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Linear equations over noncommutative graded rings

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Abstract

We call a graded connected algebra R *effectively coherent*, if for every linear equation over R with homogeneous coefficients of degrees at most d , the degrees of generators of its module of solutions are bounded by some function $D(d)$. For commutative polynomial rings, this property has been established by Hermann in 1926. We establish the same property for several classes of noncommutative algebras, including the most common class of rings in noncommutative projective geometry, that is, strongly Noetherian rings, which includes Noetherian PI algebras and Sklyanin algebras. We extensively study so-called *universally coherent* algebras, that is, such that the function $D(d)$ is bounded by $2d$ for $d \gg 0$. For example, finitely presented monomial algebras belong to this class, as well as many algebras with finite Groebner basis of relations.

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1. Introduction

1.1. Overview

Let R be a finitely generated connected graded associative algebra over a field k . We discuss the solutions of a linear equation

$$a_1x_1 + \cdots + a_nx_n = 0 \quad (1.1)$$

over R , where a_1, \dots, a_n are homogeneous elements of R or of a (free) R -module M , and x_1, \dots, x_n are indeterminates. When and how can such an equation be solved, and how does one describe the solutions?

For general finitely generated R , there is no algorithm even to check if a solution exists (at least in the nonhomogeneous case for a free module M , see [38]). Also, the set of solutions Ω may be infinitely generated as a submodule of the free module R^n (for example, over the algebra $F \otimes F$, where F is a free associative algebra with a large number of generators). That is why it seems reasonable to restrict the class of algebras under consideration to algebras R such that the module of solutions of Eq. (1.1) is finitely generated. Such algebras are called (right) *coherent* [9,19]; this class includes all Noetherian algebras, free associative algebras, and many other examples. However, the condition of coherence does not in general give a way to find all the generators of Ω : we can find the generators one by one, but when we have to stop?

If R is a commutative affine algebra, there is an easy (but not the most effective) way to find the generators of Ω . It was established by Hermann [24] in 1926 that there exists a function $D_R : \mathbf{N} \rightarrow \mathbf{N}$ such that Ω is generated in degrees at most $D_R(d)$ provided that all coefficients a_i have degrees at most d . So, to find all the solutions, it is sufficient to find the solutions in the finite-dimensional vector space $R_{\leq D_R(d)}$: it is a standard exercise in linear algebra.

The main object of this paper is to consider noncommutative algebras R which admit such a function $D_R(d)$. We call these algebras *effectively coherent*. An analogous concept for commutative (non)graded algebras has been introduced by Soublin [36]. A commutative ring R is called uniformly coherent if there is a function $\Delta_R : \mathbf{N} \rightarrow \mathbf{N}$ such that Ω is generated by at most $\Delta_R(n)$ elements. It was shown in [21] that an affine or local Noetherian commutative ring is uniformly coherent if and only if its dimension is at most two.

Fortunately, our graded analogue of this concept is more common. We show that most rings considered in noncommutative projective geometry, that is, strongly Noetherian algebras [5] over algebraically closed fields, are effectively coherent (R is called (right) *strongly Noetherian* if $R \otimes C$ is right Noetherian for every commutative Noetherian k -algebra C). This class includes, in particular, Sklyanin algebras, Noetherian PI algebras (in particular, standard Noetherian semigroup algebras of polynomial growth [20, Theorem 3.1]), Noetherian domains of Gelfand–Kirillov dimension two, and Artin–Shelter regular algebras of dimension three [5]. Also, free associative algebras and finitely presented monomial algebras are effectively coherent as well. Every coherent algebra over a finite field k is effectively coherent; however, if k contains two algebraically independent

elements or has zero characteristic, then there exist Noetherian but not effectively coherent algebras.

The class of finitely generated effectively coherent algebras is closed under extensions by finitely presented modules, free products, direct sums, and taking Veronese subalgebras (in the degree-one generated case). Every finitely presented graded module M over such an algebra is effectively coherent as well, that is, there is a similar function $D_M(d)$ which bounds the degrees of generators in the case of Eq. (1.1) over M . This means that homogeneous linear equations over such modules are effectively solvable as well.

The degree bound function $D_R(d)$ for the commutative polynomial ring R grows as a double exponent. That is why we cannot hope that there is a wide class of noncommutative algebras R with slow growth of $D_R(d)$. However, there are interesting classes of algebras with linear growth of $D_R(d)$. We call an algebra *universally coherent* if $D(d) \leq 2d$ for all $d \gg 0$. We investigate this class of algebras and more general classes, so-called algebras with Koszul filtrations and with coherent families of ideals. In particular, it is shown that every finitely presented module over a universally coherent algebra has rational Hilbert series, including the algebra R itself, and R has finite Backelin's rate (that is, there is a number r such that every space $\text{Tor}_i^R(k, k)$ is concentrated in degrees at most ri). Free associative algebras are universally coherent, as well as finitely presented monomial algebras and *algebras with r -processing* [30], that is, algebras with finite Groebner basis such that the normal form of a product of two their elements can be calculated by the product of their normal forms via a bounded number of reductions. The main property of such algebras is that every right-sided ideal I has finite Groebner basis: it consists of elements of degree less than $d + r$ [30, Theorem 5], where I is generated in degrees at most d .

Note that the most effective modern method to solve an equation of type (1.1) is based on the theory of Groebner bases [22,26,29,30]. Let I be a submodule of M generated by the coefficients a_1, \dots, a_n . In this method, we can calculate the Groebner bases of relations of R , of relations of M , and of the submodule I up to degree $D(d)$, and then find a generating set of the relations of degrees at most $D(d)$ between these elements of Groebner basis of I , again using standard Groebner theory methods. The calculation of the Groebner bases above may be done in the same way as the usual calculation of Groebner bases of ideals in algebras, since I is an ideal in the trivial extension algebra $R' = M \oplus R$; this calculation is equivalent also to finding the two-sided Groebner basis of relations of the larger trivial extension $R'/I \oplus R'$ (a similar trick has been described in [25]).

1.2. Motivation

Despite of the famous recent progress in noncommutative projective geometry, no general noncommutative version of computational methods of algebraic geometry is known. In this paper, we try to show that a “computational noncommutative geometry” is possible. At least, if R is a “ring of noncommutative projective geometry” [28], then there exist algorithms to solve linear equations over R (since R is usually strongly Noetherian), therefore, to calculate the relations and the minimal projective resolution of a finite module (because R has often finite global dimension).

1.3. Notation and assumptions

We will deal with finitely generated \mathbf{Z}_+ -graded connected associative algebras over a fixed field k , that is, algebras of the form $R = \bigoplus_{i \geq 0} R_i$ with $R_0 = k$. Also, all algebras are assumed to be finitely generated. All modules and ideals are graded and right-sided.

A solution $x = (x_1, \dots, x_n)$ of Eq. (1.1) is called *homogeneous*, if there is $D > 0$ (the *degree* of x) such that $\deg a_i + \deg x_i = D$ for all nonzero x_i .

If a sequence $a = \{a_1, \dots, a_n\}$ of homogeneous elements in an R -module M generates a submodule I , let $b = \{a_1, \dots, a_m\}$ ($m \leq n$) be a minimal subsequence of a generating the same submodule. Let Ω^x be the module of solutions of Eq. (1.1), let Ω^y be the module of solutions of the corresponding equation for b

$$a_1 y_1 + \dots + a_m y_m = 0, \quad (1.2)$$

and let D_x and D_y be the maximal degrees of homogeneous generators of Ω^x and Ω^y . It is easy to see that $D_x \leq \max\{D_y, d\}$. Therefore, we may (and will) always assume that the coefficients in Eq. (1.1) *minimally* generate some submodule $I = a_1 R + \dots + a_n R \subset M$.

For an R -module M , we will denote by $H_i M$ the graded vector space $\text{Tor}_i^R(M, k)$. By $H_i R$ we will denote the graded vector space $\text{Tor}_i^R(k, k) = H_i k_R$. In particular, the vector space $H_1 R$ is isomorphic to the linear k -span of a minimal set of homogeneous generators of R , and $H_2 R$ is isomorphic to the k -span of a minimal set of its homogeneous relations. Analogously, the space $H_0 M$ is the span of generators of M , and $H_1 M$ is the span of its relations.

