alpha(\overline{y}-y-\overline{x}+a)])}_{f(\cdots)}. \tag{4.5}$$

Clearly, (4.5) can be written as the sum of terms of the form $c_\eta \cdot f(\cdots)$ over all $\eta \in \mathscr{A}_0(n)$, $\eta \leqslant \alpha\vee\gamma$. Let $0 \leqslant \eta \leqslant \alpha\vee\gamma$, and let us calculate c_η. We set $\mu = \alpha\vee\eta$ and note that $\alpha \leqslant \mu \leqslant \alpha\vee\gamma$, since

$$\alpha \leqslant \mu = \alpha + (1-\alpha)\eta \leqslant \alpha + (1-\alpha)(\alpha+\gamma-\alpha\gamma) = \alpha\vee(\alpha\vee\gamma) = \alpha\vee\gamma.$$

It follows that $|\alpha| \leqslant |\mu| \leqslant |\alpha\vee\gamma|$. If $|\mu| \leqslant i \leqslant |\alpha\vee\gamma|$, then, again by (3.3),

$$\#\{\beta \in \mathscr{A}(n) \mid \mu \leqslant \beta \leqslant \alpha\vee\gamma, |\beta| = i\} = C_{|\alpha\vee\gamma|-|\mu|}^{i-|\mu|}$$

and so, (4.5) implies

$$c_\eta = (-1)^{|\eta|}\sum_{i=|\mu|}^{|\alpha\vee\gamma|}(-1)^i C_{|\alpha\vee\gamma|-|\mu|}^{i-|\mu|} = (-1)^{|\eta|+|\mu|}\sum_{j=0}^{|\alpha\vee\gamma|-|\mu|}(-1)^j C_{|\alpha\vee\gamma|-|\mu|}^{j}$$
$$= \begin{cases} (-1)^{|\eta|+|\mu|} & \text{if } |\mu| = |\alpha\vee\gamma|, \\ 0 & \text{if } |\alpha| \leqslant |\mu| < |\alpha\vee\gamma|. \end{cases}$$

We have: $|\mu| = |\alpha\vee\gamma|$ if and only if $\mu = \alpha\vee\gamma$ if and only if there exists a unique $\theta \in \mathscr{A}_0(n)$ such that $\theta \leqslant \alpha$ and $\eta = \gamma(1-\alpha) + \theta$. Indeed, if $\eta \in \mathscr{A}_0(n)$ and $\alpha\vee\eta = \alpha\vee\gamma$ (recall that $\mu = \alpha\vee\eta$), then $\eta \leqslant \alpha\vee\eta = \alpha + \gamma(1-\alpha)$, and so, $\eta - \gamma(1-\alpha) \leqslant \alpha$; also, equality $\alpha\vee\eta = \alpha\vee\gamma$ yields $\eta \geqslant \gamma(1-\alpha)$, for otherwise there exists $i \in \{1, \ldots, n\}$ such that $\eta_i = 0$ and $\gamma_i(1-\alpha_i) = 1$, so that $\alpha_i \vee \eta_i = \alpha_i \neq \alpha_i + 1 = \alpha_i \vee \gamma_i$. It remains to define $0 \leqslant \theta \leqslant \alpha$ by $\theta = \eta - \gamma(1-\alpha)$.

Now, let $\theta \in \mathscr{A}_0(n)$, $\theta \leqslant \alpha$ (as in (4.4)). Setting $\eta = \gamma(1-\alpha) + \theta$, we find

$$|\eta| = |\gamma(1-\alpha)| + |\theta| \quad \text{and} \quad |\mu| = |\alpha\vee\eta| = |\alpha\vee\gamma| = |\alpha| + |\gamma(1-\alpha)|$$

and so, $c_\eta = (-1)^{|\alpha|}(-1)^{|\theta|}$; it also follows that

$$\begin{aligned} &a + \alpha(\overline{x} - a) + \eta[y - a + \alpha(\overline{y} - y - \overline{x} + a)] \\ &\quad = a + \gamma(1-\alpha)(y-a) + \alpha(\overline{x} - a) + \theta(\overline{y} - \overline{x}). \end{aligned}$$

On the other hand, if $\eta \in \mathscr{A}_0(n)$, $\eta \leqslant \alpha \vee \gamma$ and $\eta \neq \gamma(1-\alpha)+\theta$, then $c_\eta = 0$. In this way we have proved that (4.5) is equal to (4.4).

4. Suppose $f : I_a^b \to \mathbb{R}$ is totally monotone. Given $\alpha \in \mathscr{A}(n)$, we put

$$\mathscr{I}_\alpha = \{I = I_{\overline{x}}^{\overline{y}} \lfloor \alpha \,|\, \overline{x}, \overline{y} \in I_a^b, \overline{x} < \overline{y}\}$$

and define a function (on rectangles) $Q_\alpha : \mathscr{I}_\alpha \to \mathbb{R}$ as follows: if $I \in \mathscr{I}_\alpha$ is of the form $I = I_{\overline{x}}^{\overline{y}} \lfloor \alpha$ for some $\overline{x}$, $\overline{y} \in I_a^b$ with $\overline{x} < \overline{y}$, we set:

$$Q_\alpha(I) = (-1)^{|\alpha|}\mathrm{md}_{|\alpha|}(f_\alpha^{a+\gamma(y-a)}, I_{\overline{x}}^{\overline{y}} \lfloor \alpha).$$

