

# On the existence of mixed fiber bodies.

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The notion of the mixed fiber polytope is a natural generalization of the mixed volume and the Minkowski integral (see [BS] or Definition 1.1). It is closely related to elimination theory, see [EKh], [ST], or Theorem 4.5. The existence of mixed fiber polytopes was predicted in [McD] and proved in [McM]. One can extend the notion of the mixed fiber polytope to convex bodies by continuity. We present a direct proof of the existence of mixed fiber bodies, which does not exploit the reduction to polytopes by continuity (see the proof of Theorem 1.2). It is based on an explicit formula (\*) for the support function of a mixed fiber body. One can also use this formula to compute the support function of a Newton polytope of a multidimensional resultant (Theorem 4.7) and to prove a certain monotonicity property for mixed fiber bodies (Theorem 4.1).

Key words: fiber polytope, mixed volume, convexity, resultant.

MSC: 52A20, 52A39, 52B20.

## 1 Mixed fiber bodies.

Let  $L$  and  $M$  be real vector spaces of dimension  $l$  and  $m$  respectively, and let  $\mu$  be a volume form on  $M$ . Denote the projections of  $L \oplus M$  to  $L$  and  $M$  by  $u$  and  $v$  respectively. Let  $\Delta \subset L \oplus M$  be a convex body, i. e. a compact set, which contains all the line segments connecting any pair of its points. For a convex body and a point  $a \in M$ , denote the *fiber*  $u(\Delta \cap v^{-1}(a))$  of  $\Delta$  by  $\Delta_a$ . Recall that the *support function*  $B(\cdot) : L^* \rightarrow \mathbb{R}$  of a convex body  $B \subset L$  is defined as  $B(\gamma) = \max_{b \in B} \langle \gamma, b \rangle$  for every covector  $\gamma \in L^*$ .

**DEFINITION 1.1.** For a convex body  $\Delta \subset L \oplus M$ , its *Minkowski integral* is the convex body  $B \subset L$ , such that its support function equals the integral of the support functions of the fibers  $\Delta_a$ , where  $a$  runs over  $v(\Delta)$ :

$$B(\gamma) = \int_{v(\Delta)} \Delta_a(\gamma) \mu \text{ for every } \gamma \in L^*.$$

The Minkowski integral is denoted by  $\int \Delta \mu$ .

This definition is slightly different from the original one (see [BS]). We discuss this difference in Section 4.

Denote the set of all convex bodies in a real vector space  $K$  by  $\mathcal{C}(K)$ . This set is a semigroup with respect to the Minkowski summation  $A + B = \{a + b \mid a \in A, b \in B\}$ .

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**THEOREM 1.2.** *There exists a unique symmetric Minkowski-multilinear map  $\text{MF}_\mu : \underbrace{\mathcal{C}(L \oplus M) \times \dots \times \mathcal{C}(L \oplus M)}_{m+1} \rightarrow \mathcal{C}(L)$ , such that*

$$\text{MF}_\mu(\Delta, \dots, \Delta) = \int \Delta \mu \text{ for every convex body } \Delta \subset L \oplus M.$$

**DEFINITION 1.3.** The convex body  $\text{MF}_\mu(\Delta_0, \dots, \Delta_m)$  is called the *mixed fiber body* of bodies  $\Delta_0, \dots, \Delta_m$ .

Regarding the proof, it is rather easy to see that the Minkowski integral is a homogeneous polynomial, and, thus, admits such a polarization in the class of virtual convex bodies (see Definition 2.1). The fact that this polarization gives actual convex bodies rather than virtual convex bodies is the most important part of the assertion.

**PROOF OF THEOREM 1.2.**

**DEFINITION 1.4.** The *shadow volume*  $S_\mu(B)$  of a convex body  $B \subset \mathbb{R} \oplus M$  is defined as the integral  $\int_{B_0} \varphi \mu$ , where  $B_0$  is the projection of  $B$  to  $M$ , and  $\varphi$  is the maximal function on  $B_0$  such that its graph is contained in  $B$  (in other words,  $\varphi(a) = \max_{(t,a) \in B} t$  for every  $a \in B_0$ ).

One can reformulate the definition of the Minkowski integral as follows. For a covector  $\gamma \in L^*$  and a convex body  $\Delta \subset L \oplus M$ , denote the image of  $\Delta$  under the projection  $(\gamma, \text{id}) : L \oplus M \rightarrow \mathbb{R} \oplus M$  by  $\Gamma_\Delta(\gamma)$ .

**LEMMA 1.5.** *The value of the support function of the Minkowski integral  $\int \Delta \mu$  at a covector  $\gamma \in L^*$  equals the shadow volume of the body  $\Gamma_\Delta(\gamma) \subset \mathbb{R} \oplus M$ .*

This lemma implies that, instead of constructing mixed fiber bodies, it is enough to construct the mixed shadow volume in the following sense.