Let $m(M) = m_0(M)$ denote the supremum of degrees of minimal homogeneous generators of M : if M is just a vector space with the trivial module structure, it is simply the supremum of degrees of elements of M . For $i \geq 0$, let us also put $m_i(M) := m(H_i M) = \sup\{j \mid \text{Tor}_i^R(M, k)_j \neq 0\}$. Similarly, let us put $m_i(R) = m(H_i R) = m_i(k_R)$. For example, $m(R) = m_1(R)$ is the supremum of degrees of the generators of R , and $m_2(R)$ (respectively $m_1(M)$) is the supremum of degrees of the relations of R (respectively of M). In other words, if a module I is minimally generated by the coefficients a_1, \dots, a_n of Eq. (1.1) and Ω is the module of solutions of this equation, then $m(\Omega) = m_1(I)$.

Note that the symbols $H_i R$ and $m_i(R)$ for an algebra have different meaning than the respective symbols $H_i R_R$ and $m_i(R_R)$ for R considered as a module over itself; however, the homologies $H_i R_R$ are trivial, so that there is no place for confusion.

Definition 1.1. For a finitely generated module M , let us define a function $D_M : \mathbf{N} \rightarrow \mathbf{N} \cup \{\infty\}$ by taking $D_M(d) = \sup\{m_1(L) \mid L \subset M, m_0(L) \leq d\}$.

This means that every submodule $L \subset M$ generated in degrees at most d has relations in degrees at most $D = D_M(d)$, and that the module of solutions of every linear equation (1.1) with coefficients of degrees at most d in M is generated in degrees at most $\max\{D(d), d\}$.

For a graded locally finite vector space (algebra, module...) V , its Hilbert series is defined as the formal power series $V(z) = \sum_{i \in \mathbf{Z}} (\dim V_i) z^i$. For example, the Euler char-

acteristics of a minimal free resolution of the trivial module k_R leads to the formula

$$R(z)^{-1} = \sum_{i \geq 0} (-1)^i H_i R(z). \quad (1.3)$$

As usual, we write $\sum_{i \geq 0} a_i z^i = o(z^n)$ iff $a_i = 0$ for $i \leq n$.

Let us introduce a lexicographical total order on the set of all power series with integer coefficients, i.e., we put $\sum_{i \geq 0} a_i z^i >_{\text{lex}} \sum_{i \geq 0} b_i z^i$ iff there is $q \geq 0$ such that $a_i = b_i$ for $i < q$ and $a_q > b_q$. This order extends the coefficient-wise partial order given by $\sum_{i \geq 0} a_i z^i \geq \sum_{i \geq 0} b_i z^i$ iff $a_i \geq b_i$ for all $i \geq 0$.

1.4. Results

Our technique is based on the investigation of Hilbert series of algebras and modules. We begin with recalling a classical theorem of Anick on the Hilbert series of finitely presented algebras: the set of Hilbert series of all n -generated algebras R with $m_1(R), m_2(R) < \text{Const}$ satisfies the ascending chain condition with respect to the order $>_{\text{lex}}$. Then we improve this theorem for the algebras with additional condition $m_3(R) < \text{Const}$: that is, we state

Theorem 1.2 (Theorem 2.2). *Given four integers n, a, b, c , let $D(n, a, b, c)$ denote the set of all connected algebras R over a fixed field k with at most n generators such that $m_1(R) \leq a$, $m_2(R) \leq b$, and $m_3(R) \leq c$. Then the set of Hilbert series of algebras from $D(n, a, b, c)$ is finite.*

This additional restriction $m_3(R) < \text{Const}$ (the weakest among all considered in this paper) is discussed in Section 2.1. We give also a version of both these theorems for modules (Section 2.3): in particular, if $m_i(R) < \text{Const}$ for $i = 1, 2, 3$, then the set of Hilbert series of ideals $I \subset R$ with $m_0(I), m_1(I) < \text{Const}$ is finite.

In Section 3, we introduce and study effectively coherent rings. First, we give several criteria for a ring to be effectively coherent and show that finitely presented extensions, free products, free sums, and Veronese subrings of effectively coherent rings are effectively coherent as well, and give appropriate estimates for the function $D(d)$. Further, we introduce other effectiveness properties of graded algebras, related to Hilbert series of their finitely presented modules. Let M be a finitely presented module, and let L run through the set of all its finitely generated submodules. We say that M is *effective for generators* (respectively *effective for series*), if, given the Hilbert series $L(z)$ (respectively given $m(L)$), there are only finite number of possibilities for $m(L)$ (respectively for $L(z)$). An algebra R is said to be effective for generators (respectively for series), if every finitely presented R -module satisfies this property. The relations of these properties to effective coherence are the following: if a coherent algebra R is effective both for series and for generators, then it is effectively coherent, and every effectively coherent algebra is effective for series. Also, we show that the properties of Hilbert series of finitely generated modules over strongly Noetherian algebras established in [5, Section E4] imply both effectiveness for generators and for series: in particular, we obtain

Theorem 1.3 (Corollary 3.17). *Every strongly Noetherian algebra over an algebraically closed field is effectively coherent.*

However, there are Noetherian algebras which do not satisfy any of our effectiveness properties: for example, one of Noetherian but not strongly Noetherian algebras from [35] (namely, the graded algebra $R_{p,q}$ generated by two Eulerian derivatives introduced in [27]).

In Section 4, we study algebras, not necessary coherent, but having a lot of finitely presented ideals. Such special families of ideals were first introduced for quadratic commutative algebras as *Koszul filtrations* [12,13]; then this notion has been generalized to nonquadratic commutative [16] and to quadratic noncommutative [32] algebras. Here we consider the most general version, which is called *coherent family* of ideals. A family \mathbf{F} of finitely generated ideals in R is said to be coherent if $0 \in \mathbf{F}$, $R_{\geq 1} \in \mathbf{F}$, and for every $0 \neq I \in \mathbf{F}$ there are $J \in \mathbf{F}$ and $x \in I$ such that $I \neq J$, $I = J + xR$, $m(J) \leq m(I)$, and the ideal $(x : J) := \{a \in R \mid xa \in J\}$ also belongs to \mathbf{F} . A *degree* of \mathbf{F} is the supremum of degrees of generators of ideals $I \in \mathbf{F}$. Coherent families of degree one are called Koszul filtrations; they do exist in many commutative quadratic rings (such as coordinate rings of some common varieties), in algebras with generic relations, and in quadratic monomial algebras. We show that if an algebra admits a coherent family \mathbf{F} of finite degree, then it has finite Backelin's rate (generalizing an analogous result in the commutative case [16, Proposition 1.2]), and its Hilbert series is a rational function (generalizing similar result for the algebras with Koszul filtrations [32, Theorem 3.3]), and the same is true for every ideal $I \in \mathbf{F}$.

Every ideal in a Koszul filtration is a Koszul module, and an algebra is coherent if and only if all its finitely generated ideals form a coherent family. If all ideals of an algebra R , generated in degrees at most d , form a coherent family, we call R universally d -coherent. An algebra R is called *universally coherent* if it is universally d -coherent for all $d \gg 0$. In fact, these properties are the properties of the function $D_R(d)$:

Theorem 1.4 (Proposition 4.6, Corollary 4.7). *Let R be a finitely generated graded algebra.*

- (a) *R is universally d -coherent iff $D_R(t) \leq t + d$ for all $t \leq d$.*
- (b) *R is universally coherent iff $D_R(d) \leq 2d$ for all $d \gg 0$.*

In particular, any universally coherent algebra is effectively coherent.

Commutative 1-universally coherent algebras are called *universally Koszul*; they have been studied in [14,15]. We show that the d th Veronese subring of a generated in degree one universally d -coherent algebra R is universally Koszul, therefore, such an algebra is (up to a shift of grading) a Koszul module over a universally Koszul algebra.

Some noncommutative examples of universally coherent algebras are considered in Section 4.4, that is, finitely presented monomial algebras and, more generally, a class of algebras with a finite Groebner basis of relations (algebras with r -processing), which were introduced in [30].

2. Sets of Hilbert series

2.1. The condition $\dim \text{Tor}_3^R(k, k) < \infty$

In this section, we sometimes consider finitely presented algebras R such that $m_3(R) < \infty$. Before studying their Hilbert series, let us say a few words about this inequality.

First, all common finitely presented algebras (such as Noetherian, coherent, Koszul, etc.) do satisfy this condition. In fact, it is the weakest restriction on R among all that are considering in this paper. In a coherent ring, the module of solutions of any linear equation over a free module is finitely generated; in general, there is a particular linear equation in a free module which has finite basis of solutions if and only if $m_3(R) < \infty$.