Since f is totally monotone, it follows from (4.3) that Q_α is nonnegative, i.e., $Q_\alpha(I) \geqslant 0$ for all $I \in \mathscr{I}_\alpha$. On the other hand, (4.4) implies that Q_α is additive in the sense that if a rectangle $I \in \mathscr{I}_\alpha$ is a finite union of nonoverlapping rectangles $I_j \in \mathscr{I}_\alpha$, then $Q_\alpha(I) = \sum_j Q_\alpha(I_j)$. Consequently, Q_α is monotone, i.e., if $I', I'' \in \mathscr{I}_\alpha$ and $I' \subset I''$, then $Q_\alpha(I') \leqslant Q_\alpha(I'')$. Let $\mathscr{I}_\alpha(a+\gamma(y-a))$ be the subset of $\mathscr{I}_\alpha$ of all rectangles I of the form $I = I_{a+\gamma(x-a)}^{x+\gamma(y-x)} \lfloor \alpha$, where $x \in I_a^b$ is such that $x_i < y_i$ if $\gamma_i = 1$ and $a_i < x_i$ if $\gamma_i = 0$, $i \in \{1, \ldots, n\}$. All rectangles $I \in \mathscr{I}_\alpha(a+\gamma(y-a))$ have the point $(a+\gamma(y-a))\lfloor \alpha$ as a common vertex and, as $x \to (a+0)+\gamma((y-0)-(a+0))$, the rectangles $I_{a+\gamma(x-a)}^{x+\gamma(y-x)} \lfloor \alpha$ shrink to the point $(a+\gamma(y-a))\lfloor \alpha$. Considering $\mathscr{I}_\alpha(a+\gamma(y-a))$ as a directed set, the rectangles in it being directed by inclusion, we find, due to the monotonicity of Q_α, that the Moore–Smith limit (e.g., [6, Chapter 1]) of $Q_\alpha(I)$ over rectangles $I \in \mathscr{I}_\alpha(a+\gamma(y-a))$ shrinking to $(a+\gamma(y-a))\lfloor \alpha$ exists and is equal to

$$q_\alpha(a+\gamma(y-a)) \equiv \lim_{I \in \mathscr{I}_\alpha(a+\gamma(y-a))} Q_\alpha(I) = \inf\{Q_\alpha(I) \,|\, I \in \mathscr{I}_\alpha(a+\gamma(y-a))\}.$$

It follows that

$$\lim_{x \to (a+0)+\gamma((y-0)-(a+0))} (-1)^{|\alpha|}\mathrm{md}_{|\alpha|}(f_\alpha^{a+\gamma(y-a)}, I_{a+\gamma(x-a)}^{x+\gamma(y-x)} \lfloor \alpha) = q_\alpha(a+\gamma(y-a))$$

and so, by (4.2), the desired limit $f^*(a+\gamma(y-a))$ exists and is equal to

$$f^*(a+\gamma(y-a)) = f(a+\gamma(y-a)) + \sum_{0 \neq \alpha \leqslant 1} (-1)^{|\alpha(1-\gamma)|+|\alpha|} \cdot q_\alpha(a+\gamma(y-a)).$$

Now, if $f \in \mathrm{BV}(I_a^b; \mathbb{R})$, then, by Corollary 10, the total variation function v of f and $\pi = v - f$ are totally monotone, and so, as we have just seen, they admit the limits $v^*(a+\gamma(y-a))$ and $\pi^*(a+\gamma(y-a))$, respectively, and it remains to note that $f^*(a+\gamma(y-a)) = v^*(a+\gamma(y-a)) - \pi^*(a+\gamma(y-a))$. □

Proof of Theorem 2. First we show that f^* is left continuous. It is known ([1], [6, III.5.4], [11]) that the set of discontinuity points of any totally monotone function on I_a^b lies on at most a countable set of hyperplanes of dimension $n-1$ parallel to the coordinate axes. By Corollary 10, the same is true for our function f. Therefore for each $y \in I_a^b$ with $a < y \leqslant b$ there exists a sequence $\{x^k\}_{k=1}^{\infty} \subset I_a^b$ of points of continuity of f such that $x^k < y$ for all $k \in \mathbb{N}$ and $x^k \to y$ in $\mathbb{R}^n$ as $k \to \infty$. It then follows that

$$\lim_{x \to y-0} f^*(x) = \lim_{k \to \infty} f^*(x^k) = \lim_{k \to \infty} f(x^k) = \lim_{x \to y-0} f(x) = f^*(y).$$

Now we prove inequalities (1.7) and (1.8). Let $\alpha \in \mathscr{A}(n)$ and $\mathscr{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ be a partition of I_a^b, so that $\kappa \in \mathbb{N}^n$, $x[0] = a$, $x[\kappa] = b$ and $x[\sigma-1] < x[\sigma]$ for all $\sigma \in \mathbb{N}^n$, $1 \leqslant \sigma \leqslant \kappa$. We set $\kappa^\alpha = \kappa_1^{\alpha_1} \cdots \kappa_n^{\alpha_n}$. By definition of f^*, for each $\sigma \in \mathbb{N}_0^n$, $1 \leqslant \sigma \leqslant \kappa$, we have:

$$f^*(a + \alpha(x[\sigma] - a)) = \lim_{x \to (a+0)+\alpha((x[\sigma]-0)-(a+0))} f(x)$$

and

$$f^*(a) = \lim_{x \to a+0} f(x)$$

and so, given $\varepsilon > 0$, there exist $x'[\sigma] \in I_a^b$, $x[\sigma-1] < x'[\sigma] < x[\sigma]$ and $a' \equiv x'[0] \in I_a^b$, $a < a' < x'[1]$ (note that $x'[\sigma]$ and a' depend on α and ε) such that

$$\begin{aligned} &|f^*(a + \alpha(x[\sigma] - a)) - f(a' + \alpha(x'[\sigma] - a'))| \leqslant \varepsilon/(2^{|\alpha|}\kappa^\alpha), \\ &|f^*(a) - f(a')| \leqslant \varepsilon/(2^{|\alpha|}\kappa^\alpha). \end{aligned} \tag{4.6}$$

Taking into account (1.10) we have:

$$\begin{aligned} &|\mathrm{md}_{|\alpha|}((f^*)_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]} \lfloor \alpha)| \\ &\quad = \left| \sum_{0 \leqslant \theta \leqslant \alpha} (-1)^{|\theta|} f^*(a + \alpha(x[\sigma-1] - a) + \theta(x[\sigma] - x[\sigma-1])) \right| \\ &\quad \leqslant \left| \sum_{0 \leqslant \theta \leqslant \alpha} (-1)^{|\theta|} [f^*(a + \alpha(x[\sigma-1] - a) + \theta(x[\sigma] - x[\sigma-1])) \right. \\ &\qquad \left. - f(a' + \alpha(x'[\sigma-1] - a') + \theta(x'[\sigma] - x'[\sigma-1]))] \right| \\ &\qquad + \left| \sum_{0 \leqslant \theta \leqslant \alpha} (-1)^{|\theta|} f(a' + \alpha(x'[\sigma-1] - a') + \theta(x'[\sigma] - x'[\sigma-1])) \right|. \end{aligned} \tag{4.7}$$