**THEOREM 1.6.** *There exists a unique symmetric Minkowski-multilinear function  $\text{MS}_\mu$  of  $m + 1$  convex bodies in  $\mathbb{R} \oplus M$  such that  $\text{MS}_\mu(B, \dots, B) = S_\mu(B)$  for every convex body  $B \subset \mathbb{R} \oplus M$ .*

The proof is given in Section 2 and is based on an explicit formula for the function  $\text{MS}_\mu$ , which is later used in the proof of Theorem 1.2 (see Lemma 3.6). Theorem 4.10 gives another way to prove the existence of this function and to compute it.

**DEFINITION 1.7.** The number  $\text{MS}_\mu(B_0, \dots, B_m)$  is called the *mixed shadow volume* of convex bodies  $B_0, \dots, B_m \subset \mathbb{R} \oplus M$ .

If the existence of mixed fiber bodies is proved, then Lemma 1.5 and Theorem 1.6 imply that the value of the support function of the mixed fiber body  $\text{MF}_\mu(\Delta_0, \dots, \Delta_m)$  at a covector  $\gamma \in L^*$  equals the mixed shadow volume  $\text{MS}_\mu(\Gamma_{\Delta_0}(\gamma), \dots, \Gamma_{\Delta_m}(\gamma))$ . We reverse this argument, using the following fact. Recall that a function  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  is said to be *positively homogeneous*, if  $f(ta) = tf(a)$  for all  $a \in \mathbb{R}^k$  and  $t \geq 0$ .

**THEOREM 1.8.** *For any collection of convex bodies  $\Delta_0, \dots, \Delta_m \subset L \oplus M$ , the expression  $\text{MS}_\mu(\Gamma_{\Delta_0}(\gamma), \dots, \Gamma_{\Delta_m}(\gamma))$  is a convex positively homogeneous function of a covector  $\gamma \in L^*$ .*

The proof is given in Section 3.

Theorem 1.8 implies that, for any collection of convex bodies  $\Delta_0, \dots, \Delta_m \subset L \oplus M$ , the expression  $\text{MS}_\mu(\Gamma_{\Delta_0}(\gamma), \dots, \Gamma_{\Delta_m}(\gamma))$  defines the support function of some convex body  $B \in L$ :

$$\text{MS}_\mu(\Gamma_{\Delta_0}(\gamma), \dots, \Gamma_{\Delta_m}(\gamma)) = B(\gamma) \text{ for every } \gamma \in L^*. \quad (*)$$

This body  $B$  satisfies the definition of the mixed convex body of  $\Delta_0, \dots, \Delta_m$  by Lemma 1.5 and Theorem 1.6.  $\square$

## 2 Mixed shadow volume. Proof of Theorem 1.6.

The shadow volume of a convex body  $\Delta \subset \mathbb{R} \oplus M$  is equal to the volume of some virtual convex body  $[\Delta]$  associated with  $\Delta$  (see Definition 2.5 and Lemma 2.7 below). Since the correspondence  $\Delta \rightarrow [\Delta]$  is Minkowski-linear, one can define the mixed shadow volume of convex bodies  $\Delta_0, \dots, \Delta_m \subset \mathbb{R} \oplus M$  as the mixed volume of virtual bodies  $[\Delta_0], \dots, [\Delta_m]$ , which implies the existence of the mixed shadow volume. To formulate this in detail, recall the definition of a virtual convex body.

The *Grothendieck group*  $\Lambda_G$  of a commutative semigroup  $\Lambda$  with the cancellation law ( $a + c = b + c \Rightarrow a = b$ ) is the group of formal differences of elements from  $\Lambda$ . In more detail, it is the quotient of the set  $\Lambda \times \Lambda$  by the equivalence relation  $(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$ , with operations  $(a, b) + (c, d) = (a + c, b + d)$  and  $-(a, b) = (b, a)$ . The map, which carries every  $a \in \Lambda$  to  $(a + a, a) \in \Lambda_G$ , is an inclusion  $\Lambda \hookrightarrow \Lambda_G$ . An element of the form  $(a + a, a) \in \Lambda_G$  is said to be *proper* and is usually identified with  $a \in \Lambda$ . Under this convention, one can write  $(a, b) = a - b$ .

**DEFINITION 2.1.** *The group of virtual bodies in a real vector space  $K$  is the Grothendieck group of the semigroup of convex bodies in  $K$  (with respect to the operation of Minkowski summation).*

The classical operation of taking the mixed volume can be extended to virtual bodies by linearity. This extension is unique, but fails to be increasing (for example,  $MV(-A, A) > MV(-A, 2A)$  for a convex polygon  $A$ ).