Indeed, let $a = \{a_1, \dots, a_g\}$ be a minimal set of homogeneous generators of R , and let $f = \{f_1, \dots, f_r\}$ be a minimal set of its homogeneous relations. Let $f_j = \sum_{i=1}^g a_i b_j^i$ for $j = 1, \dots, r$. In the minimal free resolution of k_R

$$\dots \rightarrow H_3(R) \otimes R \rightarrow H_2(R) \otimes R \rightarrow H_1(R) \otimes R \rightarrow R \rightarrow k \rightarrow 0$$

we see that $H_1(R)$ is the span of a , and $H_2(R)$ is the span of f . Let $\tilde{f}_j = \sum_{i=1}^g a_i \otimes b_j^i \in ka \otimes R$ be the image of $f_j \otimes 1$ in the free module $M = ka \otimes R$. Consider the following equation with coefficients in M :

$$\tilde{f}_1 x_1 + \dots + \tilde{f}_r x_r = 0.$$

Since the resolution above is minimal, every minimal space of generators of the solution module \mathcal{Q} of this equation is isomorphic to $H_3(R)$.

However, in general, given a presentation (a, f) of an algebra R , there does not exist an algorithm to decide if the condition $m_3(R) < \infty$ holds. This has been shown in [2] for Roos algebras, that is, universal enveloping algebras of quadratic graded Lie superalgebras.

2.2. Hilbert series of finitely presented algebras

The following well-known theorem describes an interesting property of Hilbert series of finitely presented algebras.

Theorem 2.1 [3, Theorem 4.3]. *Given three integers n, a, b , let $C(n, a, b)$ be the set of all n -generated connected algebras R with $m_1(R) \leq a$ and $m_2(R) \leq b$ and let $\mathcal{H}(n, a, b)$ be the set of Hilbert series of such algebras. Then the ordered set $(\mathcal{H}(n, a, b), >_{\text{lex}})$ admits no infinite ascending chains.*

The example of an infinite *descending* chain of Hilbert series in the set $C(7, 1, 2)$ is constructed in [3, Example 7.7]. All algebras in this chain have global dimension three, but in the vector spaces $H_3 R$ there are elements of arbitrary high degree.

The following theorem shows, in particular, that the last property is essential for such examples.

Theorem 2.2. *Given four integers n, a, b, c , let $D(n, a, b, c)$ denote the set of all connected algebras A over a fixed field k with at most n generators such that $m_1(A) \leq a$, $m_2(A) \leq b$, and $m_3(A) \leq c$. Then the set of Hilbert series of algebras from $D(n, a, b, c)$ is finite.*

For the set $D(n, 1, 2, 3)$, Theorem 2.2 was proved in [34, Section 4.2, Corollary 2 and Remark 1] (in a different way, using a geometrical technique). In particular, given a number n , the set of Hilbert series of n -generated quadratic Koszul algebras is finite. For the class of degree-one generated algebras of bounded Backelin's rate (that is, when there is a number r such that for every i the vector space $H_i R$ is concentrated in degrees at most ri), a similar statement has been proved by L. Positselski (unpublished).

We need the following standard version of Koenig lemma.

Lemma 2.3. *Let P be a totally ordered set satisfying both ACC and DCC. Then P is finite.*

Proof of Theorem 2.2. Consider a connected algebra A with a minimal space of generators V and a minimal space of relations $R \subset T(V)$. Choose a homogeneous basis $f = \{f_i\}$ in R . Let I be the ideal in $T(V)$ generated by R , and let G be the graded algebra associated to the I -adic filtration on $T(V)$. By [31, Theorem 3.2], we have $H_j G = H_j A \oplus H_{j+1} A$ for all $j \geq 1$; in particular, the space of generators of G is isomorphic to $V \oplus R$. Let $\tilde{f} = \{\tilde{f}_i\}$ be the set of generators of the second summand, corresponding to the basis $f = \{f_i\}$ in R .

Note that G is generated by its subsets A and \tilde{f} , and its grading extends the grading of A if $\deg \tilde{f}_i = \deg f_i$. Moreover, the algebra G is the quotient of the free product $A * k\langle \tilde{f} \rangle$ by the ideal generated by some elements of $A \otimes \tilde{f} \otimes A$ [31, Section 3], so that we can consider another of G given by $\deg' a = \deg a$ for $a \in A$ and $\deg' \tilde{f}_i = \deg f_i - 1$. Let us denote the same algebra G with this new grading by $C = C(A)$. It follows from the consideration in [31, proof of Lemma 5.5] that its homology groups are given by the formula $H_j C = H_j A \oplus H_{j+1} A[1]$ for all $j \geq 1$. By the formula (1.3), we have

$$\begin{aligned} C(z)^{-1} &= \sum_{i \geq 0} (-1)^i H_i C(z) = \\ &= \sum_{i \geq 0} (-1)^i H_i A(z) + z^{-1} \left(1 - V(z) - \sum_{i \geq 0} (-1)^i H_i A(z) \right) \\ &= A(z)^{-1} (1 - z^{-1}) + z^{-1} (1 - V(z)). \end{aligned}$$

Now suppose that two connected algebras A, B have the same graded vector space of generators V . Suppose $A(z) >_{\text{lex}} B(z)$, that is, $A(z) - B(z) = pz^q + o(z^q)$ for some $p > 0$, $q > 1$. We have

$$\begin{aligned} C(A)(z) - C(B)(z) &= (C(B)(z)^{-1} - C(A)(z)^{-1}) C(A)(z) C(B)(z) \\ &= (B(z)^{-1} - A(z)^{-1}) (1 - z^{-1}) (1 + o(1)) \\ &= \frac{B(z) - A(z)}{z A(z) B(z)} (1 + o(1)) = -pz^{q-1} + o(z^{q-1}), \end{aligned}$$

that is, $C(A)(z) <_{\text{lex}} C(B)(z)$.

Now, we are ready to prove the theorem. Assume (ad absurdum) that the set $D(n, a, b, c)$ is infinite for some n, a, b, c . By Lemma 2.3, there is an infinite descending chain Ch_0

$$A^{(1)}(z) >_{\text{lex}} A^{(2)}(z) >_{\text{lex}} \dots$$

Since for algebras in the set $D(n, a, b, c)$ there are only finite number of possibilities for the number and degrees of generators, the chain Ch_0 contains an infinite subchain Ch_1

$$A^1(z) >_{\text{lex}} A^2(z) >_{\text{lex}} \dots,$$

where all algebras A^i are generated by the same graded vector space V . It follows that we have an ascending chain $C(Ch_1)$:

$$C(A^1)(z) <_{\text{lex}} C(A^2)(z) <_{\text{lex}} \dots.$$

For every algebra $C(A^i)$, its generators are concentrated in degrees at most $a' = \max(a, b - 1)$ and relations are concentrated in degrees at most $b' = \max(b, c - 1)$. Moreover, for the number of its generators we have the following estimate:

$$\dim H_1 C(A^i) = \dim H_1 A^i + \dim H_2 A^i \leq n + n^b =: n'.$$

We deduce that an infinite ascending chain $C(Ch_1)$ consists of algebras from the set $C(n', a', b')$, in contradiction to Theorem 2.1. \square

2.3. Modules and ideals

The following gives module versions of Theorems 2.1 and 2.2 for modules.

Proposition 2.4. *Let $n, a, b, c, m, p_1, p_2, q, r$ be 9 integers.*

- (a) *Let R be an algebra from $C(n, a, b)$, and let $CM = CM(m, p_1, p_2, q)$ denote the set of all graded right R -modules M with at most m generators such that $M_i = 0$ for $i < p_1$, $m_0(M) \leq p_2$, and $m_1(M) \leq q$. Then the ordered set of Hilbert series of modules from CM satisfies ACC.*
- (b) *Let $DM = DM(n, a, b, c, m, p_1, p_2, q, r)$ denote the set of all graded right modules over algebras from $D(n, a, b, c)$ with at most m generators such that $M_i = 0$ for $i < p_1$, $m_0(M) \leq p_2$, $m_1(M) \leq q$, and $m_2(M) \leq r$. Then the set of Hilbert series of modules from DM is finite.*

Proof. If $p_1 < 0$, let us shift the grading of all modules by $1 - p_1$ and consider the sets of Hilbert series of the modules from $CM(m, 1, p_2 - p_1 + 1, q - p_1 + 1)$ (respectively $DM(n, a, b, c, m, 1, p_2 - p_1 + 1, q - p_1 + 1, r - p_1 + 1)$). Since these new sets are in bijections with CM and DM , we may assume that $p_1 > 0$, that is, that all our modules are generated in strictly positive degrees.