Noting that for $\theta \leqslant \alpha$ the expression $a + \alpha(x-a) + \theta(y-x)$ in the arguments of f^* and f can also be written in the form

$$a + \alpha(x-a) + \theta(y-x) = [a + \alpha(x-a)] + \theta([a + \alpha(y-a)] - [a + \alpha(x-a)])$$

(and so, inequalities (4.6) apply) and that, by (1.10),

$$\sum_{0\leqslant\theta\leqslant\alpha}(-1)^{|\theta|}f(a+\alpha(x-a)+\theta(y-x))=\mathrm{md}_{|\alpha|}(f_\alpha^a,I_x^y\lfloor\alpha),$$

applying (4.6) and (1.11), we proceed from (4.7) as follows:

$$\begin{aligned}&\leqslant\left(\sum_{0\leqslant\theta\leqslant\alpha}\frac{\varepsilon}{2^{|\alpha|}\kappa^\alpha}\right)+|\mathrm{md}_{|\alpha|}(f_\alpha^{a'+\alpha(x'[\sigma-1]-a')},I_{a'+\alpha(x'[\sigma-1]-a')}^{x'[\sigma]}\lfloor\alpha)|\\&\quad\equiv(\varepsilon/\kappa^\alpha)+M_\alpha(\sigma).\end{aligned}\tag{4.8}$$

By Lemma 7 with $x=a'+\alpha(x'[\sigma-1]-a')$ and $y=x'[\sigma]$, we get

$$M_\alpha(\sigma)\leqslant\sum_{\alpha\leqslant\beta\leqslant 1}|\mathrm{md}_{|\beta|}(f_\beta^a,I_{a+\alpha(x'[\sigma-1]-a)}^{a'+\alpha(x'[\sigma]-a')}\lfloor\beta)|.$$

Taking this into account, summing inequalities (4.7)–(4.8) over all $\sigma\lfloor\alpha$ (i.e., over those σ_i, $1\leqslant\sigma_i\leqslant\kappa_i$, for which $\alpha_i=1$) and using the additivity property of $V_{|\beta|}$ in the last inequality below, we find

$$\begin{aligned}&\sum_{\sigma\lfloor\alpha}|\mathrm{md}_{|\alpha|}((f^*)_\alpha^a,I_{x[\sigma-1]}^{x[\sigma]}\lfloor\alpha)|\\&\quad\leqslant\sum_{\sigma\lfloor\alpha}\frac{\varepsilon}{\kappa^\alpha}+\sum_{\alpha\leqslant\beta\leqslant 1}\sum_{\sigma\lfloor\alpha}|\mathrm{md}_{|\beta|}(f_\beta^a,I_{a+\alpha(x'[\sigma-1]-a)}^{a'+\alpha(x'[\sigma]-a')}\lfloor\beta)|\\&\quad\leqslant\varepsilon+\sum_{\alpha\leqslant\beta\leqslant 1}\sum_{\sigma\lfloor\alpha}V_{|\beta|}(f_\beta^a,I_{a+\alpha(x'[\sigma-1]-a)}^{a'+\alpha(x'[\sigma]-a')}\lfloor\beta)\\&\quad=\varepsilon+\sum_{\alpha\leqslant\beta\leqslant 1}V_{|\beta|}(f_\beta^a,I_{a+\alpha(x'[0]-a)}^{a'+\alpha(x'[\kappa]-a')}\lfloor\beta)\\&\quad\leqslant\varepsilon+\sum_{\alpha\leqslant\beta\leqslant 1}V_{|\beta|}(f_\beta^a,I_{a+\alpha(a'-a)}^{a'+\alpha(b-a')}\lfloor\beta),\end{aligned}$$

whence, taking the supremum over all partitions $\mathscr{P}$ of I_a^b,

$$V_{|\alpha|}((f^*)_\alpha^a,I_a^b\lfloor\alpha)\leqslant\varepsilon+\sum_{\alpha\leqslant\beta\leqslant 1}V_{|\beta|}(f_\beta^a,I_a^b\lfloor\beta)$$

and (1.7) follows due to the arbitrariness of $\varepsilon>0$. The first inequality in (1.8) follows from (1.7) with $\alpha=1$ and the second inequality in (1.8) is a consequence of (1.7), (1.2) and the facts that the right-hand side of (1.7) does not exceed the total variation $\mathrm{TV}(f,I_a^b)$ and $\#\mathscr{A}(n)=2^n-1$. $\square$

5. Lipschitzian superposition operators

This section is devoted to the proof of Theorem 3 and construction of an example (cf. Theorem 14) showing that one cannot in general replace h^* by h in the representation (1.9).

Proof of Theorem 3. Since $H : \mathrm{BV}(I_a^b; \mathbb{R}) \to \mathrm{BV}(I_a^b; \mathbb{R})$ is Lipschitzian, there exists a constant $L = L(H) > 0$ such that for all $f_1, f_2 \in \mathrm{BV}(I_a^b; \mathbb{R})$ we have $\|Hf_1 - Hf_2\| \leqslant L\|f_1 - f_2\|$ or, more explicitly, using definitions (1.3) and (1.2),

$$|(Hf_1 - Hf_2)(a)| + \sum_{0 \neq \alpha \leqslant 1} V_{|\alpha|}((Hf_1 - Hf_2)_\alpha^a, I_a^b \lfloor \alpha)$$
$$\leqslant L \left[|(f_1 - f_2)(a)| + \sum_{0 \neq \alpha \leqslant 1} V_{|\alpha|}((f_1 - f_2)_\alpha^a, I_a^b \lfloor \alpha) \right]. \tag{5.1}$$

Given $r, s \in \mathbb{R}$, $r < s$, we define an auxiliary Lipschitz function $\Psi_{r,s} : \mathbb{R} \to [0, 1]$ by

$$\Psi_{r,s}(t) = \begin{cases} 0 & \text{if } t \leqslant r, \\ (t - r)/(s - r) & \text{if } r \leqslant t \leqslant s, \\ 1 & \text{if } t \geqslant s. \end{cases}$$

1. First we show that the family $\{h(\bar{x}, \cdot) \mid \bar{x} \in I_a^b\}$ is uniformly Lipschitzian:

$$|h(\bar{x}, u_1) - h(\bar{x}, u_2)| \leqslant 2L|u_1 - u_2|, \quad \bar{x} \in I_a^b, \; u_1, u_2 \in \mathbb{R}. \tag{5.2}$$

Given $\bar{x} \in I_a^b$, we consider two cases: (i) $\bar{x} \neq a$, and (ii) $\bar{x} = a$.