**DEFINITION 2.2.** Let  $\mu$  be a translation invariant volume form on a real vector space  $K$  of dimension  $n$ . *The mixed volume  $MV_\mu$  is the symmetric Minkowski-multilinear function of  $n$  virtual bodies in  $K$ , such that  $MV_\mu(\Delta, \dots, \Delta)$  equals the volume of  $\Delta$  in the sense of the form  $\mu$  for every convex body  $\Delta \subset K$ .*

**DEFINITION 2.3.** For the difference  $\Delta$  of two convex bodies  $\Delta_1$  and  $\Delta_2$  in  $K$ , the *support function*  $\Delta(\cdot) : K^* \rightarrow \mathbb{R}$  is defined as  $\Delta(\gamma) = \Delta_1(\gamma) - \Delta_2(\gamma)$ .

One can reformulate the definition of the group of virtual bodies more explicitly as follows. A function  $f : K \rightarrow \mathbb{R}$  is called a *DC function* if it can be represented as the difference of two convex functions.

**LEMMA 2.4.** *The map which carries every virtual body  $\Delta$  to its support function  $\Delta(\cdot)$  is an isomorphism between the group of virtual bodies in a real vector space  $K$  and the group of positively homogenous DC functions on the dual space  $K^*$ .*

**PROOF OF THEOREM 1.6.** Let  $M$  be an  $m$ -dimensional real vector space. Denote the ray  $\{(t, 0) \mid t \leq 0\} \subset \mathbb{R} \oplus M$  by  $l_-$ , and denote the half-space  $\{(t, x) \mid t \geq a\} \subset \mathbb{R} \oplus M$  by  $H_a$ .

**DEFINITION 2.5.** *The shadow  $[\Delta]$  of a convex body  $\Delta \subset \mathbb{R} \oplus M$  is defined as the difference of the convex bodies  $(\Delta + l_-) \cap H_a$  and  $l_- \cap H_a$ , where  $a$  is a negative number such that  $H_a \supset \Delta$ .*

This definition does not depend on the choice of  $a$ , because one can reformulate it in terms of support functions as follows. For every covector  $\gamma = (t, \gamma_0) \in (\mathbb{R} \oplus M)^*$ , denote the covector  $(\max\{0, t\}, \gamma_0) \in (\mathbb{R} \oplus M)^*$  by  $[\gamma]$ . Then  $[\Delta](\gamma) = \Delta([\gamma])$  for every  $\gamma$ .

Denote the unit volume form on  $\mathbb{R}$  by  $dt$ . The function which assigns the number  $MV_{dt \wedge \mu}([\Delta_0], \dots, [\Delta_m])$  to every collection of convex bodies  $\Delta_0, \dots, \Delta_m$  in  $\mathbb{R} \oplus M$  is symmetric Minkowski-multilinear by Lemma 2.6

below and assigns the shadow volume  $S_\mu(\Delta)$  to the collection  $(\Delta, \dots, \Delta)$  for every convex body  $\Delta$  by Lemma 2.7 below. Thus, it satisfies the definition of the mixed shadow volume. Its uniqueness follows from Lemma 2.9 below.  $\square$

**LEMMA 2.6.**  $[\Delta_1 + \Delta_2] = [\Delta_1] + [\Delta_2]$ .

**PROOF.**  $[\Delta_1 + \Delta_2](\gamma) = \Delta_1([\gamma]) + \Delta_2([\gamma]) = [\Delta_1](\gamma) + [\Delta_2](\gamma)$  for every covector  $\gamma$ .  $\square$

**LEMMA 2.7.**  $S_\mu(\Delta) = \text{MV}_{dt \wedge \mu}([\Delta], \dots, [\Delta])$ .

**PROOF.** If  $\Delta \subset H_0$  in the notation of Definition 2.5, then the shadow  $[\Delta]$  equals the convex body  $(\Delta + l_-) \cap H_0$  by definition, and both  $S_\mu(\Delta)$  and  $\text{MV}_{dt \wedge \mu}([\Delta], \dots, [\Delta])$  equal the volume of  $(\Delta + l_-) \cap H_0$ . One can reduce the statement of Lemma 2.7 to this special case: substitute an arbitrary body  $\Delta$  by a shifted body  $\Delta + \{v\} \subset H_0$ , where  $v$  is a vector of the form  $(s, 0) \in \mathbb{R} \oplus M$ , and note that both  $S_\mu(\Delta)$  and  $\text{MV}_{dt \wedge \mu}([\Delta], \dots, [\Delta])$  increase by  $s$  times the volume of the projection of  $\Delta \subset \mathbb{R} \oplus M$  to  $M$ . The last fact follows from the definition for the shadow volume  $S_\mu(\Delta)$ , and follows from Lemma 2.8 for the mixed volume  $\text{MV}_{dt \wedge \mu}([\Delta], \dots, [\Delta])$ .  $\square$

**LEMMA 2.8.**  $\text{MV}_{dt \wedge \mu}([\Delta_0 + \{(s, 0)\}], [\Delta_1], \dots, [\Delta_m]) =$

$$= \text{MV}_{dt \wedge \mu}([\Delta_0], \dots, [\Delta_m]) + \frac{s}{m+1} \text{MV}_\mu(B_1, \dots, B_m),$$

where the convex body  $B_j \subset M$  is the projection of  $\Delta_j \subset \mathbb{R} \oplus M$  to  $M$  and  $s$  is a positive number.