Let R be an algebra in $C(n, a, b)$ (respectively in $D(n, a, b, c)$), and let M be an R -module contained in CM (respectively in DM). Consider its trivial extension $C_M = M \oplus R$. By the classical formula [23] for Poincaré series of trivial extensions, we have

$$P_{C_M}(s, t) = \frac{P_R(s, t)}{1 - sP_M(s, t)} = P_R(s, t)(1 + sP_M(s, t) + s^2P_M(s, t)^2 + \dots)$$

(where $P_-(s, t) = \sum_{i \geq 0} s^i H_i(-)(t)$), hence $C_M \in C(N, A, B)$ (respectively $C_M \in D(N, A, B, C)$) for some $N, A, B, (C)$ depending on $n, a, b, m, p_1, p_2, q, c, r$. In the case (a), we apply Theorem 2.1 and conclude that the set of Hilbert series of such algebras C_M satisfies ACC; in the case (b), we also apply Theorem 2.2 and find that there is only a finite number of possibilities for $C_M(z)$. Since $M(z) = C_M(z) - R(z)$, the same is true for the set of Hilbert series $M(z)$. \square

Corollary 2.5. *Let $D > 0$ be an integer.*

- (a) *If R is a connected finitely presented algebra, then the set of Hilbert series of right-sided ideals in R generated in degree at most D satisfies DCC.*
- (b) *If, in addition, $m_3(R) < \infty$, then the set of Hilbert series of right-sided ideals in R having generators and relations in degrees at most D is finite.*

Proof. If I is an ideal in R and $M = R/I$, then the exact sequence

$$0 \rightarrow I \rightarrow R \rightarrow M \rightarrow 0$$

implies that $I(z) = R(z) - M(z)$ and $m_{i+1}(M) = m_i(I)$. It remains to apply Proposition 2.4 to the set of such modules M . \square

There is a class of algebras for which the finiteness of the set of Hilbert series of right ideals can be proved without the assumption on degrees of relations. Such algebras will be considered in the next section.

3. Effective coherence

3.1. Effectively coherent rings

All algebras below are connected graded, all modules (and ideals) are right and graded, as before.

Recall that a ring R is called *(right) coherent* if every map $M \rightarrow N$ of two finitely generated (right) free R -modules has finitely generated kernel, or, equivalently, if every finitely generated right ideal in R is finitely presented. Other equivalent conditions may be found in [9,10,19]. If the algebra R is graded, two versions of coherence can be considered, “affine” (general) and “projective” (where all maps and modules are assumed to be graded):

the author does not know whether these concepts are equivalent or not for connected algebras. By coherent algebra we will mean here a projective coherent one. In particular, every Noetherian ring is coherent, while the free associative algebras are coherent but not Noetherian.

Let us give an effective version of this definition.

Definition 3.1. Let A be an algebra. A finitely generated A -module M is called *effectively coherent* if $D_M(d)$ is finite for every $d > 0$.

The algebra A is called *effectively coherent*, if it satisfies either of the following equivalent conditions:

- (i) A is effectively coherent as a module over itself;
- (ii) every finitely presented A -module is effectively coherent;
- (iii) for every finitely presented A -module M there is a sequence of functions $\{D_i : \mathbf{N} \rightarrow \mathbf{N}\}$ such that, whenever a submodule $L \subset M$ is generated in degrees at most d , the graded vector spaces $\text{Tor}_i^A(L, k)$ are concentrated in degrees at most $D_i(d)$ for all $i \geq 0$.

Proof of equivalence. We begin with

Lemma 3.2. *Let A be an algebra. In an exact triple of A -modules*

$$0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0,$$

if any two of these three modules are effectively coherent, then the third is.

In this case

$$D_K(d) \leq D_M(d) \leq D'_K(D'_N(d))$$

and

$$D_N(d) \leq D'_M(\max\{d, m(K)\}),$$

where $D'_-(n) = \max\{D_-(n), n\}$.

This lemma can be shown in the same way as an analogous statement for coherent modules, but involving appropriate estimates for the functions D_M, D_N, D_K . Following N. Bourbaki [9, Exercise 10 to §3], we leave the proof to the reader.

Let us return to the proof of equivalence in Definition 3.1. Since every finite free module of rank greater than one is a direct sum of free modules of smaller ranks, every such free module is effectively coherent provided that A_A and all its submodules are. In particular, a finite presentation

$$F'' \xrightarrow{\phi} F' \rightarrow M \rightarrow 0$$

gives the following exact sequence

$$0 \rightarrow \ker \phi \rightarrow F' \rightarrow M \rightarrow 0$$

with effectively coherent first two terms. This proves (i) \Rightarrow (ii).

To prove (ii) \Rightarrow (iii), just show by induction that i th syzygy module is effectively coherent.

It remains to point out that the implication (iii) \Rightarrow (i) is trivial. \square

A Noetherian effectively coherent algebra is called *effectively Noetherian*: in such algebras, all finite (= finitely generated) modules are effectively coherent. Note that the same property of commutative affine algebras is well known at least since 1926 [24] when the first (double-exponential) bound for $D(d)$ for polynomial rings was found. It is a particular case of effective Nullstellensatz and effective division problem, and there are many papers (MathSciNet gives about 50) concerning syzygy degree bounds $D(d)$ and Betti number degree bounds $D_i(d)$ for ideals in commutative affine algebras.

We will see in the last subsection that every finitely presented monomial algebra is effectively coherent, as well as coherent algebras with finite Groebner bases introduced in [30]. A class of effectively Noetherian algebras (over an algebraically closed field) includes so-called strongly Noetherian algebras (in particular, Noetherian PI algebras and 3-dimensional Sklyanin algebras), as we will show in the next subsection.

Several methods to construct coherent algebras work as well for effectively coherent ones.

The next criterion is a variation of the classical criterion of coherence [9,10].

Proposition 3.3. *An algebra R is effectively coherent if and only if there are two functions $D^\cap, D^{\text{Ann}} : \mathbf{N} \rightarrow \mathbf{N}$ such that for every $a \in R$ we have $m(\text{Ann}_R a) \leq D^{\text{Ann}}(\deg a)$ and for every two right-sided ideals I, J with $m(I) \leq d, m(J) \leq d$ we have $m(I \cap J) \leq D^\cap(d)$.*

In this case we have

$$D^{\text{Ann}}(d) \leq D(d), \quad D^\cap(d) \leq \max\{d, D(d)\}, \quad \text{and} \quad D(d) \leq \max\{D^{\text{Ann}}(d), D^\cap(d)\}.$$

We begin with

Lemma 3.4. *Let M be a graded module, and let K, L be two its submodules generated in degrees at most d . Then*

$$m(K \cap L) \leq \max\{d, D_M(d)\}.$$

Proof. Consider the following exact triple

$$0 \rightarrow K \cap L \rightarrow K \oplus L \rightarrow K + L \rightarrow 0.$$

The exact sequence of Tor's gives:

$$\cdots \rightarrow \text{Tor}_1^R(K + L, k) \rightarrow \text{Tor}_0^R(K \cap L, k) \rightarrow \text{Tor}_0^R(K \oplus L, k) \rightarrow \cdots.$$

Thus

$$m(K \cap L) \leq \max\{m_1(K + L), m(K), m(L)\} \leq \max\{d, D_M(d)\}. \quad \square$$

Proof of Proposition 3.3. The “only if” part follows from Lemma 3.4. To show the “if” part, we will show that $D(d) \leq \max\{D^\cap(d), D^{\text{Ann}}(d)\}$. Let us proceed by induction in the number of generators t of an ideal $K \subset R$ with $m(K) \leq d$. For $t = 1$, we have $m_1(K) \leq D^{\text{Ann}}(d)$. For $t > 1$, we may assume that $K = I + J$, where I and J are generated by at most $(t - 1)$ elements. By exact triple

$$0 \rightarrow I \cap J \rightarrow I \oplus J \rightarrow I + J \rightarrow 0,$$

we have

$$m_1(K) \leq \max\{m_1(I), m_1(J), m(I \cap J)\}.$$

Here $m(I \cap J) \leq D^\cap(d)$ by definition and $m_1(I), m_1(J) \leq \max\{D^\cap(d), D^{\text{Ann}}(d)\}$ by induction hypothesis, so the claim follows. \square

The following claim is standard for coherent algebras (see [1, Proposition 10], [33, Proposition 1.3] for two its generalizations).

Proposition 3.5. *Let $A \rightarrow B$ be a map of connected algebras. Suppose that A is (effectively) coherent and the module B_A is finitely presented. Then B is (effectively) coherent.*

Proof. Let $b = m(B_A)$, let J be a right-sided ideal in B with $m(J) = d$, and let $0 \rightarrow K \rightarrow F \rightarrow J \rightarrow 0$ be its minimal presentation with a free B -module F . Here $m(F_A) \leq m(F_B) + m(B_A) = d + b$ and $m(J_A) \leq d + b$, hence $m_1(J_A) \leq D_{B_A}(d + b)$.