Case (i): There exist $a < x \leqslant b$ and $\gamma \in \mathscr{A}(n)$ such that $\bar{x} = a + \gamma(x - a)$. If $u_1, u_2 \in \mathbb{R}$, we define two functions $f_1, f_2 \in \mathrm{BV}(I_a^b; \mathbb{R})$ by

$$f_j(y) = \frac{u_j}{|\gamma|} \sum_{i=1}^{n} \gamma_i \Psi_{a_i, x_i}(y_i), \quad a \leqslant y = (y_1, \ldots, y_n) \leqslant b, \quad j = 1, 2. \tag{5.3}$$

Let us calculate $\|f_1 - f_2\|$ (cf. (5.1)). Clearly, $f_j(a) = 0$, $j = 1, 2$, and if we set $v(\alpha) = V_{|\alpha|}((f_1 - f_2)_\alpha^a, I_a^b \lfloor \alpha)$ for $\alpha \in \mathscr{A}(n)$, then $v(\alpha) = \gamma_i |u_1 - u_2|/|\gamma|$ if $|\alpha| = \alpha_i = 1, i = 1, \ldots, n$ (note that $v(\alpha)$ for $|\alpha| = 1$ is the usual Jordan variation of the function $(f_1 - f_2)_\alpha^a$ of one variable on the closed interval $I_a^b \lfloor \alpha$). Also, $v(\alpha) = 0$ if $|\alpha| \geqslant 2$; in fact, given $p, q \in I_a^b$, $p < q$, by (1.12) and (5.3) for $j = 1, 2$ we have:

$$\mathrm{md}_{|\alpha|}((f_j)_\alpha^p, I_p^q \lfloor \alpha) = \frac{u_j}{|\gamma|} \sum_{i=1}^{n} \gamma_i \sum_{0 \leqslant \theta \leqslant \alpha} (-1)^{|\theta|} \Psi_{a_i, x_i}(p_i + \theta_i(q_i - p_i))$$

and so, if $i \in \{1, \ldots, n\}$ and $\alpha_i = 0$, then $\theta_i = 0$ and the last sum $\sum_{0 \leqslant \theta \leqslant \alpha} \cdots$ is equal to

$$\left[\sum_{0 \leqslant \theta \leqslant \alpha} (-1)^{|\theta|} \right] \Psi_{a_i, x_i}(p_i) = \left[\sum_{k=0}^{|\alpha|} (-1)^k C_{|\alpha|}^k \right] \Psi_{a_i, x_i}(p_i)$$
$$= (1 - 1)^{|\alpha|} \Psi_{a_i, x_i}(p_i) = 0$$

and if $i \in \{1, \dots, n\}$ and $\alpha_i = 1$, then it is equal to

$$\left[\sum_{0\leqslant\theta\leqslant\alpha,\theta_i=0}(-1)^{|\theta|}\right]\Psi_{a_i,x_i}(p_i)+\left[\sum_{0\leqslant\theta\leqslant\alpha,\theta_i=1}(-1)^{|\theta|}\right]\Psi_{a_i,x_i}(q_i)$$
$$=\left[\sum_{k=0}^{|\alpha|-1}(-1)^k C_{|\alpha|-1}^k\right]\Psi_{a_i,x_i}(p_i)+\left[\sum_{k=1}^{|\alpha|}(-1)^k C_{|\alpha|-1}^{k-1}\right]\Psi_{a_i,x_i}(q_i)=0.$$

Thus,

$$\|f_1-f_2\|=\sum_{i=1}^n \gamma_i|u_1-u_2|/|\gamma|=|u_1-u_2|.$$

Noting that $(Hf_1)(a)=(Hf_2)(a)$ and $f_j(\overline{x})=f_j(a+\gamma(x-a))=u_j$, $j=1,\ 2$, applying Lemma 6 and taking into account (5.1) we get:

$$\begin{aligned}|h(\overline{x},u_1)-h(\overline{x},u_2)|&=|h(\overline{x},f_1(\overline{x}))-h(\overline{x},f_2(\overline{x}))|=|(Hf_1-Hf_2)(\overline{x})|\\&=|(Hf_1-Hf_2)(a+\gamma(x-a))-(Hf_1-Hf_2)(a)|\\&=\left|\sum_{0\neq\alpha\leqslant\gamma}(-1)^{|\alpha|}\mathrm{md}_{|\alpha|}((Hf_1-Hf_2)^a_\alpha, I_a^x\lfloor\alpha)\right|\\&\leqslant\sum_{0\neq\alpha\leqslant\gamma}V_{|\alpha|}((Hf_1-Hf_2)^a_\alpha, I_a^b\lfloor\alpha)\\&\leqslant\mathrm{TV}(Hf_1-Hf_2, I_a^b)=\|Hf_1-Hf_2\|\\&\leqslant L\|f_1-f_2\|=L|u_1-u_2|\end{aligned}$$

and inequality (5.2) follows.