**PROOF.**  $\text{MV}_{dt \wedge \mu}([\{(s, 0)\}], [\Delta_1], \dots, [\Delta_m]) = \frac{s}{m+1} \text{MV}_\mu(B_1, \dots, B_m)$ , which is a corollary of the following well known formula (one can consider  $N = \mathbb{R} \oplus M$  and  $L = \mathbb{R} \times \{0\}$ ). Let  $A_1, \dots, A_n$  be convex bodies in an  $n$ -dimensional real vector space  $N$ , suppose that  $A_1, \dots, A_l$  are contained in an  $l$ -dimensional subspace  $L \subset N$ , and denote the projection  $N \rightarrow N/L$  by  $p$ . Then  $n! \text{MV}_{\mu \wedge \mu'}(A_1, \dots, A_n) = (n-l)! \text{MV}_\mu(pA_{l+1}, \dots, pA_n) \cdot l! \text{MV}_{\mu'}(A_1, \dots, A_l)$ , where  $\mu$  and  $\mu'$  are volume forms on  $N/L$  and  $L$ .  $\square$

**LEMMA 2.9.**

$$\text{MS}_\mu(\Delta_0, \dots, \Delta_m) = \frac{1}{(m+1)!} \sum_{0 \leq i_1 < \dots < i_p \leq m} (-1)^{m+1-p} S_\mu(\Delta_{i_1} + \dots + \Delta_{i_p}).$$

PROOF. To prove the identity

$$nl(a_1, \dots, a_n) = \sum_{1 \leq i_1 < \dots < i_p \leq n} (-1)^{n-p} l(a_{i_1} + \dots + a_{i_p}, \dots, a_{i_1} + \dots + a_{i_p})$$

for every symmetric multilinear function  $l$ , open the brackets and collect similar terms in the right hand side.  $\square$

The following lemma describes how the mixed shadow volume changes under translation and dilatation of arguments along the line  $\mathbb{R} \times \{0\} \subset \mathbb{R} \oplus M$ .

**LEMMA 2.10.** 1) Let  $D_s : \mathbb{R} \oplus M \rightarrow \mathbb{R} \oplus M$  be a dilatation along  $\mathbb{R} \times \{0\}$ , i. e.  $D_s(t, x) = (st, x)$  for all  $t \in \mathbb{R}$  and  $x \in M$ . Then  $\text{MS}_\mu(D_s \Delta_0, \dots, D_s \Delta_m) = s \text{MS}_\mu(\Delta_0, \dots, \Delta_m)$  for every non-negative  $s$ .

2) Let  $T_s : \mathbb{R} \oplus M \rightarrow \mathbb{R} \oplus M$  be a translation,  $T_s(t, x) = (t + s, x)$  for all  $t \in \mathbb{R}$  and  $x \in M$ . Then  $\text{MS}_\mu(T_s \Delta_0, \Delta_1, \dots, \Delta_m) = \text{MS}_\mu(\Delta_0, \dots, \Delta_m) + \frac{s}{m+1} \text{MV}_\mu(B_1, \dots, B_m)$ , where the convex body  $B_j \subset M$  is the projection of  $\Delta_j \subset \mathbb{R} \oplus M$  to  $M$ .

In particular, the mixed shadow volume is not translation invariant.

PROOF. Part 1 follows from the definition of shadows and the equality  $\text{MV}_{dt \wedge \mu}(D_s \Delta_0, \dots, D_s \Delta_m) = s \text{MV}_{dt \wedge \mu}(\Delta_0, \dots, \Delta_m)$  for convex bodies  $\Delta_0, \dots, \Delta_m$ . Part 2 follows Lemma 2.8.  $\square$

### 3 Convexity of mixed shadow volume. Proof of Theorem 1.8.

**DEFINITION 3.1.** A set with a convexity structure is a pair  $(U, C)$ , where  $U$  is an arbitrary set and  $C$  is an arbitrary map  $U \times (0, 1) \times U \rightarrow U$ .

**EXAMPLE.** The pair  $(\mathbb{R}^k, C_{\mathbb{R}^k})$ , where  $C_{\mathbb{R}^k}(u, t, v) = tu + (1 - t)v$ , is a set with a convexity structure.

**DEFINITION 3.2.** Let  $(U, C)$  and  $(V, D)$  be two sets with convexity structures, and let  $\leq$  be a partial order on  $V$ . A map  $f : U \rightarrow V$  is said to be a convex map from  $(U, C)$  to  $(V, D, \leq)$ , if  $f(C(u, t, v)) \leq D(f(u), t, f(v))$  for all triples  $(u, t, v) \in U \times (0, 1) \times U$ .

**EXAMPLE.** For a map from  $(\mathbb{R}^k, C_{\mathbb{R}^k})$  to  $(\mathbb{R}^1, C_{\mathbb{R}^1}, \leq)$ , this definition coincides with the classical one.