From the exact sequence of Tor 's we have

$$\text{Tor}_1^A(J, k) \rightarrow \text{Tor}_0^A(K, k) \rightarrow \text{Tor}_0^A(F, k).$$

Therefore,

$$m_1(J) = m(K_B) \leq m(K_A) \leq \max\{m(F_A), m_1(J_A)\} \leq \max\{d + b, D_{B_A}(d + b)\}. \quad \square$$

Corollary 3.6. *A singular extension of an (effectively) coherent algebra along a finitely presented module is (effectively) coherent.*

Proposition 3.7. *Let A and B be two (effectively) coherent algebras. Then their direct sum with common unit $A \oplus B$ and their free product $A * B$ are (effectively) coherent as well.*

The coherence of the free product of two coherent algebras has been proved in [11, Theorem 2.1] (and, in more general settings, in [1, Theorem 12]).

Proof. Let $C = A \oplus B$ and $E = A * B$. Let X_A and X_B be minimal homogeneous sets of generators of A and B .

Let $I = \sum_{i=1}^s y_i C$ be a finitely generated ideal in C , where $y_i = a_i + b_i$, $a_i \in A$, $b_i \in B$ with $\deg y_i \leq d$. Then I is the factor module of the free module F with generators $\tilde{y}_1, \dots, \tilde{y}_s$ by a syzygy submodule K . Then an element $w = \sum_{i=1}^s \tilde{y}_i(\alpha_i + \beta_i) \in F$ belongs to K if and only if $\sum_{i=1}^s a_i \alpha_i = 0$ and $\sum_{i=1}^s b_i \beta_i = 0$. Let K_A, K_B be the syzygy modules of the ideals in A and B generated by a_1, \dots, a_s and b_1, \dots, b_s , respectively, and let R_A, R_B be minimal sets of generators of these syzygy modules. Then K is generated by $R_A \cup R_B$, thus $m_1(I) = \max\{m(K_A), m(K_B)\} \leq \max\{D_A(d), D_B(d)\}$. So, C is (effectively) coherent if and only if both A and B are.

Now, let J be a right-sided ideal in E with $m(J) = d$, and let $M = E/J$. The Mayer–Vietoris long sequence [17, Theorem 6]

$$\cdots \rightarrow \text{Tor}_p^k(M, k) \rightarrow \text{Tor}_p^A(M, k) \oplus \text{Tor}_p^B(M, k) \rightarrow \text{Tor}_p^E(M, k) \rightarrow \text{Tor}_{p-1}^k(M, k) \rightarrow \cdots$$

gives an isomorphism

$$0 \rightarrow \text{Tor}_2^A(M, k) \oplus \text{Tor}_2^B(M, k) \rightarrow \text{Tor}_2^E(M, k) \rightarrow 0.$$

So, $m_1(J) = m_2(M) = \max\{m_2(M_A), m_2(M_B)\} = \max\{m_1(J_A), m_1(J_B)\}$.

For instance, let us estimate $m_1(J_A)$. The set R of generators of J lies in $E_{\leq d}$. Let J^A be the span of all elements of the form ux , where $u \in J$, $x \in X_A$, and $\deg u \leq d$ but $\deg ux > d$. The subset J^B is defined in the same way by replacing A by B . Then we have a direct sum decomposition $J = J' \oplus J''$, where $J' = J_{\leq d} \oplus J^A A = J_{\leq d} A$ and $J'' = J^A E X_B E \oplus J^B E$. Here both summands are A -modules, moreover, the second summand J'' is a free A -module. That is why the relations between the elements of $R \subset J'$ are the same as the relations of the submodule J' . But J' is a submodule of a finitely generated free A -module $P = E_{\leq d} A$, so that $m_1(J_A) = m_1(J') \leq D_P(d)$. Involving the analogous estimate for $m_1(J_B)$, we have finally

$$m_1(J) \leq \max\{D_{E_{\leq d} A}(d), D_{E_{\leq d} B}(d)\}. \quad \square$$

The following corollary will be generalized later in Section 4.4.

Corollary 3.8. *Any free algebra with finitely many generators is effectively coherent.*

The following criterion of coherence of Veronese subrings has been proved by Polishchuk [33, Proposition 2.6]. Here we give its effective version. Notice that by n th Veronese subring of a graded algebra A we mean the subalgebra $A^{(n)} = \bigoplus_{i \geq 0} A_{in}$ with the induced grading (the degrees are not divided by d).

Proposition 3.9. *Let A be a connected finitely presented algebra generated in degree one. For every $n \geq 2$, the Veronese subring $A^{(n)}$ is (effectively) coherent if and only if A is (effectively) coherent.*

In the notation of Proposition 3.3, we have

$$D_{A^{(n)}}^{\cap}(d) \leq D_A^{\cap}(d) + n - 1, \quad D_{A^{(n)}}^{\text{Ann}}(d) \leq D_A^{\text{Ann}}(d) + n - 1.$$

Proof. By [33, Proposition 2.6], A is a finitely presented module over $A^{(n)}$. By Proposition 3.5, it follows that if $A^{(n)}$ is (effectively) coherent then so is A .

Assume that A is effectively coherent. To show that $A^{(n)}$ is also effectively coherent, we are going to apply Proposition 3.3.

Let $x \in A^{(n)}$ be an element of degree d . If $\text{Ann}_A(x) = \sum_{i=1}^s y_i A$ with $\deg y_i = nq_i - r_i$, where $0 \leq r_i < n$, then $\text{Ann}_{A^{(n)}}(x) = \sum_{i=1}^s y_i A_{r_i} A^{(n)}$, hence $m_1(x A^{(n)}) \leq D_A^{\text{Ann}}(d) + n - 1$. Therefore,

$$D_{A^{(n)}}^{\text{Ann}}(d) \leq D_A^{\text{Ann}}(d) + n - 1 < \infty.$$

Now, let $I = \sum_{i=1}^u a_i A^{(n)}$ and $J = \sum_{i=1}^v b_i A^{(n)}$ be two ideals in $A^{(n)}$ generated in degrees at most d , and let

$$K = \left(\sum_{i=1}^u a_i A \right) \cap \left(\sum_{i=1}^v b_i A \right) \subset A.$$

Let $K = \sum_{i=1}^w c_i A$, where $\deg c_i = nq'_i - r'_i \leq D_A^{\cap}(d)$ with $0 \leq r'_i < n$. Then $I \cap J = K^{(n)} = \sum_{i=1}^w c_i A_{r'_i} A^{(n)}$, hence $m(I \cap J) \leq D_A^{\cap}(d) + n - 1$. Thus

$$D_{A^{(n)}}^{\cap}(d) \leq D_A^{\cap}(d) + n - 1 < \infty. \quad \square$$

3.2. Strongly Noetherian algebras are effectively Noetherian

Effective coherence implies some properties of Hilbert series. For example, we will see in Section 4 below that if $D_A(d) \leq 2d$ for $d \gg 0$, then the Hilbert series $A(z)$ is a rational function. Let us introduce three other properties of Hilbert series which are closely related to effective coherence: the first two are dual to each other, but the third is stronger.

Definition 3.10. Let A be an algebra, M be a finite module, and L runs through the set of its finitely generated submodules.

- (1) M is said to be *effective for series* if, given $m(L)$, there are only finite number of possibilities for Hilbert series $L(z)$.
- (2) M is said to be *effective for generators* if, given a Hilbert series $L(z)$, there are only finite number of possibilities for $m(L)$.
- (3) M is said to be *Artin–Zhang* if for every formal power series $h(z)$ there is $d > 0$ with the following property: for every L with $L(z) = h(z)$ we have $m(L) \leq d$, and for every L with $m(L) \leq d$ and $L(z) = h(z) + o(z^d)$ we have $L(z) = h(z)$.

A finitely presented algebra A is said to be effective for series (respectively effective for generators, Artin–Zhang), if every finitely presented A -module satisfies such a property.

For commutative algebras, the effectiveness for series follows from the fact that every ideal (or submodule) I has a Groebner basis of bounded degree. Similar property has been established also for several classes of ideals in noncommutative rings (for example, for torsion free finite modules over 3-dimensional quadratic Artin–Shelter regular algebras [18, Theorem A]). The “effective for generators” property is an obvious part of the Artin–Zhang condition; the property of being effective for series is naturally dual to effectiveness for generators.

The Artin–Zhang property itself first appeared in [6] in the following context. A connected algebra A is said to be (right) *strongly Noetherian* if $A \otimes R$ is (right) Noetherian for every Noetherian commutative k -algebra R [5]. In particular, Noetherian affine PI algebras, Sklyanin algebras, Noetherian domains of Gelfand–Kirillov dimension 2, Artin–Shelter regular algebras of global dimension three, and some twisted homogeneous coordinate rings are strongly Noetherian [5].