Case (ii): Given $u_1,\ u_2\in\mathbb{R}$, we set

$$f_j(y)=\left(1-\frac{1}{n}\sum_{i=1}^n\Psi_{a_i,b_i}(y_i)\right)u_j,\quad a\leqslant y=(y_1,\dots,y_n)\leqslant b,\quad j=1,2$$

and note that $f_j(a)=u_j$, $j=1,\ 2$ and so, $|(f_1-f_2)(a)|=|u_1-u_2|$. We also have (see Case (i) for the definition of $v(\alpha)$): $v(\alpha)=|u_1-u_2|/n$ if $|\alpha|=1$, and $v(\alpha)=0$ if $|\alpha|\geqslant 2$. Thus, $\|f_1-f_2\|=2|u_1-u_2|$. Since $(Hf_1)(b)=(Hf_2)(b)$, it follows from Lemma 6 (with $\gamma=1$) and (5.1) that

$$\begin{aligned}|h(a,u_1)-h(a,u_2)|&=|(Hf_1-Hf_2)(b)-(Hf_1-Hf_2)(a)|\\&\leqslant\sum_{0\neq\alpha\leqslant 1}|\mathrm{md}_{|\alpha|}((Hf_1-Hf_2)^a_\alpha, I_a^b\lfloor\alpha)|\\&\leqslant\sum_{0\neq\alpha\leqslant 1}V_{|\alpha|}((Hf_1-Hf_2)^a_\alpha, I_a^b\lfloor\alpha)\\&\leqslant 2L|u_1-u_2|,\end{aligned}$$

which completes the proof of (5.2).

2. Now we prove the validity of the representation (1.9). For this, we fix $\overline{x} \in I_a^b$, so that $\overline{x} = a + \gamma(x - a)$ for some $a < x \leqslant b$ and $\gamma \in \mathscr{A}_0(n)$. We let $m \in \mathbb{N}$, and for $k \in \{1, \ldots, m\}$ we set $x[k] = (x_1(k), \ldots, x_n(k))$ and $y[k] = (y_1(k), \ldots, y_n(k))$, where, given $i \in \{1, \ldots, n\}$, the ith coordinates of $x[k]$ and $y[k]$ satisfy the following inequalities: $a_i < x_i(1) < y_i(1) < x_i(2) < y_i(2) < \cdots < x_i(m-1) < y_i(m-1) < x_i(m) < y_i(m) < b_i + \gamma_i(x_i - b_i)$. Also, we define n auxiliary Lipschitz functions $\Psi_m^i : [a_i, b_i] \to [0, 1]$ $(i = 1, \ldots, n)$ as follows:

$$\Psi_m^i(t) = \begin{cases} 0 & \text{if } a_i \leqslant t \leqslant x_i(1), \\ \Psi_{x_i(k), y_i(k)}(t) & \text{if } x_i(k) \leqslant t \leqslant y_i(k), k = 1, \ldots, m, \\ 1 - \Psi_{y_i(k), x_i(k+1)}(t) & \text{if } y_i(k) \leqslant t \leqslant x_i(k+1), k = 1, \ldots, m-1, \\ 1 & \text{if } y_i(m) \leqslant t \leqslant b_i. \end{cases}$$

For arbitrary numbers $u_1,\ u_2 \in \mathbb{R}$ and $j = 1,\ 2$ we set

$$f_j(y) = \frac{u_j}{n} \sum_{i=1}^{n} \Psi_m^i(y_i) + (2 - j)u_2, \quad a \leqslant y = (y_1, \ldots, y_n) \leqslant b$$

and note that $f_1,\ f_2 \in \mathrm{BV}(I_a^b; \mathbb{R})$, $(f_1 - f_2)(y) = u_2$ for all $y \in I_a^b$ and so, $\|f_1 - f_2\| = |u_2|$.

Setting $\mathscr{H} = Hf_1 - Hf_2$, let us estimate the sum of $|\mathscr{H}(y[k]) - \mathscr{H}(x[k])|$ over all $k = 1, \ldots, m$. Applying (2.1), Lemma 8 and (5.1), we get:

$$\begin{aligned} \sum_{k=1}^{m} |\mathscr{H}(y[k]) - \mathscr{H}(x[k])| &\leqslant \sum_{k=1}^{m} [\mathrm{TV}(\mathscr{H}, I_a^{y[k]}) - \mathrm{TV}(\mathscr{H}, I_a^{x[k]})] \\ &= \mathrm{TV}(\mathscr{H}, I_a^{y[m]}) - \mathrm{TV}(\mathscr{H}, I_a^{x[1]}) \\ &\quad - \sum_{k=1}^{m-1} [\mathrm{TV}(\mathscr{H}, I_a^{x[k+1]}) - \mathrm{TV}(\mathscr{H}, I_a^{y[k]})] \\ &\leqslant \mathrm{TV}(\mathscr{H}, I_a^{y[m]}) \leqslant \mathrm{TV}(\mathscr{H}, I_a^b) \\ &\leqslant \|Hf_1 - Hf_2\| \leqslant L|u_2|. \end{aligned}$$

Since, for $k \in \{1, \ldots, m\}$,

$$f_1(y[k]) = f_1(y_1(k), \ldots, y_n(k)) = \frac{u_1}{n} \sum_{i=1}^{n} \Psi_m^i(y_i(k)) + u_2 = u_1 + u_2,$$

$f_2(y[k]) = u_1,\ f_1(x[k]) = u_2$ and $f_2(x[k]) = 0$, we have:

$$\mathscr{H}(y[k]) = h(y[k],\ f_1(y[k])) - h(y[k],\ f_2(y[k])) = h(y[k], u_1 + u_2) - h(y[k], u_1)$$

and similarly, $\mathscr{H}(x[k]) = h(x[k], u_2) - h(x[k], 0)$, and so, the last inequality implies

$$\sum_{k=1}^{m} |h(y[k], u_1 + u_2) - h(y[k], u_1) - h(x[k], u_2) + h(x[k], 0)| \leqslant L|u_2|. \tag{5.4}$$