**LEMMA 3.3** (tautological). *If maps  $f : (U, C) \rightarrow (V, D, \leq)$  and  $g : (V, D) \rightarrow (W, E, \leq)$  are convex, and  $g$  is increasing, then their composition is convex.*

**PROOF OF THEOREM 1.8.** Positive homogeneity follows from part 1 of Lemma 2.10. To prove convexity, represent the map which assigns the mixed shadow volume  $\text{MS}_\mu(\Gamma_{\Delta_0}(\gamma), \dots, \Gamma_{\Delta_m}(\gamma))$  to every covector  $\gamma \in L^*$  as a composition of simpler maps (see the diagram (\*\*)) below), prove convexity and monotonicity for each of them, and apply Lemma 3.3. The implementation of this plan occupies the rest of Section 3.  $\square$

For a convex body  $B \subset M$ , let  $\mathcal{C}(B)$  be the set of all convex bodies  $\Delta \subset \mathbb{R} \oplus M$  such that the projection of  $\Delta$  to  $M$  equals  $B$ . Introduce the shadow and the Minkowski convexity structures  $C_S$  and  $C_M$  on  $\mathcal{C}(B)$  as follows. Consider convex bodies  $\Delta_i = \{(t, a) \mid a \in B, t \in [\psi_i(a), \varphi_i(a)]\}$ ,  $i = 1, 2$ , in  $\mathcal{C}(B)$ , where  $\varphi_i$  and  $-\psi_i$  are concave functions on  $B$ . Then, by definition,

$$C_S(\Delta_1, \alpha, \Delta_2) = \left\{ (t, a) \mid a \in B, t \in [\alpha\psi_1(a) + (1-\alpha)\psi_2(a), \alpha\varphi_1(a) + (1-\alpha)\varphi_2(a)] \right\},$$

$$C_M(\Delta_1, \alpha, \Delta_2) = \alpha\Delta_1 + (1-\alpha)\Delta_2, \quad \Delta_1 \leq \Delta_2 \Leftrightarrow \varphi_1 \leq \varphi_2.$$

For convex bodies  $\Delta_0, \dots, \Delta_m \subset L \oplus M$ , denote the projection of  $\Delta_j$  to  $M$  by  $B_j$ , and consider the maps

$$\begin{aligned} (L^*, C_{L^*}) &\xrightarrow{(\Gamma_{\Delta_0}, \dots, \Gamma_{\Delta_m})} (\mathcal{C}(B_0) \times \dots \times \mathcal{C}(B_m), C_S, \leq) \xrightarrow{(\text{id}, \dots, \text{id})} \\ &\rightarrow (\mathcal{C}(B_0) \times \dots \times \mathcal{C}(B_m), C_M, \leq) \xrightarrow{\text{MS}_\mu} (\mathbb{R}, C_{\mathbb{R}}, \leq), \end{aligned} \quad (**)$$

where the map  $\text{id}$  carries every convex body to itself.

These three maps are convex and increasing. Namely, the convexity of the map  $\Gamma_{\Delta_j} : (L^*, C_{L^*}) \rightarrow (\mathcal{C}(B_j), C_S, \leq)$  is proved in Lemma 3.4 below. The increasing monotonicity of  $\text{id} : (\mathcal{C}(B_j), C_S, \leq) \rightarrow (\mathcal{C}(B_j), C_M, \leq)$  is tautological, the convexity follows from Lemma 3.5. The convexity of the mixed shadow volume follows from its linearity, and the increasing monotonicity of the mixed shadow volume is the statement of Lemma 3.6 below.

**LEMMA 3.4.** *If a convex body  $B \subset M$  is the projection of a convex body  $\Delta \subset L \oplus M$ , then the map  $\Gamma_\Delta : (L^*, C_{L^*}) \rightarrow (\mathcal{C}(B), C_S, \leq)$  is convex.*

**PROOF.** If  $\dim M = 0$ , then the body  $\Gamma_\Delta(\gamma) \subset \mathbb{R} \oplus \{0\}$  is the segment  $[-\Delta(-\gamma), \Delta(\gamma)]$  for every  $\gamma \in L^*$ , and the convexity of  $\Gamma_\Delta$  follows from the

convexity of the support function  $\Delta(\cdot)$ . One can reduce the general statement to this special case, because the convexity of the map  $\Gamma_\Delta : (L^*, C_{L^*}) \rightarrow (\mathcal{C}(B), C_S, \leq)$  is equivalent to the convexity of the maps  $\Gamma_{\Delta_a} : (L^*, C_{L^*}) \rightarrow (\mathcal{C}(\{0\}), C_S, \leq)$  for all points  $a \in B$  (see the first paragraph of the paper for the definition of the fiber  $\Delta_a$  of the body  $\Delta$ ).  $\square$

**LEMMA 3.5.** *If  $\Delta_1$  and  $\Delta_2$  are convex bodies from  $\mathcal{C}(B)$ , and  $\alpha \in (0, 1)$ , then  $C_S(\Delta_1, \alpha, \Delta_2) \leq C_M(\Delta_1, \alpha, \Delta_2)$ .*