Theorem 3.11 [6, Corollary E4.5]. *Let A be a strongly Noetherian algebra over an algebraically closed field k . Then A is Artin–Zhang.*

A partial converse is given by

Proposition 3.12. *Every finitely generated Artin–Zhang module is Noetherian. In particular, every Artin–Zhang algebra is Noetherian.*

The proof consists of the following two lemmas.

Lemma 3.13. *Let M be a finitely generated Artin–Zhang module. Then the set $\mathbf{H} = \mathbf{H}^M$ of Hilbert series of finitely generated submodules of M satisfies ACC with respect to the order “ $<_{\text{lex}}$ ”.*

Notice that for the partial order “ $<$ ”, the same is proved in [6, Corollary E4.13].

Proof. Assume the converse, that is, that there is an infinite sequence

$$L^1(z) <_{\text{lex}} L^2(z) <_{\text{lex}} \dots$$

of Hilbert series of finite submodules of M . Since $L^i(z) < M(z)$, there exists $\lim_{i \rightarrow \infty} L^i(z) = h(z)$. Let d be the same as in the definition of Artin–Zhang module. For $i \gg 0$ we have $L^i(z) = h(z) + o(z^d)$. Let N^i be a submodule of L^i generated by all elements of L^i having degree at most d , i.e.,

$$N^i = L^i_{\leq d} A.$$

By Artin–Zhang condition, for $i \gg 0$ we have $N^i(z) = h(z)$ because $N^i(z) = h(z) + o(z^d)$. Then we have $h(z) = N^i(z) \leq_{\text{lex}} L_i(z) <_{\text{lex}} h(z)$ for $i \gg 0$, a contradiction. \square

Lemma 3.14. *Let M be a finitely generated module such that $(\mathbf{H}^M, <_{\text{lex}})$ satisfies ACC. Then M is Noetherian.*

Proof. Assume that $L \subset M$ is minimally generated by an infinite sequence x_1, x_2, \dots , and let L^i denote its finitely generated submodule $x_1A + \dots + x_iA$. Then we obtain an infinite chain in \mathbf{H}^M :

$$L^1(z) < L^2(z) < \dots \quad \square$$

Question 3.15. Is every Artin–Zhang algebra over an algebraically closed field strongly Noetherian?

Our next purpose is to prove that every strongly Noetherian algebra over an algebraically closed field is also effectively Noetherian. To do this, we establish the following relations between our effectiveness properties.

Theorem 3.16. *Let A be a coherent algebra. Consider the following properties:*

- (AZ) A is Artin–Zhang;
- (ES + EG) A is both effective for series and effective for generators;
- (EC) A is effectively coherent;
- (ES) A is effective for series.

Then there are implications:

$$(\text{AZ}) \Rightarrow (\text{ES} + \text{EG}) \Rightarrow (\text{EC}) \Rightarrow (\text{ES}).$$

Note that the assumption that A is coherent is actually used only in the implication $(\text{ES} + \text{EG}) \Rightarrow (\text{EC})$.

Proof. Fix a finitely presented A -module P . For $d \in \mathbf{Z}$, let \mathbf{H}_d be the set of Hilbert series of its submodules generated in degrees at most d , and let $\mathbf{H} = \bigcup_d \mathbf{H}_d$.

$(\text{AZ}) \Rightarrow (\text{ES} + \text{EG})$. Assume that P is Artin–Zhang, and let $h(z)$ be a formal power series. Let $d = d(h)$ be as in Definition 3.10(3).

If $L(z) = h(z)$, then $L_{\leq d}A(z) = L(z)$, hence $m(L) \leq d$. So, P is effective for generators.

For any $m \geq 0$, the subset $\mathbf{H}_m \subset \mathbf{H}$ satisfies ACC by Lemma 3.13. By Proposition 2.4(a), the set $\{P(z) - h(z) \mid h(z) \in \mathbf{H}_m\}$ of Hilbert series of quotient modules P/L satisfies ACC, so, \mathbf{H}_m satisfies DCC. By Koenig Lemma 2.3, this means that every set \mathbf{H}_m is finite.

$(\text{ES} + \text{EG}) \Rightarrow (\text{EC})$. For $d > 0$, let P^d be the set of all submodules in P generated in degrees at most d . Let $p(z)$ denote the polynomial $\sum_{i \leq d} z^i \dim P_i$. Every submodule $L \in P^d$ is isomorphic to a quotient module of a free module $M_L = X_L \otimes A$, where $X_L(z) \leq p(z)$, by a submodule K_L generated by a minimal set of relations of L . Since P is coherent, L is finitely presented, that is, K_L is finitely generated. The condition $X_L(z) \leq p(z)$ implies that there are only finite number of possibilities for $X_L(z)$, so, there are only finite number of isomorphism classes of M_L . Since P is effective for series, there are also only finite number of possibilities for Hilbert series $L(z)$. Finite free module M_L is effective for generators, so, given a Hilbert series $K_L(z) = M_L(z) - L(z)$, there are only finite number of

possibilities for $m_1(L) = m_0(K_L)$. The set of such Hilbert series $K_L(z)$ is finite, therefore, P is effectively coherent.

(EC) \Rightarrow (ES). Since A is effectively coherent, it is coherent, and we have $\dim \text{Tor}_i^A(k, k) < \infty$ for all $i \geq 0$, hence we can apply to the set of Hilbert series of the quotient modules P/M (which has the same cardinality as \mathbf{H}_d) Proposition 2.4(b) with $m = \dim H_0 P$, $p_1 = \min\{i \mid P_i \neq 0\}$, $p_2 = m(P)$, $q = d$, and $r = D_P(d)$. \square

Corollary 3.17. *Every strongly Noetherian algebra over an algebraically closed field is effectively Noetherian.*

We do not know, in what (geometrical?) terms the function $D(d)$ could be estimated for general strongly Noetherian algebras.

Question 3.18. For a Noetherian PI algebra A over an algebraically closed field, are there estimates for the syzygy degree function $D_A(d)$ in terms of its generators, relations, and identities?

A partial converse to the implication (EC) \Rightarrow (ES) of Theorem 3.16 is given by

Proposition 3.19. *Let A be a coherent algebra of global dimension 2. Then A is effective for series if and only if it is effectively coherent.*

Proof. The “if” part is proved in Theorem 3.16; let us proof the “only if” part.

Let I be a right ideal of A with $m(I) \leq d$. Since I has projective dimension at most one, its minimal free resolution has the form

$$0 \rightarrow V_2 \otimes A \rightarrow V_1 \otimes A \rightarrow I \rightarrow 0,$$

where V_1, V_2 are finite-dimensional vector spaces. Given d , there are only finite number of possibilities for the Hilbert series $V_1(z)$ of the space of generators of I . Since A is effective for series, there is also only a finite number of possibilities for

$$V_2(z) = A(z)^{-1} (V_1(z)A(z) - I(z)).$$

Thus, there exists a constant $D = D(d)$ such that $m_1(I) = m(V_2) \leq D$. \square

However, in general an effective for series algebra of global dimension two may not be effectively coherent, e.g., every finitely generated algebra over a finite field is effective for series, but is not necessarily coherent (for example, the algebra $k\langle x, y, z, t \mid zy - tz, zx \rangle$ has global dimension two but is not coherent [30, Proposition 10]).

Examples of Noetherian but not strongly Noetherian algebras has been found by Rogalski in [35]. The simplest example is the ring $R_{p,q}$ generated by two Eulerian derivatives with two parameters $p, q \in k^*$ [27].

Proposition 3.20. *Assume that k is an algebraically closed field with either $\text{char } k = 0$ or $\text{tr deg } k \geq 2$. Then for some $p, q \in k^*$ the algebra $R_{p,q}$ is Noetherian but neither effective for generators nor effective for series; in particular, $R_{p,q}$ is not effectively coherent.*

Proof. By [35, Example 12.8], there are p, q such that the algebra $R = R_{p,q}$ is Noetherian. To show that $R_{p,q}$ is not effective in any sense, we will use some properties of its point modules [27, Section 7]. Recall that a *point module* is a cyclic module $M = R/I$ with Hilbert series $M(z) = (1 - z)^{-1}$; the ideal I is called a *point ideal*. It is shown in [27, Section 7] that for every $d > 0$ there are two nonisomorphic point modules such that their truncations $M/M_{<d}$ are isomorphic. In particular, a point ideal I can have generators of arbitrary high degree. Since the Hilbert series $I(z) = h(z) := R(z) - (1 - z)^{-1}$ is the same for all point ideals, it follows that R is not effective for generators.