Because constant functions on I_a^b lie in $\mathrm{BV}(I_a^b;\mathbb{R})$ and the operator H maps $\mathrm{BV}(I_a^b;\mathbb{R})$ into itself, the function $h(\cdot,u)=[z\mapsto h(z,u)]$ also belongs to $\mathrm{BV}(I_a^b;\mathbb{R})$ for all $u\in\mathbb{R}$, and hence, by Theorem 2, its left regularization $h^*(\cdot,u)$ is in $\mathrm{BV}^*(I_a^b;\mathbb{R})$ for all $u\in\mathbb{R}$. Passing to the limit as

$$y[m]+\gamma(x[1]-y[m])\to(a+0)+\gamma((x-0)-(a+0))$$

(i.e., $x_i(1)\to x_i-0$ if $i\in\{1,\dots,n\}$ and $\gamma_i=1$, and $y_i(m)\to a_i+0$ for those $i\in\{1,\dots,n\}$ for which $\gamma_i=0$) in the inequality (5.4) and noting that

$$x[1]\leqslant x[k]<y[k]\leqslant y[m],\quad k=1,\dots,m$$

so that $x[k]$ and $y[k]$ tend to $(a+0)+\gamma((x-0)-(a+0))=\overline{x}$ as well, we find

$$m|h^*(\overline{x},u_1+u_2)-h^*(\overline{x},u_1)-h^*(\overline{x},u_2)+h^*(\overline{x},0)|\leqslant L|u_2|.$$

Due to the arbitrariness of $m\in\mathbb{N}$ for all $x\in I_a^b$ and $u_1,\ u_2\in\mathbb{R}$, we have:

$$h^*(x,u_1+u_2)-h^*(x,u_1)-h^*(x,u_2)+h^*(x,0)=0. \tag{5.5}$$

The rest of the proof of (1.9) is standard (cf. [2,8]). For the reader's convenience we recall the details. Given $x\in I_a^b$, we define the function $T_x:\mathbb{R}\to\mathbb{R}$ by $T_x(u)=h^*(x,u)-h^*(x,0)$, $u\in\mathbb{R}$, so that equality (5.5) can be written as $T_x(u_1+u_2)=T_x(u_1)+T_x(u_2)$ for all $u_1,\ u_2\in\mathbb{R}$, which shows that T_x is an additive function. By (5.2) and the definition of $h^*(\cdot,u)$, we have $|T_x(u_1)-T_x(u_2)|\leqslant 2L|u_1-u_2|$ for all $u_1,\ u_2\in\mathbb{R}$, and so, T_x is (Lipschitz) continuous on $\mathbb{R}$. Therefore there exists a function $h_1:I_a^b\to\mathbb{R}$ such that $T_x(u)=h_1(x)u$ for all $x\in I_a^b$ and $u\in\mathbb{R}$. Setting $h_0(x)=h^*(x,0)$, $x\in I_a^b$, we obtain the representation (1.9). It remains to note that, since $h_0(\cdot)=h^*(\cdot,0)$ and $h_1(\cdot)=h^*(\cdot,1)-h^*(\cdot,0)$, then, by Theorem 2, $h_0,\ h_1\in\mathrm{BV}^*(I_a^b;\mathbb{R})$. □

Remark 13. A theorem similar to Theorem 3 holds for the right regularization of $h(\cdot,u)$, $u\in\mathbb{R}$ (the definition of the right regularization of a function $f\in\mathrm{BV}(I_a^b;\mathbb{R})$ is analogous to (1.6)).

In general the left regularization h^* in (1.9) cannot be replaced by h itself. The first example in this direction for $n=1$ was constructed in [9, p. 157]; it was extended for $n=2$ in [2, Theorem 3]. Here we present an example of a general nature for $n\in\mathbb{N}$. Also, the next theorem shows that a general Lipschitzian superposition operator H from $\mathrm{BV}(I_a^b;\mathbb{R})$ into itself need not be generated by $h(x,u)$ of the form $h_1(x)u+h_0(x)$, $x\in I_a^b$, $u\in\mathbb{R}$, for some $h_0,h_1\in\mathrm{BV}(I_a^b;\mathbb{R})$ (cf. the characterization in Corollary 4 and the sufficient condition in [5, Corollary 3]).

Theorem 14. *Let $\{p_i(k)\}_{k=1}^{\infty}$ be a sequence of all distinct rational numbers from the interval $[a_i,b_i]$, $i=1,\dots,n$. Let the function $h:I_a^b\times\mathbb{R}\to\mathbb{R}$ be defined for $x=(x_1,\dots,x_n)\in I_a^b$ and $u\in\mathbb{R}$ by*

$$h(x_1,\dots,x_n,u)=\begin{cases}2^{-(\ell_1+\cdots+\ell_n)}\sin u & \text{if } x_i=p_i(\ell_i) \text{ with } \ell_i\in\mathbb{N},\, i=1,\dots,n,\\ 0 & \text{otherwise}.\end{cases}$$

Then the superposition operator H generated by h satisfies the conditions:

(a) $\|Hf\| \leqslant 3^n$ *for all* $f \in \mathrm{BV}(I_a^b; \mathbb{R})$, *i.e., H maps* $\mathrm{BV}(I_a^b; \mathbb{R})$ *into itself*;
(b) $\|Hf_1 - Hf_2\| \leqslant 3^n \|f_1 - f_2\|$ *for all* $f_1, f_2 \in \mathrm{BV}(I_a^b; \mathbb{R})$, *i.e., H is Lipschitzian*;
(c) *the left regularization of h is given by* $h^*(x,u) = 0$ *for all* $x \in I_a^b$ *and* $u \in \mathbb{R}$, *i.e., it is of the form* (1.9).

Proof. (a) Given $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$, we set $p[\ell] = (p_1(\ell_1), \ldots, p_n(\ell_n))$. Let $\mathscr{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ be a partition of I_a^b. Taking into account (1.10), for any $f \in \mathrm{BV}(I_a^b; \mathbb{R})$ and $\alpha \in \mathscr{A}(n)$ we have (below the sum over $1 \leqslant \sigma \lfloor \alpha$ means, as usual, the sum only over those $\sigma_i \in \{\sigma_1, \ldots, \sigma_n\}$ in the range $1 \leqslant \sigma_i \leqslant \kappa_i$ for which $\alpha_i = 1$, and the same applies to the sum over $0 \leqslant \sigma \lfloor \alpha$, but in the range $0 \leqslant \sigma_i \leqslant \kappa_i$):

$$\sum_{1 \leqslant \sigma \lfloor \alpha} |\mathrm{md}_{|\alpha|}((Hf)_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]} \lfloor \alpha)| \leqslant \sum_{1 \leqslant \sigma \lfloor \alpha} \sum_{0 \leqslant \theta \leqslant \alpha} |(Hf)(a + \alpha(x[\sigma-1] - a) + \theta(x[\sigma] - x[\sigma-1]))|.$$