**PROOF.** Every point  $(t, a)$  of the left hand side can be represented as  $(\alpha s_1 + (1 - \alpha)s_2, a)$ , where  $(s_1, a) \in \Delta_1$  and  $(s_2, a) \in \Delta_2$ . Thus, it equals  $\alpha(s_1, a) + (1 - \alpha)(s_2, a)$ , which is contained in the right hand side.  $\square$

**LEMMA 3.6.** *If  $\Delta_j^0 \in C(B_j)$ ,  $\Delta_j^1 \in C(B_j)$  and  $\Delta_j^0 \leq \Delta_j^1$  for  $j = 0, \dots, m$ , then  $MS_\mu(\Delta_0^0, \dots, \Delta_m^0) \leq MS_\mu(\Delta_0^1, \dots, \Delta_m^1)$ .*

**PROOF.** Denote the convex body  $\{0\} \times B_j \subset \mathbb{R} \oplus M$  by  $\tilde{B}_j$ , and denote the half-space  $\mathbb{R}_{\geq 0} \times M$  by  $H$ . Shifting the bodies  $\Delta_j^i$  and using part 2 of Lemma 2.10, we can assume without loss of generality, that  $\Delta_j^i \subset H$  for all  $i = 1, 2$  and  $j = 0, \dots, m$ . Under this assumption, the shadows  $[\Delta_j^i]$  of bodies  $\Delta_j^i$  (see Definition 2.5) are convex hulls  $\text{conv}(\Delta_j^i \cup \tilde{B}_j)$ . In particular, the shadows  $[\Delta_j^i]$  are convex bodies, and  $[\Delta_j^0] \subset [\Delta_j^1]$ . Since the mixed shadow volume equals the mixed volume of shadows, the inequality  $MS_\mu(\Delta_0^0, \dots, \Delta_m^0) \leq MS_\mu(\Delta_0^1, \dots, \Delta_m^1)$  follows from the monotonicity of the mixed volume of convex bodies.  $\square$

## 4 Remarks.

**MONOTONICITY OF MIXED FIBER BODIES.** Proof of Theorem 1.8 gives the following fact as a byproduct.

**THEOREM 4.1.** *In the notation of Theorem 1.2, consider convex bodies  $\Delta_0, \dots, \Delta_m, \Delta'_0, \dots, \Delta'_m$  in the space  $L \oplus M$ . If  $\Delta_i \subset \Delta'_i$  and  $v(\Delta_i) = v(\Delta'_i)$  for every  $i$ , where  $v$  is the projection  $L \oplus M \rightarrow M$ , then  $MF_\mu(\Delta_0, \dots, \Delta_m) \subset MF_\mu(\Delta'_0, \dots, \Delta'_m)$ .*

**REMARK.** If  $v(\Delta_i) \neq v(\Delta'_i)$ , then the statement is not true in general, but  $MF_\mu(\Delta_0, \dots, \Delta_m) \subset MF_\mu(\Delta'_0, \dots, \Delta'_m) + a$  for some  $a \in L$  (see [EKH]).



**BILLERA-STURMFELS VERSION OF MINKOWSKI INTEGRAL.** The original definition of the fiber integral is slightly different from Definition 1.1. Let  $p : N \rightarrow K$  be a projection of an  $n$ -dimensional real vector space to a  $k$ -dimensional one, and let  $\mu$  be a volume form on  $K$ .

**DEFINITION 4.2 ([BS]).** For a convex body  $\Delta \subset N$ , the set of all points of the form  $\int_{p(\Delta)} s\mu \in N$ , where  $s : p(\Delta) \rightarrow \Delta$  is a continuous section of the projection  $p$ , is called the *Minkowski integral* of  $\Delta$  and is denoted by  $\int^{BS} \Delta \mu$ .

Definitions 1.1 and 4.2 are related as follows. If, combining notation from these definitions,  $N = L \oplus M$  and  $p$  is the projection  $L \oplus M \rightarrow M$ , then the convex body  $\int^{BS} \Delta \mu$  is contained in a fiber of  $p$ , and  $\int \Delta \mu$  is the image of  $\int^{BS} \Delta \mu$  under the projection  $L \oplus M \rightarrow L$ .

One can reduce Definition 4.2 to Definition 1.1 as well. This time, combining notation from these definitions, suppose that  $L = N$ ,  $M = K$ , and the body  $\Delta^{diag}$  consists of points  $(a, p(a)) \in L \oplus M$ , where  $a$  runs over all points of a convex body  $\Delta \subset N$ . Then  $\int^{BS} \Delta \mu = \int \Delta^{diag} \mu$ . In the same way, one can denote  $\text{MF}_\mu(\Delta_0^{diag}, \dots, \Delta_m^{diag})$  by  $\text{MF}_\mu^{BS}(\Delta_0, \dots, \Delta_m)$  and reformulate Theorem 1.2 for the Billera-Sturmfels version of mixed fiber bodies.