More precisely, there is a family of point modules $M(n, c) = R/I(n, c)$ (where $n \geq 0$ is an integer, $c \in \mathbb{P}^1(k)$), such that for $n \geq 2$ they have several relations of degree at most 3 and one relation of degree $(n + 1)$ [27, Section 7]. Let $I'(n, c) = I(n, c)_{\leq 3} R$. Then $I'(n, c)(z) = h(z) + o(z^n)$ while $I'(n, c)(z) \neq h(z) + o(z^{n+1})$, therefore, the set of Hilbert series of the ideals $I'(n, c)$ is infinite. Since all these ideals are generated in degrees at most 3, R is not effective for series, whence is not effectively coherent by Theorem 3.16. \square

4. Coherent families and universally coherent algebras

4.1. Algebras with coherent families

Definition 4.1. Let R be a finitely generated graded algebra, and let \mathbf{F} be a set of finitely generated right ideals in R . The family \mathbf{F} is said to be coherent if:

- (1) the zero ideal and the maximal homogeneous ideal \bar{R} belong to \mathbf{F} , and
- (2) for every $0 \neq I \in \mathbf{F}$ there is an ideal $J \in \mathbf{F}$ and a homogeneous element $x \in I$ such that $I = J + xR$, $m(J) \leq m(I)$, and the ideal $N = (x : J) := \{a \in R \mid xa \in J\}$ belongs to \mathbf{F} .

A coherent family \mathbf{F} is said to be of degree d , if $m(I) \leq d$ for all $I \in \mathbf{F}$, i.e., all its members are generated in degrees at most d .

If R is commutative and standard (i.e., degree-one generated), coherent families are called *generalized Koszul filtrations* [16]. In particular, it is shown in [16, Theorem 2.1] that coordinate rings of certain sets of points of the projective space \mathbf{P}^n admit generalized Koszul filtrations of finite degrees. In the noncommutative setting I cannot imagine a coherent family as a filtration, so this new notion is introduced.

A coherent family of degree one is called *Koszul filtration*: in this case, every ideal $I \in \mathbf{F}$ is generated by linear forms. This concept has been introduced for commutative algebras and investigated in several papers [8,12–15]. In particular, every coordinate ring of an algebraic curve admits a Koszul filtration provided that it is quadratic. The noncommutative version of Koszul filtrations has been considered in [32]; for example, every generic

n -generated quadratic algebra R admits a Koszul filtration if either it has less than n relations or is such that $\dim R_2 < n$.

One of equivalent definitions of a coherent ring is as follows [10, Theorem 2.2]: a ring R is coherent iff, for every finitely generated ideal $J = JR$ and every element $x \in R$, the ideal $N = (x : J)$ is finitely generated. The similar criterion holds for projective coherence. This definition is similar to our definition of coherent family, as shows the following

Proposition 4.2. *For a connected graded algebra R , the following two statements are equivalent:*

- (i) R is coherent;
- (ii) all finitely generated homogeneous ideals in R form a coherent family.

Proof. The implication (i) \Rightarrow (ii) follows from the criterion above. The dual implication (ii) \Rightarrow (i) follows from Proposition 4.3(a). \square

Note that the existence of a coherent family (even a Koszul filtration) is not sufficient for an algebra to be coherent. Indeed, the algebra $A = k\langle x, y, z, t \mid zy - tz, zx \rangle$ admits a Koszul filtration (it is initially Koszul with the Groebner flag (x, y, t, z) , see [32, Section 5]), but it is not coherent, since the annihilator $\text{Ann}_A z$ is not finitely generated [30, Proposition 10].

The following property of coherent families gives linear bounds for degrees of solutions of some linear equations.

Let us recall some notations. By definition, the *rate* [7] of a (degree-one generated) algebra R is the number

$$\text{rate } R = \sup_{i \geq 2} \left\{ \frac{m_i(R) - 1}{i - 1} \right\}.$$

For commutative standard algebras [4,7] as well as for noncommutative algebras with finite Groebner basis of relations [4] the rate is always finite. The rate is equal to 1 if and only if R is Koszul. If an algebra has finite rate, then its Veronese subring of sufficiently high order is Koszul [7].

The following proposition (part (b)) was originally proved for commutative algebras in [16, Proposition 1.2]. In fact, it holds for noncommutative ones as well.

Proposition 4.3.

- (a) *Let \mathbf{F} be a coherent family in an algebra R . Then the trivial R -module k_A and every ideal $I \in \mathbf{F}$ have free resolutions of finite type.*
- (b) *Assume in addition, that the coherent family \mathbf{F} has degree at most d . Then*

$$m_i(I) \leq m(I) + di$$

for all $i \geq 1$ and $I \in \mathbf{F}$. In particular, if R is generated in degree one, then

$$\text{rate } R \leq d.$$

Proof. Following the arguments of [16, Proposition 1.2], we proceed by induction in i and in I (by inclusion). First, note that the degree c of x in Definition 4.1 cannot be greater than $m(I)$, and so, in the case (b) it does not exceed d . Taking J and N as in Definition 4.1, we get the exact sequence

$$0 \rightarrow J \rightarrow I \rightarrow R/N[-c] \rightarrow 0,$$

which gives to the following fragments of the exact sequence of Tor's:

$$H_0(J)_j \rightarrow H_0(I)_j \rightarrow k_{j-c}$$

and

$$H_i(J)_j \rightarrow H_i(I)_j \rightarrow H_{i-1}(N)_{j-c}$$

for $i \geq 1$ (since N is the first syzygy module for R/N , we have isomorphisms $H_i(R/N) = H_{i-1}(N)$ for all $i \geq 1$).

By induction, the first and the last terms in these triples are finite-dimensional k -modules, so the middle one is. This proves (a). Also, in the case (b), the first term vanishes for $j \geq m(J) + di$, and the third term vanishes for $j - c \geq m(N) + d(i - 1)$. Since $m(J) \leq m(I)$ and $m(N) \leq d$, they both vanish for all $j \geq m(I) + di$, so that the middle term vanishes too. \square

4.2. Hilbert series and coherent families

Proposition 4.4. *Let \mathbf{F} be a coherent family of degree d in an algebra R . Then the set of all Hilbert series of ideals $I \in \mathbf{F}$ is finite.*

Proof. By Proposition 4.3, for every ideal $I \in \mathbf{F}$ we have $m(I) \leq d$ and $m_1(I) \leq 2d$. Since $\bar{R} \in \mathbf{F}$, we have also $m_1(R) \leq d$, $m_2(R) \leq 2d$, and $m_3(R) \leq 3d$. Thus, we can apply Corollary 2.5. \square

The following property of algebras with coherent families seems to be the most interesting. Its analogue for Koszul filtrations has been proved in [32, Theorem 3.3].

Theorem 4.5. *Suppose that an algebra R has a coherent family \mathbf{F} of degree d . Then R has rational Hilbert series, as well as every ideal $I \in \mathbf{F}$.*

If the set \mathcal{Hilb} of all Hilbert series of ideals $I \in \mathbf{F}$ contains at most s nonzero elements, then the degrees of numerators and denominators of these rational functions are not greater than ds .

Proof. Let $\mathcal{Hilb} = \{I_0(z) = 0, I_1(z), \dots, I_s(z)\}$, where I_1, \dots, I_s are some nonzero ideals in \mathbf{F} . By definition, for every nonzero ideal $I = I_i$ there are ideals $J = J(I)$, $N = N(I) \in \mathbf{F}$ such that $J \subset I$, $J \neq I$, and for some positive $c = c(I)$ the following triple is exact:

$$0 \rightarrow J \rightarrow I \rightarrow R/N[-c] \rightarrow 0.$$

Taking the Euler characteristics, we deduce

$$I(z) = J(I)(z) + z^c(R(z) - N(I)(z)). \quad (4.1)$$

Let us put $J^{(1)} := J(I)$ and $J^{(n+1)} := J(J^{(n)})$. Since all ideals $J^{(n)}$ are generated by subspaces of a finite-dimensional space $R_{\leq d}$, the chain

$$I \supset J^{(1)} \supset J^{(2)} \supset \dots$$

contains only a finite number of nonzero terms. Applying the formula (4.1) to the ideals $J^{(n)}$, we obtain a finite sum presentation

$$\begin{aligned} I(z) = & z^{c(I)}(R(z) - N(I)(z)) + z^{c(J^{(1)})}(R(z) - N(J^{(1)})(z)) \\ & + z^{c(J^{(2)})}(R(z) - N(J^{(2)})(z)) + \dots \end{aligned}$$

Thus

$$I_i(z) = \sum_{j=1}^s a_{ij}(R(z) - I_j(z)) + a_{i0}R(z),$$

where $a_{ij} \in z\mathbf{Z}[z]$ (the last term corresponds to the cases $N(J^{(t)}) = 0$).