Note that the ith component in the argument of Hf is equal to a_i if $\alpha_i = 0$ (so that $\theta_i = 0$), $x_i[\sigma_i - 1]$ if $\alpha_i = 1$ and $\theta_i = 0$, and $x_i[\sigma_i]$ if $\alpha_i = 1$ and $\theta_i = 1$. And so, since $\#\{\theta \in \mathscr{A}_0(n) \mid \theta \leqslant \alpha\} = 2^{|\alpha|}$, we can proceed the above inequality in the following way:

$$\begin{aligned}
&\leqslant 2^{|\alpha|} \sum_{0 \leqslant \sigma \lfloor \alpha} |(Hf)(a + \alpha(x[\sigma] - a))| \\
&= 2^{|\alpha|} \sum_{0 \leqslant \sigma \lfloor \alpha} |h(a + \alpha(x[\sigma] - a), f(a + \alpha(x[\sigma] - a)))| \\
&\leqslant 2^{|\alpha|} \sum_{1 \leqslant \ell \lfloor \alpha}^{\infty} |h(a + \alpha(p[\ell] - a), f(a + \alpha(p[\ell] - a)))| \\
&\leqslant 2^{|\alpha|} \sum_{1 \leqslant \ell \lfloor \alpha}^{\infty} 2^{-|\ell \lfloor \alpha|} |\sin\, f(a + \alpha(p[\ell] - a))| \\
&\leqslant 2^{|\alpha|} \sum_{1 \leqslant \ell \lfloor \alpha}^{\infty} 2^{-|\ell \lfloor \alpha|} = 2^{|\alpha|}
\end{aligned}$$

and it follows that

$$V_{|\alpha|}((Hf)_\alpha^a, I_a^b \lfloor \alpha) \leqslant 2^{|\alpha|}, \quad \alpha \in \mathscr{A}(n).$$

Since $\#\{\alpha \in \mathscr{A}(n) \mid |\alpha| = j\} = C_n^j$, $1 \leqslant j \leqslant n$, we find

$$\begin{aligned}
\mathrm{TV}(Hf, I_a^b) &= \sum_{\alpha \in \mathscr{A}(n)} V_{|\alpha|}((Hf)_\alpha^a, I_a^b \lfloor \alpha) \leqslant \sum_{\alpha \in \mathscr{A}(n)} 2^{|\alpha|} \\
&= \sum_{j=1}^{n} 2^j C_n^j = (2+1)^n - 1 = 3^n - 1.
\end{aligned}$$

Noting that $|(Hf)(a)| \leqslant 1$, we get $\|Hf\| \leqslant 3^n$.

(b) Given $f_1,\ f_2 \in \mathrm{BV}(I_a^b;\mathbb{R})$, $\alpha \in \mathscr{A}(n)$ and a partition $\mathscr{P}=\{x[\sigma]\}_{\sigma=0}^{\kappa}$ of I_a^b, we have (cf. the calculations in (a) and take into account (1.3) and (2.1)):

$$\begin{aligned}
&\sum_{1\leqslant\sigma\lfloor\alpha} |\mathrm{md}_{|\alpha|}((Hf_1 - Hf_2)_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]}\lfloor\alpha)| \\
&\quad\leqslant 2^{|\alpha|}\sum_{0\leqslant\sigma\lfloor\alpha} |(Hf_1 - Hf_2)(a+\alpha(x[\sigma]-a))| \\
&\quad\leqslant 2^{|\alpha|}\sum_{1\leqslant\ell\lfloor\alpha}^{\infty} |h(a+\alpha(p[\ell]-a),\ f_1(a+\alpha(p[\ell]-a))) \\
&\qquad - h(a+\alpha(p[\ell]-a),\ f_2(a+\alpha(p[\ell]-a)))| \\
&\quad\leqslant 2^{|\alpha|}\sum_{1\leqslant\ell\lfloor\alpha}^{\infty} 2^{-|\ell\lfloor\alpha|}|\sin\ f_1(a+\alpha(p[\ell]-a)) - \sin f_2(a+\alpha(p[\ell]-a))| \\
&\quad\leqslant 2^{|\alpha|}\sum_{1\leqslant\ell\lfloor\alpha}^{\infty} 2^{-|\ell\lfloor\alpha|}|(f_1 - f_2)(a+\alpha(p[\ell]-a))| \\
&\quad\leqslant 2^{|\alpha|}\sum_{1\leqslant\ell\lfloor\alpha}^{\infty} 2^{-|\ell\lfloor\alpha|}\|f_1 - f_2\| = 2^{|\alpha|}\|f_1 - f_2\|.
\end{aligned}$$

It follows that

$$V_{|\alpha|}((Hf_1 - Hf_2)_\alpha^a, I_a^b\lfloor\alpha) \leqslant 2^{|\alpha|}\|f_1 - f_2\|, \quad \alpha\in\mathscr{A}(n).$$

Finally, noting that

$$\begin{aligned}
|(Hf_1 - Hf_2)(a)| &= |h(a,\ f_1(a)) - h(a,\ f_2(a))| \leqslant |\sin\ f_1(a) - \sin\ f_2(a)| \\
&\leqslant |(f_1 - f_2)(a)| \leqslant \|f_1 - f_2\|,
\end{aligned}$$

we conclude that $\|Hf_1 - Hf_2\| \leqslant 3^n\|f_1 - f_2\|$.