**THEOREM 4.3.** *There exists a unique symmetric Minkowski-multilinear map  $\text{MF}_\mu^{BS} : \underbrace{\mathcal{C}(N) \times \dots \times \mathcal{C}(N)}_{k+1} \rightarrow \mathcal{C}(N)$ , such that  $\text{MF}_\mu^{BS}(\Delta, \dots, \Delta) = \int^{BS} \Delta \mu$  for every convex body  $\Delta \subset N$ .*

**MIXED FIBER POYTOPES AND ELIMINATION THEORY.** Consider Laurent polynomials  $f_0, \dots, f_k$  on the complex torus  $(\mathbb{C} \setminus \{0\})^k \oplus (\mathbb{C} \setminus \{0\})^{n-k}$ , such that the set  $\{f_0 = \dots = f_k = 0\}$  has codimension  $k + 1$ , and denote the projection of this set to  $(\mathbb{C} \setminus \{0\})^{n-k}$  by  $\Sigma$ .

**DEFINITION 4.4.** If  $\text{codim } \Sigma = 1$ , then the *eliminant* of the polynomials  $f_0, \dots, f_k$  is defined as the polynomial  $g^d$  on  $(\mathbb{C} \setminus \{0\})^{n-k}$ , where  $g$  is the equation of the closure of  $\Sigma$ , and  $d$  is the topological degree of the projection  $\{f_0 = \dots = f_k = 0\} \rightarrow \Sigma$ . Otherwise the eliminant of  $f_0, \dots, f_k$  is set to 1 by definition.

**REMARK.** We could change some details of this definition to make it more natural (for example, we could take the multiplicities of components

of the complete intersection  $f_0 = \dots = f_k = 0$  into account, see [EKh]). However, the assertion of the following theorem does not depend on these details.

**THEOREM 4.5 ([EKh]).** *If polytopes  $N_0, \dots, N_k$  in  $\mathbb{Z}^k \oplus \mathbb{Z}^{n-k}$  are the Newton polytopes of Laurent polynomials  $f_0, \dots, f_k$  on the complex torus  $(\mathbb{C} \setminus \{0\})^k \oplus (\mathbb{C} \setminus \{0\})^{n-k}$ , and the coefficients of  $f_0, \dots, f_k$  are in general position, then the eliminant of  $f_0, \dots, f_k$  is well defined, and its Newton polytope equals up to a shift  $(k+1)! \cdot \text{MF}_\mu(N_0, \dots, N_k)$ , where  $\mu$  is the standard volume form on  $\mathbb{Z}^k$ .*

**NEWTON POLYTOPES OF RESULTANTS.** Since the resultant is a special case of the eliminant, its Newton polytope is a special case of the mixed fiber polytope, and one can use formula (\*) to compute the support function of the Newton polytope of the resultant in terms of mixed shadow volumes. To formulate this in detail, we introduce necessary notation and recall the definition of the multidimensional resultant.

For a finite set  $A \subset \mathbb{Z}^k$ , denote the space of complex Laurent polynomials of the form  $\sum_{a \in A} c_a x^a$  by  $\mathbb{C}[A]$  (we denote a monomial  $x_1^{a_1} \dots x_k^{a_k}$  by  $x^a$ , where  $x = (x_1, \dots, x_k) \in (\mathbb{C} \setminus \{0\})^k$  and  $a = (a_1, \dots, a_k) \in \mathbb{Z}^k$ ). Let  $\Sigma$  be the set of all collections  $(f_0, \dots, f_k) \in \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$  such that  $f_0(x) = \dots = f_k(x) = 0$  for some  $x \in (\mathbb{C} \setminus \{0\})^k$ . For a generic collection  $(f_0, \dots, f_k) \in \Sigma$ , let  $d$  be the number of points  $x \in (\mathbb{C} \setminus \{0\})^k$  such that  $f_0(x) = \dots = f_k(x) = 0$ .

**DEFINITION 4.6.** If  $\text{codim } \Sigma = 1$ , then the  $(A_0, \dots, A_k)$ -resultant  $R_{A_0, \dots, A_k}$  is defined as  $R^d$ , where  $R$  is the equation of the closure of the hypersurface  $\Sigma$ . Otherwise,  $R_{A_0, \dots, A_k} = 1$  by definition.

For a collection of polynomials  $(f_0, \dots, f_k) \in \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$ , denote the coefficient of the monomial  $x^a$  in the polynomial  $f_i$  by  $c_{a,i}(f_0, \dots, f_k)$ . The functions  $c_{a,i} : \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k] \rightarrow \mathbb{C}$  form a coordinate system on  $\mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$ . The resultant  $R_{A_0, \dots, A_k}$  is a polynomial of variables  $c_{a,i}$ . Its Newton polytope  $N$  is contained in the lattice  $\Lambda$  of monomials  $\prod c_{a,i}^{\lambda_{a,i}}$ . Covectors  $\lambda_{a,i}$  form a basis of the dual lattice  $\Lambda^*$ . For a covector  $\gamma = \sum \gamma_{a,i} \lambda_{a,i} \in \Lambda^*$  with coordinates  $\gamma_{a,i}$  in this basis, denote the the convex hull  $\text{conv}\{(\gamma_{a,i}, a) \mid a \in A_i\} \subset \mathbb{R} \oplus \mathbb{R}^k$  by  $A_{i,\gamma}$ .