Let $H = H(z)$ be the column vector $[I_1(z), \dots, I_s(z)]^t$, let A be the matrix $(a_{ij}) \in M_s(z\mathbf{Z}[z])$, let $H_0 = H_0(z)$ be the column vector $[a_{1,0}, \dots, a_{s,0}]^t$, and let e be the unit s -dimensional column vector. Then we have

$$H = A(R(z)e - H) + R(z)H_0,$$

or

$$(A + E)H = R(z)(Ae + H_0),$$

where E is the unit matrix.

The determinant $D(z) = \det(A + E) \in \mathbf{Z}[z]$ is a polynomial of degree at most sd . It is invertible in $\mathbf{Q}[[z]]$. Then $(A + E)^{-1} = D(z)^{-1}B$ with $B \in M_s(\mathbf{Z}[z])$. The elements of B are $(s-1) \times (s-1)$ minors of $(A + E)$, so, their degrees do not exceed $d(s-1)$. We have

$$H = R(z)D(z)^{-1}C,$$

where $C = B(Ae + H_0) \in z\mathbf{Z}[z]^s$. So, for every $1 \leq i \leq s$ we have

$$I_i(z) = R(z)C_i(z)D(z)^{-1}.$$

Assume that $\bar{R} = I_s$. Then

$$\bar{R}(z) = R(z) - 1 = R(z)C_s(z)D(z)^{-1},$$

therefore,

$$R(z) = \frac{D(z)}{D(z) - C_s(z)},$$

so, $R(z)$ is a quotient of two polynomials of degrees at most sd . By the above,

$$I_i(z) = R(z)C_i(z)D(z)^{-1} = \frac{C_i(z)}{D(z) - C_s(z)},$$

so, the same is true for the Hilbert series $I_i(z)$. \square

4.3. Universally coherent algebras

An algebra R is called *universally d -coherent* if all ideals in R generated in degrees at most d form a coherent family. Universally 1-coherent algebras are called *universally Koszul* (because all their ideals generated by linear forms are Koszul modules). Commutative universally Koszul algebras have been considered in [14,15]. In particular, commutative monomial universally Koszul algebras are completely classified in [15].

The following criterion is a consequence of Proposition 4.3.

Proposition 4.6. *The following two conditions for a connected algebra R are equivalent:*

- (i) R is universally d -coherent;
- (ii) $D_R(t) \leq t + d$ for all $t \leq d$.

Proof. (i) \Rightarrow (ii). By Proposition 4.3(b), we have $m_1(I) \leq m(I) + d$ if $t = m(I) \leq d$.

(ii) \Rightarrow (i). By (ii), we have $m_1(I) \leq m(I) + d$ if $t = m(I) \leq d$. Let $x = \{x_1, \dots, x_r\}$ be a minimal system of generators for I with $\deg x_1 = t$, let $J = x_2R + \dots + x_rR$, and let $N = N(I) = (x_1 : J)$. Obviously, N is the (shifted by t) projection of the syzygy module $\Omega = \Omega(x_1, \dots, x_r)$ onto the first component, so that $m(N) + t \leq m(\Omega) = m_1(I) \leq t + d$. Thus, $m(N(I)) \leq d$ for every I with $m(I) \leq d$. By Definition 4.1, this means that all such ideals I form a coherent family. \square

Corollary 4.7. *The following two conditions for a connected algebra R are equivalent:*

- (i) R is universally d -coherent for all $d \gg 0$;
- (ii) $D_R(d) \leq 2d$ for all $d \gg 0$.

In particular, in this case R is effectively coherent.

We call an algebra R satisfying either of the equivalent conditions of this Corollary 4.7 simply *universally coherent*.

Every generated in degree one universally d -coherent algebra is a finite Koszul module over a universally Koszul algebra, namely, over its Veronese subalgebra. This follows from

Proposition 4.8. *Let R be a universally d -coherent algebra generated in degree one. Then its Veronese subalgebra $R^{(d)}$ (with grading divided by d) is universally Koszul.*

Proof. Let B be the algebra $R^{(d)}$ with grading divided by d . By Proposition 4.6, we have to check that $D_B(1) \leq 2$. By Propositions 3.9 and 3.3, we have

$$D_B^\cap(1) \leq d^{-1} D_{R^{(d)}}^\cap(d) \leq d^{-1} (D_R^\cap(d) + d - 1) \leq 3 - d^{-1},$$

and, analogously,

$$D_B^{\text{Ann}}(t) \leq 3 - d^{-1}.$$

By Proposition 3.3, we have

$$D_B(1) \leq \max\{D_B^\cap(1), D_B^{\text{Ann}}(1)\} \leq 2. \quad \square$$

In particular, in a universally coherent algebra all Veronese subrings of sufficiently high order are universally Koszul, thus the algebra itself is a finite direct sum of Koszul modules over every such Veronese subalgebra.

Proposition 4.9. *Every finitely presented module over a universally coherent algebra has rational Hilbert series.*

Proof. Let R be a universally coherent algebra. Since every finitely generated ideal in R is a member of a coherent family, it has rational Hilbert series by Theorem 4.5, as well as the algebra R itself.

Let M be a finitely presented R -module, and let

$$0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$$

be its finite presentation with free F . Since $M(z) = F(z) - K(z)$, it is sufficient to show that the finitely generated submodule K of a free module F has rational Hilbert series.

Let $d = m(K)$, and let $K = L + xR$, where $\deg x = d$. By the induction in the number of generators of K , we can assume that $L(z)$ is rational.

Let $J = (x : L)$. Since R is coherent, this ideal is finitely generated, therefore, its Hilbert series is rational. The exact triple

$$0 \rightarrow L \rightarrow K \rightarrow R/J[-d] \rightarrow 0$$

gives the formula $K(z) = L(z) + z^d(R(z) - J(z))$. Thus $K(z)$ is rational. \square

4.4. Examples: monomial algebras and their generalizations

Universal coherence is the strongest property among all considered above. It is a fortunate surprise that some interesting classes of noncommutative algebras are indeed universally coherent.

All finitely presented algebras whose relations are *monomials* on generators are universally coherent (see below). It is an important point in noncommutative computer algebra that many properties of finitely presented monomial algebras are inherited by the algebras with finite Groebner bases (for example, they have finite rate, and quadratic ones are necessary Koszul). However, the latter are not in general coherent, see examples in [30]; the reason is that while the algebra itself has finite Groebner basis, a finitely generated one-sided ideal in it sometimes can have only an infinite one (in contrast to the monomial case [29, Theorem 1]). That is why a new class of algebras between these two has been introduced [30], so-called *algebras with r-processing*.

In general, these algebras are not assumed to be graded but still finitely generated. Let A be a quotient algebra of a free algebra F by a two-sided ideal I with a Groebner basis $G = \{g_1, \dots, g_s\}$. For every element $f \in F$, there is a well-defined normal form $N(f)$ of f with respect to G [37, Section 2.3].

The algebra A is called *algebra of r-processing* for some $r \geq 0$, if for any pair $p, q \in F$ of normal monomials, where $q = q_1 q_2$, $\deg q_1 \leq r$, we have

$$N(pq) = N(pq_1)q_2.$$

The simplest example is an algebra A whose relations are monomials of degree at most $r + 1$.

A simple sufficient condition for an algebra to have this property is as follows. Consider a graph Γ with s vertices marked by g_1, \dots, g_s such that an arrow $g_i \rightarrow g_j$ exists iff there is an overlap between any *nonleading* term of g_i and *leading* term of g_j . If Γ is acyclic, then A is an algebra with r -processing for some r . The simplest case is when the monomials in the decompositions of the relations of the algebra do not overlap each other. See [30] for other sufficient conditions and for a way how to calculate the number r for given Γ .

The main property of algebras with r -processing is that every finitely generated right ideal in such algebra has finite Groebner basis. In the case of standard degree-lexicographical order on monomials, the degrees of its elements do not exceed $m(I) + r$. Moreover, the degrees of relations of I do not exceed the number $m(I) + 2r - 1$; in particular, they are coherent. Therefore, we have

Proposition 4.10. *Let R be a connected algebra with r-processing (with a degree-lexicographical order on monomials). Then $D_R(d) < d + 2r$. In particular, it is universally coherent.*

For example, any algebra with 1-processing is universally Koszul (algebras with 1-processing were separately considered in [26]). So, all quadratic monomial algebras are universally Koszul (unlike the commutative case, see [15]).

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