Since item (c) is clear, the proof is complete. □

6. More generalizations

1. Let $N\in\mathbb{N}$ and $(\mathbb{R}^N)^{I_a^b} = (\mathbb{R}^{I_a^b})^N$ be the algebra of all functions f mapping I_a^b into $\mathbb{R}^N$. If $h: I_a^b\times\mathbb{R}^N\to\mathbb{R}$ is a given function of $n+N$ variables, $h=h(x_1,\ldots,x_n,u_1,\ldots,u_N)$, we denote by $H_h:(\mathbb{R}^N)^{I_a^b}\to\mathbb{R}^{I_a^b}$ the *superposition operator* defined by

$$(H_h f)(x) = h(x,\ f_1(x),\ldots,\ f_N(x)), \quad x\in I_a^b, \quad f=(f_1,\ldots,f_N)\in(\mathbb{R}^N)^{I_a^b}.$$

The Cartesian product

$$\mathrm{BV}(I_a^b;\mathbb{R})^N = \underbrace{\mathrm{BV}(I_a^b;\mathbb{R})\times\cdots\times\mathrm{BV}(I_a^b;\mathbb{R})}_{N\ \text{times}}$$

is endowed with the product norm $\|f\|_N = \sum_{i=1}^N \|f_i\|$ for $f = (f_1, \ldots, f_N) \in \mathrm{BV}(I_a^b; \mathbb{R})^N$, in which case it is a Banach algebra with respect to the componentwise pointwise operations, and, by virtue of Theorem 1, the following inequality holds: $\|f \cdot g\|_N \leqslant 2^n \|f\|_N \|g\|_N$ for all $f, g \in \mathrm{BV}(I_a^b; \mathbb{R})^N$. We have the following converse to Corollary 13 from [5]:

Corollary 15. *If H_h maps the space $\mathrm{BV}(I_a^b; \mathbb{R})^N$ into $\mathrm{BV}(I_a^b; \mathbb{R})$ and is Lipschitzian (in the obvious sense), then for the left regularization of $h(\cdot, u_1, \ldots, u_N)$ we have: $h^*(x, u_1, \ldots, u_N) = h_0(x) + \sum_{j=1}^N h_j(x) u_j$ for all $(u_1, \ldots, u_N) \in \mathbb{R}^N$ and $x \in I_a^b$ where functions $h_0, h_1, \ldots, h_N$ belong to $\mathrm{BV}^*(I_a^b; \mathbb{R})$.*

2. Let $(\mathbb{U}, |\cdot|_{\mathbb{U}})$ and $(\mathbb{V}, |\cdot|_{\mathbb{V}})$ be normed linear spaces. The definition of the corresponding space $\mathrm{BV}(I_a^b; \mathbb{U})$ of functions $f : I_a^b \to \mathbb{U}$ of bounded variation in the sense of Vitali, Hardy and Krause is straightforward (cf. Section 1). Let us denote by $L(\mathbb{U}; \mathbb{V})$ the normed linear space of all linear continuous operators from $\mathbb{U}$ into $\mathbb{V}$, and let $\mathbb{U}^I$ be the set of all functions mapping $I = I_a^b$ into $\mathbb{U}$. Given $h : I \times \mathbb{U} \to \mathbb{V}$, the *superposition operator* $H : \mathbb{U}^I \to \mathbb{V}^I$ is defined as in (1.4) with $f \in \mathbb{R}^I$ replaced by $f \in \mathbb{U}^I$. Furthermore, let $\mathrm{P}_i([a_i, b_i]; \mathbb{U})$ be the family of functions from $[a_i, b_i]$ into $\mathbb{U}$ having the following property: if $m \in \mathbb{N}$, $u_1, u_2 \in \mathbb{U}$ and $a_i < x_i(1) < y_i(1) < x_i(2) < y_i(2) < \cdots < x_i(m) < y_i(m) < b_i$, then the polygonal function $[a_i, b_i] \ni t \mapsto \Psi_m^i(t) u_1 + u_2 \in \mathbb{U}$ belongs to $\mathrm{P}_i([a_i, b_i]; \mathbb{U})$, $i = 1, \ldots, n$ (for the definition of Ψ_m^i see step 2 in the proof of Theorem 3). We have: $\mathrm{P}(I_a^b; \mathbb{U}) = \sum_{i=1}^n \mathrm{P}_i([a_i, b_i]; \mathbb{U})$ is a subspace of $\mathrm{BV}(I_a^b; \mathbb{U})$, which we endow with the norm $\|\cdot\|$ from $\mathrm{BV}(I_a^b; \mathbb{U})$. The analysis of the proofs of Theorems 2 and 3 shows that the following counterpart and generalization of Theorem 3 holds:

Theorem 16. *Suppose that the superposition operator $H : \mathbb{U}^I \to \mathbb{V}^I$ is generated by a function $h : I \times \mathbb{U} \to \mathbb{V}$ with $I = I_a^b$. If $\mathbb{U}$ is a real normed linear space, $\mathbb{V}$ is a Banach space and H maps the space $\mathrm{P}(I_a^b; \mathbb{U})$ into $\mathrm{BV}(I_a^b; \mathbb{V})$ and is Lipschitzian (in the sense of the norms in these spaces), then there exists a constant $L_0 > 0$ such that $|h(x, u_1) - h(x, u_2)|_{\mathbb{V}} \leqslant L_0 |u_1 - u_2|_{\mathbb{U}}$, $x \in I_a^b$, $u_1, u_2 \in \mathbb{U}$, and there exist two functions $h_0 \in \mathrm{BV}^*(I_a^b; \mathbb{V})$ and $h_1 : I_a^b \to L(\mathbb{U}; \mathbb{V})$ with the property that $h_1(\cdot)u \in \mathrm{BV}^*(I_a^b; \mathbb{V})$ for all $u \in \mathbb{U}$such that $h^*(x, u) = h_1(x)u + h_0(x)$ in $\mathbb{V}$ for all $x \in I_a^b$ and $u \in \mathbb{U}$.*

The proof of Theorem 16 is the same as that of Theorem 3, except the last paragraph: since $\mathbb{U}$ is real, the additivity and continuity of the mapping T_x imply $T_x \in L(\mathbb{U}; \mathbb{V})$ for all $x \in I_a^b$, and so, if we set $h_1(x)u = T_x(u)$, $x \in I_a^b$, $u \in \mathbb{U}$, then we find that h_1 maps I_a^b into $L(\mathbb{U}; \mathbb{V})$ and $h_0, h_1(\cdot)u \in \mathrm{BV}^*(I_a^b; \mathbb{V})$ for all $u \in \mathbb{U}$. For the partial converse of Theorem 16 cf. [5, Theorem 14].

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