**THEOREM 4.7.** *For every covector  $\gamma \in \Lambda^*$ ,*

$$N(\gamma) = (k+1)! \text{MS}_\mu(A_{0,\gamma}, \dots, A_{k,\gamma}),$$

where  $N(\cdot)$  is the support function of the Newton polytope  $N$  of the  $(A_0, \dots, A_k)$ -resultant,  $\mu$  is the unit volume form on  $\mathbb{R}^k$ , and the polytopes  $A_{i,\gamma}$  are as defined above.

PROOF. Consider the expression  $\sum_{a \in A_i} c_{a,i} x^a$  as a polynomial  $g_i$  of variables  $x_1, \dots, x_k$  and  $c_{a,i}$ , where  $a$  runs over  $A_i$ . Then the resultant is the eliminant of the polynomials  $g_0, \dots, g_k$  (see Definition 4.4), and its Newton polytope  $N$  equals  $(k+1)!$  times the mixed fiber polytope of the Newton polytopes of  $g_0, \dots, g_k$  by Theorem 4.5. Applying formula (\*) to compute the value of the support function  $N(\gamma)$ , note that the  $i$ -th argument of the left hand side of this formula equals  $A_{i,\gamma}$  in this special case.  $\square$

MIXED VOLUMES OF PAIRS. Mixed shadow volume is a special case of the following relative version of the classical mixed volume.

DEFINITION 4.8. Polyhedra  $\Delta_1$  and  $\Delta_2$  in  $\mathbb{R}^n$  are said to be *parallel* if  $a + \Delta_1 \subseteq \Delta_1 \Leftrightarrow a + \Delta_2 \subseteq \Delta_2$  for every point  $a \in \mathbb{R}^n$ .

DEFINITION 4.9. ([E05], [E06]) 1) A pair of polyhedra  $\Delta_1, \Delta_2$  in  $\mathbb{R}^n$  is called *bounded*, if both  $\Delta_1 \setminus \Delta_2$  and  $\Delta_2 \setminus \Delta_1$  are bounded. The set of all bounded pairs of polyhedra parallel to a given convex cone  $C \subset \mathbb{R}^n$  is denoted by  $\text{BP}_C$ .

2) *The Minkovsky sum*  $(\Delta_1, \Delta_2) + (\Gamma_1, \Gamma_2)$  of two pairs from  $\text{BP}_C$  is the pair  $(\Delta_1 + \Gamma_1, \Delta_2 + \Gamma_2) \in \text{BP}_C$ .

3) *The volume*  $\text{Vol}(\Delta_1, \Delta_2)$  of a bounded pair  $(\Delta_1, \Delta_2)$  is the difference of volumes  $\text{Vol}(\Delta_1 \setminus \Delta_2) - \text{Vol}(\Delta_2 \setminus \Delta_1)$ .

4) *The mixed volume* is a symmetric Minkowski-multilinear function  $\text{MV}_C : \underbrace{\text{BP}_C \times \dots \times \text{BP}_C}_n \rightarrow \mathbb{R}$  such that  $\text{MV}_C(A, \dots, A) = \text{Vol}(A)$  for each pair  $A \in \text{BP}_C$ .

See [E06] for the existence and other basic properties of the mixed volume of pairs. Let  $\mu$  be the unit volume form on  $\mathbb{R}^k$ , let  $p$  be the projection  $\mathbb{R} \oplus \mathbb{R}^k \rightarrow \{0\} \times \mathbb{R}^k$ , and let  $l_-$  be the ray  $\{(t, 0, \dots, 0) \mid t \leq 0\} \subset \mathbb{R} \oplus \mathbb{R}^k$ . For a convex body  $\Delta \subset \mathbb{R} \oplus \mathbb{R}^k$ , denote the pair  $(\Delta + l_-, p(\Delta) + l_-) \in \text{BP}_{l_-}$  by  $\tilde{\Delta}$ .

THEOREM 4.10.  $\text{MS}_\mu(\Delta_0, \dots, \Delta_k) = \text{MV}_{l_-}(\tilde{\Delta}_0, \dots, \tilde{\Delta}_k)$  for every collection of convex bodies  $\Delta_0, \dots, \Delta_k$  in  $\mathbb{R} \oplus \mathbb{R}^k$ .

PROOF. This equality follows from definitions if  $\Delta_0 = \dots = \Delta_k$ . The general statement follows from this special case by uniqueness of the mixed shadow volume.  $\square$

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